BINARY SIMPLE HOMOGENEOUS STRUCTURES

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ABSTRACT. We describe all binary simple homogeneous structures $\mathcal{M}$ in terms of $\emptyset$-definable equivalence relations on M, which "coordinatize" $\mathcal{M}$ and control dividing, and extension properties that respect these equivalence relations.
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1. Introduction

We describe the fine structure of binary simple homogeneous structures to the extent that seems feasible without further assumptions and with known concepts and methods from infinite model theory. In this respect, this article completes the earlier work on this topic by Aranda Lopéz [3], Ahlman [2] and the present author [2, 19, 20, 21]. Before discussing the results, we explain what "homogeneity" means here, and give some background.

We call a structure $\mathcal{M}$ *homogeneous* if it is countable, has a finite relational vocabulary (also called signature) and every isomorphism between finite substructures of $\mathcal{M}$ can be extended to an automorphism of $\mathcal{M}$. For a countable structure $\mathcal{M}$ with finite relational vocabulary, being homogeneous is equivalent to having elimination of quantifiers [16, Corollary 7.42]; it is also equivalent to being a *Fraïssé limit* of an *amalgamation class* of finite structures [10, 16]. A structure with a relational vocabulary will be called *binary* if every relation symbol is unary or binary. Certain kinds of homogeneous structures have been classified. This holds for homogeneous partial orders, graphs, directed graphs, finite 3-hypergraphs, and coloured multipartite graphs [4, 12, 13, 22, 25, 27, 26, 30, 31]. For a survey about homogeneous structures, including their connections to permutation groups, Ramsey theory, topological dynamics and constraint satisfaction problems, see [29] by Macpherson.

A detailed theory, due to Lachlan, Cherlin, Harrington, Knight and Shelah [5, 17, 22, 23, 24], exists for stable infinite homogeneous structures, for any finite relational language, which describes them in terms of (finitely many) dimensions and $\emptyset$-definable indiscernible sets (which may live in $\mathcal{M}^{\text{eq}}$); see [23] for a survey. This theory also sheds light on finite homogeneous structures. But we seem to be a very long way from a classification of (even binary) finite homogeneous structures. This has consequences for (eventual) classifications of infinite homogeneous structures, for the following reason. Suppose that $\mathcal{N}$ is a finite (binary) homogeneous structure. Let $\mathcal{M}$ be the disjoint union of ω copies of $\mathcal{N}$ and add an equivalence relation such that each equivalence class is exactly the set of elements in some copy of $\mathcal{N}$. Then $\mathcal{M}$ is a (binary) stable homogeneous structure. Hence a classification of all (binary) stable homogeneous structures presupposes an equally detailed classification of all (binary) finite homogeneous structures. Thus we ignore the inner structure of such ("very local") finite "blocks" as the copies of $\mathcal{N}$ in the example, and focus on the "global fine structure" of an infinite structure $\mathcal{M}$.

The notion of simplicity generalizes stability and implies that there is a quite useful notion of independence. Moreover, there are interesting (binary) simple homogeneous structures which are unstable, such as the Rado graph and (other) homogeneous metric spaces with a finite distance set. (More about this is Section 7.4.) From this point of

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view it is natural, and seems feasible, to study simple homogeneous structures. From now on when saying that a structure is simple we assume that it is infinite, so "simple and homogeneous" implies that it is countably infinite. The theory of binary simple homogeneous structures has similarities to the theory of stable homogeneous structures, but also differences. Every stable (infinite) homogeneous structure is ω-stable, hence superstable, with finite SU-rank (which is often called U-rank in the context of stable structures). Analogously, every binary simple homogeneous structure is supersimple with finite SU-rank (which is bounded by the number of 2-types over $\emptyset$) [19]. However, the rank considered in the work on stable homogeneous structures is Shelah's "$CR(\ , 2)$-rank" [32, p. 55]. This rank is finite for stable homogeneous structures, but it is infinite for the Rado graph. If $\mathcal{M}$ is stable (infinite) and homogeneous and $C \subseteq M^{\mathrm{eq}}$ is $\emptyset$-definable and such that, on C, there is no $\emptyset$-definable nontrivial equivalence relation, then C is an indiscernible set. This is not true in general for (binary) simple homogeneous structures, as witnessed again by the Rado graph.

