BINARY PRIMITIVE HOMOGENEOUS SIMPLE STRUCTURES

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ABSTRACT. Suppose that $\mathcal{M}$ is countable, binary, primitive, homogeneous, and simple, and hence 1-based. We prove that the SU-rank of the complete theory of $\mathcal{M}$ is 1. It follows that $\mathcal{M}$ is a random structure. The conclusion that $\mathcal{M}$ is a random structure does not hold if the binarity condition is removed, as witnessed by the generic tetrahedron-free 3-hypergraph. However, to show that the generic tetrahedron-free 3-hypergraph is 1-based requires some work (it is known that it has the other properties) since this notion is defined in terms of imaginary elements. This is partly why we also characterize equivalence relations which are definable without parameters in the context of ω-categorical structures with degenerate algebraic closure. Another reason is that such characterizations may be useful in future research about simple (nonbinary) homogeneous structures.
Keywords: model theory, homogeneous structure, simple theory, 1-based theory, random structure.

1. INTRODUCTION

This article is part of a study of (in particular binary) homogeneous and simple structures. In order not to be too repetitive we refer to the introductory sections of [2, 13] for more background concerning homogeneous structures, simple structures and the conjunction of both. But in general the idea is that, although some particular classes of homogeneous structures have been classified, the class of all homogeneous structures is too large and diverse to be understood in a relatively uniform way.[1] So we like to impose some extra conditions that give us tools to work with. Given the existing and model theoretically important framework of simple structures [18] it is natural to consider structures which are both homogeneous and simple. The probably most well known example in this class is the Rado graph, an example of a random structure in the sense of Definition 2.1 below. The study of homogeneous simple structures is also an extension of the work of Lachlan and others about stable homogeneous structures [14].

Here a structure $\mathcal{M}$ is called *homogeneous* if it has a finite relational vocabulary (signature) and every isomorphism between finite substructures can be extended to an automorphism of $\mathcal{M}$. If $\mathcal{M}$ is a countable structure with finite relational vocabulary, then $\mathcal{M}$ is homogeneous if and only if it has elimination of quantifiers. When assuming that a structure is simple we automatically assume that it is infinite. A structure $\mathcal{M}$ is called *primitive* if there is no *nontrivial* equivalence relation on its universe M which is $\emptyset$-definable, i.e. definable without parameters. (By a nontrivial equivalence relation we mean one which has at least two equivalence classes and at least one equivalence class contains more than one element.) A reason why primitive homogeneous structures are of interest is the following: Suppose that $\mathcal{M}$ is a homogeneous structure with a nontrivial $\emptyset$-definable equivalence relation on M. Let A be any one of the equivalence classes. Then it is easy to see that $\mathcal{M} \restriction A$, the substructure of $\mathcal{M}$ with universe A, is homogeneous. If

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[1] For a survey of homogeneous structures, including applications to permutation groups, constraint satisfaction problems, Ramsey theory and topological dynamics, see [16]. For a classification of all homogeneous digraphs, see [4]. Both sources contain many references, for example to classifications of various kinds of homogeneous structures.

$\mathcal{M}$ is, in addition, simple and A is infinite, then $\mathcal{M}{\restriction}A$ is also simple. Thus we cannot understand $\mathcal{M}$ any better than we can understand $\mathcal{M}{\restriction}A$. If, in particular, A is an equivalence class which cannot be "split" into two nonempty parts by some other $\emptyset$-definable equivalence relation, then $\mathcal{M}{\restriction}A$ is a primitive structure.

A structure $\mathcal{M}$ is called *binary* if its vocabulary contains only unary and/or binary relation symbols. For basics about simple structures see for example [18]. Our first (and main) result is the following:

Theorem 1.1. *Suppose that $\mathcal{M}$ is countable, binary, primitive, homogeneous and simple. Then the SU-rank of $Th(\mathcal{M})$ is 1.*

By a result of Aranda Lopez [3, Proposition 3.3.3], if $\mathcal{M}$ is binary, homogeneous, primitive and simple and $Th(\mathcal{M})$ has SU-rank 1, then $\mathcal{M}$ is a random structure in the sense of Definition 2.1 below.[2] Hence we get the following consequence, which gives a positive answer to the leading question asked by the author in [13]:

Corollary 1.2. *If $\mathcal{M}$ is countable, binary, primitive, homogeneous and simple, then $\mathcal{M}$ is a random structure.*

As shown by Example 2.7, the binarity assumption cannot be removed from Theorem 1.1. Since random structures (according to Definition 2.1) have SU-rank 1, it follows that the binarity condition cannot be removed from Corollary 1.2.

The proof of Theorem 1.1 relies on the following claim: *If $\mathcal{M}$ is countable, binary, homogeneous and simple, then it is 1-based.* This is also stated as (part of) Fact 2.6 below, and a justification for this claim is given in Remark 6.6. Actually, all currently known homogeneous simple structures are 1-based, or appear to be so. (We will see that verifying 1-basedness is not necessarily straightforward, even if the dividing/forking behaviour on real elements is as simple as it can be.)

Since the binarity condition cannot be removed from Corollary 1.2 one may ask the following question: If $\mathcal{M}$ is countable, primitive, homogeneous, supersimple with SU-rank 1 and 1-based, must $\mathcal{M}$ be a random structure? The answer is no, as witnessed by the generic tetrahedron-free 3-hypergraph from Definition 2.3 below:

Proposition 1.3. *The generic tetrahedron-free 3-hypergraph is primitive, homogeneous, supersimple with SU-rank 1 and 1-based, but not a random structure.*

It is known that the generic tetrahedron-free 3-hypergraph is primitive, homogeneous (by construction) and supersimple with SU-rank 1, but not a random structure (in the sense of Definition 2.1 below). See Remark 2.4 for further explanations. However, as far as the author knows, the claim of Proposition 1.3 that it is 1-based has never been verified before and is not a trivial matter, because we need to deal with imaginary elements, in other words with elements defined by $\emptyset$-definable equivalence relations on tuples of elements from the structure. (Independently of the present author, Conant has recently proved a result, about homogeneous structures whose age has "free amalgamation", which implies that the generic tetrahedron-free 3-hypergraph is 1-based [5].)

Thus we prove two results, Theorems 5.1 and 6.1, which characterize $\emptyset$-definable equivalence relations on tuples of elements, starting from a bit different assumptions. Besides being used to justify the 1-basedness claim of Proposition 1.3, via Example 6.4 (ii) and Proposition 6.5 below, these theorems may be useful in the future for understanding ω-categorical structures with additional properties, such as being homogeneous and simple (but not necessarily binary). Corollary 6.2 shows that, under the same hypotheses as in Theorem 6.1, the algebraic closure and definable closure in $\mathcal{M}^{\mathrm{eq}}$ of any $A \subseteq M$ are identical.

[2] Proposition 3.3.3 in [3] does not use the terminology "random structure" but formulates the result in terms of "Alice's restaurant property", also known as "extension axioms/properties".

Now follows an outline of this article and of the proof of Theorem 1.1. Section 2 gives a few definitions and remarks of relevance for this article. Since the work here takes place within the same context as [2, 13] we refer to the preliminary section of any one of these articles for more detailed explanations of notions and known results that will be used (concerning homogeneous and ω-categorical structures with simple theories and about imaginary elements).

Suppose that $\mathcal{M}$ is countable, binary, primitive, homogeneous and simple. Then $\mathcal{M}$ is supersimple with finite SU-rank, 1-based and has trivial dependence. (See Fact 2.6 and the discussion just before and after it.) Hence the results about coordinatization developed in [8, Section 3] are applicable to $\mathcal{M}$. These results and [2, Theorem 5.1] were used in [13] to show that $\mathcal{M}$ can be "strongly interpreted" in a binary random structure. This "strong interpretation" can also be seen as a coordinatization of $\mathcal{M}$ by a binary random structure and constitutes the framework within which we will prove Theorem 1.1. This framework is explained in Section 3. In Section 4 we prove Theorem 1.1.

As already mentioned, the main results of Sections 5 and 6 (Theorems 5.1 and 6.1) characterize $\emptyset$-definable equivalence relations on n-tuples ($0 < n < \omega$) under assumptions including ω-categoricity and degenerate algebraic closure. These sections do not depend on Sections 3 or 4 and can be read separately. Theorem 6.1 is used to prove Proposition 1.3, via Proposition 6.5.

2. Preliminaries

The notation and terminology used here is more or less standard, but we nevertheless begin with clarifying some notation. First-order structures (the only kind considered) are denoted $\mathcal{A}, \mathcal{B}, \ldots, \mathcal{M}, \mathcal{N}, \ldots$ and their universes are denoted $A, B, \ldots, M, N, \ldots$, respectively. Finite sequences are denoted by $\bar{a}, \bar{b}, \ldots, \bar{x}, \bar{y}, \ldots$. We may denote the concatenation of $\bar{a}$ and $\bar{b}$ by $\bar{a}\bar{b}$. The set of elements occuring in $\bar{a}$ is denoted by $\mathrm{rng}(\bar{a})$, "the range of $\bar{a}$". We often write '$\bar{a} \in A$' as shorthand for '$\mathrm{rng}(\bar{a}) \subseteq A$'. A structure is called ω-categorical, (super)simple or 1-based if its complete theory has the corresponding property. The SU-rank of a supersimple structure is (by definition) the SU-rank of its complete theory; and the SU-rank of a supersimple complete theory T is the supremum (if it exists) of the SU-ranks of all 1-types over $\emptyset$ with respect to T. In this article it is often important to distinguish in which structure a complete type, the algebraic closure etcetera, is taken, so we use subscripts (or superscripts) such as in '$\mathrm{tp}_{\mathcal{M}}$' or '$\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}$' to indicate this. It will also be convenient to occasionally use the notation $\bar{a} \equiv_{\mathcal{M}} \bar{b}$ as shorthand for $\mathrm{tp}_{\mathcal{M}}(\bar{a}) = \mathrm{tp}_{\mathcal{M}}(\bar{b})$.

The context of this article is the same as that of [2, 13] and therefore we refer to those articles (any one of them will do) for basics and relevant facts about homogeneous structures, simple structures and the extension $\mathcal{M}^{\mathrm{eq}}$ of $\mathcal{M}$ by imaginaries. However we repeat the following definitions here: we say that $\mathcal{N}$ is *canonically embedded* in $\mathcal{M}^{\mathrm{eq}}$ if N is a $\emptyset$-definable subset of M^{eq} and for every $0 < n < \omega$ and every relation $R \subseteq N^n$ that is $\emptyset$-definable in $\mathcal{M}^{\mathrm{eq}}$ there is a relation symbol in the vocabulary of $\mathcal{N}$ which is interpreted as R, and the vocabulary of R contains no other symbols. If $\mathcal{M}$ and $\mathcal{N}$ are structures (possibly with different vocabularies) then $\mathcal{N}$ is a *reduct* of $\mathcal{M}$ if the following holds: $M = N$ and if $0 < k < \omega$ and $R \subseteq N^k$ is $\emptyset$-definable in $\mathcal{N}$, then R is $\emptyset$-definable in $\mathcal{M}$.

Also, it is important to distinguish between two distinct, but related, notions of "triviality". A pregeometry (or matroid, see [11, Chapter 4.6] for a definition) (A, cl) will be called *trivial* if, for all $a \in A$ and $B \subseteq A$, $a \in \mathrm{cl}(B)$ implies that $a \in \mathrm{cl}(\{b\})$ for some $b \in B$. A structure $\mathcal{M}$ has *degenerate algebraic closure* if for every $A \subseteq M$, $\mathrm{acl}_{\mathcal{M}}(A) = A$; in this case we may also say that $\mathrm{acl}_{\mathcal{M}}$ is *degenerate*.

The question of what a "truly" random structure is does not have an obvious answer, but here is the definition that we will use:

Definition 2.1. (i) Let V be a vocabulary and let $\mathcal{M}$ be a V-structure. We call a finite V-structure $\mathcal{A}$ a *forbidden structure with respect to* $\mathcal{M}$ if $\mathcal{A}$ cannot be embedded into $\mathcal{M}$. If, in addition, there is no proper substructure of $\mathcal{A}$ which is forbidden with respect to $\mathcal{M}$, then we call $\mathcal{A}$ a *minimal forbidden structure with respect to* $\mathcal{M}$.
(ii) If $W \subseteq V$ are vocabularies and $\mathcal{M}$ is a V-structure, then $\mathcal{M}\restriction W$ denotes the reduct of $\mathcal{M}$ to W.
(iii) Let V be a finite relational vocabulary with maximal arity r, where $r \geq 2$. We say that a V-structure $\mathcal{M}$ is a *random structure* if $\mathcal{M}$ is infinite, countable, homogeneous and, for every $k = 2, \ldots, r$, there does *not* exist a minimal forbidden structure $\mathcal{A}$ with respect to

$$\mathcal{M}\restriction\{P \in V : \text{ the arity of } P \text{ is } \leq k\}$$

such that $|A| \geq k+1$. If $\mathcal{M}$ is a random structure and the maximal arity of its vocabulary is 2, then we may call $\mathcal{M}$ a *binary random structure*.

Remark 2.2. The definition of *binary random structure* above coincides with the one given in [13] and is equivalent to the definition given in [2]. Clearly, the Rado graph is a binary random structure according to the definition given here.

Definition 2.3. (i) A *3-hypergraph* is a structure $\mathcal{M}$ whose vocabulary contains one ternary relation symbol, say P, (and no other symbols) and which satisfies the following for any permutation π of $\{1, 2, 3\}$:

$$\forall x_1, x_2, x_3 \Bigg(P(x_1, x_2, x_3) \ \rightarrow \ \Big[\bigwedge_{i \neq j} x_i \neq x_j \ \wedge \ P(x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)}) \Big] \Bigg).$$

(ii) By a *tetrahedron* we mean a 3-hypergraph $\mathcal{M}$ such that $|M| = 4$ and for all distinct $a, b, c \in M$, $\mathcal{M} \models P(a, b, c)$. A 3-hypergraph $\mathcal{M}$ is called *tetrahedron-free* if no tetrahedron can be embedded into it.
(ii) Let $\mathbf{K}$ be the class of all finite tetrahedron-free 3-hypergraphs. Then $\mathbf{K}$ has the hereditary property and amalgamation property and therefore $\mathbf{K}$ has a (unique) Fraïssé limit, which is an infinite countable homogeneous structure.[3]

Remark 2.4. Let $\mathcal{M}$ be the generic tetrahedron-free 3-hypergraph. It is known that $\mathcal{M}$ is supersimple with SU-rank 1 and an argument showing this is found in [7, Section 3] where the same structure is called the "random pyramid-free (3)-hypergraph". Moreover (see [7]), if $A, B, C \subseteq M$ then $A \underset{C}{\downarrow} B$ if and only if $A \cap (B \cup C) = A \cap C$. Since all pairs of distinct elements have the same type it follows that $\mathcal{M}$ is also primitive. Clearly, $\mathcal{M}$ is not a random structure. Proposition 6.5 implies that $\mathcal{M}$ is 1-based. Its proof uses Theorem 6.1, the proof of which is a slight variation of the proof of Theorem 5.1.

