A LIMIT LAW OF ALMOST l-PARTITE GRAPHS

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Abstract. For integers $l \geq 1$, $d \geq 0$ we study (undirected) graphs with vertices $1, \ldots, n$ such that the vertices can be partitioned into l parts such that every vertex has at most d neighbours in its own part. The set of all such graphs is denoted $\mathbf{P}_n(l, d)$. We prove a labelled first-order limit law, i.e., for every first-order sentence φ, the proportion of graphs in $\mathbf{P}_n(l, d)$ that satisfy φ converges as $n \to \infty$. By combining this result with a result of Hundack, Prömel and Steger [12] we also prove that if $1 \leq s_1 \leq \ldots \leq s_l$ are integers, then $\mathbf{Forb}(\mathcal{K}_{1,s_1,\ldots,s_l})$ has a labelled first-order limit law, where $\mathbf{Forb}(\mathcal{K}_{1,s_1,\ldots,s_l})$ denotes the set of all graphs with vertices $1, \ldots, n$, for some n, in which there is no subgraph isomorphic to the complete $(l+1)$-partite graph with parts of sizes $1, s_1, \ldots, s_l$. In the course of doing this we also prove that there exists a first-order formula ξ, depending only on l and d, such that the proportion of $\mathcal{G} \in \mathbf{P}_n(l, d)$ with the following property approaches 1 as $n \to \infty$: there is a unique partition of $\{1, \ldots, n\}$ into l parts such that every vertex has at most d neighbours in its own part, and this partition, viewed as an equivalence relation, is defined by ξ.

Keywords: finite model theory, limit law, random graph, forbidden subgraph.

1. Introduction

Over the last four decades a large number of logical limit laws and zero-one laws, as well as some non-convergence results, have been proved, for various collections of finite structures, various probability measures and various logics. One of the main directions of research has considered random graphs with vertex set $[n] = \{1, \ldots, n\}$ such that, for some $0 < \alpha < 1$, an edge appears between two vertices with probability $n^{-\alpha}$, independently of other edges. See [18, 19] for this line of research. This article deals with the following context. For a first-order language L and every positive integer n, let $\mathbf{K}_n$ be a set of L-structures with universe $[n]$, so we are dealing with 'labelled' structures. Give all members of $\mathbf{K}_n$ the same probability $1/|\mathbf{K}_n|$, so the probability that a random member of $\mathbf{K}_n$ belongs to $\mathbf{X} \subseteq \mathbf{K}_n$ equals the proportion $|\mathbf{X}|/|\mathbf{K}_n|$. We say that $\mathbf{K} = \bigcup_{n \in \mathbb{N}^+} \mathbf{K}_n$ has a *limit law* if for every L-sentence φ, the proportion of $\mathcal{G} \in \mathbf{K}_n$ in which φ is true converges as n tends to infinity. If the limit is always 0 or 1 then we say that $\mathbf{K}$ has a *zero-one law*. Such a result was first proved by Glebskii, Kogan, Liogonkii, and Talanov [10] and independently by Fagin [9] in the case when $\mathbf{K}_n$ contains all L-structures with universe $[n]$ and L has finite relational vocabulary and every relation symbol has arity at least 2. Suppose that we keep the assumptions on the language, but restrict membership in $\mathbf{K}_n$ to L-structures with universe $[n]$ which satisfy some constraints. What can we say about limit laws in this case? In general, dividing lines for when a limit law holds, or not, are not known. But a number of results have been obtained for various $\mathbf{K}$. Compton [5] has proved that if $\mathbf{K}_n$ is the set of partial orders, then $\mathbf{K}$ satisfies a zero-one law. Compton [4] and others have also developed a theory of limit laws (with emphasis on 'unlabelled' structures) when $\mathbf{K}$ is, up to isomorphism, closed under forming disjoint unions and extracting connected components and the growth of $|\mathbf{K}_n|$ is slow as n grows. A book by Burris [3] treats this theory, based on number theory. Kolaitis, Prömel and Rothschild [13] have proved a zero-one law in the case when $\mathbf{K}_n$ is the set of $(l+1)$-clique free graphs ($l \geq 2$). In the process of doing this they proved that if $\mathbf{K}_n$ is the set of l-partite (or l-colourable) graphs, then $\mathbf{K}$ satisfies a zero-one law. This result was generalised by the author who proved that whenever the vocabulary of L is finite, relational and all relation symbols have arity at least 2, then, with $\mathbf{K}_n$ being the set of l-colourable L-structures, $\mathbf{K}$ has a zero-one law [14]. Lynch [16] has proved a limit law when (for every n) $\mathbf{K}_n$ consists of all graphs with a degree sequence that satisfies certain conditions; in particular his result implies that $\mathbf{K}$ has a limit law when it is the set of d-regular graphs (d fixed) with vertex set $[n]$ for some n. More results about limit laws when $\mathbf{K}_n$ is the set of $d(n)$-regular graphs and $d(n)$ is a growing function appear in work of Haber and Krivelevich [11]. In the case when $\mathbf{K}_n$ is the set of graphs with vertex set $[n]$ in which every vertex has degree at most d, a limit law also holds [15]. That will be used in this paper.

The author has two viewpoints on the present work. One is that it adds more examples of collections of structures for which a limit law holds. In particular, we get more examples of graphs $\mathcal{H}$ for which the set of $\mathcal{H}$-free graphs satisfy a limit law (but in general not a zero-one law). The only previously known example appears to be when $\mathcal{H}$ is an $(l+1)$-clique for $l \geq 2$ [13]. The addition of more concrete examples may be of help in attempts to understand dividing lines between $\mathbf{K}$ with a limit law and $\mathbf{K}$ without it.

Another viewpoint is that the work presented here seeks to develop methods for understanding limit laws in the case when members of $\mathbf{K}$ can be decomposed into simpler substructures (in some sense) and where the interaction between these substructures is known (at least in a probabilistic sense). In particular, we will use knowledge from [15] about the typical structure of graphs with maximum degree d when studying $\mathbf{K}_n = \mathbf{P}_n(l,d)$, the set of graphs with vertex set $[n]$ such that $[n]$ can be partitioned into l parts such that every vertex has at most d neighbours in its own part. This approach to understanding asymptotic properties is inspired by infinite model theory, where one often tries to understand structures in terms of simpler building blocks (strongly minimal sets, rank one sets, etc.) and how these blocks are "glued" together.

When proving a limit law for $\mathcal{H}$-free graphs where $\mathcal{H}$ is as in Theorem 1.3, below, we use a structure result for almost all $\mathcal{H}$-free graphs by Hundack, Prömel and Steger [12] (when $\mathcal{H}$ has a colour critical vertex, defined below). More structural results for other choices of $\mathcal{H}$ and almost all $\mathcal{H}$-free graphs have been proved by Balogh, Bollobás and Simonovits [1, 2]. These may be useful in further studies of limit laws.

We now describe the main results this paper. By 'graph' we always mean 'undirected graph'. For integers $l \geq 1$ and $d \geq 0$ let $\mathbf{P}_n(l,d)$ be the set of graphs with vertex set $[n] = \{1,\ldots,n\}$ such that $[n]$ can be partitioned into l parts such that every vertex has at most d neighbours in its own part. Let $\mathbf{P}(l,d) = \bigcup_{n\in\mathbb{N}^+} \mathbf{P}_n(l,d)$. Note that $\mathbf{P}_n(l,0)$ is the set of l-partite, or l-colourable, graphs with vertex set $[n]$, and that $\mathbf{P}_n(1,d)$ is the set of graphs with vertex set $[n]$ in which every vertex has degree at most d. For integers $1 \leq s_1 \leq s_2 \leq \ldots \leq s_l$ let $\mathcal{K}_{1,s_1,s_2,\ldots,s_l}$ denote the complete $(l+1)$-partite graph with parts of sizes $1, s_1, s_2, \ldots, s_l$. So if $s_1 = \ldots = s_l = 1$ then $\mathcal{K}_{1,s_1,s_2,\ldots,s_l}$ is an $(l+1)$-clique, i.e. a complete graph on $l+1$ vertices. For a graph $\mathcal{H}$ let $\mathbf{Forb}_n(\mathcal{H})$ be the set of graphs with vertex set $[n]$ which contain no subgraph that is isomorphic to $\mathcal{H}$, and let $\mathbf{Forb}(\mathcal{H}) = \bigcup_{n\in\mathbb{N}^+} \mathbf{Forb}_n(\mathcal{H})$. Note that $\mathbf{P}_n(2,0) \subseteq \mathbf{Forb}_n(\mathcal{K}_{1,1,1})$. In an article from 1976 [8], Erdös, Kleitman and Rothschild proved that the proportion of $\mathcal{G} \in \mathbf{Forb}_n(\mathcal{K}_{1,1,1})$ which are bipartite, i.e., belong to $\mathbf{P}_n(2,0)$, approaches 1 as $n \to \infty$. Later, Kolaitis, Prömel and Rothschild [13] generalised this by proving that, for every $l \geq 2$, if $s_1 = s_2 = \ldots = s_l = 1$, then $|\mathbf{P}_n(l,0)| \big/ |\mathbf{Forb}_n(\mathcal{K}_{1,s_1,\ldots,s_l})| \to 1$ as $n \to \infty$ and $\mathbf{P}(l,0)$ satisfies a zero-one law; hence also $\mathbf{Forb}_n(\mathcal{K}_{1,s_1,\ldots,s_l})$ satisfies a zero-one law if $s_1 = s_2 = \ldots = s_l = 1$.

We say that a vertex v of a graph $\mathcal{H}$ is *colour-critical* if one can obtain a graph with smaller chromatic number than $\mathcal{H}$ by removing some edges of $\mathcal{H}$ which contain v, and only such edges. The *criticality* of a colour-critical vertex v is the minimal number of edges which contain v that must be removed to produce a graph with smaller chromatic number. Prömel and Steger [17] and then Hundack, Prömel and Steger [12] have generalised the result of Kolaitis, Prömel and Rothschild that almost all $(l+1)$-clique-free graphs are l-partite to the following:

Theorem 1.1. [12] *Suppose that $\mathcal{H}$ is a graph with chromatic number $l+1$ and with a colour critical vertex v with criticality d and suppose that no other colour-critical vertex has smaller criticality than v. Then*

$$\frac{|\mathbf{Forb}_n(\mathcal{H}) \cap \mathbf{P}_n(l, d-1)|}{|\mathbf{Forb}_n(\mathcal{H})|} \to 1 \quad \text{as } n \to \infty.$$

The main result of this article is the following, where the 'language of graphs' refers to the first-order language built up from a vocabulary (also called signature) which consists only of a binary relation symbol, besides the identity symbol:

Theorem 1.2. *Suppose that $l \geq 1$ and $d \geq 0$ are integers. For every first-order sentence φ in the language of graphs, the proportion of $\mathcal{G} \in \mathbf{P}_n(l, d)$ in which φ is true converges as $n \to \infty$. If $d = 0$ or $d = 1$ then the this proportion always converges to either 0 or 1; if $d > 1$ then it may converge to some $0 < c < 1$.*

In the case $d = 0$ Theorem 1.2 states the same thing as one of the main results of [13] (described above). In the case $l = 1$ Theorem 1.2 states the same thing as the main result of [15]. Therefore we focus on the case when $d \geq 1$ and $l \geq 2$. Theorems 1.1 and 1.2 will be used to prove the following result, in the last section.

Theorem 1.3. *Suppose that $l \geq 2$, $1 \leq s_1 \leq s_2 \leq \ldots \leq s_l$ are integers.*
(i) For every sentence φ in the language of graphs, the proportion of $\mathcal{G} \in \mathbf{Forb}_n(\mathcal{K}_{1,s_1,\ldots,s_l})$ in which φ is true converges as $n \to \infty$.
(ii) If $s_1 \leq 2$ then this proportion converges to 0 or 1 for every sentence φ.
(iii) If $s_1 > 2$ then there are infinitely many mutually contradictory sentences φ_i, $i \in \mathbb{N}$, in the language of graphs such that the proportion of $\mathcal{G} \in \mathbf{Forb}_n(\mathcal{K}_{1,s_1,\ldots,s_l})$ in which φ_i is true approaches some α_i such that $0 < \alpha_i < 1$.

This article is organised as follows. Section 2 considers the possibly different ways in which the vertex set of $\mathcal{G} \in \mathbf{P}_n(l, d)$ can be partitioned into l parts such that every vertex has at most d neighbours in its own part. We show that there is $\mu > 0$, depending only on l, such that the proportion of $\mathcal{G} \in \mathbf{P}_n(l, d)$ with the following property approaches 1 as $n \to \infty$: for every partition $V_1, \ldots, V_l$ of the vertex set such that every vertex has at most d neighbours in its own part, $|V_i| \geq \mu n$ for all $i \in [l]$. In Section 3 we consider the following sort of question, the probability of an "extension property", where $\mathcal{H}_1$ is assumed to be an induced subgraph of $\mathcal{H}_2$: Given $\mathcal{G} \in \mathbf{P}_n(l, d)$, what is the probability that every induced subgraph of $\mathcal{G}$ that is isomorphic to $\mathcal{H}_1$ is contained in an induced subgraph of $\mathcal{G}$ which is isomorphic to $\mathcal{H}_2$? In Section 4 we use the results from sections 2 and 3 to prove that the proportion of $\mathcal{G} \in \mathbf{P}_n(l, d)$ with the following property approaches 1 as $n \to \infty$: there is exactly one way in which the vertex set can be partitioned into l (non-empty) parts such that every vertex has at most d neighbours in its own part. In Section 5 we use the results from Sections 3 and 4, the main results from [15] and an Ehrenfeucht-Fraïssé game argument to prove Theorem 1.2. In Section 6 we consider "forbidden subgraphs" of the type $\mathcal{K}_{1,s_1,\ldots,s_l}$ and prove Theorem 1.3 with the help of Theorem 1.2.