Suppose that $\mathcal{M}$ is binary, simple, and homogeneous. We already mentioned that $Th(\mathcal{M})$, the complete theory of $\mathcal{M}$, is supersimple with finite SU-rank. It is also known that $Th(\mathcal{M})$ is 1-based and has trivial dependence/forking [21, Fact 2.6 and Remark 6.6]. If $\mathcal{M}$ is, in addition, *primitive*, then $\mathcal{M}$ has SU-rank 1 and is a random structure [21]. (See Section 2.3 for a definition of 'primitive structure'.) Before stating the main results of this article, we note that, although the definition (above) of 'homogeneous structure' involves the assumption that the structure is countable, the main results hold for *every* model of $Th(\mathcal{M})$. The reason is that, $\mathcal{M}$ (being homogeneous) is ω-categorical and hence ω-saturated. So if elements could be found in some $\mathcal{N} \models Th(\mathcal{M})$ such that one of the statements (a)–(d) below fails in $\mathcal{N}$, then such elements could also be found in $\mathcal{M}$.

Main results (Theorems 5.1 and 6.2). *Suppose that $\mathcal{M}$ is binary, simple, and homogeneous (hence supersimple with finite SU-rank and trivial dependence). Let $\mathbf{R}$ be the (finite) set of all $\emptyset$-definable equivalence relations on M. If $a \in M$ and $R \in \mathbf{R}$, then a_R denotes the R-equivalence class of a as an element of M^{eq}.*

(a) Coordinatization by equivalence relations: *For every $a \in M$, if $\mathrm{SU}(a) = k$, then there are $R_1, \ldots, R_k \in \mathbf{R}$, depending only on $\mathrm{tp}(a)$, such that $a \in \mathrm{acl}(a_{R_k})$, $\mathrm{SU}(a_{R_1}) = 1$, $R_{i+1} \subset R_i$ and $\mathrm{SU}(a_{R_{i+1}}/a_{R_i}) = 1$ for all $1 \leq i < k$ (or equivalently, $\mathrm{SU}(a/a_{R_i}) = k - i$ for all $1 \leq i \leq k$).*

(b) Characterization of dividing: *Suppose that $a, b, \bar{c} \in M$ and $a \not\perp_{\bar{c}} b$. Then there is $R \in \mathbf{R}$ such that $a \not\perp_{\bar{c}} a_R$ and $a_R \in \mathrm{acl}(b)$ (and thus $a_R \notin \mathrm{acl}(\bar{c})$).*

(c) Characterization of dividing in the symmetric case: *Suppose that all binary $\emptyset$-definable relations on M are symmetric. If $a, b, \bar{c} \in M$ and $a \not\perp_{\bar{c}} b$, then there is $R \in \mathbf{R}$ such that $a \not\perp_{\bar{c}} a_R$ and $R(a, b)$ (hence $a_R \in \mathrm{acl}(b)$, $a_R \notin \mathrm{acl}(\bar{c})$ and thus $\neg R(a, c)$ for every $c \in \bar{c}$).*

(d) Extension properties: *Let $a, b, c, \bar{d} \in M$.*
(i) There is $R \in \mathbf{R}$ such that $c \perp_{c_R} \bar{d}$.
(ii) If for some R as in part (i),
$$a \perp_{c_R} c, \quad b \perp_{c_R} \bar{d} \text{ and } \mathrm{tp}(a/\mathrm{acl}(c_R)) = \mathrm{tp}(b/\mathrm{acl}(c_R)),$$
where 'acl' is taken in M^{eq}, then there is $e \in M$ such that $\mathrm{tp}(e, c) = \mathrm{tp}(a, c)$ and $\mathrm{tp}(e, \bar{d}) = \mathrm{tp}(b, \bar{d})$. Otherwise such e may not exist (in any elementary extension of $\mathcal{M}$), not even when $\bar{d}$ is a single element.