The useful consequence of 1-basedness in the context of homogeneous simple structures is that dependence is trivial, in the sense of the following definition:

Definition 2.5. Let $\mathcal{M}$ be a simple structure. We say that $\mathcal{M}$ has *trivial dependence* if whenever $\mathcal{N} \models Th(\mathcal{M})$, $A, B, C_1, C_2 \subseteq N^{\text{eq}}$ and $A \underset{B}{\not\downarrow} (C_1 \cup C_2)$, then $A \underset{B}{\not\downarrow} C_i$ for $i = 1$ or $i = 2$.

[3] See [11, Chapter 7] for the involved notions and relevant results. It is straightforward to see that $\mathbf{K}$ has the hereditary property and amalgamation property. The joint embedding property follows from the amalgamation property since the vocabulary is relational.

If $\mathcal{M}$ is countable, homogeneous and simple, then (by results of Macpherson [15], De Piro and Kim [6] and Hart, Kim and Pillay [10]) $\mathcal{M}$ is 1-based if and only if it has trivial dependence and finite SU-rank. By [12, Theorem 1], every countable, binary, homogeneous simple structure is supersimple with finite SU-rank. Therefore, if $\mathcal{M}$ is countable, binary, homogeneous and simple, then it is 1-based if and only if it has trivial dependence.

Fact 2.6. *If $\mathcal{M}$ is countable, binary, homogeneous and simple then it is supersimple with finite SU-rank, 1-based and has trivial dependence.*

From what has been said it follows that the only thing that needs to be proved is that $\mathcal{M}$, as in Fact 2.6, has trivial dependence. We postpone this to Remark 6.6 at the very end, because then we can "reuse" a part of the argument in the proof of Proposition 6.5, rather than repeating that argument.

Example 2.7. The necessity of the binarity assumption in Theorem 1.1 is shown by the following example which also appears as Example 3.3.2 in [16].

Let $\mathcal{M}$ be a countable infinite structure with empty vocabulary, so $\mathcal{M}$ is just a set, hence $\mathcal{M}$ is homogeneous and ω-stable (thus supersimple) of SU-rank 1. Trivially, $\mathcal{M}$ is binary, so by Fact 2.6 $\mathcal{M}$ is 1-based and has trivial dependence. Let $\mathcal{G} = (V, E)$ be the graph where V is the set of all (unordered) 2-subsets of M and let two vertices of G be adjacent if and only if they intersect in exactly one point (of M). Then $\mathcal{G}$ is interpretable in M (without parameters), so it is ω-categorical and stable. However $\mathcal{G}$ is not homogeneous. Let $\mathcal{G}'$ be the expansion of $\mathcal{G}$ by adding a ternary relation symbol Q, where $Q(a,b,c)$ holds in $\mathcal{G}'$ if and only if a, b and c are distinct and the intersection of all three is nonempty. Then $\mathcal{G}'$ is homogeneous (we leave the proof to the reader). Moreover, Q is definable in $\mathcal{G}$ without parameters, so $\mathcal{G}'$ is stable. Observe that every permutation of M naturally induces an automorphism of $\mathcal{G}'$ (if distinct $a, b \in M$ are mapped to a', b', respectively, then let $\{a, b\}$ be mapped to $\{a', b'\}$). Therefore $\mathcal{G}'$ has a unique 1-type over $\emptyset$ and there are exactly 2 different 2-types of distinct elements over $\emptyset$ (adjacent or nonadjacent vertices). By using the definition of dividing it is straightforward to show that the unique 1-type (over $\emptyset$) of $\mathcal{G}'$ has SU-rank 2, so $\mathcal{G}'$ has SU-rank 2.[4]

The vocabulary of $\mathcal{G}'$ has only the symbols E and Q and it is easy to see that E is not an equivalence relation. As $\mathcal{G}'$ has elimination of quantifiers (being homogeneous) it follows that it is primitive. Furthermore, $\mathcal{G}'$ is 1-based. To show this, it is, by [10, Corollary 4.7], sufficient to show that *for every complete type of SU-rank 1 (possibly realized by imaginary elements), the pregeometry on its realizations (given by algebraic closure) is trivial.* Since $\mathcal{M}$ has trivial dependence this is true for $\mathcal{M}$. As $\mathcal{G}'$ is definable without parameters in $\mathcal{M}^{\mathrm{eq}}$ it follows that $\mathcal{G}'^{\mathrm{eq}}$ is definable without parameters in $(\mathcal{M}^{\mathrm{eq}})^{\mathrm{eq}}$. Since $\mathcal{M}^{\mathrm{eq}}$ has elimination of imaginaries it follows that $\mathcal{G}'^{\mathrm{eq}}$ is definable without parameters in $\mathcal{M}^{\mathrm{eq}}$. Hence the statement in italics holds for $\mathcal{G}'$.[5]

3. Coordinatization by a random structure

In this section and in Section 4 we assume that $\mathcal{M}$ ***is countable, binary, primitive, homogeneous and simple***. By Fact 2.6, $\mathcal{M}$ ***is supersimple with finite SU-rank, 1-based and has trivial dependence. Let the SU-rank of $\mathcal{M}$ be ρ.***

Note that the primitivity of $\mathcal{M}$ implies that for all $a, b \in M$, $\mathrm{tp}_{\mathcal{M}}(a) = \mathrm{tp}_{\mathcal{M}}(b)$.

[4] The following argument shows that the SU-rank is at least 2. Let $a, b \in V$ and $E(a,b)$. Then $\mathrm{tp}_{\mathcal{G}'}(a/b)$ is nonalgebraic so $\mathrm{SU}(a/b) \geq 1$. Take $b_i \in V$, $i < \omega$, such that $b = b_0$ and whenever $i \neq j$ then $\neg E(b_i, b_j)$. Then $\{E(x, b_i) : i < \omega\}$ is 3-inconsistent, so $\mathrm{tp}_{\mathcal{G}'}(a/b)$ divides over $\emptyset$. Hence $\mathrm{SU}(a) \geq 2$.

[5] Strictly speaking, the statement in italics should be proved in the context when the parameters of the type come from an arbitrary model of $Th(\mathcal{G}')$, so the verification of 1-basedness is not quite complete. However, the beginning of the proof of Proposition 6.5 shows how to overcome this slight obstacle.

Fact 3.1. *(*[13, Section 3]*, which uses the "coordinatization" from* [8]*) There is $C \subseteq M^{\text{eq}}$ such that:*

(i) *C is $\emptyset$-definable in $\mathcal{M}^{\text{eq}}$ and only finitely many sorts are represented in C,*
(ii) $\text{SU}(c) = 1$ *for every $c \in C$ (where SU-rank of elements/types is taken with respect to $\mathcal{M}^{\text{eq}}$),*
(iii) $\text{acl}_{\mathcal{M}^{\text{eq}}}$ *restricted to C is* degenerate*, by which we mean that* $\text{acl}_{\mathcal{M}^{\text{eq}}}(A) \cap C = A$ *for every $A \subseteq C$, and*
(iv) *for every $a \in M$, $a \in \text{acl}_{\mathcal{M}^{\text{eq}}}(\text{crd}(a))$ where we define* $\text{crd}(a) = \text{acl}_{\mathcal{M}^{\text{eq}}}(a) \cap C$ *(so in particular $M \subseteq \text{acl}_{\mathcal{M}^{\text{eq}}}(C)$).*
(v) *For every $c \in C$ there is $a \in M$ such that $c \in \text{crd}(a)$.*[6]

We call C as in Fact 3.1 a set of *coordinates* of $\mathcal{M}$ and for each $a \in M$, $\text{crd}(a)$ may be called the (set of) *coordinates of a*. Moreover, if $A \subseteq C$ and there is $a \in M$ such that $\text{crd}(a) = A$, then we call A a *line*. From the assumptions about $\mathcal{M}$ and the properties of C and crd from Fact 3.1 one easily derives the following[7]:

Fact 3.2. *Let C and* crd *be as in Fact 3.1. Then:*

(i) *For all $a, a' \in M$, $\text{SU}(a) = |\text{crd}(a)| = |\text{crd}(a')| = \text{SU}(a')$, so in particular all lines have the same cardinality, which is ρ.*
(ii) *For all $a \in M$, $a \in \text{dcl}_{\mathcal{M}^{\text{eq}}}(\text{crd}(a))$. Hence, for every line $A \subseteq C$ there is a unique $a \in M$ such that $\text{crd}(a) = A$.*
(iii) *for all $a, a' \in M$, $\text{crd}(a)$ and $\text{crd}(a')$ can be ordered as $\bar{c}$ and $\bar{c}'$, respectively, so that $\text{tp}_{\mathcal{M}^{\text{eq}}}(\bar{c}) = \text{tp}_{\mathcal{M}^{\text{eq}}}(\bar{c}')$.*

Assumption 3.3. *For the rest of this section and in Section 4 we assume that C* ***is a coordinatizing set as in Fact 3.1 and $\mathcal{C}$ is the canonically embedded structure in $\mathcal{M}^{\text{eq}}$ with universe C.***

From [2, Theorem 5.1] we immediately get the following:

Fact 3.4. *There is a binary random structure $\mathcal{R}$ such that $\mathcal{C}$ is a reduct of $\mathcal{R}$.*

Moreover, by [13, Lemma 3.9]:

Fact 3.5. *Suppose that $\mathcal{R}$ is like in Fact 3.4. Let $a \in M$ and $\text{crd}(a) = \{c_1, \ldots, c_\rho\}$ where the elements are enumerated without repetition. Then for every nontrivial permutation π of $\{1, \ldots, \rho\}$, $(c_1, \ldots, c_\rho) \not\equiv_{\mathcal{R}} (c_{\pi(1)}, \ldots, c_{\pi(\rho)})$.*

Unfortunately, the information given by these facts is not quite enough for the purpose of proving Theorem 1.1. Therefore the next result gives the strengthening of Fact 3.4 that we need.

Fact 3.6. *There is a binary random structure $\mathcal{R}$ such that:*

(i) *$\mathcal{C}$ is a reduct of $\mathcal{R}$.*
(ii) *For every $p(x) \in S_1^{\mathcal{M}^{\text{eq}}}(\text{acl}_{\mathcal{M}^{\text{eq}}}(\emptyset))$ that is realized in C there is a unique unary relation symbol R_p in the vocabulary of $\mathcal{R}$ such that for all $c \in C$, $\mathcal{R} \models R_p(c)$ if and only if $\mathcal{M}^{\text{eq}} \models p(c)$.*
(iii) *For every $q(x, y) \in S_2^{\mathcal{M}^{\text{eq}}}(\emptyset)$ that is realized in C^2 there is a unique binary relation symbol R_q in the vocabulary of $\mathcal{R}$ such that for all $c_1, c_2 \in C$, $\mathcal{R} \models R_q(c_1, c_2)$ if and only if $\mathcal{M}^{\text{eq}} \models q(c_1, c_2)$.*

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[6] This is the only part which may not be immediate from [13, Section 3]. However, if C has all properties (i)–(iv) but not (v), then we can let $C' = \{c \in C : c \in \text{crd}(a) \text{ for some } a \in M\}$ and it is straightforward to verify (using that $\mathcal{M}$ is ω-categorical) that C' satisfies (i)–(v), because for every $a \in M$, $\text{crd}(a)$ is the same whether computed with respect to C or with respect to C'.

[7] For part (ii), use [13, Lemma 3.5], primitivity of $\mathcal{M}$, and the fact that (by ω-categoricity) the equivalence relation '$\text{acl}_{\mathcal{M}}(x) = \text{acl}_{\mathcal{M}}(y)$' has infinitely many classes

(iv) *The vocabulary of* $\mathcal{R}$ *has no other symbols than those mentioned in (ii) and (iii). Consequently, every type in* $S_n^{\mathcal{R}}(\emptyset)$ *(for any* $0 < n < \omega$*) is isolated by a conjunction of such formulas* $R_p(x)$ *and* $R_q(x, y)$ *mentioned in (ii) and (iii).*

Remark: In (i) we consider 1-types over $\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(\emptyset)$ and in (ii) we consider 2-types over $\emptyset$; this is the intention and not a mistake.

Proof. The proof is a modification of the proof of Theorem 5.1 in [2] and we only explain how to modify that proof. By Lemma 4.5 in [2] there is a $\emptyset$-definable $D \subseteq M^{\mathrm{eq}}$ in which only finitely many sorts are represented and such that

(a) D is acl-complete (see Definition 4.4 in [2]),
(b) for every $c \in C$ there is $d \in D$ such that $c \in \mathrm{dcl}_{\mathcal{M}^{\mathrm{eq}}}(d)$ and $d \in \mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(c)$, and
(c) for every $d \in D$ there is (a unique) $c \in C$ such that $c \in \mathrm{dcl}_{\mathcal{M}^{\mathrm{eq}}}(d)$ and $d \in \mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(c)$.

Note that if c and d are as in (b) (or (c)), then $\mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}(c/\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(\emptyset))$ is determined by $\mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}(d/\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(\emptyset))$, because $c \in \mathrm{dcl}_{\mathcal{M}^{\mathrm{eq}}}(d)$. If, in addition, c' and d' satisfy the same conditions as c and d, then $\mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}(c, c')$ is determined by $\mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}(d, d')$.

Just as in the proof of Theorem 5.1 in [2, p 244], let $p_1, \ldots, p_r$ be all complete 1-types over $\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(\emptyset)$ which are realized in D, and let $p_{r+1}, \ldots, p_s$ be all complete 2-types over $\emptyset$ which are realized in D^2. For each $i = 1, \ldots, r$ let R_i be a unary relation symbol and for each $i = r+1, \ldots, s$ let R_i be a binary relation symbol. Let $V = \{R_1, \ldots, R_s\}$ and let $\mathcal{D}$ be the V-structure with universe D such that, for every $i = 1, \ldots, s$ and every $\bar{d} \in D$ of appropriate length, $\mathcal{D} \models R_i(\bar{d})$ if and only if $\mathcal{M}^{\mathrm{eq}} \models p_i(\bar{d})$.