Terminology and notation 1.4. See for example [7] for an introduction to first-order logic and first-order structures and [6] for basics about graph theory. By *graph* we mean undirected graph without loops. By the *first-order language of graphs* we mean the set of first-order formulas over a vocabulary (also called signature) with the identity symbol

'$=$' and a binary relation symbol 'E' (for the edge relation). When speaking of a formula or sentence we will always mean a formula, or sentence, in the language of graphs. We view graphs as first-order structures $\mathcal{G} = (V, E^{\mathcal{G}})$ for the language of graphs. Since we only consider undirected graphs without loops, the interpretation of E, $E^{\mathcal{G}}$, will always be symmetric and irreflexive, so we may, if convenient, view $E^{\mathcal{G}}$ as a set of 2-subsets of V. Let $\mathcal{G} = (V, E^{\mathcal{G}})$ be a graph. If $\varphi(x_1, \ldots, x_m)$ is a formula with free variables $x_1, \ldots, x_m$ and $v_1, \ldots, v_m \in V$, then the notation '$\mathcal{G} \models \varphi(v_1, \ldots, v_m)$' means that $v_1, \ldots, v_m$ satisfies the statement $\varphi(x_1, \ldots, x_m)$ in $\mathcal{G}$, and for a sentence ψ, $\mathcal{G} \models \psi$ means that ψ is satisfied by $\mathcal{G}$ (or in other words, that $\mathcal{G}$ has the property expressed by ψ). For $v, w \in V$, the notation $v \sim_{\mathcal{G}} w$ means that v and w are adjacent in $\mathcal{G}$; so $v \sim_{\mathcal{G}} w$ expresses the same thing as $\mathcal{G} \models E(v, w)$. We say that $\mathcal{H} = (W, E^{\mathcal{H}})$ is a *subgraph* of $\mathcal{G} = (V, E^{\mathcal{G}})$ if $W \subseteq V$ and $E^{\mathcal{H}} \subseteq E^{\mathcal{G}}$. If, in addition, for all $a \in W$, $a \sim_{\mathcal{H}} b$ if and only if $a \sim_{\mathcal{G}} b$, then we call $\mathcal{H}$ an *induced subgraph* of $\mathcal{G}$. Hence, $\mathcal{H}$ is an induced subgraph of $\mathcal{G}$ if and only if $\mathcal{H}$ is a substructure of $\mathcal{G}$ in the sense of model theory. For $X \subseteq V$, $\mathcal{G}[X]$ denotes the induced subgraph of $\mathcal{G}$ with vertex set X. In model theoretic terms, $\mathcal{G}[X]$ is the substructure of $\mathcal{G}$ with universe X. The distance between two vertices $v, w \in V$ in $\mathcal{G}$ is denoted $\mathrm{dist}_{\mathcal{G}}(v, w)$, and for sets of vertices A and B, $\mathrm{dist}_{\mathcal{G}}(A, B) = \min\{\mathrm{dist}_{\mathcal{G}}(v, w) : v \in A,\ w \in B\}$. If $\mathcal{G} = (V, E^{\mathcal{G}})$ and $\mathcal{H} = (W, E^{\mathcal{H}})$ are graphs and $f : V \to W$ is injective and has the property that, for all $a, b \in V$, $a \sim_{\mathcal{G}} b$ if and only if $f(a) \sim_{\mathcal{H}} f(b)$, then we call f a *strong embedding* of $\mathcal{G}$ into $\mathcal{H}$. We say that functions $f, g : \mathbb{N} \to \mathbb{R}$ are *asymptotic*, written $f \sim g$, if $f(n)/g(n) \to 1$ as $n \to \infty$.

2. Decompositions

Let $l \geq 1$ and $d \geq 0$ be integers. Let $\mathbf{P}_n(l, d)$ be the set of graphs with vertex set $[n] = \{1, \ldots, n\}$ such that $[n]$ can be partitioned into l parts in such a way that every vertex has at most d neighbours in its own part. In general, for $\mathcal{G} \in \mathbf{P}_n(l, d)$ there may be more than one partition of the vertex set into l parts such that every vertex has at most d neighbours in its own part. In this section we show that there is $\mu > 0$ depending only on l such that for almost all $\mathcal{G} \in \mathbf{P}_n(l, d)$ (for large enough n) every such partition $V_1, \ldots, V_l$ has the property that $|V_i| \geq \mu n$ for all $i = 1, \ldots, l$.

Definition 2.1. Let $\mathcal{G} = (V, E^{\mathcal{G}}) \in \mathbf{P}_n(l, d)$, so $V = [n]$. By the definition of $\mathbf{P}_n(l, d)$, there exists a partition $V_1, \ldots, V_l$ of V, which we denote by π, such that the following holds:

(i) $E^{\mathcal{G}} = E_1 \cup E_2$,
(ii) The graph (V, E_1) is l-colourable and the partition π defines an l-colouring of it.
(iii) $E_2 = E'_1 \cup \ldots \cup E'_l$ and for every $i \in [l]$, $E'_i \subseteq V_i^{(2)}$ and every vertex of the graph (V_i, E'_i) has degree $\leq d$.

A pair (E_1, E_2) such that (i)–(iii) hold is called a *decomposition of* $\mathcal{G}$ *based on* π, or *a* π*-based decomposition of* $\mathcal{G}$. A pair (E_1, E_2) is called a *decomposition of* $\mathcal{G}$ if, for some partition π of V into l parts, it is a π-based decomposition of $\mathcal{G}$.

Partitions of $V = [n]$ into l parts will be denoted by π, sometimes with an index. Note that if $\mathcal{G} = (V, E^{\mathcal{G}}) \in \mathbf{P}_n(l, 0)$ and (E_1, E_2) is a decomposition of $\mathcal{G}$ which is based on a partition π of V into l parts $V_1, \ldots, V_l$, then $E_2 = \emptyset$, $E_1 = E^{\mathcal{G}}$ and π induces an l-colouring of $\mathcal{G}$, in the sense that all elements in V_i can be assigned the colour i, for all $i \in [l]$. It is straightforward to verify the following:

Observation 2.2. Let $\mathcal{G} = (V, E^{\mathcal{G}}) \in \mathbf{P}_n(l, d)$.
(a) If (E_1, E_2) is a decomposition of $\mathcal{G}$, then E_1 and E_2 are disjoint.
(b) By the definition of $\mathbf{P}_n(l, d)$, in the beginning of the section, $\mathcal{G}$ has a decomposition based on some partition of $V = [n]$ into l parts.

(c) For every partition π of V into l parts, there is at most one decomposition of $\mathcal{G}$ which is based on π.
(d) In general, it is possible that there are different partitions of V into l parts, say π_1 and π_2, a decomposition of $\mathcal{G}$ based on π_1 and another decomposition of $\mathcal{G}$ based on π_2.

Part (d) of the observation might look discouraging because, in general, there is not a unique way to present a graph $\mathcal{G} \in \mathbf{P}(l,d)$ by its decomposition. However, the next couple of lemmas together with the results in Sections 3, 4 show that, as $n \to \infty$, the proportion of graphs $\mathcal{G} \in \mathbf{P}_n(l,d)$ which have a unique decomposition approaches 1.

Definition 2.3. (i) Let $\alpha \in \mathbb{R}$. An l-colouring $f : [n] \to [l]$ of a graph $\mathcal{G} \in \mathbf{P}_n(l,0)$ is called *α-rich* if $|f^{-1}(i)| \geq \alpha$ for every $i \in [l]$; that is, for every colour i, at least α vertices are assigned the colour i by f.
(ii) Similarly as in (i), a partition of $[n]$ into l parts is called *α-rich* if each one of the l parts contains at least α elements.
(iii) Let π denote any partition of $[n]$ into l parts. By $\mathbf{P}_{n,\pi}(l,d)$ we denote the set of all $\mathcal{G} \in \mathbf{P}_n(l,d)$ which have a π-based decomposition.

Theorem 10.5 in [14] has the following as an immediate consequence:

Fact 2.4. [14] *For every $l \geq 2$ and every sufficiently small $\mu > 0$, there is $\lambda > 0$ such that for all sufficiently large n,*

$$\frac{\big|\{\mathcal{G} \in \mathbf{P}_n(l,0) : \ \mathcal{G} \text{ has an } l\text{-colouring which is not } \mu n\text{-rich}\}\big|}{|\mathbf{P}_n(l,0)|} \ \leq \ 2^{-\lambda n^2 \pm O(n)}. \tag{1}$$

An analysis of the proof of Theorem 10.5 (ii) in [14] shows that if $0 < \mu < \frac{1}{2l(l-1)}$, then there is $\lambda > 0$ such that the conclusion of Fact 2.4 holds. By applying Fact 2.4 we get the following:

Corollary 2.5. *Let $l \geq 1$ and $d \geq 0$ be integers. If $\mu > 0$ is sufficiently small and $\widehat{\mathbf{P}}_n(l,d)$ denotes the set of all $\mathcal{G} \in \mathbf{P}_n(l,d)$ which have a decomposition which is based on a partition that is* not *μn-rich, then there is $\lambda > 0$ (depending on μ) such that for all sufficiently large n,*

$$\frac{\big|\widehat{\mathbf{P}}_n(l,d)\big|}{\big|\mathbf{P}_n(l,d)\big|} \ \leq \ 2^{-\lambda n^2 + O(n \log n)} \ \to \ 0 \quad \text{as } n \to \infty.$$

Proof. If $l = 1$ then for every $0 < \mu \leq 1$ we have $\widehat{\mathbf{P}}_n(l,d) = \emptyset$ for all n, so the conclusion of the lemma follows trivially. Now suppose that $l > 1$. By Fact 2.4, we can choose $\mu > 0$ small enough so that there exists $\lambda > 0$ such that (1) holds for all sufficiently large n. Recall that $\mathbf{P}_n(l,0)$ is the set of all l-colourable graphs with vertices $1, \ldots, n$. Also, observe that $\mathbf{P}_n(1,d)$ is the set of all graphs with vertices $1, \ldots, n$ such that every vertex has degree $\leq d$. Note that if $V = [n]$ and (E_1, E_2) is a decomposition of $\mathcal{G} \in \mathbf{P}_n(l,d)$, then $(V, E_1) \in \mathbf{P}_n(l,0)$ and $(V, E_2) \in \mathbf{P}_n(1,d)$. Observe that $\mathcal{G} \in \mathbf{P}_n(l,0)$ has an l-colouring which is *not* μn-rich if and only if $\mathcal{G}$ has a decomposition based on a partition which is *not* μn-rich. Since every $\mathcal{G} \in \widehat{\mathbf{P}}_n(l,d)$ has a decomposition which is based on a partition of $V = [n]$ into l parts which is *not* μn-rich, it follows that

$$\big|\widehat{\mathbf{P}}_n(l,d)\big| \ \leq \ \big|\widehat{\mathbf{P}}_n(l,0)\big| \cdot \big|\mathbf{P}_n(1,d)\big|, \tag{2}$$

where $\widehat{\mathbf{P}}_n(l,0)$ is the set of $\mathcal{G} \in \mathbf{P}_n(l,0)$ that have an l-colouring which is not μn-rich. Next, we estimate an upper bound of $|\mathbf{P}_n(1,d)|$. For all sufficiently large n, each vertex of a graph in $\mathbf{P}_n(1,d)$ can be connected to the other vertices in at most $\sum_{i=0}^{d-1} \binom{n}{i} \leq dn^{d-1}$ ways. Therefore,

$$\big|\mathbf{P}_n(1,d)\big| \ \leq \ \Big(dn^{d-1}\Big)^n \ \leq \ 2^{(d-1)n \log n \ + \ O(n)}. \tag{3}$$

Hence, for all sufficiently large n,

$$\begin{aligned}
\frac{\big|\widehat{\mathbf{P}}_n(l,d)\big|}{\big|\mathbf{P}_n(l,d)\big|} &\leq \frac{\big|\widehat{\mathbf{P}}_n(l,0)\big| \cdot \big|\mathbf{P}_n(1,d)\big|}{\big|\mathbf{P}_n(l,d)\big|} && \text{by (2)} \\
&\leq \frac{\big|\widehat{\mathbf{P}}_n(l,0)\big| \cdot \big|\mathbf{P}_n(1,d)\big|}{\big|\mathbf{P}_n(l,0)\big|} && \text{because } \mathbf{P}_n(l,0) \subseteq \mathbf{P}_n(l,d) \\
&\leq 2^{-\lambda n^2 \, \pm \, O(n)} \cdot \big|\mathbf{P}_n(1,d)\big| && \text{by (1)} \\
&\leq 2^{-\lambda n^2 \, + \, (d-1)n\log n \, + \, O(n)} && \text{by (3)} \\
&= 2^{-\lambda n^2 \, + \, O(n \log n)}. \quad \square
\end{aligned}$$

3. Extension properties

Fix an integer $d \geq 0$. In this section we prove some technical lemmas about extension properties which will be used in Sections 4 and 5.

Assumption 3.1. (until Definition 3.5) Let $\mathcal{H} = (X_1 \cup X_2 \cup Y, E^{\mathcal{H}})$ be a graph where X_1, X_2 and Y are mutually disjoint and $v \not\sim_{\mathcal{H}} w$ whenever $v \in X_1$ and $w \in Y$.

Lemma 3.2. *For $i = 1, 2$, let $\mathcal{K}_i = (W_i, E^{\mathcal{K}_i})$ be graphs such that $W_1 \cup W_2 = V = [n]$, $\mathcal{K}_1$ has maximum degree at most d and W_1 has at least $n^{1/4}$ subsets Z_i, $i = 1, \ldots, n^{1/4}$, such that $\mathcal{K}_1[Z_i] \cong \mathcal{H}[Y]$. Let $\mathbf{P}(\mathcal{K}_1, \mathcal{K}_2)$ be the set of graphs $\mathcal{G} = (V, E^{\mathcal{G}})$ such that $\mathcal{G}[W_i] = \mathcal{K}_i$ for $i = 1, 2$. Then the proportion of $\mathcal{G} \in \mathbf{P}(\mathcal{K}_1, \mathcal{K}_2)$ such that every strong embedding $h_0 : \mathcal{H}[X_1 \cup X_2] \to \mathcal{G}$ satisfying $h_0(X_i) \subseteq W_i$, for $i = 1, 2$, can be extended to a strong embedding $h : \mathcal{H} \to \mathcal{G}$ satisfying $h(Y) \subseteq W_1$, is at least*

$$1 - n^{|X_1| + |X_2|} \, \alpha^{\beta n^{1/4}},$$

where $0 < \alpha, \beta < 1$ are constants that depend only on $|X_1|, |X_2|, |Y|$ and d.

Proof. Assume that $h_0 : X_1 \cup X_2 \to V = [n]$ is injective and $h_0(X_i) \subseteq W_i$ for $i = 1, 2$. Let $\mathbf{P}_{h_0}$ be the set of $\mathcal{G} \in \mathbf{P}(\mathcal{K}_1, \mathcal{K}_2)$ such that h_0 is a strong embedding of $\mathcal{H}[X_1 \cup X_2]$ into $\mathcal{G}$. Then:

Claim. The proportion of $\mathcal{G} \in \mathbf{P}_{h_0}$ such that h_0 cannot be extended to a strong embedding $h : \mathcal{H} \to \mathcal{G}$ such that $f(Y) \subseteq W_1$ is at most $(1 - 2^{-p})^{\beta n^{1/4}}$ where $p \geq 1$ and $0 < \beta < 1$ depend only on $|X_1|, |X_2|, |Y|$ and d.

Proof of the claim. By one of the assumptions of the lemma, for every $i = 1, \ldots, n^{1/4}$, h_0 can be extended to a function $h_i : X_1 \cup X_2 \cup Y \to V$ such that $h_i{\restriction}Y$ is an isomorphism from $\mathcal{H}[Y]$ onto $\mathcal{K}_1[Z_i]$. Since the maximum degree of $\mathcal{K}_1$ is at most d, there are different $i_1, \ldots, i_{\beta n^{1/4}} \in \{1, \ldots, n^{1/4}\}$, where $0 < \beta < 1$ depends only on $|X_1| + |X_2| + |Y|$ and d, such that $\mathrm{dist}_{\mathcal{K}_1}(h_0(X_1 \cup X_2), Z_{i_j}) > 1$ for all j and $\mathrm{dist}_{\mathcal{K}_1}(Z_{i_j}, Z_{i_{j'}}) > 1$ whenever $j \neq j'$. By the assumptions about $\mathcal{H}$ (Assumption 3.1), $h_{i_j}{\restriction}X_1 \cup Y$ is a strong embedding of $\mathcal{H}[X_1 \cup Y]$ into $\mathcal{K}_1$, for every $j = 1, \ldots \beta n^{1/4}$. From the definitions of $\mathbf{P}(\mathcal{K}_1, \mathcal{K}_2)$ and $\mathbf{P}_{h_0}$ it follows that for every $\mathcal{G} \in \mathbf{P}_{h_0}$ and every $j = 1, \ldots, \beta n^{1/4}$, $h_{i_j}{\restriction}X_1 \cup Y$ is a strong embedding of $\mathcal{H}[X_1 \cup Y]$ into $\mathcal{G}$.