In parts (b) and (c) we only consider singletons a and b because $\mathcal{M}$ has trivial dependence. We will show (in Section 7.2) that the "symmetry condition" in part (c) cannot be removed; in other words, the conclusion in part (b) cannot be strengthened so that it becomes identical to the conclusion in part (c). Further remarks on (a)–(c) are made in Remark 5.2. Regarding part (d)(ii), the conditions that $a \underset{c_R}{\perp} c$, $b \underset{c_R}{\perp} \bar{d}$ and $\mathrm{tp}(a/\mathrm{acl}(c_R)) = \mathrm{tp}(b/\mathrm{acl}(c_R))$ are just the premisses (in the present context) of the independence theorem for simple theories. So the interesting part, with respect to (d)(ii), is that if (for every R as in (i)) these premisses are not satisfied, then a "common extension" may not exist. Thus we do not, in general, get anything "for free" beyond what the independence theorem guarantees. From this, one may get the impression that common extensions of types like in (d) are unusual. But note that, by part (i) of (d), we can always find a $\emptyset$-definable equivalence relation R such that $c \underset{c_R}{\perp} \bar{d}$. Therefore I would say that (by part (ii) of (d)), in a binary simple structure, common extensions of two types do exist as long as we respect all $\emptyset$-definable equivalence relations and some other "reasonable" conditions related to them. The examples in sections 7.1 – 7.3 show that these conditions are, in fact, necessary. The reason that (d) only considers an extension of two 1-types (one of which has only one parameter c) is that, since $\mathcal{M}$ is binary with elimination of quantifiers, the problem of extending more than two k-types (with finite parameter sets) can be reduced to a finite sequence of "extension problems", each of which involves only two 1-types and one of the types has only one parameter. More about this is said in the beginning of Section 6.

From the proofs of the main results, one can extract information about ω-categorical (not necessarily binary or homogeneous) supersimple structures with finite SU-rank and trivial dependence. This information is presented in Corollaries 5.3 and 5.4, and may be useful in future studies of nonbinary simple homogeneous structures.

Now we turn to problems about simple homogeneous structures. If $\mathcal{M}$ is stable and homogeneous, then $\mathcal{M}$ has the *finite submodel property*, which means that every sentence which is satisfied by $\mathcal{M}$ is satisfied by a finite substructure of it, and $Th(\mathcal{M})$ is decidable. (For the first result, see [23, Proposition 5.1] or [17, Lemma 7.1]; for the second, see the proof of Theorem 5.2 in [23].) It is still not settled whether every binary simple homogeneous structure has the finite submodel property, nor whether its theory must be decidable. But my guess is that the answer is 'yes' to both questions.

Regarding nonbinary simple homogeneous structures, I would say that all core problems are unsolved. The answer is unknown to each of these questions, where we assume that $\mathcal{M}$ is (nonbinary) simple and homogeneous: Must $Th(\mathcal{M})$ be supersimple? If $Th(\mathcal{M})$ is supersimple, must it have finite SU-rank?. Must $Th(\mathcal{M})$ be 1-based? Must $Th(\mathcal{M})$ have trivial dependence? (If $\mathcal{M}$ is supersimple, the last two problems are tightly connecteds to the problem of which kinds of definable pregeometries, induced by algebraic closure, there can be on the realizations, in M^{eq}, of types of SU-rank 1.) If $\mathcal{M}$ is supersimple with SU-rank 1, what possibilities are there for the fine structure of $\mathcal{M}$ (according to some "reasonably" informative classification)? Even if we add 'primitivity' and 'trivial dependence' to the assumptions of the last question, the answer is unknown.

Here follows an outline of the article. Section 2 explains the notation and terminology that will be used, and gives background regarding homogeneous (or just ω-categorical) simple structures. Section 3 describes the "coordinatization" developed in [9, Section 3] for ω-categorical, supersimple structures with finite SU-rank and trivial dependence (or equivalently, ω-categorical simple 1-based structures with trivial dependence). This coordinatization will be the framework in Sections 4 and 5. In Section 4 we prove the main technical lemmas, on which the main results rest. In Section 5 we prove (a)–(c) from the main results above. (This involves proving that every "coordinate" in the sense of Section 3 is interalgebraic with a new coordinate a_R where $a \in M$ and R is a $\emptyset$-definable equivalence relation on M.) In Section 6 we partially prove part (d) above, with the help of part (b). To complete the proof of (d), we also need to construct "counterexamples", which is done in Sections 7.1 – 7.3. Section 7.4 is an exposition of results by Conant [7] about homogeneous metric spaces, which concretize the main results of this article in that context.