So far we have followed the proof of Theorem 5.1 in [2]. The difference comes now when we define a subvocabulary $V' \subseteq V$ and then a class $\mathbf{K}$ of finite V'-structures, instead of a class $\mathbf{K}$ of finite V-structures as in [2]. Let $I \subseteq \{1, \ldots, s\}$ be minimal (with respect to inclusion) such that the following hold:

- For every $q \in S_1^{\mathcal{M}^{\mathrm{eq}}}(\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(\emptyset))$ that is realized in C there is p_i, $1 \leq i \leq r$ and $i \in I$, such that whenever $c \in C$ and $\mathcal{M}^{\mathrm{eq}} \models q(c)$, then there is $d \in D$ satisfying (b) and $\mathcal{M}^{\mathrm{eq}} \models p_i(d)$.
- For every $q \in S_2^{\mathcal{M}^{\mathrm{eq}}}(\emptyset)$ that is realized in C^2 there is p_i, $r < i \leq s$ and $i \in I$, such that whenever $c_1, c_2 \in C$ and $\mathcal{M}^{\mathrm{eq}} \models q(c_1, c_2)$, then there are $d_1, d_2 \in D$ satisfying (b) (d_i with respect to c_i) and $\mathcal{M}^{\mathrm{eq}} \models p_i(d_1, d_2)$.

Now let $V' = \{R_i : i \in I\}$ and let $\mathbf{K}$ be the class of all finite V'-structures $\mathcal{N}$ such that

- for all $a \in N$, $\mathcal{N} \models R_i(a)$ for some $R_i \in V'$, and
- there is an embedding $f : \mathcal{N} \to \mathcal{D}{\restriction}V'$ such that $f(N)$ is an independent set (where independence is with respect to $\mathcal{M}^{\mathrm{eq}}$).

Now we can define $\mathbf{P}_2$ and $\mathbf{RP}_2$ from $\mathbf{K}$ in the same way as in [2, p 244]. The rest of the proof, starting from Lemma 5.3 in [2], is like the proof of Theorem 5.1 in [2]; although $\mathbf{K}$ (and consequently $\mathbf{P}_2$ and $\mathbf{RP}_2$) is defined differently here the same arguments work out in the present context. Hence we find a binary random V'-structure $\mathcal{R}$ such that (i)–(iii) of this lemma hold. This modification of the proof of [2, Theorem 5.1] thus amounts to showing that the vocabulary V may have redundant symbols (for the purpose of making $\mathcal{C}$ a reduct of $\mathcal{R}$) and we can always do with the vocabulary V' as defined above. □

4. Proof of Theorem 1.1

As in the previous section we assume that $\mathcal{M}$ ***is countable, binary, primitive, homogeneous and simple***, so by Fact 2.6, $\mathcal{M}$ ***is supersimple with finite SU-rank, 1-based and has trivial dependence.*** Just as in the previous section we ***assume that the SU-rank of*** $\mathcal{M}$ ***is*** ρ. Furthermore, we assume that $\rho \geq 2$ ***and*** $C \subseteq M^{\mathrm{eq}}$ ***is as in Assumption 3.3***. In addition, we adopt the following:

Assumption 4.1. *For the rest of this section we assume that $\mathcal{R}$* ***is a binary random structure such that (i)–(iv) of Lemma 3.6 are satisfied****, so in particular $\mathcal{C}$ is a reduct of $\mathcal{R}$, which implies that $C = R$ (where R is the universe of $\mathcal{R}$).*

Remark 4.2. (i) A direct consequence of Assumption 4.1 is that if $\bar{c}, \bar{c}' \in C$ and $\mathrm{tp}_{\mathcal{R}}(\bar{c}) = \mathrm{tp}_{\mathcal{R}}(\bar{c}')$ then $\mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}(\bar{c}) = \mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}(\bar{c}')$.
(ii) Let $p \in S_1^{\mathcal{R}}(\emptyset)$ and let $X = \{c \in R : \mathcal{R} \models p(c)\}$. By Theorem 6.1, it follows that there is no nontrivial equivalence relation on X which is $\emptyset$-definable in $\mathcal{R}$. This can also be proved directly by a straightforward argument. (The existence of a nontrivial $\emptyset$-definable equivalence relation that is not definable by a unary formula would contradict the defining property of a binary random structure.)

Given this framework, including the facts of the previous section, Theorem 1.1 is a consequence of Proposition 4.4 (via its corollary) and Lemma 4.3. Most of the work is devoted to proving Proposition 4.4.

Lemma 4.3. *(i) There is no formula $\varphi(x, y)$ (without parameters) such that for some $a \in M$ there are $c, c' \in \mathrm{crd}(a)$ such that $\mathcal{M}^{\mathrm{eq}} \models \varphi(c, a) \wedge \neg\varphi(c', a)$.*
(ii) For every $a \in M$ and every $c \in \mathrm{crd}(a)$, $c \notin \mathrm{dcl}_{\mathcal{M}^{\mathrm{eq}}}(a)$.
(iii) For every $a \in M$ and all $c, c' \in \mathrm{crd}(a)$ there are an ordering $c_1, \ldots, c_{\rho-1}$ of $\mathrm{crd}(a) \setminus \{c\}$ and an ordering $c'_1, \ldots, c'_{\rho-1}$ of $\mathrm{crd}(a) \setminus \{c'\}$ such that

$$(a, c, c_1, \ldots, c_{\rho-1}) \equiv_{\mathcal{M}^{\mathrm{eq}}} (a, c', c'_1, \ldots, c'_{\rho-1}).$$

Proof. (i) Suppose that there are a formula $\varphi(x, y)$, $a \in M$ and $c, c' \in \mathrm{crd}(a)$ such that $\mathcal{M}^{\mathrm{eq}} \models \varphi(c, a) \wedge \neg\varphi(c', a)$. Since $\mathcal{M}$ is primitive it follows that $\mathrm{tp}_{\mathcal{M}}(b) = \mathrm{tp}_{\mathcal{M}}(b')$ for all $b, b' \in M$ and therefore the previous statement holds for *all* $a \in M$. Moreover, by primitivity it follows that for all $a, b \in M$,

$$|\{c \in \mathrm{crd}(a) : \mathcal{M}^{\mathrm{eq}} \models \varphi(c, a)\}| \;=\; |\{c \in \mathrm{crd}(b) : \mathcal{M}^{\mathrm{eq}} \models \varphi(c, b)\}|.$$

This implies that the following is an equivalence relation on M, which is $\emptyset$-definable in $\mathcal{M}$:

$$x \sim y \iff \forall z\Big(\big(z \in \mathrm{crd}(x) \;\wedge\; \varphi(z, x)\big) \;\rightarrow\; \big(z \in \mathrm{crd}(y) \;\wedge\; \varphi(z, y)\big)\Big).$$

Take $a, b \in M$ such that $a \mathop{\smile\!\!\!\!\!\;|} b$. By Fact 3.1 (ii)–(iv), $\mathrm{crd}(a) \cap \mathrm{crd}(b) = \emptyset$, so $a \not\sim b$. Hence '$\sim$' has at least two classes (actually infinitely many). Let $\varphi(\mathcal{M}^{\mathrm{eq}}, a) \cap C = \{c_1, \ldots, c_k\}$ and $\mathrm{crd}(a) = \{c_1, \ldots, c_\rho\}$, where by assumption $\rho > k > 0$. From Fact 3.1 it follows that $a, c_{k+1}, \ldots, c_\rho \notin \mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(c_1, \ldots, c_k)$, so there are (by the existence of nonforking extensions, for example) $a', c'_{k+1}, \ldots, c'_\rho \notin \mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(a)$ such that

$$(a, c_1, \ldots, c_\rho) \equiv_{\mathcal{M}^{\mathrm{eq}}} (a', c_1, \ldots, c_k, c'_{k+1}, \ldots, c'_\rho).$$

Then $a' \neq a$ and $a' \sim a$, so '$\sim$' is nontrivial. This contradicts that $\mathcal{M}$ is primitive. Hence (i) is proved.

Part (ii) follows directly from (i) because we assume that that the SU-rank ρ is at least two and hence $|\mathrm{crd}(a)| = \rho \geq 2$ for every $a \in M$.

(iii) Let $a \in M$ and $c, c' \in \mathrm{crd}(a)$. Let $\varphi(x, y)$ isolate $\mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}(c, a)$. By (i), $\mathcal{M}^{\mathrm{eq}} \models \varphi(c', a)$, so $(a, c) \equiv_{\mathcal{M}^{\mathrm{eq}}} (a, c')$. Since $\mathrm{crd}(a)$ is finite and $\{a\}$-definable it follows that there are orderings $c_1, \ldots, c_{\rho-1}$ and $c'_1, \ldots, c'_{\rho-1}$ of $\mathrm{crd}(a) \setminus \{c\}$ and $\mathrm{crd}(a) \setminus \{c'\}$, respectively, such that

$$(a, c, c_1, \ldots, c_{\rho-1}) \equiv_{\mathcal{M}^{\mathrm{eq}}} (a, c', c'_1, \ldots, c'_{\rho-1}).$$

□

The main part of the proof of Theorem 1.1 consists of proving the following:

Proposition 4.4. *For all $c, c' \in C$, $\mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}(c/\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(\emptyset)) = \mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}(c'/\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(\emptyset))$.*

We postpone the proof of Proposition 4.4 for a little while and first show how it is used to prove Theorem 1.1 via the following corollary.

Corollary 4.5. *For all $0 < n < \omega$ and all $c_1, \ldots, c_n, c'_1, \ldots, c'_n \in C$,*

$$(c_1, \ldots, c_n) \equiv_{\mathcal{M}^{\mathrm{eq}}} (c'_1, \ldots, c'_n) \ \text{ if and only if } \ (c_1, \ldots, c_n) \equiv_{\mathcal{R}} (c'_1, \ldots, c'_n).$$

Proof. By Proposition 4.4, there is only one $p \in S_1^{\mathcal{M}^{\mathrm{eq}}}(\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(\emptyset))$ which is realized in C. From part (ii) of Fact 3.6 it follows that the vocabulary of $\mathcal{R}$ has only one unary relation symbol P and $\mathcal{R} \models \forall x P(x)$. Suppose that $(c_1, \ldots, c_n) \equiv_{\mathcal{M}^{\mathrm{eq}}} (c'_1, \ldots, c'_n)$, so in particular $(c_i, c_j) \equiv_{\mathcal{M}^{\mathrm{eq}}} (c'_i, c'_j)$ for all i and j. By Fact 3.6 (iii) it follows that $(c_i, c_j) \equiv_{\mathcal{R}} (c'_i, c'_j)$ for all i and j. Since $\mathcal{R}$ is binary and has elimination of quantifiers we get $(c_1, \ldots, c_n) \equiv_{\mathcal{R}} (c'_1, \ldots, c'_n)$. The other direction follows from the assumption that $\mathcal{C}$ is a reduct of $\mathcal{R}$ (and was stated in Remark 4.2). $\square$

We now show how Lemma 4.3 and Corollary 4.5 imply our main result:

Theorem 1.1 *Suppose that $\mathcal{M}$ is countable, binary, homogeneous, primitive and simple. Then the SU-rank of $Th(\mathcal{M})$ is 1.*

Proof. By Fact 2.6, the premises of the theorem imply that $\mathcal{M}$ satisfies all conditions assumed in this section. Suppose, as in this whole section, that the SU-rank of $Th(\mathcal{M})$ is $\rho \geq 2$. Then, for every $a \in M$, $\mathrm{SU}(a) = \rho$ and therefore (by Fact 3.2 (i)) $|\mathrm{crd}(a)| = \rho$. By Fact 3.5 and Corollary 4.5 we get:

For every line $\{c_1, \ldots, c_\rho\} \subseteq C$ and every nontrivial permutation π of $\{1, \ldots, \rho\}$, $(c_1, \ldots, c_\rho) \not\equiv_{\mathcal{M}^{\mathrm{eq}}} (c_{\pi(1)}, \ldots, c_{\pi(\rho)})$.

This implies that for every $a \in M$ and every $c \in \mathrm{crd}(a)$, $c \in \mathrm{dcl}_{\mathcal{M}^{\mathrm{eq}}}(a)$. But this contradicts Lemma 4.3. $\square$

The rest of this section is devoted to proving Proposition 4.4. We begin with a sequence of lemmas, numbered from 4.7 to 4.9, which deal with properties of coordinates in the present context and of the equivalence relation $\mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}(x/\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(\emptyset)) = \mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}(y/\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(\emptyset))$ restricted to C. Then we have the tools to finish the proof of Proposition 4.4; this part begins with Notation 4.10.
For the rest of this section we use the following definition and notation:

Definition 4.6. For all $c, c' \in C$,

$$E(c, c') \iff \mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}(c/\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(\emptyset)) = \mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}(c'/\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(\emptyset)).$$

Observe that, since $\mathcal{M}$ is ω-categorical, E is $\emptyset$-definable in $\mathcal{M}^{\mathrm{eq}}$.

Lemma 4.7. *Suppose that $c_1, \ldots, c_\rho, c'_1, \ldots, c'_\rho \in C$. If*

$$(c_1, \ldots, c_\rho) \equiv_{\mathcal{M}^{\mathrm{eq}}} (c'_1, \ldots, c'_\rho) \ \text{ and } \ E(c_i, c'_i) \text{ for all } i = 1, \ldots, \rho,$$

then $(c_1, \ldots, c_\rho) \equiv_{\mathcal{R}} (c'_1, \ldots, c'_\rho)$.

Proof. From part (ii) of Fact 3.6 we get $c_i \equiv_{\mathcal{R}} c'_i$ for every $i = 1, \ldots, \rho$. This together with parts (iii) and (iv) of Fact 3.6 gives $(c_i, c_j) \equiv_{\mathcal{R}} (c'_i, c_j)$ for all $i = 1, \ldots, \rho$. Since the vocabulary of $\mathcal{R}$ is binary and $\mathcal{R}$ has elimination of quantifiers we get $(c_1, \ldots, c_\rho) \equiv_{\mathcal{R}} (c'_1, \ldots, c'_\rho)$. $\square$

Lemma 4.8. *Let $A \subseteq C$ be a line and let $X, Y \subseteq C$ be E-classes. If $A \cap X \neq \emptyset$ and $A \cap Y \neq \emptyset$, then $|A \cap X| = |A \cap Y|$.*

Proof. Suppose for a contradiction that there are a line A and E-classes X, Y such that $A \cap X \neq \emptyset$, $A \cap Y \neq \emptyset$ and $k = |A \cap X| \neq |A \cap Y|$. Let $a \in M$ be such that $\mathrm{crd}(a) = A$, let $c \in A \cap X$ and $c' \in A \cap Y$. Let $\varphi(x, y)$ be the formula which expresses that

"$x \in \text{crd}(y)$ and there are exactly k different elements $z \in \text{crd}(y)$ such that $E(x, z)$".