Recall that there are no restrictions on the existence (or nonexistence) of edges going between W_1 and W_2; in other words we can see each such edge as existing with probability $1/2$ independently of the other edges of the graph. Therefore, if all $\mathcal{G} \in \mathbf{P}_{h_0}$ have the same probability and such $\mathcal{G}$ is chosen at random, then the probability that h_{i_j} is a strong embedding of $\mathcal{H}$ into $\mathcal{G}$ is 2^{-p} where $p = |X_2| \cdot |Y|$; and this holds independently of whether $h_{i_{j'}}$ is a strong embedding of $\mathcal{H}$ into $\mathcal{G}$ for $j' \neq j$. It follows that the proportion of $\mathcal{G} \in \mathbf{P}_{h_0}$ such that, for every $j = 1, \ldots, \beta n^{1/4}$, h_{i_j} is not a strong embedding of $\mathcal{H}$ into $\mathcal{G}$ is $\left(1 - 2^{-p}\right)^{\beta n^{1/4}}$. □

There are not more than $n^{|X_1|+|X_2|}$ injective functions from $X_1 \cup X_2$ into V. By the claim, for every such function, say h_0, the proportion of $\mathcal{G} \in \mathbf{P}_{h_0}$ such that h_0 cannot be extended to a strong embedding h of $\mathcal{H}$ into $\mathcal{G}$ that satisfies $h(Y) \subseteq W_1$ is at most $\left(1 - 2^{-p}\right)^{\beta n^{1/4}}$. Therefore the proportion of $\mathcal{G} \in \mathbf{P}(\mathcal{K}_1, \mathcal{K}_2)$ such that there is a strong embedding $h_0 : \mathcal{H}[X_1 \cup X_2] \to \mathcal{G}$ with $h_0(X_i) \subseteq W_i$, for $i = 1, 2$, which cannot be extended to a strong embedding $h : \mathcal{H} \to \mathcal{G}$ with $h(Y) \subseteq W_1$ is at most $n^{|X_1|+|X_2|} \left(1 - 2^{-p}\right)^{\beta n^{1/4}}$.
□

Assumption 3.3. For the rest of this section we fix $\mu > 0$ small enough so that there exists $\lambda > 0$ such that (1) holds for all sufficiently large n and let π denote an arbitrary μn-rich partition of $V = [n]$ into parts $V_1, \ldots, V_l$.

Recall Definition 2.3 (iii) of $\mathbf{P}_{n,\pi}(l, d)$, for a partition π of $[n]$.

Lemma 3.4. *Let $p \in [l]$. Then there are constants $0 < \alpha, \beta < 1$, depending only on $|X_1|, |X_2|, |Y|$ and d, such that the proportion of $\mathcal{G} \in \mathbf{P}_{n,\pi}(l, d)$ that satisfies the following condition is at least $1 - n^{|X_1|+|X_2|}\ \alpha^{\beta n^{1/4}}$:*

- $(*)$ *If $\mathcal{G}[V_p]$ has at least $n^{1/4}$ different induced subgraphs which are isomorphic to $\mathcal{H}[Y]$ and $h_0 : \mathcal{H}[X_1 \cup X_2] \to \mathcal{G}$ is a strong embedding such that $h_0(X_1) \subseteq V_p$ and $h_0(X_2) \subseteq V \setminus V_p$, then h_0 can be extended to a strong embedding $h : \mathcal{H} \to \mathcal{G}$ such that $h(Y) \subseteq V_p$.*

Proof. We will reduce the proof to an application of Lemma 3.2. Let $W_1 = V_p$ and $W_2 = V \setminus V_p$. Suppose that $\mathcal{K}_1$ is a graph with vertex set W_1, maximum degree at most d and such that $\mathcal{K}_1$ has at least $n^{1/4}$ different induced subgraphs which are isomorphic to $\mathcal{H}[Y]$. Also suppose that $\mathcal{K}_2$ is a graph such that $\mathcal{K}_2 = \mathcal{G}[W_2]$ for some $\mathcal{G} \in \mathbf{P}_{n,\pi}(l, d)$. Then let $\mathbf{P}(\mathcal{K}_1, \mathcal{K}_2)$ be the set of graphs $\mathcal{G}$ such that $\mathcal{G}[W_1] = \mathcal{K}_1$ and $\mathcal{G}[W_2] = \mathcal{K}_2$. Finally let $\mathbf{Q}$ be the set of $\mathcal{G} \in \mathbf{P}_{n,\pi}(l, d)$ such that $\mathcal{G}[W_1]$ (where $W_1 = V_p$) has less than $n^{1/4}$ different induced subgraphs which are isomorphic to $\mathcal{H}[Y]$. Then we have

$$\mathbf{P}_{n,\pi}(l, R) \;=\; \mathbf{Q} \;\cup\; \bigcup_{\mathcal{K}_1, \mathcal{K}_2} \mathbf{P}(\mathcal{K}_1, \mathcal{K}_2),$$

where the union ranges over all pairs $(\mathcal{K}_1, \mathcal{K}_2)$ of graphs as described above. Moreover, if $(\mathcal{K}_1, \mathcal{K}_2) \neq (\mathcal{K}_1', \mathcal{K}_2')$, then $\mathbf{P}(\mathcal{K}_1, \mathcal{K}_2) \cap \mathbf{P}(\mathcal{K}_1', \mathcal{K}_2') = \emptyset$. We also have $\mathbf{Q} \cap \mathbf{P}(\mathcal{K}_1, \mathcal{K}_2) = \emptyset$ for all pairs $(\mathcal{K}_1, \mathcal{K}_2)$ as described. Hence we get

$$|\mathbf{P}_{n,\pi}(l, d)| = |\mathbf{Q}| + \sum_{\mathcal{K}_1, \mathcal{K}_2} |\mathbf{P}(\mathcal{K}_1, \mathcal{K}_2)|$$

where the sum ranges over all pairs $(\mathcal{K}_1, \mathcal{K}_2)$ of graphs as described above. Therefore it suffices to prove that there are constants $0 < \alpha, \beta < 1$ depending only on $|X_1|$, $|X_2|$, $|Y|$ and d such that

(a) The proportion of $\mathcal{G} \in \mathbf{Q}$ which satisfy $(*)$ is at least $1 - n^{|X_1|+|X_2|}\ \alpha^{\beta n^{1/4}}$.
(b) For every pair $(\mathcal{K}_1, \mathcal{K}_2)$ as described above, the proportion of $\mathcal{G} \in \mathbf{P}(\mathcal{K}_1, \mathcal{K}_2)$ which satisfy $(*)$ is at least $1 - n^{|X_1|+|X_2|}\ \alpha^{\beta n^{1/4}}$

But (a) is trivially true because every $\mathcal{G} \in \mathbf{Q}$ has less than $n^{1/4}$ different induced subgraphs which are isomorphic to $\mathcal{H}[Y]$; hence $(*)$ holds for every $\mathcal{G} \in \mathbf{Q}$. And (b) is obtained by an application of Lemma 3.2 to every pair $(\mathcal{K}_1, \mathcal{K}_2)$ as described above. □

Definition 3.5. (i) If Assumption 3.1 holds and $|X_1| + |X_2| + |Y| \leq k$, then, *for every* $p \in [l]$, we call the condition $(*)$ of Lemma 3.4 a *k-extension property with respect to π.* Note that for every $k \in \mathbb{N}$, there are only finitely many (non-equivalent) k-extension properties with respect to π.
(ii) We say that a graph $\mathcal{G} \in \mathbf{P}_{n,\pi}(l,d)$ *has the k-extension property* if it satisfies every k-extension property with respect to π.

Corollary 3.6. *For every $k \in \mathbb{N}$, there is $\varepsilon_k : \mathbb{N} \to \mathbb{R}$, depending only on k, such that* $\lim_{n\to\infty} \varepsilon_k(n) = 0$ *and the proportion of $\mathcal{G} \in \mathbf{P}_{n,\pi}(l,d)$ which have the k-extension property with respect to π is at least $1 - \varepsilon_k(n)$.*

Proof. There are only finitely many, say m, non-equivalent k-extension properties. For each of these (and large enough n), the proportion of $\mathcal{G} \in \mathbf{P}_{n,\pi}(l,d)$ which does *not* have it is, by Lemma 3.4, at most $n^{2k}\ \alpha^{\beta n^{1/4}}$ where $0 < \alpha, \beta < 1$ depend only on k and d, so $n^{2k}\ \alpha^{\beta n^{1/4}} \to 0$ as $n \to \infty$. Hence $\varepsilon_k(n)$ can be taken as the sum of m terms of the form $n^{2k}\ \alpha^{\beta n^{1/4}}$. □

The next lemma will be used in Section 5.

Lemma 3.7. *Suppose that $\mathcal{G} \in \mathbf{P}_{n,\pi}(l,d)$ and let $\mathcal{G}'$ be the graph that results from removing (from $\mathcal{G}$) or adding (to $\mathcal{G}$) at most s edges $\{v,w\}$ such that for some $i \neq j$, $v \in V_i$ and $w \in V_j$. If $\mathcal{G}$ has the $(k+2s)$-extension property with respect to π, then $\mathcal{G}'$ has the k-extension property with respect to π.*

Proof. Straightforward consequence of the definition of k-extension property. □

4. Unique decomposition: expressing a partition in the language of graphs

The goal in this section is to show that the proportion of $\mathcal{G} \in \mathbf{P}_n(l,d)$ which have a unique decomposition approaches 1 as $n \to \infty$. For $l = 1$ this is trivially true, so *we assume that $l \geq 2$ in this section.* As in Assumption 3.3, we fix a sufficiently small $\mu > 0$ such that there exists $\lambda > 0$ such that (1) holds for all sufficiently large n and we let π denote an arbitrary μn-rich partition π of $[n]$ into parts $V_1, \ldots, V_l$. Recall the definition of $\mathbf{P}_{n,\pi}(l,d)$ from Definition 2.3 (iii). First we show that there are $q \in \mathbb{N}$ and a first-order formula $\xi(x,y)$ in the language of graphs such that if $\mathcal{G} \in \mathbf{P}_{n,\pi}(l,d)$ has the q-extension property, then $\mathcal{G} \models \xi(v,w)$ if and only if v and w belong to the same part of the partition π. This result is then used to show that the proportion of $\mathcal{G} \in \mathbf{P}_n(l,d)$ which have a unique decomposition approaches 1 as $n \to \infty$ (Theorem 4.6).

Lemma 4.1. *Let $m_1, m_2 \in \mathbb{N}$. For all large enough n, if $\mathcal{G} = (V, E^{\mathcal{G}}) \in \mathbf{P}_{n,\pi}(l,d)$ has the (m_1+m_2)-extension property with respect to π, then $\mathcal{G}$ has the following property:*

> *Whenever $p \in [l]$, $X \subseteq V \setminus V_p$, $|X| \leq m_1$, then there are (at least) m_2 distinct vertices $v_1, \ldots, v_{m_2} \in V_p$ such that v_i is adjacent to every member in X, for $i = 1, \ldots, m_2$.*

Proof. Let $m_1, m_2 \in \mathbb{N}$. Let $k = m_1 + m_2$ and suppose that $\mathcal{G} = (V, E^{\mathcal{G}}) \in \mathbf{P}_{n,\pi}(l,d)$ has the k-extension property. Let $p \in [l]$ and let $X \subseteq V \setminus V_p$ satisfy $|X| \leq m_1$. Now we define a suitable $\mathcal{H} = (X_1 \cup X_2 \cup Y, E^{\mathcal{H}})$ satisfying Assumption 3.1, and then apply Lemma 3.4.

Let $X_1 = \emptyset$, $X_2 = X$ and $Y = \{a_1, \ldots, a_{m_2}\}$ where $a_1, \ldots, a_{m_2}$ are new vertices. Then let $X_1 \cup X_2 \cup Y$ be the vertex set of $\mathcal{H}$ and define the edge relation $E^{\mathcal{H}}$ as follows: $\mathcal{H}[X_2] = \mathcal{G}[X_2]$ (recall that $X_2 = X \subseteq V \setminus V_p$), $a_i \sim_{\mathcal{H}} w$ for every i and every $w \in X_2$, and $a_i \not\sim_{\mathcal{H}} a_j$ if $i \neq j$. Let h_0 denote the identity function on $X_2 = X_1 \cup X_2$ (recall that $X_1 = \emptyset$), so h_0 is a strong embedding of $\mathcal{H}[X_1 \cup X_2]$ into $\mathcal{G}$ such that $h_0(X_1) = \emptyset \subseteq V_p$ and $h_0(X_2) = X_2 = X \subseteq V \setminus V_p$. Moreover, as no vertex of $\mathcal{G}[V_p]$ has degree more than

d and Y is an independent set of cardinality m_2 it follows, for large enough n, that $\mathcal{G}$ has at least $n^{1/4}$ different induced subgraphs that are isomorphic to $\mathcal{H}[Y]$. As $\mathcal{G}$ has the k-extension property and $|X_1| + |X_2| + |Y| \leq m_1 + m_2 = k$, it follows that h_0 can be extended to a strong embedding $h : \mathcal{H} \to \mathcal{G}$ such that $h(Y) \subseteq V_p$. If $v_i = h(a_i)$ for $i = 1, \ldots, m_2$, then $v_i \sim_{\mathcal{G}} w$ for every i and every $w \in h(X_2) = X_2 = X$. □

We will use the following:

Observation 4.2. Let $v, w_1, \ldots, w_s$ be distinct vertices of a graph $\mathcal{G} \in \mathbf{P}_{n,\pi}(l, d)$. If all $w_1, \ldots, w_s$ are neighbours of v, then at least $s - d$ of the vertices $w_1, \ldots, w_s$ do *not* belong to the same π-part as v.

Definition 4.3. Let $m = (l + 1)d + 1$ and let $\xi(x, y)$ denote the formula

$$\exists z_1 \ldots z_{(l-1)m} \Bigg[\bigwedge_{1 \leq i < j \leq (l-1)m} z_i \neq z_j \;\; \wedge \bigwedge_{1 \leq i \leq (l-1)m} \Big(E(x, z_i) \wedge E(y, z_i) \Big) \\ \wedge \bigwedge_{2 \leq k \leq l-1} \; \bigwedge_{i \leq (k-1)m < j} E(z_i, z_j) \Bigg].$$

Lemma 4.4. *Let $m = (l+1)d+1$ (as in Definition 4.3) and $q = 2+lm$. If $\mathcal{G} \in \mathbf{P}_{n,\pi}(l, d)$ has the q-extension property with respect to π, then, for all vertices v and w of $\mathcal{G}$,*

$$v \text{ and } w \text{ belong to the same } \pi\text{-part} \iff \mathcal{G} \models \xi(v, w).$$

Proof. Suppose that $\mathcal{G} = (V, E^{\mathcal{G}}) \in \mathbf{P}_{n,\pi}(l, d)$ has the q-extension property with respect to π, where $q = 2 + lm$ and $m = (l + 1)d + 1$. Also assume that v and w are vertices of $\mathcal{G}$.