2. Preliminaries

2.1. **Notation and terminology.** Structures will be denoted by calligraphic letters, usually $\mathcal{M}$ or $\mathcal{N}$ in which case their universes are denoted M or N, respectively. Finite sequences (and *only finite* sequences) are denoted by $\bar{a}, \bar{b}, \ldots, \bar{x}, \bar{y}, \ldots$. The concatenation of $\bar{a}$ and $\bar{b}$ is denoted $\bar{a}\bar{b}$, but sometimes we also write $(\bar{a}, \bar{b})$ (like when using the type notation $\mathrm{tp}(\bar{a}, \bar{b})$). The set of elements that occur in $\bar{a}$ (in other words, the range/image of $\bar{a}$) is denoted $\mathrm{rng}(\bar{a})$. But when the order of $\bar{a}$ does not matter, we often abuse notation and (notationally) identify the sequence $\bar{a}$ with the set $\mathrm{rng}(\bar{a})$. So we may write things like '$a \in \bar{a}$' instead of '$a \in \mathrm{rng}(\bar{a})$'. When a, b and c are single elements we sometimes write 'ab' for the pair '(a, b)', or 'abc' for the triple '(a, b, c)', and similarly for longer tuples. Further, we often write '$\bar{a} \in A$' when meaning that $\bar{a}$ is a finite sequence such that $\mathrm{rng}(\bar{a}) \in A$. If we may emphasize that the length of $\bar{a}$ (denoted $|\bar{a}|$) is n, then we may write $\bar{a} \in A^n$.

As usual, '$\mathrm{acl}_{\mathcal{M}}$', '$\mathrm{dcl}_{\mathcal{M}}$', and '$\mathrm{tp}_{\mathcal{M}}$' denote the algebraic closure, definable closure, and type (of a set or sequence) in the structure $\mathcal{M}$; and if $A \subseteq M$, then $S_n^{\mathcal{M}}(A)$ is the set of n-types over A with respect to $Th(\mathcal{M})$, the complete theory of $\mathcal{M}$. The notation '$\bar{a} \equiv_{\mathcal{M}} \bar{b}$' means the same as '$\mathrm{tp}_{\mathcal{M}}(\bar{a}) = \mathrm{tp}_{\mathcal{M}}(\bar{b})$'. The notation '$\bar{a} \equiv_{\mathcal{M}}^{at} \bar{b}$' means that $\bar{a}$ and $\bar{b}$ satisfy exactly the same atomic formulas with respect to $\mathcal{M}$. In sections 3 – 6 the structure $\mathcal{M}$ is fixed and we work in $\mathcal{M}^{\mathrm{eq}}$, so for brevity we will, in those sections, omit the subscript '$\mathcal{M}^{\mathrm{eq}}$' and write for example 'tp' instead of '$\mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}$'. We remind about this again in Notation 3.1. If $p(\bar{x})$ is a type (or formula), then $p(\mathcal{M})$ denotes the set of realizations of p in $\mathcal{M}$.

If R is a $\emptyset$-definable equivalence relation on M^n for some $n < \omega$, then we may also call R a *sort*. For every such R and every $\bar{a} \in M^n$, $[\bar{a}]_R$ denotes the R-equivalence class of $\bar{a}$. When we view $[\bar{a}]_R$ as an *element* of M^{eq} we write $\bar{a}_R$ to emphasize this. If $A \subseteq M^{\mathrm{eq}}$ then we say that *only finitely many types are represented in* A if there are only finitely many sorts R such that for some $n < \omega$ and $\bar{a} \in M^n$, $\bar{a}_R \in A$.

When saying that $\mathcal{M}$ is ω-categorical, (super)simple, 1-based, or that $\mathcal{M}$ has finite SU-rank, then we mean that $Th(\mathcal{M})$ is ω-categorical, (super)simple, 1-based, or that $Th(\mathcal{M})$ has finite SU-rank, respectively.

A *pregeometry* (or *matroid*) is a pair (X, cl) where X is a set and $\mathrm{cl} : \mathcal{P}(X) \to \mathcal{P}(X)$ satisfies certain conditions (see [16, Chapter 4.6]). We say that a pregeometry (X, cl) is *trivial* if for all $Y \subseteq X$, $\mathrm{cl}(Y) = \bigcup_{a \in Y} \mathrm{cl}(\{a\})$.