Then $\mathcal{M}^{\text{eq}} \models \varphi(c, a) \wedge \neg\varphi(c', a)$, which contradicts Lemma 4.3 (i). □

Lemma 4.9. *There is a number $s \geq 1$ such that for every line $A \subseteq C$ and every E-class $X \subseteq C$, $|A \cap X| = s$.*

Proof. By Lemma 4.8 and since all elements of M have the same type it suffices to show that every line has nonempty intersection with every E-class. Let l be the number of E-classes (so $l < \omega$). Let A be any line and let k be the number of E-classes X such that $A \cap X \neq \emptyset$. Since all elements of $\mathcal{M}$ realize the same 1-type over $\emptyset$ it follows that for every line A' there are exactly k E-classes X such that $A' \cap X \neq \emptyset$.

Consider then following $\emptyset$-definable (in $\mathcal{M}$) equivalence relation on M :

$$x \sim y \iff \text{for all } u \in \text{crd}(x) \text{ and all } v \in \text{crd}(y) \text{ there are } u' \in \text{crd}(y) \text{ and } v' \in \text{crd}(x) \text{ such that } E(u, u') \text{ and } E(v, v').$$

We will show that if $l > k$ then '$\sim$' is nontrivial, contradicting that $\mathcal{M}$ is primitive. So suppose that $l > k$. Take any $a \in M$ and let $\text{crd}(a) = \{c_1, \dots, c_\rho\}$. As $l > k$ there is $c' \in C$ such that $\neg E(c', c_i)$ for all $i = 1, \dots, \rho$. By Fact 3.1 (v) there is $a' \in M$ such that $c' \in \text{crd}(a')$. Then $a \not\sim a'$ so '$\sim$' has at least two classes. Let X be the E-class of c_1. As $c_1 \notin \text{acl}_{\mathcal{M}^{\text{eq}}}(c_2, \dots, c_\rho)$ there is $c_1' \in X \setminus \text{acl}_{\mathcal{M}^{\text{eq}}}(c_1, \dots, c_\rho)$ such that

$$(c_1', c_2, \dots, c_\rho) \equiv_{\mathcal{M}^{\text{eq}}} (c_1, c_2, \dots, c_\rho).$$

Then $\{c_1', c_2, \dots, c_\rho\}$ is a line so $\{c_1', c_2, \dots, c_\rho\} = \text{crd}(a')$ for some $a' \in M$. It follows that $a' \neq a$ and $a' \sim a$. Thus '$\sim$' is nontrivial, a contradiction.

Hence we conclude that $l = k$ which implies that for every line $A \subseteq C$ and every E-class $X \subseteq C$, $A \cap X \neq \emptyset$. □

Notation 4.10. For the rest of this section we will use the following notation:

(i) Let $X_1, \dots, X_l$ enumerate all E-classes (without repetition).

(ii) Let A_0 be any line. By Lemma 4.9, there is $s \geq 1$ such that $|A_0 \cap X_i| = s$ for every $i = 1, \dots, l$. For $i = 1, \dots, l$, let $\bar{d}_i = (d_{i,1}, \dots, d_{i,s})$ enumerate $A_0 \cap X_i$. By Lemma 4.3 we may assume that $\bar{d}_i \equiv_{\mathcal{M}^{\text{eq}}} \bar{d}_j$ for all i and j.

(iii) Let

$$p(\bar{x}_1 \dots \bar{x}_l) = \text{tp}_{\mathcal{M}^{\text{eq}}}(\bar{d}_1 \dots \bar{d}_l) \text{ and}$$
$$p^+(\bar{x}_1 \dots \bar{x}_l) = \text{tp}_{\mathcal{R}}(\bar{d}_1 \dots \bar{d}_l).$$

Observe that, by Remark 4.2 (i), for all $\bar{c}_1, \dots, \bar{c}_l \in C$, if $\mathcal{R} \models p^+(\bar{c}_1 \dots \bar{c}_l)$ then $\mathcal{M}^{\text{eq}} \models p(\bar{c}_1 \dots \bar{c}_l)$. Morover, by Remark 4.2 (ii), if $\mathcal{R} \models p^+(\bar{c}_1 \dots \bar{c}_l)$ then $\text{rng}(\bar{c}_i) \subseteq X_i$ for all $i = 1, \dots, l$, and if $\mathcal{M}^{\text{eq}} \models p(\bar{c}_1 \dots \bar{c}_l)$ then, for every $i = 1, \dots, l$, all members of $\bar{c}_i$ belong to the same E-class.

Lemma 4.11. *Every line $A \subseteq C$ can be enumerated as $\bar{c}_1 \dots \bar{c}_l$, where $\bar{c}_i = (c_{i,1}, \dots c_{i,s})$ for $i = 1, \dots, l$, so that $\mathcal{M}^{\text{eq}} \models p(\bar{c}_1 \dots \bar{c}_l)$ and, for all $i = 1, \dots, l$ and all $j = 1, \dots, s$, $c_{i,j} \in X_i$.*

Proof. Define on M:

$$a \sim a' \iff \text{crd}(a) \text{ and } \text{crd}(a') \text{ can be enumerated as } \bar{c}_1 \dots \bar{c}_l \text{ and } \bar{c}_1' \dots \bar{c}_l', \text{ respectively, so that } \mathcal{M}^{\text{eq}} \models p(\bar{c}_1 \dots \bar{c}_l) \wedge p(\bar{c}_1' \dots \bar{c}_l') \text{ and, for all } i = 1, \dots, l \text{ and all } j = 1, \dots, s,\ E(c_{i,j}, c_{i,j}').$$

Then '$\sim$' is clearly $\emptyset$-definable in $\mathcal{M}$ (because by the ω-categoricity of $\mathcal{M}$, p is isolated), as well as reflexive and symmetric. We will show that '$\sim$' is transitive, hence an equivalence relation. Since $\mathcal{R}$ is a binary random structure and $\mathcal{C}$ is a reduct of $\mathcal{R}$ it is easy to see that there are distinct $a, a' \in M$ such that $a \sim a'$. As $\mathcal{M}$ is primitive it follows that $a \sim a'$ for all $a, a' \in M$. The conclusion of the lemma follows from this. Hence it remains to show that '$\sim$' is transitive.

Suppose that $a \sim a' \sim a''$. From $a \sim a'$ it follows that $\mathrm{crd}(a)$ and $\mathrm{crd}(a')$ can be enumerated as $\bar{c}_1 \dots \bar{c}_l$ and $\bar{c}'_1 \dots \bar{c}'_l$, respectively, so that

$$\text{(4.1)} \qquad \begin{aligned} &\mathcal{M}^{\mathrm{eq}} \models p(\bar{c}_1 \dots \bar{c}_l) \wedge p(\bar{c}'_1 \dots \bar{c}'_l) \text{ and,} \\ &\text{for all } i = 1, \dots, l \text{ and all } j = 1, \dots, s,\ E(c_{i,j}, c'_{i,j}). \end{aligned}$$

From $a' \sim a''$ it follows that $\mathrm{crd}(a')$ and $\mathrm{crd}(a'')$ can be enumerated as $\bar{c}^*_1 \dots \bar{c}^*_l$ and $\bar{c}''_1 \dots \bar{c}''_l$, respectively, so that

$$\text{(4.2)} \qquad \begin{aligned} &\mathcal{M}^{\mathrm{eq}} \models p(\bar{c}^*_1 \dots \bar{c}^*_l) \wedge p(\bar{c}''_1 \dots \bar{c}''_l) \text{ and,} \\ &\text{for all } i = 1, \dots, l \text{ and all } j = 1, \dots, s,\ E(c^*_{i,j}, c''_{i,j}). \end{aligned}$$

Since $\bigcup_{i=1}^l \mathrm{rng}(\bar{c}'_i) = \bigcup_{i=1}^l \mathrm{rng}(\bar{c}^*_i)$ it follows from (4.1) and (4.2) that there is a

$$\text{permutation } \pi \text{ of } \big\{(i,j) : 1 \leq i \leq l,\ 1 \leq j \leq s\big\}$$

such that

$$\text{(4.3)} \qquad \text{for all } i = 1, \dots, l \text{ and all } j = 1, \dots, s,\ c'_{\pi(i,j)} = c^*_{i,j}$$

and (since p is a complete type)

$$\mathcal{M}^{\mathrm{eq}} \models \forall x_{1,1}, \dots, x_{1,s}, \dots, x_{l,1}, \dots, x_{l,s} \Big(p(x_{1,1}, \dots, x_{1,s}, \dots, x_{l,1}, \dots, x_{l,s}) \rightarrow p(x_{\pi(1,1)}, \dots, x_{\pi(1,s)}, \dots, x_{\pi(l,1)}, \dots, x_{\pi(l,s)}) \Big).$$

This together with (4.1) and (4.2) gives

$$\text{(4.4)} \qquad \begin{aligned} \mathcal{M}^{\mathrm{eq}} \models\ & p(c_{\pi(1,1)}, \dots, c_{\pi(1,s)}, \dots, c_{\pi(l,1)}, \dots, c_{\pi(l,s)}) \\ & \wedge p(c''_{1,1}, \dots, c''_{1,s}, \dots, c''_{l,1}, \dots, c''_{l,s}). \end{aligned}$$

From (4.2) and (4.3) it follows that, for all $i = 1, \dots, l$ and all $j = 1, \dots, s$, $E(c'_{\pi(i,j)}, c''_{i,j})$. From (4.1) we get $E(c_{i,j}, c'_{i,j})$ and hence $E(c_{\pi(i,j)}, c'_{\pi(i,j)})$ for all $i = 1, \dots, l$ and all $j = 1, \dots, s$. By transitivity of E we get $E(c_{\pi(i,j)}, c''_{i,j})$ for all $i = 1, \dots, l$ and all $j = 1, \dots, s$. This and (4.4) imply that $a \sim a''$, so '$\sim$' is transitive. □

Recall that if $\mathcal{R} \models p^+(\bar{c}_1 \dots \bar{c}_l)$ then $\mathcal{M}^{\mathrm{eq}} \models p(\bar{c}_1 \dots \bar{c}_l)$ and, for all $i = 1, \dots, l$ and all $j = 1, \dots, s$, $c_{i,j} \in X_i$. Moreover, the range of every tuple that realizes p is a line.

For the rest of this section we assume that $l > 1$.

From this we will derive a contradiction and thus prove Proposition 4.4. Since $\mathcal{R}$ is a binary random structure there are

$$\text{(4.5)} \qquad \begin{aligned} &\bar{a}_i = (a_{i,1}, \dots, a_{i,s}), i = 1, \dots, l, \\ &\bar{b}_i = (b_{i,1}, \dots, b_{i,s}), i = 1, \dots, l, \quad \text{such that} \\ &\bar{a}_l = \bar{b}_l,\ \bigcup_{i=1}^{l} \mathrm{rng}(\bar{a}_i) \ \cap\ \bigcup_{i=1}^{l} \mathrm{rng}(\bar{b}_i) = \mathrm{rng}(\bar{a}_l) \quad \text{and} \\ &\mathcal{R} \models p^+(\bar{a}_1 \dots \bar{a}_l) \ \wedge\ p^+(\bar{b}_1 \dots \bar{b}_l). \end{aligned}$$

Moreover, there is a disjoint copy (up to isomorphism in $\mathcal{R}$) of the above elements. More precisely, there are

$$\begin{aligned} &\bar{a}'_i = (a'_{i,1}, \ldots, a'_{i,s}), i = 1, \ldots, l, \\ &\bar{b}'_i = (b'_{i,1}, \ldots, b'_{i,s}), i = 1, \ldots, l, \quad \text{such that} \\ &(\bar{a}'_1, \ldots, \bar{a}'_l, \bar{b}'_1, \ldots, \bar{b}'_l) \equiv_{\mathcal{R}} (\bar{a}_1, \ldots, \bar{a}_l, \bar{b}_1, \ldots, \bar{b}_l), \quad \text{and} \\ &\bigcup_{i=1}^{l} (\mathrm{rng}(\bar{a}'_i) \cup \mathrm{rng}(\bar{b}'_i)) \ \cap \ \bigcup_{i=1}^{l} (\mathrm{rng}(\bar{a}_i) \cup \mathrm{rng}(\bar{b}_i)) = \emptyset, \end{aligned} \tag{4.6}$$

and consequently

$$\begin{aligned} &\bar{a}'_l = \bar{b}'_l, \ \bigcup_{i=1}^{l} \mathrm{rng}(\bar{a}'_i) \ \cap \ \bigcup_{i=1}^{l} \mathrm{rng}(\bar{b}'_i) = \mathrm{rng}(\bar{a}'_l) \quad \text{and} \\ &\mathcal{R} \models p^+(\bar{a}'_1 \ldots \bar{a}'_l) \ \wedge \ p^+(\bar{b}'_1 \ldots \bar{b}'_l). \end{aligned}$$

Moreover (as $\mathcal{R}$ is a binary random structure), we can choose $\bar{a}'_i, \bar{b}'_i$, $i = 1, \ldots, l$, so that, in addition to (4.6), the following holds:

$$\text{for all } i = 1, \ldots, l, \ (\bar{a}'_1, \bar{a}_i) \equiv_{\mathcal{R}} (\bar{b}'_1, \bar{a}_i) \ \text{ and } \ (\bar{a}'_1, \bar{b}_i) \equiv_{\mathcal{R}} (\bar{b}'_1, \bar{b}_i). \tag{4.7}$$

If σ is a permutation of a set I, $\{e_i : i \in I\}$ is a set indexed by I and $\bar{e} = (e_{i_1}, \ldots, e_{i_k})$ where $i_1, \ldots, i_k \in I$, then we let $\sigma(\bar{e})$ denote the sequence $(e_{\sigma(i_1)}, \ldots, e_{\sigma(i_k)})$.