First suppose that v and w belong to the same π-part of $V = [n]$. For the sake of simplicity of notation, and without loss of generality, suppose that $v, w \in V_1$. In order to show that $\mathcal{G} \models \xi(v, w)$ we need to find distinct vertices $u_1, \ldots, u_{(l-1)m}$ such that

$$\mathcal{G} \models \bigwedge_{1 \leq i < j \leq (l-1)m} u_i \neq u_j \;\wedge \bigwedge_{1 \leq i \leq (l-1)m} \Big(E(v, u_i) \;\wedge\; E(w, u_i) \Big) \\ \wedge \bigwedge_{2 \leq k \leq l-1} \; \bigwedge_{i \leq (k-1)m < j} E(u_i, u_j). \tag{4}$$

This can be proved by showing, by induction, that for every $t = 1, \ldots, l - 1$, there are vertices $u_1, \ldots, u_{tm}$ such that

(5) for every $k = 1, \ldots, t$, we have $u_{(k-1)m+1}, \ldots, u_{km} \in V_{k+1}$, and

$$\mathcal{G} \models \bigwedge_{1 \leq i < j \leq tm} u_i \neq u_j \;\wedge \bigwedge_{1 \leq i \leq tm} \Big(E(v, u_i) \;\wedge\; E(w, u_i) \Big) \\ \wedge \bigwedge_{2 \leq k \leq t} \; \bigwedge_{1 \leq i \leq (k-1)m < j \leq tm} E(u_i, u_j). \tag{6}$$

In the base case $t = 1$, we need to find $u_1, \ldots, u_m$ such that (5) and (6) hold for $t = 1$. But the existence of such $u_1, \ldots, u_m \in V_2$ is guaranteed by Lemma 4.1 and the assumption that $\mathcal{G}$ has the q-extension property.

In the inductive step we assume that $t < l - 1$ and that there are $u_1, \ldots, u_{tm}$ such that (5) and (6) hold. Recall the assumption that $v, w \in V_1$. Again, Lemma 4.1 and the assumption that $\mathcal{G}$ has the q-extension property implies that there are $u_{tm+1}, \ldots, u_{(t+1)m} \in V_{t+2}$ such that (5) and (6) hold if t is replaced by $t + 1$.

It remains to prove that if $\mathcal{G} \models \xi(v, w)$, then v and w belong to the same π-part. So suppose that $\mathcal{G} \models \xi(v, w)$, which implies that there are distinct vertices $u_1, \ldots, u_{(l-1)m}$ such that (4) holds. For a contradiction, suppose that v and w do not belong to the same π-part. By Observation 4.2 and the choice of $m = (l+1)d + 1$, it follows that there are $i_1, \ldots, i_{l-1}$ such that, for every $k = 1, \ldots, l-1$ (recall that $l \geq 2$), $(k-1)m < i_k \leq km$ and u_{i_k} does not belong to the same π-part as any of $v, w, u_{i_1}, \ldots, u_{i_{k-1}}$. As v and w do not belong to the same π-part (by assumption) this contradicts that there are only l π-parts. □

Definition 4.5. Let $\mathcal{G} = (V, E^{\mathcal{G}}) \in \mathbf{P}(l, d)$.
(i) A relation $R \subseteq V^k$ is called *definable in* $\mathcal{G}$ by a formula $\varphi(x_1, \ldots, x_k)$ if for all $(v_1, \ldots, v_k) \in V^k$, $(v_1, \ldots, v_k) \in R \iff \mathcal{G} \models \varphi(v_1, \ldots, v_k)$.
(ii) A partition π of V is called *definable in* $\mathcal{G}$ by a first-order formula $\varphi(x_1, x_2)$ if the relation 'v and w belong to the same π-part' is definable by $\varphi(x_1, x_2)$.

Theorem 4.6. *For every $k \in \mathbb{N}$, the proportion of $\mathcal{G} \in \mathbf{P}_n(l, d)$ with the following properties approaches 1 as $n \to \infty$:*

(i) *$\mathcal{G}$ has a unique decomposition,*
(ii) *The formula $\xi(x, y)$ (from Definition 4.3) defines the partition on which the unique decomposition of $\mathcal{G}$ is based, and this partition is μn-rich.*
(iii) *$\mathcal{G}$ has the k-extension property with respect to the partition on which its unique decomposition is based.*

Proof. It suffices to prove the proposition for all sufficiently large k. So we assume that $k \geq 2 + lm$, where $m = (l+1)d + 1$ (which will allow us to use Lemma 4.4). For every μn-rich partition π of $[n]$ let $\mathbf{X}_{n,\pi}$ denote the set of $\mathcal{G} \in \mathbf{P}_{n,\pi}(l, d)$ which have the k-extension property with respect to π. By Corollary 3.6, there is $\varepsilon_k : \mathbb{N} \to \mathbb{R}$ such that $\lim_{n\to\infty} \varepsilon_k(n) = 0$ and

$$\begin{aligned} &\text{for every } n \text{ and } \mu n\text{-rich partition } \pi \text{ of } [n] \text{ into } l \text{ parts,} \\ &|\mathbf{X}_{n,\pi}| / |\mathbf{P}_{n,\pi}(l, d)| \geq 1 - \varepsilon_k(n). \end{aligned} \tag{7}$$

Claim. If π_1 and π_2 are different μn-rich partitions of $[n]$ into l parts, then
$\mathbf{X}_{n,\pi_1} \cap \mathbf{X}_{n,\pi_2} = \emptyset$.

Proof of Claim. Suppose that π_1 and π_2 are μn-rich partitions of $[n]$ into l parts and that $\mathcal{G} \in \mathbf{X}_{n,\pi_1} \cap \mathbf{X}_{n,\pi_2}$. Lemma 4.4 and the choice of k (being large enough) implies that, for all $v, w \in [n]$,

$$\begin{aligned} &v \text{ and } w \text{ belong to the same part with respect to } \pi_1 \\ \iff &\mathcal{G} \models \xi(v, w) \\ \iff &v \text{ and } w \text{ belong to the same part with respect to } \pi_2. \end{aligned}$$

Hence, $\pi_1 = \pi_2$. □

Let

$$\mathbf{X}_n = \bigcup_{\pi\ \mu n\text{-rich}} \mathbf{X}_{n,\pi},$$

where the union ranges over all μn-rich partitions π of $[n]$, and let $\mathbf{P}^*_n$ be the set of $\mathcal{G} \in \mathbf{P}_n(l, d)$ such that *every* decomposition of $\mathcal{G}$ is based on a partition of $[n]$ that is

μn-rich. By the claim and (7) we get

$$(8) \qquad |\mathbf{X}_n| = \sum_{\pi\ \mu n\text{-rich}} |\mathbf{X}_{n,\pi}| \geq \sum_{\pi\ \mu n\text{-rich}} (1-\varepsilon_k(n))|\mathbf{P}_{n,\pi}|$$
$$= (1-\varepsilon_k(n)) \sum_{\pi\ \mu n\text{-rich}} |\mathbf{P}_{n,\pi}| \geq (1-\varepsilon_k(n))|\mathbf{P}^*_n|,$$

where the sums range over all μn-rich partitions π of $[n]$ into l parts. By the choice of μ and Corollary 2.5,

$$(9) \qquad \lim_{n\to\infty} \frac{|\mathbf{P}^*_n|}{|\mathbf{P}_n(l,d)|} = 1.$$

We now get

$$1 \geq \frac{|\mathbf{X}_n|}{|\mathbf{P}_n(l,d)|} = \frac{|\mathbf{X}_n|}{|\mathbf{P}^*_n|} \cdot \frac{|\mathbf{P}^*_n|}{|\mathbf{P}_n(l,d)|} \overset{(8)}{\geq} (1-\varepsilon_k(n))\frac{|\mathbf{P}^*_n|}{|\mathbf{P}_n(l,d)|},$$

so by (9),

$$\lim_{n\to\infty} \frac{|\mathbf{X}_n|}{|\mathbf{P}_n(l,d)|} = 1,$$

and together with (9), this implies

$$\lim_{n\to\infty} \frac{|\mathbf{X}_n \cap \mathbf{P}^*_n|}{|\mathbf{P}_n(l,d)|} = 1.$$

Thus it suffices to prove that every $\mathcal{G} \in \mathbf{X}_n \cap \mathbf{P}^*_n$ satsifies (i)–(iii) of the proposition. So let $\mathcal{G} \in \mathbf{X}_n \cap \mathbf{P}^*_n$. As $\mathcal{G} \in \mathbf{P}^*_n$, every decomposition of $\mathcal{G}$ is based on a μn-rich partition of $[n]$ into l parts. Let π_1 and π_2 be two μn-rich partitions such that $\mathcal{G}$ has decompositions based on π_1 and on π_2. Then $\mathcal{G} \in \mathbf{X}_{n,\pi} \cap \mathbf{X}_{n,\pi}$ so by the claim, $\pi_1 = \pi_2$. Hence all decompositions of $\mathcal{G}$ are based on the same partition, and thus, by Observation 2.2 (c), $\mathcal{G}$ has a unique decomposition. As $\mathcal{G} \in \mathbf{P}^*_n$ this partition, say π, is μn-rich, and as $\mathcal{G} \in \mathbf{X}_{n,\pi}$, it follows from Lemma 4.4 and the choice of k that $\xi(x,y)$ defines π. □

5. A limit law

In this section we prove the main result of this article, Theorem 1.2, in a slightly different formulation compared with its statement in Section 1. In Section 6 we use it to get limit laws for $\mathcal{H}$-free graphs for certain types of $\mathcal{H}$.

Theorem 1.2. *Let $l \geq 1$ be an integer.*
(i) $\mathbf{P}(l,1)$ has a zero-one law.
(ii) For every $d \geq 2$, $\mathbf{P}(l,d)$ has a limit law, but not zero-one law.

We first prove part (ii) of Theorem 1.2 and then, in Section 5.2, sketch the much easier proof of part one.

5.1. Proof of part (ii) of Theorem 1.2.

Suppose that $d \geq 2$. As the case $l = 1$ is proved in [15] we also assume that $l \geq 2$. Let φ be an arbitrary first-order sentence in the language of graphs. We need to prove that the quotient

$$\frac{|\{\mathcal{G} \in \mathbf{P}_n(l,d) : \mathcal{G} \models \varphi\}|}{|\mathbf{P}_n(l,d)|}$$

converges as $n \to \infty$. Suppose that the quantifier rank of φ is at most k, where $k \geq 1$. (See for example [7] for the definition of quantifier rank.)

Definition 5.1. For graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ let $\mathcal{G}_1 \equiv_k \mathcal{G}_2$ mean that $\mathcal{G}_1$ and $\mathcal{G}_2$ satisfy exactly the same first-order sentences with quantifier rank at most k.

Note that '$\equiv_k$' is an equivalence relation on $\mathbf{P}(l,d) = \bigcup_{n\in\mathbb{N}^+} \mathbf{P}_n(l,d)$. As the language of graphs has a finite relational vocabulary it follows that $\equiv_k$ has only finitely many equivalence classes (e.g. [7]). Therefore, to prove that $\mathbf{P}(l,d)$ has a limit law, it suffices to prove that for every $\equiv_k$-class $\mathbf{E}$, the quotient

$$\frac{|\mathbf{E} \cap \mathbf{P}_n(l,d)|}{|\mathbf{P}_n(l,d)|}$$

converges as $n \to \infty$. This is done in Corollary 5.17, which finishes the proof of the first claim of part (ii) of Theorem 1.2. From the proof it is easy to deduce that a zero-one law does not hold, which is explained after the proof of Corollary 5.17.

Recall that $\mathbf{P}_n(1,d)$ (the case $l=1$) denotes the set of graphs with vertex set $[n]$ in which every vertex has degree at most d. We will use results from [15] about the asymptotic structure of graphs in $\mathbf{P}_n(1,d)$, stated as Theorems 5.2 and 5.8 below.

Theorem 5.2. [15] *Suppose that $d \geq 2$ and $s,t > 0$ are integers and that $0 < \varepsilon < d$. The proportion of $\mathcal{G} \in \mathbf{P}_n(1,d)$ with properties (1)–(6) below approaches 1 as $n \to \infty$. If $d \geq 3$ then the proportion of $\mathcal{G} \in \mathbf{P}_n(1,d)$ with properties (1)–(7) approaches 1 as $n \to \infty$.*

(1) *There is no vertex with degree degree less than $d-2$.*
(2) *There are between $\sqrt{(d-\varepsilon)n}$ and $\sqrt{(d+\varepsilon)n}$ vertices with degree $d-1$.*
(3) *If $p,q \leq s$ then there are no p-cycle and different q-cycle within distance at most t of each other.*
(4) *If $p \leq s$ then there are no vertex v with degree less than d and p-cycle within distance at most t of each other. In particular, no p-cycle contains a vertex of degree less than d.*
(5) *There do not exist distinct vertices v_1, v_2, v_3 all of which have degree at most $d-1$ such that for all distinct $i,j \in \{1,2,3\}$, $\mathrm{dist}_\mathcal{G}(v_i,v_j) \leq t$.*
(6) *There do not exist distinct vertices v and w such that* $\deg_\mathcal{G}(v) \leq d-1$, $\deg_\mathcal{G}(w) \leq d-2$ *and* $\mathrm{dist}_\mathcal{G}(v,w) \leq t$.
(7) *Every connected component has more than t vertices.*

Definition 5.3. For graphs $\mathcal{G}_i = (V_i, E^{\mathcal{G}_i})$, $i = 1,\ldots,m$, $\bigcup_{i=1}^m \mathcal{G}_i$ denotes the graph $\mathcal{G} = (V, E^\mathcal{G})$ where $V = \bigcup_{i=1}^m V_i$ and $E^\mathcal{G} = \bigcup_{i=1}^m E^{\mathcal{G}_i}$.

Remark 5.4. Let π denote the partition $V_1,\ldots,V_l$ of $V=[n]$. By definition of $\mathbf{P}_{n,\pi}(l,d)$ (Definition 2.3 (iii)) and Observation 2.2 (c), every $\mathcal{G} \in \mathbf{P}_{n,\pi}(l,d)$ can be constructed in a unique way by choosing $\mathcal{G}_1 \in \mathbf{P}_{n,\pi}(l,0)$ and then, for $i=1,\ldots,l$, choosing $\mathcal{G}'_i = (V_i, E^{\mathcal{G}'_i})$ in which every vertex has degree at most d (and every choice is independent of the previous choices) and letting

$$\mathcal{G} = \mathcal{G}_1 \cup \bigcup_{i=1}^l \mathcal{G}'_i.$$

Conversely, every graph that is constructed by this procedure belongs to $\mathbf{P}_{n,\pi}(l,d)$. Therefore,

$$|\mathbf{P}_{n,\pi}(l,d)| \;=\; |\mathbf{P}_{|V_1|}(1,d)| \cdots |\mathbf{P}_{|V_l|}(1,d)| \cdot |\mathbf{P}_{n,\pi}(l,0)|, \tag{10}$$

The set defined in the next definition contains the typical (for large enough n) graphs in $\mathbf{P}_n(l,d)$.