2.2. **ω-Categorical structures.** Since homogeneous structures have elimination of quantifiers, it follows from the well-known characterization of ω-categoricity [16, Theorem 7.3.1], that every infinite homogeneous structure is ω-categorical. We now state some basic facts about $\mathcal{M}^{\mathrm{eq}}$ when $\mathcal{M}$ is ω-categorical. These will tacitly be used throughout the article.

Fact 2.1. *Suppose that $\mathcal{M}$ is ω-categorical and assume that only finitely many sorts are represented in $A \subseteq M^{\mathrm{eq}}$.*

(i) *For every $n < \omega$ and every finite $B \subseteq M^{\mathrm{eq}}$, only finitely many types from $S_n^{\mathcal{M}^{\mathrm{eq}}}(\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(B))$ are realized by tuples in A^n.*

(ii) *For every finite $B \subseteq M^{\mathrm{eq}}$, $A \cap \mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(B)$ is finite.*

(iii) *For every $\bar{a} \in M^{\mathrm{eq}}$ and every finite $B \subseteq M^{\mathrm{eq}}$, the types $\mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}(\bar{a}/B)$ and $\mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}(\bar{a}/\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(B))$ are isolated.*

For some explanations of the above claims, see [2, Section 2.4]. Part (iii) of Fact 2.1 will usually not be used in the form stated above, but rather we use the following (namely *ω-homogeneity* and a variant of it), which are proved straightforwardly from Fact 2.1 (iii):

Fact 2.2. *Suppose that $\mathcal{M}$ is ω-categorical.*

(i) *If $\bar{a}, \bar{b}, c \in M^{\mathrm{eq}}$ and $\bar{a} \equiv_{\mathcal{M}^{\mathrm{eq}}} \bar{b}$, then there is $d \in M^{\mathrm{eq}}$ such that $\bar{a}c \equiv_{\mathcal{M}^{\mathrm{eq}}} \bar{b}d$.*

(ii) *If $\bar{a}, \bar{b}, \bar{c}, \bar{e} \in M^{\mathrm{eq}}$ and*

$$\mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}\big(\bar{a}/\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(\bar{e})\big) \;=\; \mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}\big(\bar{b}/\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(\bar{e})\big),$$

then there is $\bar{d} \in M^{\mathrm{eq}}$ such that

$$\mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}\big(\bar{a}\bar{c}/\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(\bar{e})\big) \;=\; \mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}\big(\bar{b}\bar{d}/\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(\bar{e})\big).$$

2.3. **Simple homogeneous structures.** We assume basic knowledge about simple structures as can be found in [34], for instance, but nevertheless recall a couple of things. When saying that a structure is simple we automatically assume that it is infinite.[1] Since ω-categorical simple theories have elimination of hyperimaginaries [34, Corollary 6.1.11], the independence theorem of simple theories [34, Theorem 2.5.20] takes the following form if the involved sets of parameters are finite and $\mathcal{M}$ is ω-categorical and simple:

Suppose that $\bar{a}, \bar{b} \in M^{\mathrm{eq}}$, $A, B, C \subseteq M^{\mathrm{eq}}$ are finite, $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_C A$, $\bar{b} \mathop{\smile\!\!\!\!\!\!|}_C B$, and

$$\mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}\big(\bar{a}/\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(C)\big) \;=\; \mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}\big(\bar{b}/\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(C)\big).$$

Then there is $\bar{d} \in M^{\mathrm{eq}}$ such that

$$\mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}\big(\bar{d}/A \cup \mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(C)\big) \;=\; \mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}\big(\bar{a}/A \cup \mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(C)\big) \quad \text{and}$$
$$\mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}\big(\bar{d}/B \cup \mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(C)\big) \;=\; \mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}\big(\bar{b}/B \cup \mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(C)\big).$$

Note that if $\mathcal{M}$ is ω-categorical and supersimple with finite SU-rank, then (since $S_1^{\mathcal{M}}(\emptyset)$ is finite) there is $n < \omega$ such that $\mathrm{SU}(p) \leq n$ for every $p \in S_1^{\mathcal{M}}(\emptyset)$. Before recalling what is known from before about binary simple homogeneous structures, we give the definition of trivial dependence (also called 'totally trivial forking' in [14]).

Definition 2.3. A simple complete theory T has *trivial dependence* if for all $\mathcal{M} \models T$ and all $A, B, C \subseteq M^{\mathrm{eq}}$, if $A \not\mathop{\smile\!\!\!\!\!\!|}_C B$, then $A \not\mathop{\smile\!\!\!\!\!\!|}_C b$ for some $b \in B$. We say that a simple structure $\mathcal{M}$ has *trivial dependence* if $Th(\mathcal{M})$ has it.