Lemma 4.12. *There are*

$$\begin{aligned} &\bar{a}''_i = (a''_{i,1}, \ldots, a''_{i,s}), i = 1, \ldots, l, \\ &\bar{b}''_i = (b''_{i,1}, \ldots, b''_{i,s}), i = 1, \ldots, l, \quad \textit{and} \\ &\textit{a permutation } \sigma \textit{ of } \{(i,j) : 1 \leq i \leq l, \ 1 \leq j \leq s\} \textit{ such that} \\ &\bar{a}''_1 = \bar{b}''_1, \ \bigcup_{i=1}^{l} \mathrm{rng}(\bar{a}''_i) \ \cap \ \bigcup_{i=1}^{l} \mathrm{rng}(\bar{b}''_i) = \mathrm{rng}(\bar{a}''_1), \\ &\mathcal{R} \models p^+(\bar{a}''_1 \ldots \bar{a}''_l) \ \wedge \ p^+(\bar{b}''_1 \ldots \bar{b}''_l) \quad \textit{and} \\ &(\bar{a}_1, \ldots, \bar{a}_l, \bar{b}_1, \ldots, \bar{b}_l) \equiv_{\mathcal{M}^{\mathrm{eq}}} (\sigma(\bar{a}''_1), \ldots, \sigma(\bar{a}''_l), \sigma(\bar{b}''_1), \ldots, \sigma(\bar{b}''_l)). \end{aligned} \tag{4.8}$$

Proof. From the choice of p in Notation 4.10, it follows that $\bar{a}_1 \equiv_{\mathcal{M}^{\mathrm{eq}}} \bar{a}_l$. Hence there is an automorphism f of $\mathcal{M}^{\mathrm{eq}}$ such that $f(\bar{a}_l) = \bar{a}_1$ (and since $\bar{a}_l = \bar{b}_l$ we also have $f(\bar{b}_l) = \bar{a}_1$). Let

$$\begin{aligned} &\bar{a}^*_i = (a^*_{i,1}, \ldots, a^*_{i,s}) = (f(a_{i,1}), \ldots, f(a_{i,s})) \quad \text{and} \\ &\bar{b}^*_i = (b^*_{i,1}, \ldots, b^*_{i,s}) = (f(b_{i,1}), \ldots, f(b_{i,s})) \quad \text{for } i = 1, \ldots, l. \end{aligned}$$

Then

$$(\bar{a}_1, \ldots, \bar{a}_l, \bar{b}_1, \ldots, \bar{b}_l) \ \equiv_{\mathcal{M}^{\mathrm{eq}}} \ (\bar{a}^*_1, \ldots, \bar{a}^*_l, \bar{b}^*_1, \ldots, \bar{b}^*_l) \ \text{ where } \bar{a}^*_l = \bar{b}^*_l = \bar{a}_1, \tag{4.9}$$

from which it follows that

$$\begin{aligned} &\mathcal{M}^{\mathrm{eq}} \models p(\bar{a}^*_1 \ldots \bar{a}^*_l) \ \wedge \ p(\bar{b}^*_1 \ldots \bar{b}^*_l) \quad \text{and} \\ &\text{for all } i = 1, \ldots, l \text{ and all } j = 1, \ldots, s, \ E(a^*_{i,j}, b^*_{i,j}). \end{aligned} \tag{4.10}$$

Then $\mathrm{rng}(\bar{a}^*_1) \cup \ldots \cup \mathrm{rng}(\bar{a}^*_l)$ is a line so, by Lemma 4.11, there is a

$$\text{permutation } \pi \text{ of } \{(i,j) : 1 \leq i \leq l, \ 1 \leq j \leq s\}$$

such that

$$\begin{aligned} &\mathrm{rng}(\pi(\bar{a}_i^*)) \subseteq X_i \text{ for all } i = 1, \ldots, l, \text{ and} \\ &\mathcal{M}^{\mathrm{eq}} \models p(\pi(\bar{a}_1^*) \ldots \pi(\bar{a}_l^*)). \end{aligned} \tag{4.11}$$

This and (4.10) implies that

$$\mathcal{M}^{\mathrm{eq}} \models p(\pi(\bar{b}_1^*) \ldots \pi(\bar{b}_l^*)). \tag{4.12}$$

From (4.10) and (4.11) it follows that

$$\text{for all } i = 1, \ldots, l, \ \mathrm{rng}(\pi(\bar{b}_i^*)) \subseteq X_i. \tag{4.13}$$

Thus we have

$$\big\{\pi(1, j) : 1 \leq j \leq l\big\} \ = \ \big\{(l, j) : 1 \leq j \leq s\big\},$$

so there is a permutation γ of $\big\{(l, j) : 1 \leq i \leq s\big\}$ such that

$$\bar{a}_1^* = \pi^{-1}\gamma(\bar{a}_l^*) \ \text{ and } \ \bar{b}_1^* = \pi^{-1}\gamma(\bar{b}_l^*).$$

As $\bar{a}_l^* = \bar{b}_l^*$ we get $\bar{a}_1^* = \bar{b}_1^*$ and hence

$$\pi(\bar{a}_1^*) = \pi(\bar{b}_1^*).$$

By (4.11) – (4.13) and Lemma 4.7 we get

$$\mathcal{R} \models p^+(\pi(\bar{a}_1^*) \ldots \pi(\bar{a}_l^*)) \ \wedge \ p^+(\pi(\bar{b}_1^*) \ldots \pi(\bar{b}_l^*)).$$

If we now let $\bar{a}_i'' = \pi(\bar{a}_i^*)$, $\bar{b}_i'' = \pi(\bar{b}_i^*)$, for $i = 1, \ldots, l$, and $\sigma = \pi^{-1}$, then (4.8) is satisfied, so the lemma is proved. □

By Lemma 4.12 there are $\bar{a}_i'', \bar{b}_i''$, $i = 1, \ldots, l$, so that (4.8) holds. As $\mathcal{R}$ is a binary random structure, and by (4.7), we can choose these elements so that, in addition to (4.8),

$$\begin{aligned} &\bigcup_{i=1}^{l} \big(\mathrm{rng}(\bar{a}_i'') \cup \mathrm{rng}(\bar{b}_i'')\big) \ \cap \ \bigcup_{i=1}^{l} \big(\mathrm{rng}(\bar{a}_i) \cup \mathrm{rng}(\bar{b}_i) \cup \mathrm{rng}(\bar{a}_i') \cup \mathrm{rng}(\bar{b}_i')\big) = \emptyset, \\ &(\bar{a}_1'', \ldots, \bar{a}_l'', \bar{a}_1, \ldots, \bar{a}_l) \ \equiv_{\mathcal{R}} \ (\bar{a}_1', \ldots, \bar{a}_l', \bar{a}_1, \ldots, \bar{a}_l), \\ &(\bar{a}_1'', \ldots, \bar{a}_l'', \bar{b}_1, \ldots, \bar{b}_l) \ \equiv_{\mathcal{R}} \ (\bar{a}_1', \ldots, \bar{a}_l', \bar{b}_1, \ldots, \bar{b}_l), \\ &(\bar{b}_1'', \ldots, \bar{b}_l'', \bar{a}_1, \ldots, \bar{a}_l) \ \equiv_{\mathcal{R}} \ (\bar{b}_1', \ldots, \bar{b}_l', \bar{a}_1, \ldots, \bar{a}_l), \ \text{ and} \\ &(\bar{b}_1'', \ldots, \bar{b}_l'', \bar{b}_1, \ldots, \bar{b}_l) \ \equiv_{\mathcal{R}} \ (\bar{b}_1', \ldots, \bar{b}_l', \bar{b}_1, \ldots, \bar{b}_l). \end{aligned} \tag{4.14}$$

Since $\mathcal{C}$ is a reduct of $\mathcal{R}$ (and $\mathcal{C}$ is canonically embedded in $\mathcal{M}^{\mathrm{eq}}$) it follows that

$$\text{in (4.14) we can replace `}\equiv_{\mathcal{R}}\text{' by `}\equiv_{\mathcal{M}^{\mathrm{eq}}}\text{'}. \tag{4.15}$$

Let $a, a', a'', b, b', b'' \in M$ be such that

$$\begin{aligned} \mathrm{crd}(a) &= \mathrm{rng}(\bar{a}_1) \cup \ldots \cup \mathrm{rng}(\bar{a}_l), & \mathrm{crd}(b) &= \mathrm{rng}(\bar{b}_1) \cup \ldots \cup \mathrm{rng}(\bar{b}_l), \\ \mathrm{crd}(a') &= \mathrm{rng}(\bar{a}_1') \cup \ldots \cup \mathrm{rng}(\bar{a}_l'), & \mathrm{crd}(b') &= \mathrm{rng}(\bar{b}_1') \cup \ldots \cup \mathrm{rng}(\bar{b}_l'), \\ \mathrm{crd}(a'') &= \mathrm{rng}(\bar{a}_1'') \cup \ldots \cup \mathrm{rng}(\bar{a}_l'') \ \text{ and} & \mathrm{crd}(b'') &= \mathrm{rng}(\bar{b}_1'') \cup \ldots \cup \mathrm{rng}(\bar{b}_l''). \end{aligned}$$

We have $a \in \mathrm{dcl}_{\mathcal{M}^{\mathrm{eq}}}(\mathrm{crd}(a))$ and similarly for a', a'', b, b' and b''. Therefore it follows from (4.15) and (4.14) that

$$\begin{aligned} (a', b') &\equiv_{\mathcal{M}^{\mathrm{eq}}} (a'', b''), & (a', a) &\equiv_{\mathcal{M}^{\mathrm{eq}}} (a'', a), \\ (a', b) &\equiv_{\mathcal{M}^{\mathrm{eq}}} (a'', b), & (b', a) &\equiv_{\mathcal{M}^{\mathrm{eq}}} (b'', a), \ \text{ and} \\ (b', b) &\equiv_{\mathcal{M}^{\mathrm{eq}}} (b'', b). \end{aligned}$$

Since all the involved elements belong to M we can replace '$\equiv_{\mathcal{M}^{\mathrm{eq}}}$' by '$\equiv_{\mathcal{M}}$'. As $\mathcal{M}$ is binary with elimination of quantifiers we get

$$(a, b, a', b') \equiv_{\mathcal{M}} (a, b, a'', b''). \tag{4.16}$$

Now consider a formula $\varphi(x_1, x_2, x_3, x_4)$ in the language of $\mathcal{M}^{\mathrm{eq}}$ which expresses the following:

> "$x_1, x_2, x_3, x_4 \in M$ and there is an E-class X such that $\mathrm{crd}(x_1) \cap X = \mathrm{crd}(x_2) \cap X$ and $\mathrm{crd}(x_3) \cap X = \mathrm{crd}(x_4) \cap X$."

It is straightforward to verify that

$$\mathcal{M}^{\mathrm{eq}} \models \varphi(a, b, a', b') \ \wedge \ \neg\varphi(a, b, a'', b''),$$

so $(a, b, a', b') \not\equiv_{\mathcal{M}^{\mathrm{eq}}} (a, b, a'', b'')$. Since $a, a', a'', b, b', b'' \in M$ it follows that $(a, b, a', b') \not\equiv_{\mathcal{M}} (a, b, a'', b'')$, which contradicts (4.16). This completes the proof of Proposition 4.4 and hence of Theorem 1.1.

5. Definable equivalence relations

In this section we prove a result, Theorem 5.1, about $\emptyset$-definable equivalence relations on n-tuples (for any fixed $0 < n < \omega$) in ω-categorical supersimple structures with SU-rank 1 and degenerate algebraic closure. One reason for doing this is that the author thinks that this result may be useful in future research about nonbinary simple homogeneous structures. Another reason is that a variant of Theorem 5.1, namely Theorem 6.1, gives (under certain conditions) a full characterization of the $\emptyset$-definable equivalence relations on n-tuples, for any $0 < n < \omega$. Theorem 6.1 is then used to prove Proposition 6.5 which implies that the generic tetrahedron-free 3-hypergraph is 1-based.

Throughout this section we suppose that $\mathcal{M}$ is ω-categorical, supersimple with SU-rank 1 and with degenerate algebraic closure. Let $n < \omega$ and let $p(\bar{x}) \in S_n^{\mathcal{M}^{\mathrm{eq}}}(\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(\emptyset))$ be realized by some n-tuple of elements from M.

Let

$$X = \{\bar{a} \in M^n : \mathcal{M} \models p(\bar{a})\}.$$

Suppose that E is an equivalence relation on X which is $\emptyset$-definable in $\mathcal{M}$. In other words, there is a formula $\varphi(\bar{x}, \bar{y})$ without parameters, in the language of $\mathcal{M}$, such that for all $\bar{a}, \bar{b} \in X$, $E(\bar{a}, \bar{b})$ if and only if $\mathcal{M} \models \varphi(\bar{a}, \bar{b})$.

Theorem 5.1. *Suppose that E is nontrivial, i.e. it has at least two equivalence classes and at least one of the equivalence classes has more than one element. Then there is a nonempty $I \subseteq \{1, \ldots, n\}$, a group of permutations Γ of I and $\emptyset$-definable equivalence relations E' and E'' on X such that the following hold:*

(a) *$E'' \subseteq E \subseteq E'$.*

(b) *For all $\bar{a} = (a_1, \ldots, a_n), \bar{b} = (b_1, \ldots, b_n) \in X$, $E'(\bar{a}, \bar{b})$ if and only if there is a permutation $\gamma \in \Gamma$ of I such that $a_i = b_{\gamma(i)}$ for all $i \in I$.*

(c) *For all $\bar{a} = (a_1, \ldots, a_n), \bar{b} = (b_1, \ldots, b_n) \in X$, $E''(\bar{a}, \bar{b})$ if and only if $a_i = b_i$ for all $i \in I$ and if $\{1, \ldots, n\} \setminus I = \{i_1, \ldots, i_m\}$, then*

$$\mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}(a_{i_1}, \ldots, a_{i_m}/\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(\{a_i : i \in I\})) = \mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}(b_{i_1}, \ldots, b_{i_m}/\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(\{a_i : i \in I\})).$$

The rest of this section proves this theorem. Without loss of generality we assume that $p(\bar{x})$ implies $x_i \neq x_j$ for all $1 \leq i < j \leq n$. In this section and the next we frequently abuse notation by notationally identifying '$\bar{a}$' and '$\mathrm{rng}(\bar{a})$'. ***From now on suppose that E is nontrivial.*** Several times in this section and the next we will use the following:

Observation 5.2. *(i) For all $\bar{a}, \bar{b}, \bar{c} \in M$, $\bar{a} \underset{\bar{c}}{\downarrow} \bar{b}$ if and only if $\bar{a} \cap \bar{b} \subseteq \bar{c}$.*

(ii) For all $\bar{a}, \bar{b}, \bar{c} \in M$ there is $\bar{a}' \in M$ such that $\mathrm{tp}_{\mathcal{M}}(\bar{a}'/\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(\bar{c})) = \mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}(\bar{a}/\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(\bar{c}))$ and $\bar{a}' \cap \bar{b} \subseteq \bar{c}$.