Definition 5.5. *Fix some $0 < \varepsilon < d$ for the rest of this section. Also fix some sufficiently small $\mu > 0$ such that there exists $\lambda > 0$ such that (1) of Section 1 holds for all sufficiently large n.* Let $\mathbf{P}_n^k(l,d)$ be the set of all $\mathcal{G} \in \mathbf{P}_n(l,d)$ such that

(i) $\mathcal{G}$ has a decomposition which is based on a μn-rich partition $V_1,\ldots,V_l$ of $[n]$ and this partition is defined by $\xi(x,y)$,

(ii) for $s = 5^k$, $t = 5^{k+1}$ and every $i \in [l]$ properties (1)–(6) of Theorem 5.2 hold for $\mathcal{G}[V_i]$, and
(iii) if $d \geq 3$ then, for $t = 5^{k+1}$, also property (7) of Theorem 5.2 holds for $\mathcal{G}[V_i]$ for every $i \in [l]$.

Let $\mathbf{P}^k(l,d) = \bigcup_{n=1}^{\infty} \mathbf{P}_n^k(l,d)$.

Lemma 5.6. $\lim_{n\to\infty} |\mathbf{P}_n^k(l,d)|/|\mathbf{P}_n(l,d)| = 1$.

Proof. By the choice of μ in Definition 5.5 and Theorem 4.6, the proportion of $\mathcal{G} \in \mathbf{P}_n(l,d)$ for which (i) of Definition 5.5 holds approaches 1 as $n \to \infty$. Now consider any μn-rich partition π of $V = [n]$ with parts $V_1, \ldots, V_l$, so $|V_i| \geq \mu n$ for all $i \in [l]$. From (10) and Theorem 5.2 it follows that the proportion of $\mathcal{G} \in \mathbf{P}_{n,\pi}(l,d)$ such that for every $i \in [l]$ properties (1)–(6) (and (7) if $d \geq 3$) of Theorem 5.2 hold for $\mathcal{G}[V_i]$ approaches 1 as $n \to \infty$. Since $|V_i| \geq \mu n$ for all $i \in [l]$, the rate of convergence depends only on l and d, as μ depends only on l. Therefore $|\mathbf{P}_n^k(l,d)|/|\mathbf{P}_n(l,d)| \to 1$ as $n \to \infty$. □

Now we define an equivalence relation on $\mathbf{P}(l,d)$ which distinguishes whether a graph $\mathcal{G}$ belongs to $\mathbf{P}^k(l,d)$ or not, and if the answer is 'yes' then it distinguishes the number of vertices of degree $d-2$, the number of i-cycles for $i \leq 5^k$ and the number of i-paths with endpoints of degree $d-1$ for $i \leq 5^k$ in $\mathcal{G}[V_j]$, for each part V_j of the partition on which the unique decomposition of $\mathcal{G}$ is based.

Definition 5.7. We define an equivalence relation '$\approx_k$' on $\mathbf{P}(l,d)$ as follows:
$\mathcal{G} \approx_k \mathcal{H}$ if and only if

- either $\mathcal{G}, \mathcal{H} \notin \mathbf{P}^k(l,d)$ or
- $\mathcal{G}, \mathcal{H} \in \mathbf{P}^k(l,d)$ and if $V_1, \ldots, V_l$ is the partition of the vertex set of $\mathcal{G}$ defined by $\xi(x,y)$ on which some decomposition of $\mathcal{G}$ is based and $W_1, \ldots, W_l$ is the partition of the vertex set of $\mathcal{H}$ defined by $\xi(x,y)$ on which some decomposition of $\mathcal{H}$ is based, then there is a permutation σ of $[l]$ such that, for every $i \in [l]$,
  (a) $\mathcal{G}[V_i]$ and $\mathcal{H}[W_{\sigma(i)}]$ have the same number of vertices with degree $d-2$,
  (b) for every $j = 3, \ldots, 5^k$, $\mathcal{G}[V_i]$ and $\mathcal{H}[W_{\sigma(i)}]$ have the same number of j-cycles, and
  (c) for every $j = 1, \ldots, 5^k$, $\mathcal{G}[V_i]$ and $\mathcal{H}[W_{\sigma(i)}]$ have the same number of j-paths with both endpoints of degree $d-1$.

The next result from [15] will be used to show that for every $\approx_k$-equivalence class $\mathbf{C}$, the quotient $|\mathbf{C} \cap \mathbf{P}_n(l,d)|/|\mathbf{P}_n(l,d)|$ converges when $n \to \infty$.

Theorem 5.8. [15] *Let $t \geq 3$ be an integer. There are positive $\lambda_3, \ldots, \lambda_t, \mu_1, \ldots, \mu_t \in \mathbb{Q}$ such that for all $q, r_3, \ldots, r_t$, $s_1, \ldots, s_t \in \mathbb{N}$ the proportion of $\mathcal{G} \in \mathbf{P}_n(1,d)$ such that*

(a) *$\mathcal{G}$ has exactly q vertices with degree $d-2$,*
(b) *for $i = 3, \ldots, t$, $\mathcal{G}$ has exactly r_i i-cycles, and*
(c) *for $i = 1, \ldots, t$, $\mathcal{G}$ has exactly s_i i-paths with both endpoints of degree $d-1$*

approaches

$$\frac{(d-1)^q \, e^{-(d-1)}}{q!} \left(\prod_{i=3}^{t} \frac{(\lambda_i)^{r_i} \, e^{-\lambda_i}}{r_i!} \right) \left(\prod_{i=1}^{t} \frac{(\mu_i)^{s_i} \, e^{-\mu_i}}{s_i!} \right) \quad \text{as} \quad n \to \infty.$$

In other words, Theorem 5.8 says that the random variables which, for a random $\mathcal{G} \in \mathbf{P}_n(1,d)$, count the number of vertices with degree $d-2$, the number of i-cycles for $3 \leq i \leq t$ and the number of i-paths with both endpoints of degree $d-1$ for $1 \leq i \leq t$ have independent Poisson distributions, asymptotically. Note that since the Poisson distribution is a probability distribution it follows that the sum of all numbers as in the conclusion of Theorem 5.8 when $q, r_3, \dots, r_t, s_1, \dots, s_t$ ranges over all natural numbers is 1.

Lemma 5.9. *For every equivalence class* $\mathbf{C}$ *of the relation '*$\approx_k$*' there is a constant* $0 \le c(\mathbf{C}) \le 1$ *such that*

$$\lim_{n\to\infty} \frac{|\mathbf{C} \cap \mathbf{P}_n(l,d)|}{|\mathbf{P}_n(l,d)|} \ = \ c(\mathbf{C}).$$

Moreover, $c(\mathbf{C}) = 0$ *if and only if* $\mathbf{C} = \mathbf{P}(l,d) \backslash \mathbf{P}^k(l,d)$*. If* $(\mathbf{C}_i : i \in \mathbb{N})$ *is an enumeration of all* $\approx_k$*-classes then* $\sum_{i=1}^{\infty} c(\mathbf{C}_i) = 1$.

Proof. If $\mathbf{C} = \mathbf{P}(l,d) \setminus \mathbf{P}^k(l,d)$ then the conclusion holds with $c = 0$, by Lemma 5.6. If $\mathbf{C} \neq \mathbf{P}(l,d) \setminus \mathbf{P}^k(l,d)$ then $\mathbf{C} \subseteq \mathbf{P}^k(l,d)$ and the conclusion holds because of (10), Lemma 5.6 and Theorem 5.8. □

Definition 5.10. If $\mathcal{G}$ is a graph in which every vertex has degree at most d, then let a *small Poisson object* of $\mathcal{G}$ denote any one of

(a) a vertex with degree $d-2$, or
(b) an i-cycle where $3 \le i \le 5^k$, or
(c) an i-path with both endpoints of degree $d-1$ where $1 \le i \le 5^k$.

Definition 5.11. Let $\mathcal{G} = (V, E^{\mathcal{G}})$, $A \subseteq V$ and $t \in \mathbb{N}$.
(i) $N_{\mathcal{G}}(A,t) = \{v \in V : \ \mathrm{dist}_{\mathcal{G}}(A,v) \le t\}$. Note that $N_{\mathcal{G}}(A,0) = A$.
(ii) $\mathcal{N}_{\mathcal{G}}(A,t) = \mathcal{G}\big[N_{\mathcal{G}}(A,t)\big]$. Note that $\mathcal{N}_{\mathcal{G}}(A,0) = \mathcal{G}[A]$.
(iii) If every vertex in $\mathcal{G}$ has degree at most d, then let $NP(\mathcal{G},t)$ be the set of vertices v of $\mathcal{G}$ such that the distance from v to a small Poisson object (of $\mathcal{G}$) is at most t.
(iv) If every vertex in $\mathcal{G}$ has degree at most d, then let $\mathcal{NP}(\mathcal{G},t) = \mathcal{G}[NP(\mathcal{G},t)]$.

Remark 5.12. Recall Definition 5.3 and observe that if $\mathcal{G}, \mathcal{H} \in \mathbf{P}^k(l,d)$, then $\mathcal{G} \approx_k \mathcal{H}$ if and only if there is an isomorphism

$$f : \bigcup_{i=1}^{l} \mathcal{NP}(\mathcal{G}[V_i], 5^k) \ \to \ \bigcup_{i=1}^{l} \mathcal{NP}(\mathcal{H}[W_i], 5^k)$$

where $V_1, \dots, V_l$ and $W_1, \dots, W_l$ denote the $\xi(x,y)$-classes of $\mathcal{G}$ and $\mathcal{H}$, respectively, and f preserves $\xi(x,y)$ in the sense that whenever v and w are in the domain of f, then $\mathcal{G} \models \xi(v,w)$ if and only if $\mathcal{H} \models \xi(f(v), f(w))$.

Note that if $\mathcal{G} \in \mathbf{P}^k(l,d)$ and the $V_1, \dots, V_l$ is the partition defined by $\xi(x,y)$, then the graph

$$\mathcal{G}\Big[\bigcup_{i=1}^{l} NP(\mathcal{G}[V_i], 5^k)\Big]$$

is the result of adding to $\bigcup_{i=1}^{l} \mathcal{NP}(\mathcal{G}[V_i], 5^k)$ all edges of $\mathcal{G}$ that connect $NP(\mathcal{G}[V_i], 5^k)$ and $NP(\mathcal{G}[V_j], 5^k)$ for all distinct $i, j \in [l]$.

Definition 5.13. We define an equivalence relation '$\approx_k^+$' on $\mathbf{P}(l,d)$ as follows:
$\mathcal{G} \approx_k^+ \mathcal{H}$ if and only if

- $\mathcal{G} \approx_k \mathcal{H}$ and
- if $\mathcal{G}, \mathcal{H} \in \mathbf{P}^k(l,d)$ then there is an isomorphism

$$f : \mathcal{G}\Big[\bigcup_{i=1}^{l} NP(\mathcal{G}[V_i], 5^k)\Big] \ \to \ \mathcal{H}\Big[\bigcup_{i=1}^{l} NP(\mathcal{H}[W_i], 5^k)\Big]$$

where $V_1, \dots, V_l$ and $W_1, \dots, W_l$ denote the $\xi(x, y)$-classes of $\mathcal{G}$ and $\mathcal{H}$, respectively, and f preserves $\xi(x, y)$ in the sense that whenever v and w are in the domain of f, then $\mathcal{G} \models \xi(v, w)$ if and only if $\mathcal{H} \models \xi(f(v), f(w))$.

Note that the equivalence relation $\approx_k^+$ refines $\approx_k$, that is, every $\approx_k^+$-class is included in some $\approx_k$-class. Also note that every $\approx_k$-class is divided into finitely many $\approx_k^+$-classes, and that there are an infinite (but countable) number of $\approx_k$-classes, and hence an infinite (but countable) number of $\approx_k^+$-classes.

Lemma 5.14. *Let* $\mathbf{C}$ *be an* $\approx_k$*-class such that* $\mathbf{C} \subseteq \mathbf{P}^k(l, d)$ *and let* $\mathbf{D}$ *be an* $\approx_k^+$*-class such that* $\mathbf{D} \subseteq \mathbf{C}$. *Then there is a constant* $c(\mathbf{D}, \mathbf{C}) > 0$, *depending only on* l, d, k *and* $\mathbf{D}$, *such that*

$$\lim_{n\to\infty} \frac{|\mathbf{D} \cap \mathbf{P}_n(l, d)|}{|\mathbf{C} \cap \mathbf{P}_n(l, d)|} \;=\; c(\mathbf{D}, \mathbf{C}).$$

In particular, if $\mathbf{D}_1, \dots, \mathbf{D}_m$ *enumerates all* $\approx_k^+$*-classes that are included in* $\mathbf{C}$, *then*

$$\sum_{i=1}^{m} c(\mathbf{D}_i, \mathbf{C}) = 1.$$

Proof. Suppose that $\mathbf{C}$ is an equivalence class of '$\approx_k$' such that $\mathbf{C} \subseteq \mathbf{P}^k(l, d)$ and let $\mathbf{D}$ be an equivalence class of '$\approx_k^+$' such that $\mathbf{D} \subseteq \mathbf{C}$. By Lemma 5.9, there is $c > 0$ such that $|\mathbf{C}\cap\mathbf{P}_n(l, d)|/|\mathbf{P}_n(l, d)| \to c$ as $n \to \infty$. For $\mathcal{G} \in \mathbf{D}\cap\mathbf{P}_n(l, d)$, let p be the number of ways in which edges can be added to $\bigcup_{i=1}^l \mathcal{NP}(\mathcal{G}[V_i], 5^k)$, where $V_1, \dots, V_l$ are the ξ-classes of $\mathcal{G}$, in such a way that the resulting graph is isomorphic, via an isomorphism preserving the partition $V_1, \dots, V_l$, to $\mathcal{G}\Big[\bigcup_{i=1}^l NP(\mathcal{G}[V_i], 5^k)\Big]$. Note that p depends only on $\mathbf{D}$ (and not on n or the particular graph $\mathcal{G}$ from $\mathbf{D}$). For $\mathcal{G} \in \mathbf{C} \cap \mathbf{P}_n(l, d)$ let q be the total number of ways in which edges can be added between $NP(\mathcal{G}[V_i], 5^k)$ and $NP(\mathcal{G}[V_j], 5^k)$, where the $V_1, \dots, V_l$ are the ξ-classes of $\mathcal{G}$, for all possible distinct $i, j \in [l]$. Also let $r = \Big|\bigcup_{i=1}^l NP(\mathcal{G}[V_i], 5^k)\Big|$. Then q and r depend only on $\mathbf{C}$. We show that

$$\lim_{n\to\infty} \frac{\big|\mathbf{D} \cap \mathbf{P}_n(l, d)\big|}{\big|\mathbf{C} \cap \mathbf{P}_n(l, d)\big|} \;=\; \frac{p}{q}. \tag{11}$$