Both parts of Observation 5.2 are straightforward to show and hold under the assumptions in this section. We note however that (ii) is a direct consequence of [1, Proposition 1.5 (1)] (which shows that 'algebraic independence' satisfies 'full existence'). Part (i) is *not* used in the proof of Theorem 6.1, but it *is* used in the proof of Proposition 6.5, which has even stronger assumptions than the present section. Part (ii) holds under the assumptions of Theorem 6.1 and the assumptions of Proposition 6.5.

Lemma 5.3. *Either $E(\bar{a}, \bar{b})$ holds for all disjoint $\bar{a}, \bar{b} \in X$, or $\neg E(\bar{a}, \bar{b})$ holds for all disjoint $\bar{a}, \bar{b} \in X$.*

Proof. For a contradiction suppose that $\bar{a}, \bar{b} \in X$ are disjoint and $E(\bar{a}, \bar{b})$ and that $\bar{c}, \bar{d} \in X$ are disjoint and $\neg E(\bar{c}, \bar{d})$. As $\mathcal{M}$ is ω-categorical it follows from the definition of X that there is $\bar{b}' \in X$ such that

$$\mathrm{tp}_{\mathcal{M}}(\bar{b}, \bar{b}') = \mathrm{tp}_{\mathcal{M}}(\bar{c}, \bar{d}).$$

Then $\bar{b} \cap \bar{b}' = \emptyset$. As $\mathrm{acl}_{\mathcal{M}}$ is degenerate and $\mathcal{M}$ has SU-rank 1 we get $\bar{b} \mathop{\smile\!\!\!\!\!\!\;|}\, \bar{b}'$. By assumption, $\bar{a}$ and $\bar{b}$ are disjoint, so (by Observation 5.2 (i)) $\bar{a} \mathop{\smile\!\!\!\!\!\!\;|}\, \bar{b}$. By definition of X, $\mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}(\bar{b}/\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(\emptyset)) = p = \mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}(\bar{b}'/\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(\emptyset))$. Therefore the independence theorem implies that there is $\bar{e}$ such that $\bar{e}$ realizes p, and hence $\bar{e} \in X$, $\mathrm{tp}_{\mathcal{M}}(\bar{a}, \bar{e}) = \mathrm{tp}_{\mathcal{M}}(\bar{a}, \bar{b})$ and $\mathrm{tp}_{\mathcal{M}}(\bar{b}, \bar{e}) = \mathrm{tp}_{\mathcal{M}}(\bar{b}, \bar{b}')$. This implies that $E(\bar{a}, \bar{e})$ and $\neg E(\bar{b}, \bar{e})$. Together with the assumption that $E(\bar{a}, \bar{b})$ we have a contradiction to the symmetry and transitivity of E. □

Lemma 5.4. *For all disjoint $\bar{a}, \bar{b} \in X$ we have $\neg E(\bar{a}, \bar{b})$.*

Proof. Suppose that the lemma is false. By Lemma 5.3, for all disjoint $\bar{a}, \bar{b} \in X$ we have $E(\bar{a}, \bar{b})$. Since E is nontrivial there are $\bar{b}, \bar{c} \in X$ such that $\neg E(\bar{b}, \bar{c})$. By Observation 5.2 (ii), there is $\bar{a} \in X$ which is disjoint from $\bar{b}$ and from $\bar{c}$. Then $E(\bar{a}, \bar{b})$, $E(\bar{a}, \bar{c})$ and $\neg E(\bar{b}, \bar{c})$, which contradicts the symmetry and transitivity of E. □

Lemma 5.5. *Suppose that $\bar{a}, \bar{b} \in X$, $E(\bar{a}, \bar{b})$, $\bar{a} \cap \bar{b} \neq \emptyset$, $\bar{a} \cap \bar{b} = \{a_{i_1}, \ldots, a_{i_k}\} = \{b_{j_1}, \ldots, b_{j_k}\}$ (where elements are listed without repetition) and $\{i_1, \ldots, i_k\} \neq \{j_1, \ldots, j_k\}$ (so $k \geq 1$). Then there is $\bar{c} \in X$ such that $E(\bar{a}, \bar{c})$ and $\bar{a} \cap \bar{c}$ is a proper subset of $\bar{a} \cap \bar{b}$.*

Proof. By reindexing variables if necessary we may, without loss of generality, assume that for some $0 < m \leq n - k$ and some permutation γ of $\{1, \ldots, k\}$

$$\bar{a} \cap \bar{b} = \{a_1, \ldots, a_k\} \text{ and } a_i = b_{m+\gamma(i)} \text{ for all } i = 1, \ldots, k.$$

In particular, $b_1, \ldots, b_m, b_{m+k+1}, \ldots, b_n \notin \bar{a}$. Since $\bar{a}, \bar{b} \in X$ we have $\mathrm{tp}_{\mathcal{M}}(\bar{b}) = \mathrm{tp}_{\mathcal{M}}(\bar{a})$ so (by ω-categoricity) there is $\bar{c} \in X$ such that $\mathrm{tp}_{\mathcal{M}}(\bar{b}, \bar{c}) = \mathrm{tp}_{\mathcal{M}}(\bar{a}, \bar{b})$. Then

$$\bar{b} \cap \bar{c} = \{b_1, \ldots, b_k\} \text{ and } b_i = c_{m+\gamma(i)} \text{ for all } i = 1, \ldots, k.$$

By Observation 5.2 (ii), we may assume that $c_1, \ldots, c_m, c_{m+k+1}, \ldots, c_n \notin \bar{a} \cup \bar{b}$. It follows that

$$\bar{a} \cap \bar{c} \subseteq \{c_{m+1}, \ldots, c_{m+k}\} = \{b_1, \ldots, b_k\}.$$

If $i \leq m$ then (as we concluded above) $b_i \notin \bar{a}$ and therefore we get

$$\bar{a} \cap \bar{c} \subseteq \{b_{m+1}, \ldots, b_k\}.$$

Since $\bar{a} \cap \bar{b} = \{b_{m+1}, \ldots, b_{m+k}\}$ and $m \geq 1$ it follows that $\bar{a} \cap \bar{c}$ is a proper subset of $\bar{a} \cap \bar{b}$. As $\mathrm{tp}_{\mathcal{M}}(\bar{b}, \bar{c}) = \mathrm{tp}_{\mathcal{M}}(\bar{a}, \bar{b})$ we also have $E(\bar{b}, \bar{c})$. By transitivity of E we get $E(\bar{a}, \bar{c})$. □

Let k be minimal such that there are $\bar{a}, \bar{b} \in X$ such that $E(\bar{a}, \bar{b})$ and $|\bar{a} \cap \bar{b}| = k$. By Lemma 5.4, $k > 0$.

Lemma 5.6. *There is $I \subseteq \{1, \ldots, n\}$ such that $|I| = k$ and for all $\bar{a}, \bar{b} \in X$ such that $E(\bar{a}, \bar{b})$ and $|\bar{a} \cap \bar{b}| = k$, we have $\bar{a} \cap \bar{b} = \{a_i : i \in I\} = \{b_i : i \in I\}$.*

Proof. For every $\bar{a} \in X$ let $k_{\bar{a}}$ be minimal such that there is $\bar{b} \in X$ such that $E(\bar{a}, \bar{b})$ and $|\bar{a} \cap \bar{b}| = k_{\bar{a}}$. Since all $\bar{a} \in X$ have the same complete type it follows that for all $\bar{a} \in X$ there is $\bar{b} \in X$ such that $E(\bar{a}, \bar{b})$ and $|\bar{a} \cap \bar{b}| = k$. Hence $k_{\bar{a}} = k$ for all $\bar{a} \in X$.

By the minimality of k and Lemma 5.5, if $\bar{a}, \bar{b} \in X$, $E(\bar{a}, \bar{b})$ and $|\bar{a} \cap \bar{b}| = k$, then there is $I_{\bar{a},\bar{b}} \subseteq \{1, \ldots, n\}$ such that $|I_{\bar{a},\bar{b}}| = k$ and $\bar{a} \cap \bar{b} = \{a_i : i \in I_{\bar{a},\bar{b}}\} = \{b_i : i \in I_{\bar{a},\bar{b}}\}$. Suppose that for some other $\bar{c} \in X$ we have $E(\bar{a}, \bar{c})$, $|\bar{a} \cap \bar{c}| = k$ and $I_{\bar{a},\bar{c}} \neq I_{\bar{a},\bar{b}}$. By Observation 5.2 (ii), we may assume that for every $i \notin I_{\bar{a},\bar{c}}$, $c_i \notin \bar{b}$. It follows that $|\bar{b} \cap \bar{c}| < k$ and, by transitivity of E, that $E(\bar{b}, \bar{c})$, which contradicts the choice of k. Hence we conclude that $I_{\bar{a},\bar{b}} = I_{\bar{a},\bar{c}}$ for all $\bar{b}, \bar{c} \in X$. Thus we denote $I_{\bar{a},\bar{b}}$ by $I_{\bar{a}}$ for any $\bar{b} \in X$. As all $\bar{a} \in X$ have the same complete type we have $I_{\bar{a}} = I_{\bar{b}}$ for all $\bar{a}, \bar{b} \in X$. So we denote $I_{\bar{a}}$ by I for any $\bar{a} \in X$. □

Let I be as in Lemma 5.6. To simplify notation and without loss of generality we assume that

$$I = \{1, \ldots, k\}.$$

For any $\bar{a} \in X$ let Γ be the set permutations γ of I such that for some $\bar{b} \in X$, $E(\bar{a}, \bar{b})$, $\bar{a} \cap \bar{b} = \{a_i : i \in I\}$ and $a_i = b_{\gamma(i)}$ for all $i \in I$. As all $\bar{a} \in X$ have the same complete type, Γ does not depend on $\bar{a}$. By the transitivity of E and since all $\bar{a} \in X$ have the same complete type it follows that Γ is closed under composition. By the symmetry of E, Γ is closed under inverses. Hence Γ is a *group* of permutations of I.

Lemma 5.7. *If $\bar{a}, \bar{b} \in X$ and $E(\bar{a}, \bar{b})$ then there is $\gamma \in \Gamma$ such that, for all $i \in I$, $a_i = b_{\gamma(i)}$.*

Proof. Suppose that $\bar{a}, \bar{b} \in X$ and $E(\bar{a}, \bar{b})$. Since all tuples in X have the same complete type and Γ contains the identity permutation there is $\bar{c} \in X$ such that

$$E(\bar{b}, \bar{c}), \ \bar{b} \cap \bar{c} = \{b_i : i \in I\} \ \text{ and } c_i = b_i \text{ for all } i \in I.$$

Moreover (by Observation 5.2 (ii)), we may assume that $\bar{a} \cap \bar{c} \subseteq \{c_i : i \in I\}$. By transitivity of E we have $E(\bar{a}, \bar{c})$. Therefore $\bar{a} \cap \bar{c} = \{c_i : i \in I\}$ by choice of k. Suppose that

$$\{a_i : i \in I\} \neq \bar{a} \cap \bar{c}.$$

It follows that $\{i : a_i \in \bar{c}\} \neq \{i : c_i \in \bar{a}\}$. Then Lemma 5.5 implies that there is $\bar{d} \in X$ such that $\bar{c} \cap \bar{d} \subsetneq \bar{c} \cap \bar{a}$ and $E(\bar{c}, \bar{d})$, which contradicts the choice of k. Thus we conclude that $\{a_i : i \in I\} = \bar{a} \cap \bar{c} = \{c_i : i \in I\}$, and hence there is $\gamma \in \Gamma$ such that $a_i = c_{\gamma(i)}$ for all $i \in I$. As $b_i = c_i$ for all $i \in I$ we get $a_i = b_{\gamma(i)}$ for all $i \in I$. □

From Lemma 5.7 it follows that if E' is defined as in (b) of Theorem 5.1, then $E \subseteq E'$. For the rest of this section let E'' be defined as in (c) of Theorem 5.1. It remains to prove that $E'' \subseteq E$.

For $\gamma \in \Gamma$ and $\bar{a} = (a_1, \ldots, a_k)$ we use the notation $\gamma(\bar{a}) = (a_{\gamma(1)}, \ldots, a_{\gamma(k)})$.