For $\mathcal{G} \in \mathbf{C} \cap \mathbf{P}_n(l, d)$ with ξ-classes $V_1, \dots, V_l$ (which is a μn-rich partition of $[n]$), let $[\mathcal{G}]_n$ be the set of $\mathcal{H} \in \mathbf{C}\cap\mathbf{P}_n(l, d)$ such that $\mathcal{H}$ has a decomposition based on $V_1, \dots, V_l$,

$$\bigcup_{i=1}^{l} \mathcal{NP}(\mathcal{H}[V_i], 5^k) \;=\; \bigcup_{i=1}^{l} \mathcal{NP}(\mathcal{G}[V_i], 5^k)$$

and if at least one of v or w does *not* belong to $\bigcup_{i=1}^l NP(\mathcal{G}[V_i], 5^k)$, then $v \sim_{\mathcal{G}} w \iff v \sim_{\mathcal{H}} w$. Note that $\big|[\mathcal{G}]_n\big|$ has a finite bound depending only on $\mathbf{C}$. By Lemma 4.4, there is m such that whenever $\mathcal{G} \in \mathbf{P}_n(l, d)$ has a decomposition based on a μn-rich partition π and $\mathcal{G}$ has the m-extension property with respect to π, then $\xi(x, y)$ defines π. By Lemma 3.7, if $\mathcal{G} \in \mathbf{C} \cap \mathbf{P}_n(l, d)$ and at least one member of $[\mathcal{G}]_n$ has the $(m + 2\binom{r}{2})$-extension property with respect to the partition $V_1, \dots, V_l$ defined by $\xi(x, y)$, then all members of $[\mathcal{G}]_n$ have the m-extension property with respect to $V_1, \dots, V_l$. So if $\mathcal{G} \in \mathbf{C}\cap\mathbf{P}_n(l, d)$ and at least one member of $[\mathcal{G}]_n$ has the $(m+2\binom{r}{2})$-extension property with respect to the partition defined by $\xi(x, y)$, then the proportion of $\mathcal{H} \in [\mathcal{G}]_n$ which belong to $\mathbf{D}$ is exactly p/q. By Theorem 4.6 the proportion of $\mathcal{G} \in \mathbf{P}_n(l, d)$ in which $\xi(x, y)$ defines a partition (equivalence relation) and $\mathcal{G}$ has the $(m+2\binom{r}{2})$-extension property with respect to this partition approaches 1 as $n \to \infty$, Since $|\mathbf{C} \cap \mathbf{P}_n(l, d)|/|\mathbf{P}_n(l, d)| \to c > 0$ as $n \to \infty$, it follows that the proportion of $\mathcal{H} \in \mathbf{C}\cap\mathbf{P}_n(l, d)$ which have the $(m+2\binom{r}{2})$-extension property approaches 1 as $n \to \infty$. As mentioned above, there is a finite bound, say β, depending only on $\mathbf{C}$ such that for every $\mathcal{G} \in \mathbf{C} \cap \mathbf{P}_n(l,d)$, $\big|[\mathcal{G}]_n\big| \leq \beta$. It follows that the proportion of $\mathcal{H} \in \mathbf{C} \cap \mathbf{P}_n(l,d)$ which belong to $\mathbf{D}$ approaches p/q as $n \to \infty$, so (11) is proved. □

Corollary 5.15. *Let $\mathbf{D}$ be an $\approx_k^+$-class. Then there is a constant $c(\mathbf{D})$, depending only on l, d, k and $\mathbf{D}$, such that*

$$\lim_{n\to\infty} \frac{|\mathbf{D} \cap \mathbf{P}_n(l,d)|}{|\mathbf{P}_n(l,d)|} \ = \ c(\mathbf{D}).$$

Moreover, $c(\mathbf{D}) = 0$ if and only if $\mathbf{D} = \mathbf{P}(l,d) \setminus \mathbf{P}^k(l,d)$. If $(\mathbf{D}_i : i \in \mathbb{N})$ enumerates all $\approx_k^+$-classes, then $\sum_{n=0}^{\infty} c(\mathbf{D}_i) = 1$.

Proof. Immediate from Lemmas 5.9 and 5.14. □

Lemma 5.16. *Let $\mathbf{D}$ be any $\approx_k^+$-class such that*

$$\lim_{n\to\infty} \frac{|\mathbf{D} \cap \mathbf{P}_n(l,d)|}{|\mathbf{P}_n(l,d)|} \ = \ c(\mathbf{D}) > 0.$$

Then there is an $\equiv_k$-class $\mathbf{E}$ such that

$$\lim_{n\to\infty} \frac{|\mathbf{E} \cap \mathbf{D} \cap \mathbf{P}_n(l,d)|}{|\mathbf{D} \cap \mathbf{P}_n(l,d)|} \ = \ 1,$$

and consequently

$$\lim_{n\to\infty} \frac{|\mathbf{E} \cap \mathbf{D} \cap \mathbf{P}_n(l,d)|}{|\mathbf{P}_n(l,d)|} \ = \ c(\mathbf{D}).$$

Proof. Let $\mathbf{D}$ be any $\approx_k^+$-class such that $|\mathbf{D} \cap \mathbf{P}_n(l,d)|\big/|\mathbf{P}_n(l,d)| \to c(\mathbf{D}) > 0$ as $n \to \infty$. Then, by the definitions of $\approx_k^+$ and $\approx_k$, $\mathbf{D} \subseteq \mathbf{P}^k(l,d)$, for every $i \in [l]$ there are $q_i, r_{i,3}, \ldots, r_{i,5^k}, s_{i,1}, \ldots, s_{i,5^k} \in \mathbb{N}$ such that whenever $\mathcal{G}_1 = (V, E^{\mathcal{G}_1}) \in \mathbf{D}$ and $\mathcal{G}_2 = (W, E^{\mathcal{G}_2}) \in \mathbf{D}$, then the following hold:

($*$) The parts of the partitions of V and of W defined by ξ in $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively, can be ordered as $V_1, \ldots, V_l$ and as $W_1, \ldots, W_l$ in such a way that, for every $i \in [l]$,
  - (a) both $\mathcal{G}_1[V_i]$ and $\mathcal{G}_2[W_i]$ have exactly q_i vertices with degree $d-2$,
  - (b) for all $j = 3, \ldots, 5^k$, both $\mathcal{G}_1[V_i]$ and $\mathcal{G}_2[W_i]$ have exactly $r_{i,j}$ j-cycles,
  - (c) for all $j = 1, \ldots, 5^k$, both $\mathcal{G}_1[V_i]$ and $\mathcal{G}_2[W_i]$ have exactly $s_{i,j}$ j-paths with both endpoints of degree $d-1$, and
  - (d) there is an isomorphism
$$f_0 : \mathcal{G}_1\left[\bigcup_{i=1}^{l} NP(\mathcal{G}_1[V_i], 5^k)\right] \ \to \ \mathcal{G}_2\left[\bigcup_{i=1}^{l} NP(\mathcal{G}_2[W_i], 5^k)\right]$$
such that, for all $p \in [l]$, if $v \in V_p$ then $f_0(v) \in W_p$.

Clearly there is an integer m such that if $\mathcal{G} \in \mathbf{D}$ and $V_1, \ldots, V_l$ are the parts of the partition defined by ξ in $\mathcal{G}$, then $\big|NP(\mathcal{G}[V_i], 5^k)\big| < \frac{m}{kl}$ for all i. From the assumption that $c(\mathbf{D}) > 0$ and Theorem 4.6 it follows that the proportion of $\mathcal{G} \in \mathbf{D} \cap \mathbf{P}_n(l,d)$ such that $\mathcal{G}$ has the m-extension property (with respect to the partition, defined by ξ, on which its unique decomposition is based) approaches 1 as $n \to \infty$. Therefore, it suffices to prove that whenever both $\mathcal{G}_1, \mathcal{G}_2 \in \mathbf{D} \cap \mathbf{P}_n(l,d)$ have the m-extension property and n is large enough, then $\mathcal{G}_1 \equiv_k \mathcal{G}_2$.

So suppose that both $\mathcal{G}_1, \mathcal{G}_2 \in \mathbf{D} \cap \mathbf{P}_n(l,d)$ have the m-extension property. By ($*$), the parts of the partition of $V = [n]$ on which the unique decomposition of $\mathcal{G}_1$ is based can be ordered as $V_1, \ldots, V_l$ in such a way that (a)–(d) hold, and the parts of the partition of

$W = [n]$ on which the unique decomposition of $\mathcal{G}_2$ is based can be ordered as $W_1, \dots, W_l$ in such a way that (a)–(d) hold. In order to prove that $\mathcal{G}_1 \equiv_k \mathcal{G}_2$ it suffices to prove that Duplicator has a winning strategy for the Ehrenfeucht-Fraïssé game in k steps on $\mathcal{G}_1$ and $\mathcal{G}_2$. See for example [7] for definitions and results about Ehrenfeucht-Fraïssé games. We will use the following simplified notation instead of the one given in Definition 5.11. For $i = 1, 2$ and $j = 1, \dots, l$, if v is a vertex of $\mathcal{G}_i[V_j]$, then $N(v, 5^k) = N_{\mathcal{G}_i[V_j]}(v, 5^k)$. To prove that Duplicator has a winning strategy for the Ehrenfeucht-Fraïssé game on $\mathcal{G}_1$ and $\mathcal{G}_2$ it suffices to prove the following statement.

Claim. Suppose that $i < k$, $v_1, \dots, v_i \in V$, $w_1, \dots, w_i \in W$, $v_{i+1} \in V$ (or $w_{i+1} \in W$) and that

$$f_i : \mathcal{G}_1\left[\bigcup_{j=1}^{l} NP\big(\mathcal{G}_1[V_j], 5^{k-i}\big) \ \cup\ \bigcup_{j=1}^{i} N\big(v_j, 5^{k-i}\big)\right] \to \mathcal{G}_2\left[\bigcup_{j=1}^{l} NP\big(\mathcal{G}_2[W_j], 5^{k-i}\big) \ \cup\ \bigcup_{j=1}^{i} N\big(w_j, 5^{k-i}\big)\right]$$

is an isomorphism such that f_i extends f_0 from (d), $f_i(v_j) = w_j$ for every $j = 1, \dots, i$, and whenever v is in the domain of f_i and $p \in [l]$, then $v \in V_p$ implies $f(v) \in W_p$. Then there is $w_{i+1} \in W$ (or $v_{i+1} \in V$) such that all of the above hold for '$i + 1$' in place of 'i'.

By symmetry it is enough to consider the case when $v_{i+1} \in V$ is given. If

$$N\big(v_{i+1}, 5^{k-i-1}\big) \subseteq \bigcup_{j=1}^{l} NP\big(\mathcal{G}_1[V_j], 5^{k-i} - 1\big) \ \cup\ \bigcup_{j=1}^{i} N\big(v_j, 5^{k-i} - 1\big), \tag{12}$$

then let $w_{i+1} = f(v_{i+1})$ and let f_{i+1} be the restriction of f_i to

$$\bigcup_{j=1}^{l} NP\big(\mathcal{G}_1[V_j], 5^{k-i-1}\big) \ \cup\ \bigcup_{j=1}^{i} N\big(v_j, 5^{k-i-1}\big).$$

Now suppose that (12) does *not* hold, so in particular $N\big(v_{i+1}, 5^{k-i-1}\big)$ is not included in $\bigcup_{j=1}^{i} N\big(v_j, 5^{k-i} - 1\big)$. Let $p \in [l]$ be such that $v_{i+1} \in V_p$. Then there is $u \in N\big(v_{i+1}, 5^{k-i-1}\big)$ such that, for every $j \le i$, either $v_j \notin V_p$ or the distance, in $\mathcal{G}_1[V_p]$, from u to v_j is at least 5^{k-i}. From this it follows that, for every $a \in N\big(v_{i+1}, 5^{k-i-1}\big)$ and every $j \le i$, either $v_j \notin V_p$ or the distance, in $\mathcal{G}_1[V_p]$, from a to v_j is at least

$$5^{k-i} - \mathrm{dist}_{\mathcal{G}_1[V_p]}(a, u) \ \ge\ 5^{k-i} - 2 \cdot 5^{k-i-1} \ge 3 \cdot 5^{k-i-1}.$$

Consequently, for every $a \in N\big(v_{i+1}, 5^{k-i-1}\big)$, every $j \le i$ such that $v_j \in V_p$ and every $b \in N\big(v_j, 5^{k-i-1}\big)$,

$$\mathrm{dist}_{\mathcal{G}_1[V_p]}(a, b) \ \ge\ 3 \cdot 5^{k-i-1} - 5^{k-i-1} \ =\ 2 \cdot 5^{k-i-1} \ \ge\ 2,$$

because $i < k$. It follows that

(13) if $j \le i$ and $v_j \in V_p$, then the distance, in $\mathcal{G}_1[V_p]$, between $N\big(v_{i+1}, 5^{k-i-1}\big)$ and $N\big(v_j, 5^{k-i-1}\big)$ is at least 2,

so, in particular, the two sets are disjoint. In the same way, by considering any vertex in a small Poisson object (Definition 5.10) instead of v_j for $j \le i$, it follows that

(14) the distance between $N\big(v_{i+1}, 5^{k-i-1}\big)$ and $NP\big(\mathcal{G}_1[V_p], 5^{k-i-1}\big)$ is at least 2.

Now we are ready to use the assumption that $\mathcal{G}_1$ and $\mathcal{G}_2$ have the m-extension property. We start by defining an appropriate graph $\mathcal{H}$ as in Assumption 3.1. Let

$$\begin{aligned}
X_1 &= NP\big(\mathcal{G}_1[V_p], 5^{k-i-1}\big) \cup \bigcup\big\{N\big(v_j, 5^{k-i-1}\big) : j \leq i \text{ and } v_j \in V_p\big\}, \\
X_2 &= \bigcup\big\{NP\big(\mathcal{G}_1[V_j], 5^{k-i-1}\big) : j \neq p\big\} \cup \bigcup\big\{N\big(v_j, 5^{k-i-1}\big) : j \leq i \text{ and } v_j \notin V_p\big\}, \\
Y &= N\big(v_{i+1}, 5^{k-i-1}\big).
\end{aligned}$$

Then let $\mathcal{H} = \mathcal{G}_1[X_1 \cup X_2 \cup Y]$. By (13) and (14), X_1, X_2 and Y are mutually disjoint. By assumption, $f_i \restriction X_1 \cup X_2$ is a strong embedding of $\mathcal{H}[X_1 \cup X_2]$ into $\mathcal{G}_2$ such that $f(X_1) \subseteq W_p$ and $f(X_2) \subseteq W \setminus W_p$. From (14) and the assumption that $\mathcal{G}_1 \in \mathbf{D} \subseteq \mathbf{P}^k(l,d)$ it follows that the subgraph of $\mathcal{G}_1[V_p]$ induced by $Y = N(v_{i+1}, 5^{k-i-1})$ is characterised as follows:

Case $d = 2$: A path with at least 5^{k-i-1} vertices in which the distance from v_{i+1} to at least one of the endpoints is 5^{k-i-1}.

Case $d \geq 3$: A tree in which every path from v_{i+1} to a leaf has length 5^{k-i-1} and either every non-leaf has degree d, or exactly one non-leaf has degree $d-1$ and all other non-leaves have degree d.

Since $\mathcal{G}_2 \in \mathbf{D} \subseteq \mathbf{P}^k$ it follows that there are between $\sqrt{(d-\varepsilon)n}$ and $\sqrt{(d+\varepsilon)n}$ vertices in $\mathcal{G}_2[V_p]$ with degree $d-1$ (in $\mathcal{G}_2[V_p]$). Moreover, because of the properties of f_i and f_0, for any two distinct vertices $w, w' \in W_p \setminus f_i(X_1 \cup X_2)$ with degree $d-1$ in $\mathcal{G}_2[W_p]$ the distance between them is at least $5^{k+1}+1$. Hence, for every $w \in W_p \setminus f_i(X_1 \cup X_2)$, the subgraph of $\mathcal{G}_2[W_p]$ induced by $N(w, 5^{k-i-1})$ is characterised in the same way as in the case of $Y = N(v_{i+1}, 5^{k-i-1})$, as explained above. Therefore, there are (assuming $|W|$ is large enough) at least $n^{1/4}$ different induced subgraphs of $\mathcal{G}_2[W_p]$ which are isomorphic to $\mathcal{H}[Y]$. By the choice of m we have $|X_1 \cup X_2 \cup Y| \leq m$ and since $\mathcal{G}_2$ has the m-extension property there is a strong embedding f_{i+1} of $\mathcal{H}$ into $\mathcal{G}_2$ which extends f_i and such that $f(Y) \subseteq W_p$. If we let $w_{i+1} = f_{i+1}(v_{i+1})$ then $f_{i+1}(v_j) = w_j$ for every $j = 1, \ldots, i+1$, and whenever v is in the domain of f_{i+1} and $j \in [l]$, then $v \in V_j$ implies $f_{i+1}(v) \in W_j$. $\square$

Let $(\mathbf{D}_i : i \in \mathbb{N})$ be an enumeration of all $\approx_k^+$-classes and let, by Corollary 5.15, $0 \leq d_i < 1$ be the constant

$$d_i = \lim_{n \to \infty} \frac{|\mathbf{D}_i \cap \mathbf{P}_n(l,d)|}{|\mathbf{P}_n(l,d)|}.$$

By the same corollary we have

$$\sum_{i=0}^{\infty} d_i = 1. \tag{15}$$

Also note that since, for every n, the equivalence relation $\approx_k^+$ partitions the (finite) set $\mathbf{P}_n(l,d)$, it follows that

$$\text{for every } n \text{ and every } \mathbf{A} \subseteq \mathbf{P}_n(l,d), \quad \frac{|\mathbf{A} \cap \mathbf{P}_n(l,d)|}{|\mathbf{P}_n(l,d)|} = \sum_{i=0}^{\infty} \frac{|\mathbf{A} \cap \mathbf{D}_i \cap \mathbf{P}_n(l,d)|}{|\mathbf{P}_n(l,d)|}. \tag{16}$$

Corollary 5.17. *Let $\mathbf{E}$ be any $\equiv_k$-class.*
(i) For every $i \in \mathbb{N}$,

$$\lim_{n \to \infty} \frac{|\mathbf{E} \cap \mathbf{D}_i \cap \mathbf{P}_n(l,d)|}{|\mathbf{P}_n(l,d)|} \quad \textit{equals either } 0 \textit{ or } d_i.$$

(ii) Let I be the set of $i \in \mathbb{N}$ such that the above limit equals d_i. Then

$$\lim_{n \to \infty} \frac{|\mathbf{E} \cap \mathbf{P}_n(l,d)|}{|\mathbf{P}_n(l,d)|} = \sum_{i \in I} d_i.$$

Proof. Part (i) is a consequence, using the notation 'd_i', of Lemma 5.16, so we turn to the proof of (ii). We use the abbreviation $\mathbf{P}_n = \mathbf{P}_n(l,d)$. Let $\varepsilon > 0$. We show that for I as defined above and large enough n,

$$\left| \frac{|\mathbf{E} \cap \mathbf{P}_n|}{|\mathbf{P}_n|} - \sum_{i \in I} d_i \right| \leq 5\varepsilon.$$

By (15) we can choose m large enough that

$$\sum_{i=m+1}^{\infty} d_i \leq \varepsilon \quad \text{and} \quad 1 - \varepsilon \leq \sum_{i=0}^{m} d_i \leq 1. \tag{17}$$

Then, using the definition of d_i, $i \in \mathbb{N}$, and part (i), choose n_0 so that

$$\text{for all } i \leq m \text{ such that } i \notin I \text{ and all } n > n_0, \ \frac{|\mathbf{E} \cap \mathbf{D}_i \cap \mathbf{P}_n|}{|\mathbf{P}_n|} \leq \frac{\varepsilon}{m+1}, \tag{18}$$

$$\text{for all } i \leq m \text{ such that } i \in I \text{ and all } n > n_0, \ \left| \frac{|\mathbf{E} \cap \mathbf{D}_i \cap \mathbf{P}_n|}{|\mathbf{P}_n|} - d_i \right| \leq \frac{\varepsilon}{m+1}, \tag{19}$$

and

$$\text{for all } i \leq m \text{ and all } n > n_0, \ \left| \frac{|\mathbf{D}_i \cap \mathbf{P}_n|}{|\mathbf{P}_n|} - d_i \right| \leq \frac{\varepsilon}{m+1}, \tag{20}$$

For all $n > n_0$ we get, by the use of (16)–(20),

$$\begin{aligned}
\left| \frac{|\mathbf{E} \cap \mathbf{P}_n|}{|\mathbf{P}_n|} - \sum_{i \in I} d_i \right| &= \left| \sum_{i=0}^{\infty} \frac{|\mathbf{E} \cap \mathbf{D}_i \cap \mathbf{P}_n|}{|\mathbf{P}_n|} - \sum_{i \in I} d_i \right| \\
&\leq \sum_{i \leq m, i \notin I} \frac{|\mathbf{E} \cap \mathbf{D}_i \cap \mathbf{P}_n|}{|\mathbf{P}_n|} + \sum_{i \leq m, i \in I} \left| \frac{|\mathbf{E} \cap \mathbf{D}_i \cap \mathbf{P}_n|}{|\mathbf{P}_n|} - d_i \right| \\
&\quad + \left| \sum_{i=m+1}^{\infty} \frac{|\mathbf{E} \cap \mathbf{D}_i \cap \mathbf{P}_n|}{|\mathbf{P}_n|} \right| + \left| \sum_{i=m+1}^{\infty} d_i \right| \\
&\leq 3\varepsilon + \left| \sum_{i=m+1}^{\infty} \frac{|\mathbf{D}_i \cap \mathbf{P}_n|}{|\mathbf{P}_n|} \right| = 3\varepsilon + \left| \sum_{i=0}^{\infty} \frac{|\mathbf{D}_i \cap \mathbf{P}_n|}{|\mathbf{P}_n|} - \sum_{i \leq m} \frac{|\mathbf{D}_i \cap \mathbf{P}_n|}{|\mathbf{P}_n|} \right| \\
&= 3\varepsilon + \left| 1 - \sum_{i \leq m} \frac{|\mathbf{D}_i \cap \mathbf{P}_n|}{|\mathbf{P}_n|} \right| \leq 3\varepsilon + \left| \varepsilon + \sum_{i \leq m} d_i - \sum_{i \leq m} \frac{|\mathbf{D}_i \cap \mathbf{P}_n|}{|\mathbf{P}_n|} \right| \\
&\leq 4\varepsilon + \sum_{i \leq m} \left| d_i - \frac{|\mathbf{D}_i \cap \mathbf{P}_n|}{|\mathbf{P}_n|} \right| \leq 5\varepsilon. \qquad \square
\end{aligned}$$

Corollary 5.17 concludes the proof of the limit law of $\mathbf{P}(l,d)$. The second claim of part (ii) of Theorem 1.2, that $\mathbf{P}(l,d)$ does not have a zero-one law, follows from the fact given by Lemma 5.9 that (for any $k \geq 1$) one can choose infinitely many $\approx_k$-classes $\mathbf{C}_i$ and sentences φ_i, $i \in \mathbb{N}$, such that $|\mathbf{C}_i \cap \mathbf{P}_n(l,d)| / |\mathbf{P}_n(l,d)|$ converges to a positive number, and φ_i is true in every $\mathcal{G} \in \mathbf{C}_i$ and false in every $\mathcal{G} \in \mathbf{C}_j$ if $j \neq i$. For example, let $\mathbf{C}_i$ be an $\approx_k$-class such that for every $\mathcal{G} \in \mathbf{C}_i$ and every part V_j of the partition defined by $\xi(x,y)$, $\mathcal{G}[V_j]$ has exactly i vertices with degree $d-2$, and let φ_i express this.

5.2. **Proof of part (i) of Theorem 1.2.** We now sketch the proof of part (i) of Theorem 1.2. By Lemma 2.11 in [15], for every $\varepsilon > 0$, the proportion of $\mathcal{G} \in \mathbf{P}_n(1, 1)$ such that there are between $n^{1/2-\varepsilon}$ and $n^{1/2+\varepsilon}$ vertices with degree 0, approaches 1 as $n \to \infty$. This information about $\mathbf{P}(1, 1)$ is sufficient for proving a zero-one law for $\mathbf{P}(l, 1)$.

Fix $0 < \varepsilon < 1/4$. By similar reasoning as when proving Lemma 5.6 it follows that, for every $k \in \mathbb{N}$, the proportion $\mathcal{G} \in \mathbf{P}_n(l, 1)$ with the following properties approaches 1 as $n \to \infty$:

(a) $\mathcal{G}$ has a unique decomposition which is based on a μn-rich partition $V_1, \ldots, V_l$ of $[n]$ which is defined by $\xi(x, y)$ (from Definition 4.3),
(b) $\mathcal{G}$ has the $2k$-extension property with respect to the partition defined by $\xi(x, y)$, and
(c) for every $i \in [l]$, $\mathcal{G}[V_i]$ has between $n^{1/2-\varepsilon}$ and $n^{1/2+\varepsilon}$ vertices degree 0.

Hence, it suffices to prove, for an arbitrary integer $k > 0$, that if $\mathcal{G}_1, \mathcal{G}_2 \in \mathbf{P}_n(l, 1)$ satisfy (a), (b) and (c), then $\mathcal{G}_1 \equiv_k \mathcal{G}_2$. This is done in a similar way as we proved, in the proof of Lemma 5.16, that if $\mathcal{G}_1, \mathcal{G}_2 \in \mathbf{D}$ have the m-extension property, for suitably chosen m, then $\mathcal{G}_1 \equiv_k \mathcal{G}_2$.

6. Forbidden subgraphs

Recall that for integers $1 \leq s_1 \leq \ldots \leq s_l$, $\mathcal{K}_{1,s_1,\ldots,s_l}$ denotes the complete l-partite graph with parts (or colour classes) of sizes $1, s_1, \ldots, s_l$. For any graph $\mathcal{H}$, $\mathbf{Forb}_n(\mathcal{H})$ denotes the set of graphs with vertices $1, \ldots, n$ in which there is no subgraph isomorphic to $\mathcal{H}$ and $\mathbf{Forb}(\mathcal{H}) = \bigcup_{n \in \mathbb{N}^+} \mathbf{Forb}_n(\mathcal{H})$. In this section we use Theorems 1.1 and 1.2 together with some new technical results to prove:

Theorem 1.3. *Suppose that $l \geq 2$, $1 \leq s_1 \leq s_2 \leq \ldots \leq s_l$ are integers.*
(i) For every sentence φ in the language of graphs, the proportion of $\mathcal{G} \in \mathbf{Forb}_n(\mathcal{K}_{1,s_1,\ldots,s_l})$ in which φ is true converges as $n \to \infty$.
(ii) If $s_1 \leq 2$ then this proportion converges to 0 or 1 for every sentence φ.
(iii) If $s_1 > 2$ then there are infinitely many mutually contradictory sentences φ_i, $i \in \mathbb{N}$, in the language of graphs such that the proportion of $\mathcal{G} \in \mathbf{Forb}_n(\mathcal{K}_{1,s_1,\ldots,s_l})$ in which φ_i is true approaches some α_i such that $0 < \alpha_i < 1$.

Part (i) of Theorem 1.3 is an immediate consequence of Lemmas 6.1 and 6.2 below. Part (ii) and (iii) of Theorem 1.3 require some more argumentation, which is given after the proof of Lemma 6.2.

Lemma 6.1. *Suppose that $l \geq 1$ and $1 \leq s_1 \leq \ldots \leq s_l$ are integers. If $V_1, \ldots, V_l$ is a partition of the vertex set of $\mathcal{K}_{1,s_1,\ldots,s_l}$ such that, for every $i = 1, \ldots, l$ and every $v \in V_i$, v has at most $s_1 - 1$ neighbours in V_i, then, for some i, V_i contains a 3-cycle.*

Proof. Let $C_0, \ldots, C_l$ be the "colour classes" of $\mathcal{K}_{1,s_1,\ldots,s_l}$, in other words, $\bigcup_{i=0}^{l} C_i$ is the vertex set of $\mathcal{K}_{1,s_1,\ldots,s_l}$ and vertices v and w are adjacent to each other if and only if there are $i \neq j$ such that $v \in V_i$ and $w \in V_j$. Moreover, assume that $|C_0| = 1$ and $|C_i| = s_i$ for $i = 1, \ldots, l$. We use induction on l. It is clear that $\mathcal{K}_{1,s_1}$ cannot be "partitioned" into one class such that every vertex has at most $s_1 - 1$ neighbours in its own part. This takes care of the base case $l = 1$.

Now assume that $l > 1$ and that $V_1, \ldots, V_l$ is a partition of $\bigcup_{i=0}^{l} C_i$ such that for every $i = 1, \ldots, l$ and every $v \in V_i$, v has at most $s_1 - 1$ neighbours in V_i. As C_0 is a singleton, we have $C_0 \subseteq V_i$ for some i. By reordering $V_1, \ldots, V_l$ if necessary, we may assume that $i = 1$. If there are $i > j > 0$ such that $V_1 \cap C_i \neq \emptyset$ and $V_1 \cap C_j \neq \emptyset$, then, as $C_0 \subseteq V_1$, it follows that V_1 contains a 3-cycle and we are done. So now suppose that there is at most one $i > 0$ such that $V_1 \cap C_i \neq \emptyset$. First suppose that there exists exactly one $k > 0$ such that $V_1 \cap C_k \neq \emptyset$. Since $C_0 \subseteq V_1$ and every vertex in V_1 has at most $s_1 - 1$ neighbours in V_1 it follows that $|V_1 \cap C_k| \leq s_1 - 1 \leq s_k - 1$ which implies that $C_k \setminus V_1 \neq \emptyset$. Choose any vertex $v \in C_k \setminus V_1$. Consider the subgraph $\mathcal{K}'$ of $\mathcal{K}_{1,s_1,\ldots,s_l}$ which is induced by

$$C' = \{v\} \cup \bigcup_{1 \leq i \leq l, i \neq k} C_k,$$

so in model theoretic terms, $\mathcal{K}'$ is the substructure of $\mathcal{K}_{1,s_1,\ldots,s_l}$ with universe C'. Note that $\mathcal{K}'$ is a complete l-partite graph with l colour classes $\{v\}, C_1, \ldots, C_{k-1}, C_{k+1}, \ldots, C_l$, one of which is a singleton. Also, $V_2 \cap C', \ldots, V_l \cap C'$ is a partition of C' such that for every $i = 2, \ldots, l$ and every $w \in V_i$, w has at most $s_1 - 1$ neighbours in V_i. Since $s_1 \leq s_2 \leq \ldots \leq s_l$ are the cardinalities of $C_1, \ldots, C_l$, respectively, it follows that, for every $i = 1, \ldots, l$, $|C_i| \geq s_1$. Therefore the induction hypothesis implies that for some $i \geq 2$, $V_i \cap C'$ contains a 3-cycle, and the same 3-cycle is contained in V_i.