Lemma 5.8. *There are $\bar{a}' \in M^k$, $\bar{a}^*, \bar{b}^* \in M^{n-k}$ such that $\bar{a}'\bar{a}^*, \bar{a}'\bar{b}^* \in X$, $E(\bar{a}'\bar{a}^*, \bar{a}'\bar{b}^*)$, $\bar{a}^* \cap \bar{b}^* = \emptyset$ and $\mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}(\bar{a}^*/\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(\bar{a}')) = \mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}(\bar{b}^*/\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(\bar{a}'))$.*

Proof. By the choice of k and I (and since the identity on I belongs to Γ) there are $\bar{a}' \in M^k$, $\bar{b}', \bar{c}' \in M^{n-k}$ such that $\bar{a}'\bar{b}', \bar{a}'\bar{c}' \in X$, $E(\bar{a}'\bar{b}', \bar{a}'\bar{c}')$ and $\bar{b}' \cap \bar{c}' = \emptyset$. By Observation 5.2 (ii), there are in fact $\bar{c}'_i \in M^{n-k}$ for all $i < \omega$ such that $\bar{a}'\bar{c}'_i \in X$, $E(\bar{a}'\bar{b}', \bar{a}'\bar{c}'_i)$, $\bar{b}' \cap \bar{c}'_i = \emptyset$ and $\bar{c}'_i \cap \bar{c}'_j = \emptyset$ for all $i < j < \omega$. By ω-categoricity there must be $i < j < \omega$ such that

$$\mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}(\bar{c}'_i/\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(\bar{a}')) = \mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}(\bar{c}'_j/\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(\bar{a}')).$$

By symmetry and transitivity we get $E(\bar{a}'\bar{c}_i', \bar{a}'\bar{c}_j')$, so we are done by taking $\bar{a}^* = \bar{c}_i'$ and $\bar{b}^* = \bar{c}_j'$. □

Lemma 5.9. *Let $\bar{a}' \in M^k$, $\bar{a}^*, \bar{b}^* \in M^{n-k}$ and suppose that $\bar{a}'\bar{a}^*, \bar{a}'\bar{b}^* \in X$, $E''(\bar{a}'\bar{a}^*, \bar{a}'\bar{b}^*)$ and $\bar{a}^* \cap \bar{b}^* = \emptyset$. Then $E(\bar{a}'\bar{a}^*, \bar{a}'\bar{b}^*)$.*

Proof. Let $\bar{a}', \bar{a}^*$ and $\bar{b}^*$ satisfy the assumptions of the lemma. Then, by the definition of E'',

$$\mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}(\bar{a}^*/\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(\bar{a}')) = \mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}(\bar{b}^*/\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(\bar{a}')).$$

Since all tuples in X have the same type over $\emptyset$ (in fact even over $\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(\emptyset)$) it follows from Lemma 5.8 that there are $\bar{c}^*, \bar{d}^* \in M^{n-k}$ such that $\bar{a}'\bar{c}^*, \bar{a}'\bar{d}^* \in X$, $E(\bar{a}'\bar{a}^*, \bar{a}'\bar{c}^*)$, $E(\bar{a}'\bar{b}^*, \bar{a}'\bar{d}^*)$, $\bar{c}^* \cap \bar{a}^* = \emptyset$, $\bar{d}^* \cap \bar{b}^* = \emptyset$,

$$\mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}(\bar{c}^*/\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(\bar{a}')) = \mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}(\bar{a}^*/\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(\bar{a}')) \quad \text{and}$$
$$\mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}(\bar{d}^*/\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(\bar{a}')) = \mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}(\bar{b}^*/\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(\bar{a}')).$$

Then

$$\mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}(\bar{c}^*/\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(\bar{a}')) = \mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}(\bar{d}^*/\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(\bar{a}')).$$

By Observation 5.2 (i), we get $\bar{a}^* \underset{\bar{a}'}{\downarrow} \bar{b}^*$, $\bar{c}^* \underset{\bar{a}'}{\downarrow} \bar{a}^*$ and $\bar{d}^* \underset{\bar{a}'}{\downarrow} \bar{b}^*$. By the independence theorem, there is $\bar{e} \in M^{n-k}$ such that

$$\mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}(\bar{e}^*, \bar{a}^*/\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(\bar{a}')) = \mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}(\bar{c}^*, \bar{a}^*/\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(\bar{a}')) \quad \text{and}$$
$$\mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}(\bar{e}^*, \bar{b}^*/\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(\bar{a}')) = \mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}(\bar{d}^*, \bar{b}^*/\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(\bar{a}')).$$

This implies that $\bar{a}'\bar{e}^* \in X$, $E(\bar{a}'\bar{a}^*, \bar{a}'\bar{e}^*)$ and $E(\bar{a}'\bar{b}^*, \bar{a}'\bar{e}^*)$, so by symmetry and transitivity of E we get $E(\bar{a}'\bar{a}^*, \bar{a}'\bar{b}^*)$. □

Lemma 5.10. *Let $\bar{a}' \in M^k$, $\bar{a}^*, \bar{b}^* \in M^{n-k}$ and suppose that $\bar{a}'\bar{a}^*, \bar{a}'\bar{b}^* \in X$ and $E''(\bar{a}'\bar{a}^*, \bar{a}'\bar{b}^*)$. Then $E(\bar{a}'\bar{a}^*, \bar{a}'\bar{b}^*)$.*

Proof. From $E''(\bar{a}'\bar{a}^*, \bar{a}'\bar{b}^*)$ we get

$$\mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}(\bar{a}^*/\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(\bar{a}')) = \mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}(\bar{b}^*/\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(\bar{a}')).$$

By Observation 5.2 (ii), there is $\bar{c}^*$ such that $\bar{a}'\bar{c}^* \in X$ and

$$\mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}(\bar{c}^*/\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(\bar{a}')) = \mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}(\bar{b}^*/\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(\bar{a}'))$$

and $\bar{c}^* \cap (\bar{a}^* \cup \bar{b}^*) = \emptyset$. From the definition of E'' we get $E''(\bar{a}'\bar{a}^*, \bar{a}'\bar{c}^*)$ and $E''(\bar{a}'\bar{c}^*, \bar{a}'\bar{b}^*)$. Since $\bar{a}^* \cap \bar{c}^* = \emptyset$ and $\bar{b}^* \cap \bar{c}^* = \emptyset$ we get $E(\bar{a}'\bar{a}^*, \bar{a}'\bar{c}^*)$ and $E(\bar{a}'\bar{c}^*, \bar{a}'\bar{b}^*)$ by Lemma 5.9. Hence $E(\bar{a}'\bar{a}^*, \bar{a}'\bar{b}^*)$ by the transitivity of E. □

Now Theorem 5.1 follows from Lemmas 5.7 and 5.10.

6. A variation of Theorem 5.1 and an application to the generic tetrahedron-free 3-hypergraph

Throughout this section, we assume that $\mathcal{M}$ is ω-categorical and that $\mathrm{acl}_{\mathcal{M}}$ ***is degenerate.*** Let $0 < n < \omega$, let $p(\bar{x}) \in S_n^{\mathcal{M}}(\emptyset)$ and let

$$X = \{\bar{a} \in M^n : \mathcal{M} \models p(\bar{a})\}.$$

Suppose that E is a $\emptyset$-definable nontrivial equivalence relation on X. Also assume the following:

(∗) *If $\bar{a}', \bar{a}^*, \bar{b}^*, \bar{c}^*, \bar{d}^* \in M$ are tuples such that their ranges are mutually disjoint, the starred sequences are nonemtpy (but $\bar{a}'$ is allowed to be empty) and $\bar{a}'\bar{a}^*, \bar{a}'\bar{b}^*, \bar{a}'\bar{c}^*, \bar{a}'\bar{d}^* \in X$, then there is $\bar{e}^* \in M$ such that*

$$\mathrm{tp}_{\mathcal{M}}(\bar{e}^*, \bar{a}^*, \bar{a}') = \mathrm{tp}_{\mathcal{M}}(\bar{c}^*, \bar{a}^*, \bar{a}') \quad \textit{and} \quad \mathrm{tp}_{\mathcal{M}}(\bar{e}^*, \bar{b}^*, \bar{a}') = \mathrm{tp}_{\mathcal{M}}(\bar{d}^*, \bar{b}^*, \bar{a}').$$

Let E' be defined as in part (b) of Theorem 5.1. Then Lemma 5.3 is proved with the help of $(*)$ (and empty $\bar{a}'$); there is no need to use the independence theorem or the notions of simplicity or SU-rank. The other lemmas in Section 5 up to (and including) Lemma 5.7 are proved in the same way as before.

Define E'' on X by

$$E''(\bar{a}, \bar{b}) \text{ if and only if } a_i = b_i \text{ for all } i \in I.$$

Then E'' is an equivalence relation which is $\emptyset$-definable in $\mathcal{M}$. For this E'', the statement of Lemma 5.9 is proved similarly as before, but using $(*)$ instead of the independence theorem and all occurences of '$\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(\bar{a}')$' are replaced by '$\bar{a}'$'. (The same substitutions can be made in Lemma 5.8 and its proof.) Lemma 5.10 follows from Lemma 5.9 in exactly the same way for E'' as defined in this section as was the case in Section 5. It follows that $E'' \subseteq E \subseteq E'$, where E' is defined in terms of some group Γ of permutations of I. Let Σ be the set of all $\gamma \in \Gamma$ such that there are $\bar{a}, \bar{b} \in X$ such that $E(\bar{a}, \bar{b})$ and $a_i = b_{\gamma(i)}$ for all $i \in I$. Since $E'' \subseteq E$ the identity function on I belongs to Σ. As E is symmetric and transitive it follows that Σ is closed under inverses and compositions. Hence Σ is a group of permutations. Thus we get the following version of Theorem 5.1:

Theorem 6.1. *Suppose that $\mathcal{M}$ is ω-categorical and that $\mathrm{acl}_{\mathcal{M}}$ is degenerate. Moreover suppose that $(*)$ holds for X as defined in this section and that E is a nontrivial $\emptyset$-definable equivalence relation on X. Then there is a nonempty $I \subseteq \{1, \ldots, n\}$ and a group Γ of permutations of I such that for all $\bar{a} = (a_1, \ldots, a_n), \bar{b} = (b_1, \ldots, b_n) \in X$, $E(\bar{a}, \bar{b})$ if and only if there is $\gamma \in \Gamma$ such that $a_i = b_{\gamma(i)}$ for all $i \in I$.*

Corollary 6.2. *Suppose that $\mathcal{M}$ is ω-categorical and that $\mathrm{acl}_{\mathcal{M}}$ is degenerate. Moreover, assume that $(*)$ holds for any choice of $0 < n < \omega$ and any $p \in S_n^{\mathcal{M}}(\emptyset)$. Then, for every $A \subseteq M$, $\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(A) = \mathrm{dcl}_{\mathcal{M}^{\mathrm{eq}}}(A)$.*

Remark: For the rest of this section we use the following notation. For every $n < \omega$, every $\emptyset$-definable equivalence relation E on M^n and every $\bar{a} \in M^n$, let $[\bar{a}]_E$ denote the E-equivalence class of $\bar{a}$ *as an element of* M^{eq} (and not as a subset of M^n).

Proof. Suppose that $A \subseteq M$, $b \in M^{\mathrm{eq}}$ and $b \in \mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(A)$. Without loss of generality we may assume that A is finite. For some $n < \omega$, some $\emptyset$-definable equivalence relation E on M^n and some $\bar{b} = (b_1, \ldots, b_n) \in M^n$ we have $b = [\bar{b}]_E$.

Let $p(\bar{x}) = \mathrm{tp}(\bar{b})$ and $X = \{\bar{a} \in M^n : \mathcal{M} \models p(\bar{a})\}$. Without loss of generality we may assume that $p(\bar{x})$ implies $x_i \neq x_j$ for all $1 \leq i < j \leq n$.[8] If for all $\bar{a}, \bar{a}' \in X$, $E(\bar{a}, \bar{a}')$, then $[\bar{b}]_E \in \mathrm{dcl}_{\mathcal{M}^{\mathrm{eq}}}(\emptyset)$. If for all $\bar{a} \in X$ such that $\bar{a} \neq \bar{b}$ we have $\neg E(\bar{a}, \bar{b})$, then $\bar{b} \in \mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}([\bar{b}]_E) \subseteq \mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(A)$ and hence $\bar{b} \in \mathrm{acl}_{\mathcal{M}}(A) = A$. It follows that $[\bar{b}]_E \in \mathrm{dcl}_{\mathcal{M}^{\mathrm{eq}}}(A)$.

Hence, from now on we can assume that E is nontrivial on X. By Theorem 6.1, there is a nonempty $I \subseteq \{1, \ldots, n\}$ and a group Γ of permutations of I such that for all $\bar{a} = (a_1, \ldots, a_n), \bar{a}' = (a'_1, \ldots, a'_n) \in X$, $E(\bar{a}', \bar{a}')$ and only if for some $\gamma \in \Gamma$ and all $i \in I$, $a_i = a'_{\gamma(i)}$. Without loss of generality assume that $I = \{1, \ldots, k\}$ where $k \leq n$.

First suppose that $b_1, \ldots, b_k \in A$. Let $\psi(x, b_1, \ldots, b_k)$ be the formula (with no other parameters than $b_1, \ldots, b_k$) which expresses:

> "x has sort E and if $x = [(y_1, \ldots, y_n)]_E$ then there is a permutation $\gamma \in \Gamma$ such that for all $i = 1, \ldots, k$, $b_i = y_{\gamma(i)}$."

Clearly, $b = [\bar{b}]_E$ satisfies $\psi(x, b_1, \ldots, b_k)$. And if $b' = [(b'_1, \ldots, b'_n)]_E$ satisfies $\psi(x, b_1, \ldots, b_k)$, then for some permutation $\gamma \in \Gamma$, $b_i = b'_{\gamma(i)}$ for all $i = 1, \ldots, k$ and hence $E(\bar{b}, (b'_1, \ldots, b'_n))$, so $b = b'$. Thus $b \in \mathrm{dcl}_{\mathcal{M}^{\mathrm{eq}}}(b_1, \ldots, b_k) \subseteq \mathrm{dcl}_{\mathcal{M}^{\mathrm{eq}}}(A)$.

Now suppose that for some $i \in I = \{1, \ldots, k\}$, $b_i \notin A$ (and we will derive a contradiction from this). To simplify notation, and without loss of generality, assume that $i = 1$.

[8] Because otherwise b is interdefinable with some b' for which this assumption holds.

Then $b_1 \notin \text{acl}_{\mathcal{M}^{\text{eq}}}(A \cup \{b_2, \ldots, b_n\})$ so for every $j < \omega$ there is $b_{1,j} \in M \setminus (A \cup \{b_{1,l} : l < j\})$ such that

$$\text{tp}_{\mathcal{M}}(b_{1,j}, b_2, \ldots, b_n/A) = \text{tp}_{\mathcal{M}}(b_1, b_2, \ldots, b_n/A).$$

Let $\bar{b}^j = (b_{1,j}, b_2, \ldots, b_n)$ for all $j < \omega$. Then for every $j < \omega$ there is an automorphism of $\mathcal{M}^{\text{eq}}$ which sends $\bar{b}$ to $\bar{b}^j$ and fixes A pointwise. As this automorphism sends $[\bar{b}]_E$ to $[\bar{b}^j]_E$ we get $\text{tp}_{\mathcal{M}^{\text{eq}}}([\bar{b}^j]_E/A) = \text{tp}_{\mathcal{M}^{\text{eq}}}([\bar{b}]_E/A)$ for all $j < \omega$. By the choice of $b_{1,j}$ we have $[\bar{b}^j]_E \neq [\bar{b}^l]_E$ whenever $j \neq l$, so this contradicts the assumption that $b = [\bar{b}]_E \in \text{acl}_{\mathcal{M}^{\text{eq}}}(A)$. □

Remark 6.3. Note that Theorems 5.1, 6.1 and Corollary 6.2 are really results about $Th(\mathcal{M})$ (rather than just $\mathcal{M}$). More precisely, the characterizations of equivalence relations in Theorems 5.1 and 6.1 hold if $\mathcal{M}$ (in the respective theorem) is replaced by any model of $Th(\mathcal{M})$. It follows (from its proof) that $\mathcal{M}$ in Corollary 6.2 can be replaced by any model of $Th(\mathcal{M})$.

Example 6.4. (i) It is easy to see that if $\mathcal{M}$ is a binary random structure, then $(*)$ is satisfied for any choice of $0 < n < \omega$ and any $p \in S_n^{\mathcal{M}}(\emptyset)$.
(ii) Let $\mathcal{M}$ be the generic tetrahedron-free 3-hypergraph given by Definition 2.3. It is straightforward to verify that $\mathcal{M}$ satisfies $(*)$ for any choice of $0 < n < \omega$ and any $p \in S_n^{\mathcal{M}}(\emptyset)$. The reason is essentially that (with the notation of $(*)$) we can find $\bar{e}^*$ such that

$$\text{tp}_{\mathcal{M}}(\bar{e}^*, \bar{a}^*, \bar{a}') = \text{tp}_{\mathcal{M}}(\bar{c}^*, \bar{a}^*, \bar{a}') \quad \text{and} \quad \text{tp}_{\mathcal{M}}(\bar{e}^*, \bar{b}^*, \bar{a}') = \text{tp}_{\mathcal{M}}(\bar{d}^*, \bar{b}^*, \bar{a}').$$

and if $e \in \text{rng}(\bar{e}^*)$, $a \in \text{rng}(\bar{a}^*)$ and $b \in \text{rng}(\bar{b}^*)$ then $\{e, a, b\}$ is *not* a hyperedge.

Recall that Remark 2.4 lists some known properties of the generic tetrahedron-free 3-hypergraph. The following result implies that it is also 1-based[9]:

Proposition 6.5. *Suppose that $\mathcal{M}$ is countable, homogeneous and supersimple with SU-rank 1. Moreover, assume that $\text{acl}_{\mathcal{M}}$ is degenerate and that $(*)$ holds. Then $\mathcal{M}$ is 1-based.*

Proof. For any type p and structure $\mathcal{N}$, let $\text{real}_{\mathcal{N}}(p)$ be the set of tuples of elements from $\mathcal{N}$ which realize p. By [10, Corollary 4.7], where 1-based theories are called "modular", it suffices to prove the following:

(†) If $\mathcal{N} \models Th(\mathcal{M})$, $A \subseteq N^{\text{eq}}$ and $p(x)$ is a complete type over A (possibly realized by imaginary elements) with SU-rank 1, then $(\text{real}_{\mathcal{N}^{\text{eq}}}(p), \text{cl})$, where $\text{cl}(B) = \text{acl}_{\mathcal{N}^{\text{eq}}}(B \cup A) \cap \text{real}_{\mathcal{N}^{\text{eq}}}(p)$ for all $B \subseteq \text{real}_{\mathcal{N}^{\text{eq}}}(p)$, is a trivial pregeometry (i.e. if $a \in \text{cl}(B)$ then $a \in \text{cl}(b)$ for some $b \in B$).[10]

The first step is to show that it suffices to prove (†) in the case when A is finite.

Suppose that $\mathcal{N} \models Th(\mathcal{M})$, $A \subseteq N^{\text{eq}}$ and that $p(x)$ is a complete type over A with SU-rank 1. Moreover, suppose that $(\text{real}_{\mathcal{N}^{\text{eq}}}(p), \text{cl})$ is nontrivial. So there are finite $B \subseteq \text{real}_{\mathcal{N}^{\text{eq}}}(p)$ and $a \in \text{real}_{\mathcal{N}^{\text{eq}}}(p)$ such that $a \in \text{acl}_{\mathcal{N}^{\text{eq}}}(B \cup A) \setminus \text{acl}_{\mathcal{N}^{\text{eq}}}(\{b\} \cup A)$ for all $b \in B$. Then there is finite $A_0 \subseteq A$ such that $a \in \text{acl}_{\mathcal{N}^{\text{eq}}}(B \cup A_0) \setminus \text{acl}_{\mathcal{N}^{\text{eq}}}(\{b\} \cup A_0)$ for all $b \in B$. Since $Th(\mathcal{M})$ is supersimple there is a finite $A_1 \subseteq A$ such that p does not divide over A_1. Let p' be the restriction of p to formulas with parameters from $A' = A_0 \cup A_1$. Then p' has SU-rank 1 and $a \in \text{acl}_{\mathcal{N}^{\text{eq}}}(B \cup A') \setminus \text{acl}_{\mathcal{N}^{\text{eq}}}(\{b\} \cup A')$ for all $b \in B$, so $(\text{real}_{\mathcal{N}^{\text{eq}}}(p'), \text{cl}')$ is a nontrivial pregeometry, where $\text{cl}'(B) = \text{acl}_{\mathcal{N}^{\text{eq}}}(B \cup A') \cap \text{real}_{\mathcal{N}^{\text{eq}}}(p')$ for all $B \subseteq \text{real}_{\mathcal{N}^{\text{eq}}}(p')$. Hence it suffices to prove (†) for finite A.

The next step is to show that it suffices to prove (†) in the case when A is finite *and* a subset of N (so that only "real elements" occur in A). Let $A \subseteq N^{\text{eq}}$ be finite and suppose

[9] The fact that the generic tetrahedron-free 3-hypergraph is 1-based also follows from a result by Conant in [5].

[10] If $(\text{real}_{\mathcal{N}^{\text{eq}}}(p), \text{cl})$ is a trivial pregeometry then p is "modular" in the sense of [10].

that $(\mathrm{real}_{\mathcal{N}^{\mathrm{eq}}}(p), \mathrm{cl})$ is nontrivial (where $p(x)$ is a complete type over A with SU-rank 1). Suppose that $a, \bar{b} \in \mathrm{real}_{\mathcal{N}^{\mathrm{eq}}}(p)$ are such that $a \in \mathrm{acl}_{\mathcal{N}^{\mathrm{eq}}}(\mathrm{rng}(\bar{b}) \cup A) \setminus \mathrm{acl}_{\mathcal{N}^{\mathrm{eq}}}(\{b\} \cup A)$ for all $b \in \mathrm{rng}(\bar{b})$. There is finite $C \subseteq N$ such that $A \subseteq \mathrm{dcl}_{\mathcal{N}^{\mathrm{eq}}}(C)$. Let $\bar{c}$ enumerate C. By considering a realization of a nondividing extension of $\mathrm{tp}_{\mathcal{N}^{\mathrm{eq}}}(\bar{c}/A)$ to $A \cup \{a\} \cup \mathrm{rng}(\bar{b})$ we may assume that $a\bar{b} \underset{A}{\downarrow} C$. Since $A \subseteq \mathrm{dcl}_{\mathcal{N}^{\mathrm{eq}}}(C)$ we have $a \in \mathrm{acl}_{\mathcal{N}^{\mathrm{eq}}}(\mathrm{rng}(\bar{b}) \cup C)$. Let $b \in \mathrm{rng}(\bar{b})$. As $\mathrm{SU}(p) = 1$ and $a \notin \mathrm{acl}_{\mathcal{N}^{\mathrm{eq}}}(\{b\} \cup A)$ we have $a \underset{A}{\downarrow} b$ and hence $\mathrm{SU}(a, b/A) = 2$. By the choice of C we have $ab \underset{A}{\downarrow} C$ and therefore $\mathrm{SU}(a, b/C) = 2$ which implies that $a \underset{C}{\downarrow} b$. Hence $a \notin \mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(\{b\} \cup C)$.

Thus it suffices to prove (†) for finite $A \subseteq N$. In fact, since $\mathcal{M}$ is ω-saturated it suffices to prove that if $A \subseteq M$ is finite and $p(x)$ is a complete type over A with SU-rank 1, then $(\mathrm{real}_{\mathcal{M}^{\mathrm{eq}}}(p), \mathrm{cl})$ is a trivial pregeometry. This is the last step of the proof.

Let $A \subseteq M$ be finite and let $p(x)$ be a complete type over A with SU-rank 1. Suppose, towards a contradiction, that $(\mathrm{real}_{\mathcal{M}^{\mathrm{eq}}}(p), \mathrm{cl})$ is a nontrivial pregeometry. Then there are $1 < m < \omega$ and distinct $a, b_1, \ldots, b_m \in \mathrm{real}_{\mathcal{M}^{\mathrm{eq}}}(p)$ such that

$$a \in \mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(\{b_1, \ldots, b_m\} \cup A) \quad \text{and} \tag{6.1}$$

$$a \notin \mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(\{b_k\} \cup A) \quad \text{for every } k = 1, \ldots, m. \tag{6.2}$$

Let $0 < n < \omega$ and $E^+ \subseteq M^{2n}$ be a $\emptyset$-definable equivalence relation such that all elements that realize p are of sort E^+. Furthermore, there is $q \in S_n^{\mathcal{M}}(\emptyset)$ such that for every $a \in \mathrm{real}_{\mathcal{M}^{\mathrm{eq}}}(p)$ there is $\bar{a} \in M^n$ such that $a = [\bar{a}]_{E^+}$ and $\mathcal{M} \models q(\bar{a})$. Let

$$X = \{\bar{a} : \mathcal{M} \models q(\bar{a})\}$$

and let E be the restriction of E^+ to X. Since $a, b_1, \ldots, b_m$ are distinct elements it follows that E has at least two classes. So either E is nontrivial or E is the identity relation on X. In either case, and using Theorem 6.1 in the first case, there are a nonempty $I \subseteq \{1, \ldots, n\}$ and a group Γ of permutations of I such that for all $\bar{a} = (a_1, \ldots, a_n), \bar{b} = (b_1, \ldots, b_n) \in X$, $E(\bar{a}, \bar{b})$ if and only if there is $\gamma \in \Gamma$ such that $a_i = b_{\gamma(i)}$ for all $i \in I$. For every $k = 1, \ldots, m$, choose any $\bar{b}_k = (b_{k,1}, \ldots, b_{k,n}) \in X$ such that $[\bar{b}_k]_{E^+} = b_k$. By the characterization of E and Observation 5.2 (ii) it follows that there is $\bar{a} = (a_1, \ldots, a_n) \in X$ such that

$$\begin{aligned} &a = [\bar{a}]_{E^+}, \\ &\{a_i : i \notin I\} \cap \{b_{k,i} : i \notin I\} = \emptyset \quad \text{for all } k = 1, \ldots, m \text{ and} \\ &\{a_i : i \notin I\} \cap A = \emptyset. \end{aligned}$$

We now divide the argument into two cases, both of which will lead to contradictions.

Case 1. Suppose that for every $k \in \{1, \ldots, m\}$,

$$\big(\{a_i : i \in I\} \setminus A\big) \cap \big(\{b_{k,i} : i \in I\} \setminus A\big) = \emptyset.$$

By Observation 5.2 (i), $\bar{a} \underset{A}{\downarrow} \{b_{k,i} : k = 1, \ldots, m,\ i = 1, \ldots, n\}$ which (since $a \in \mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(\bar{a})$ and $b_k \in \mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(\bar{b}_k)$) implies that $a \underset{A}{\downarrow} \{b_1, \ldots, b_m\}$ and hence (since $\mathcal{M}$ has SU-rank 1)

$$a \notin \mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(\{b_1, \ldots, b_m\} \cup A).$$

This contradicts (6.1).

Case 2. Suppose that for some $k \in \{1, \ldots, m\}$,

$$\big(\{a_i : i \in I\} \setminus A\big) \cap \big(\{b_{k,i} : i \in I\} \setminus A\big) \neq \emptyset.$$

Then, by the characterization of E, there is $i \in I$ such that

- $a_i \notin A$ and

- if $\bar{a}' \in X$ is such that $[\bar{a}']_{E^+} = a$ and $\bar{b}'_k \in X$ is such that $[\bar{b}'_k]_{E^+} = b_k$, then $a_i \in \mathrm{rng}(\bar{a}') \cap \mathrm{rng}(\bar{b}'_k)$.

Now it is straightforward to prove (for example by using the definition of dividing; details are left to the reader) that $a \not\mathop{\smile}\limits_{A} b_k$ and hence $a \in \mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(\{b_k\} \cup A)$. But this contradicts (6.2). □

Remark 6.6. Finally we explain why Fact 2.6 holds. Suppose that $\mathcal{M}$ is countable, binary, homogeneous and simple, so $\mathcal{M}$ is supersimple with finite SU-rank (by [12]). We will show that $\mathcal{M}$ has trivial dependence.

Consider the following statement for any simple ω-saturated $\mathcal{N}$:

(★) if $A \subseteq N$ is finite, $0 < n < \omega$ and $a_1, \ldots, a_n \in N^{\mathrm{eq}}$ are pairwise independent over A, then $\{a_1, \ldots, a_n\}$ is an independent set over A.

Lemma 1 in [9] says that if $\mathcal{N}$ is stable and (★) holds in the special case when $a_1, \ldots, a_n \in N$, then it also holds in the generality stated above. Its proof uses only basic properties of forking/dividing which also hold in simple theories/structures, as observed by Palacín [17]. Therefore we conclude that if $\mathcal{N}$ is simple and (★) holds in the special case when $a_1, \ldots, a_n \in N$, then it also holds in the generality stated above.

Now suppose that $\mathcal{N} \models Th(\mathcal{M})$, so $\mathcal{N}$ is ω-saturated (since $Th(\mathcal{M})$ is ω-categorical). By [12, Corollary 6], (★) holds for any $0 < n < \omega$ and any $a_1, \ldots, a_n \in N$. Hence, (★) holds in the generality stated above.

In order to prove that $\mathcal{M}$ has trivial dependence it suffices, according to the argument in the beginning of the proof of Proposition 6.5, to prove the following:

(♠) Suppose that $\mathcal{N} \models Th(\mathcal{M})$, $A \subseteq N$ is finite and $p(x)$ is a complete type (possibly realized by imaginary elements) over A with SU-rank 1. Then $(\mathrm{real}_{\mathcal{N}^{\mathrm{eq}}}(p), \mathrm{cl})$, where $\mathrm{cl}(B) = \mathrm{acl}_{\mathcal{N}^{\mathrm{eq}}}(B \cup A) \cap \mathrm{real}_{\mathcal{N}^{\mathrm{eq}}}(p)$ for all $B \subseteq \mathrm{real}_{\mathcal{N}^{\mathrm{eq}}}(p)$, is a trivial pregeometry

But (♠) is a direct consequence of (★), so we are done.[11]

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[11] It is tempting to claim that one could use Lemmas 1, 4 and Proposition 5 of [9] to directly conclude that $\mathcal{M}$ has trivial dependence. But the proof of Proposition 5 refers to the proof Proposition 2 (in [9]) which uses the notion of "heir", which is one reason why it is not clear to me how the argument would be translated to the context of simple structures.

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