Now suppose that $C_i \cap V_1 = \emptyset$ for every $i > 0$, so $V_1 = C_0$ (by the assumption that $C_0 \subseteq V_1$). Then we can choose any $v \in C_1$ and consider the subgraph $\mathcal{K}'$ of $\mathcal{K}_{1,s_1,\ldots,s_l}$ which is induced by the set $C' = \{v\} \cup \bigcup_{i=2}^{l} C_i$. Note that $V_2 \cap C', \ldots, V_l \cap C'$ is a partition of C' such that for all $i = 2, \ldots, l$ and every $v \in V_i \cap C'$, v has at most $s_1 - 1 \leq s_2 - 1$ neighbours in V_i. Therefore the induction hypothesis implies that for some $i \geq 2$, $V_i \cap C'$ contains a 3-cycle. □

Recall the definition of a colour-critical vertex and criticality of such a vertex, defined before the statement of Theorem 1.1.

Lemma 6.2. *Suppose that $l \geq 2$, $d \geq 0$ and that $\mathcal{H}$ is a graph with the following properties:*

- *$\mathcal{H}$ has chromatic number $l + 1$.*
- *$\mathcal{H}$ has a colour-critical vertex with criticality $d + 1$ and no colour-critical vertex has criticality smaller than $d + 1$.*
- *If $V_1, \ldots, V_l$ is a partition of the vertex set of $\mathcal{H}$ such that for, every $i = 1, \ldots, l$, and every $v \in V_i$, v has at most d neighbours in V_i, then, for some j, V_j contains a 3-cycle.*

Then, for every sentence φ in the language of graphs, the proportion of graphs $\mathcal{G} \in \mathbf{Forb}_n(\mathcal{H})$ such that $\mathcal{G} \models \varphi$ converges as $n \to \infty$.

Proof. Let $\mathcal{H}$ be a graph with the listed properties. Let $\mathbf{F}_n(\mathcal{H}) = \mathbf{Forb}_n(\mathcal{H})$, $\mathbf{P}_n = \mathbf{P}_n(l, d)$ and let $\mathbf{X}_n$ be the set of $\mathcal{G} \in \mathbf{P}_n$ such that $\mathcal{G}$ has a unique decomposition based on a partition $V_1, \ldots, V_l$ definable by $\xi(x, y)$, as in Theorem 4.6, and for every $i = 1, \ldots, l$, V_i contains *no* 3-cycle. Observe that it follows from the third property of $\mathcal{H}$ (listed in the lemma) that $\mathbf{X}_n \subseteq \mathbf{Forb}_n(\mathcal{H})$ for all n.

Until further notice, assume that $d \geq 2$ (as in Section 5.1) Choose any $k \geq 1$. Recall Definition 5.7 of the equivalence relation '$\approx_k$'. Let $\mathbf{C} \subseteq \mathbf{P}(l, d)$ be the union of all $\approx_k$-classes included in $\bigcup_{n \in \mathbb{N}^+} \mathbf{P}_n^k(l, d)$ (see Definition 5.5) and such that if $\mathcal{G} \in \mathbf{C}$, then *no* part of the partition of the vertex set of $\mathcal{G}$ defined by $\xi(x, y)$ contains a 3-cycle. By Lemma 5.9, there is $c > 0$ such that

$$\lim_{n \to \infty} \frac{|\mathbf{C} \cap \mathbf{P}_n|}{|\mathbf{P}_n|} = c. \tag{21}$$

By Theorem 4.6, the proportion of $\mathcal{G} \in \mathbf{P}_n$ which have a unique decomposition and the partition on which it is based is defined by $\xi(x, y)$, approaches 1 as $n \to \infty$. From this, (21) and since $c > 0$ it follows that

$$\lim_{n \to \infty} \frac{|\mathbf{X}_n|}{|\mathbf{P}_n|} = c. \tag{22}$$

Let $\psi_{\mathcal{H}}$ be a sentence which expresses that "there is no subgraph isomorphic to $\mathcal{H}$", so for every graph $\mathcal{G}$ with vertices $1, \ldots, n$, $\mathcal{G} \in \mathbf{Forb}_n(\mathcal{H})$ if and only if $\mathcal{G} \models \psi_{\mathcal{H}}$. Then $\mathbf{F}_n(\mathcal{H}) \cap \mathbf{P}_n = \big\{\mathcal{G} \in \mathbf{P}_n : \mathcal{G} \models \psi_{\mathcal{H}}\big\}$ and from Theorem 1.2 it follows that for some $0 \leq b \leq 1$,

$$\lim_{n\to\infty} \frac{\big|\mathbf{F}_n(\mathcal{H}) \cap \mathbf{P}_n\big|}{\big|\mathbf{P}_n\big|} = \frac{\big|\{\mathcal{G} \in \mathbf{P}_n : \mathcal{G} \models \psi_{\mathcal{H}}\}\big|}{\big|\mathbf{P}_n\big|} = b. \tag{23}$$

Since $c > 0$ and $\mathbf{X}_n \subseteq \mathbf{F}_n(\mathcal{H}) \cap \mathbf{P}_n$, it follows from (22) and (23) that $b \geq c > 0$. We have arrived at this conclusion under the assumption that $d \geq 2$. If $d = 0$ or $d = 1$ and the vertex set of $\mathcal{G} \in \mathbf{P}_n(l, d)$ is partitioned into l parts $V_1, \ldots, V_l$ such that no vertex has more than d neighbours in its own part, then V_i clearly does not contain a 3-cycle for any i. So if $d = 0$ or $d = 1$, then $|\mathbf{X}_n|\big/|\mathbf{P}_n(l,d)|$ converges to 1 as $n \to \infty$, by Theorem 4.6. Hence we get (23) for some $b > 0$ also in the case $d \in \{0, 1\}$. The rest of the proof does not depend on whether $d \leq 1$ or $d \geq 2$.

Let φ be any sentence in the language of graphs. Then, for large enough n,

$$\begin{aligned}
&\frac{\big|\{\mathcal{G} \in \mathbf{F}_n(\mathcal{H}) : \mathcal{G} \models \varphi\}\big|}{\big|\mathbf{F}_n(\mathcal{H})\big|} \\
&= \frac{\big|\{\mathcal{G} \in \mathbf{F}_n(\mathcal{H}) \cap \mathbf{P}_n : \mathcal{G} \models \varphi\}\big|}{\big|\mathbf{F}_n(\mathcal{H})\big|} + \frac{\big|\{\mathcal{G} \in \mathbf{F}_n(\mathcal{H}) \setminus \mathbf{P}_n : \mathcal{G} \models \varphi\}\big|}{\big|\mathbf{F}_n(\mathcal{H})\big|} \\
&= \frac{\big|\{\mathcal{G} \in \mathbf{P}_n : \mathcal{G} \models \psi_{\mathcal{H}} \wedge \varphi\}\big|}{\big|\mathbf{F}_n(\mathcal{H})\big|} + \frac{\big|\{\mathcal{G} \in \mathbf{F}_n(\mathcal{H}) \setminus \mathbf{P}_n : \mathcal{G} \models \varphi\}\big|}{\big|\mathbf{F}_n(\mathcal{H})\big|} \\
&= \frac{\big|\{\mathcal{G} \in \mathbf{P}_n : \mathcal{G} \models \psi_{\mathcal{H}} \wedge \varphi\}\big|}{\big|\mathbf{P}_n\big|} \cdot \frac{\big|\mathbf{P}_n\big|}{\big|\mathbf{F}_n(\mathcal{H}) \cap \mathbf{P}_n\big|} \cdot \frac{\big|\mathbf{F}_n(\mathcal{H}) \cap \mathbf{P}_n\big|}{\big|\mathbf{F}_n(\mathcal{H})\big|} \\
&\quad + \frac{\big|\{\mathcal{G} \in \mathbf{F}_n(\mathcal{H}) \setminus \mathbf{P}_n : \mathcal{G} \models \varphi\}\big|}{\big|\mathbf{F}_n(\mathcal{H})\big|} \\
&\to \left(\lim_{n\to\infty} \frac{\big|\{\mathcal{G} \in \mathbf{P}_n : \mathcal{G} \models \psi_{\mathcal{H}} \wedge \varphi\}\big|}{\big|\mathbf{P}_n\big|}\right) \cdot \frac{1}{b} \quad \text{as } n \to \infty,
\end{aligned}$$

because of (23), where $b > 0$ as explained above, and Theorems 1.1 and 1.2. □

Part (i) of Theorem 1.3 follows directly from Lemmas 6.1 and 6.2. Now we consider part (ii) of Theorem 1.3. Suppose that $l \geq 2$ and $1 \leq s_1 \leq \ldots \leq s_l$ are integers and that $s_1 \leq 2$. Then $\mathcal{H} = \mathcal{K}_{1,s_1,\ldots,s_l}$ has the three properties listed in Lemma 6.2. The proof of Lemma 6.2 shows (also in the case $s_1 \leq 2$) that the proportion of $\mathcal{G} \in \mathbf{P}_n(l, s_1 - 1)$ which are $\mathcal{K}_{1,s_1,\ldots,s_l}$-free converges to a positive number. From Theorem 1.1 and Theorem 1.2 (i) it follows that, for every sentence φ, the proportion of $\mathcal{G} \in \mathbf{Forb}_n(\mathcal{H})$ in which φ is true converges to either 0 or 1.

Now we prove part (iii) of Theorem 1.3. Suppose that $s_1 \geq 3$. Choose any integer $k \geq 1$ and note that $5^k > 4$. Then pick any integers $p > q \geq 0$. We argue similarly as in the proof of Lemma 6.2. Let $\mathbf{C}, \mathbf{D} \subseteq \mathbf{P}(l, s_1 - 1)$ be $\approx_k$-equivalence classes (see Definition 5.7) such that $\mathbf{C}, \mathbf{D} \subseteq \bigcup_{n\in\mathbb{N}^+} \mathbf{P}_n^k(l, s_1 - 1)$ and the following hold:

(a) If $\mathcal{G} \in \mathbf{C}$ and $V_1, \ldots, V_l$ is the partition of the vertex set of $\mathcal{G}$ which is defined by $\xi(x, y)$, then for every $i \in [l]$, $\mathcal{G}[V_i]$ has no 3-cycle and exactly p vertices with degree $s_1 - 3$.

(b) If $\mathcal{G} \in \mathbf{D}$ and $V_1, \ldots, V_l$ is the partition of the vertex set of $\mathcal{G}$ which is defined by $\xi(x, y)$, then for every $i \in [l]$, $\mathcal{G}[V_i]$ has no 3-cycle and exactly q vertices with degree $s_1 - 3$.

Note that $\mathbf{C}$ and $\mathbf{D}$ are distinct $\approx_k$-equivalence classes, so $\mathbf{C} \cap \mathbf{D} = \emptyset$. By Lemma 5.9, $|\mathbf{C} \cap \mathbf{P}_n(l, s_1 - 1)| / |\mathbf{P}_n(l, s_1 - 1)|$ converges to a positive number as $n \to \infty$, and the same holds for $\mathbf{D}$ in place of $\mathbf{C}$. By Theorem 4.6, the proportion of graphs $\mathcal{G} \in \mathbf{C} \cap \mathbf{P}_n(l, s_1 - 1)$ (respectively $\mathcal{G} \in \mathbf{D} \cap \mathbf{P}_n(l, s_1, -1)$) such that $\mathcal{G}$ has a unique decomposition and this decomposition is based on a partition defined by $\xi(x, y)$, approaches 1 as $n \to \infty$. It follows, by using Lemma 6.1, that the proportion of $\mathcal{G} \in \mathbf{C} \cap \mathbf{P}_n(l, s_1 - 1)$ (respectively $\mathcal{G} \in \mathbf{D} \cap \mathbf{P}_n(l, s_1 - 1)$) that are $\mathcal{K}_{1,s_1,\ldots,s_l}$-free approaches 1 as $n \to \infty$. In the proof of Lemma 6.2, which is applicable to $\mathcal{H} = \mathcal{K}_{1,s_1,\ldots,s_l}$ (by Lemma 6.1), it was shown, see (23), that

$$\frac{\big|\mathbf{P}_n(l, s_1 - 1) \cap \mathbf{Forb}_n(\mathcal{K}_{1,s_1,\ldots,s_l})\big|}{\big|\mathbf{P}_n(l, s_1 - 1)\big|} \quad \text{converges to a positive number.}$$

These conclusions together with Theorem 1.1 imply that both the quotients

$$\frac{\big|\mathbf{C} \cap \mathbf{Forb}_n(\mathcal{K}_{1,s_1,\ldots,s_l})\big|}{\big|\mathbf{Forb}_n(\mathcal{K}_{1,s_1,\ldots,s_l})\big|} \quad \text{and} \quad \frac{\big|\mathbf{D} \cap \mathbf{Forb}_n(\mathcal{K}_{1,s_1,\ldots,s_l})\big|}{\big|\mathbf{Forb}_n(\mathcal{K}_{1,s_1,\ldots,s_l})\big|}$$

converge to positive numbers. Since $\mathbf{C} \cap \mathbf{D} = \emptyset$ it follows that none of these numbers can be 1. As $p > q \geq 0$ where arbitrary and the property "the induced subgraph on every part of the partition defined by $\xi(x, y)$ has no 3-cycle and exactly p vertices with degree $s_1 - 3$" can be expressed with the (first-order) language of graphs this completes the proof of part (iii) of Theorem 1.3, and hence the proof of that theorem is finished.

Remark 6.3. Suppose that $l \geq 2$ and $1 \leq s_1 \leq \ldots \leq s_l$ are integers. One can prove, by a combinatorial argument, that '$s_1 \leq 2$ or $s_2 \geq 2(s_1 - 1)$' is a necessary and sufficient condition for $\mathcal{K}_{1,s_1,\ldots,s_l}$ having the property: there is a partition of the vertex set such that every vertex has at most $s_1 - 1$ neighbours in its own part. Consequently, $\mathbf{P}(l, s_1 - 1) \subseteq \mathbf{Forb}_n(\mathcal{K}_{1,s_1,\ldots,s_l})$ if and only if $s_1 \leq 2$ or $s_2 \geq 2(s_1 - 1)$. It also follows that $|\mathbf{P}_n(l, s_1 - 1) \cap \mathbf{Forb}_n(\mathcal{K}_{1,s_1,\ldots,s_l})| / |\mathbf{P}_n(l, s_1 - 1)|$ converges to 1, as $n \to \infty$, if and only if $s_1 \leq 2$ or $s_2 \geq 2(s_1 - 1)$; otherwise this ratio converges to a positive number less than 1.

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