Fact 2.4. *Suppose that $\mathcal{M}$ is binary, simple, and homogeneous. Then:*

(i) *$\mathcal{M}$ is supersimple with finite SU-rank (which is bounded by the number of complete 2-types over $\emptyset$).*

(ii) *$\mathcal{M}$ has trivial dependence.*

(iii) *$\mathcal{M}$ is 1-based.*

Part (i) is given by [19, Theorem 1]. Parts (ii) and (iii) are consequences of [19, Corollary 6], [14, Lemma 1], [15, Corollary 4.7], [8, Corollary 3.23] and [28, Theorem 1.1]; this is explained in more detail in the text surrounding Fact 2.6 in [21] and in Remark 6.6 of the same article.

We call a structure $\mathcal{M}$ *primitive* if there there is no nontrivial $\emptyset$-definable equivalence relation on M (where by nontrivial we mean that there are at least two equivalence classes and at least one equivalence class has at least two elements).

Fact 2.5. *Suppose that $\mathcal{M}$ is binary, primitive, simple and homogeneous. Then:*

(i) *$\mathcal{M}$ has SU-rank 1.*

(ii) *$\mathcal{M}$ is a random structure in the sense of* [21, Definition 2.1].

[1] Thus we do not follow the terminology of the work on stable homogeneous structures, where every finite structure is considered to be stable.

Part (i) is given by [21, Theorem 1.1]. Part (ii) is a consequence of part (i) and [3, Proposition 3.3.3], where the later result says that every binary simple homogeneous structure of SU-rank 1 is a random structure. From Theorem 5.1 (i) (i.e. part (a) of the 'main results' in the introduction), it follows that part (i) of Fact 2.5 still holds if the assumption about 'primitivity' is replaced by the condition that there is no $\emptyset$-definable equivalence relation on M which has infinitely many infinite equivalence classes.

Fact 2.5 (i) fails without the binarity condition as shown by Example 2.7 in [21], which is primitive, homogeneous, and superstable with SU-rank 2 (but nonbinary). It is also *not* a random structure. Consequently also part (ii) of Fact 2.5 fails without the binarity condition. But in fact it fails (without the binarity condition) in a stronger sense. Because the generic tetrahedron-free 3-hypergraph is primitive, homogeneous, supersimple with SU-rank 1 and 1-based, but not a random structure. All mentioned properties of the generic tetrahedron-free 3-hypergraph, except for the 1-basedness, have been known for a long time. Results which imply that it is 1-based were recently proved by Conant [6] and by the present author [21].

3. Coordinatization

Throughout this section we assume that $\mathcal{M}$ is ω-categorical, supersimple with finite SU-rank and trivial dependence (hence it is 1-based). Then the "coordinatization" results of Section 3 in [9] apply to $\mathcal{M}$. We will now go through these results, since they are the framework in which the arguments of sections 4 – 6 take place.

Notation 3.1. In this section and Sections 4 – 6, 'tp', '$\equiv$', 'acl', and 'dcl' will abbreviate '$\text{tp}_{\mathcal{M}^{\text{eq}}}$', '$\equiv_{\mathcal{M}^{\text{eq}}}$', '$\text{acl}_{\mathcal{M}^{\text{eq}}}$', and '$\text{dcl}_{\mathcal{M}^{\text{eq}}}$', respectively.

The basic idea with a coordinatization of $\mathcal{M}$ is that we want to find a fixed set of "coordinates" such that only finitely many 1-types over $\emptyset$ are realized in it and if $a \in M$ and $\text{SU}(a) = n$, then there are coordinates (of a) $c_1, \ldots, c_n$ such that the "place" of a in $\mathcal{M}$ is approximated with higher and higher precision by the sequences c_1, c_1c_2, $c_1c_2c_3, \ldots, c_1c_2c_3 \ldots c_n$. More technically speaking, we wish to find $C \subseteq M^{\text{eq}}$ such that $M \subseteq C$, only finitely many 1-types over $\emptyset$ are realized in C and if $a \in C$ has SU-rank n, then there are $c_1, \ldots, c_n \in C$ such that $\text{SU}(a/c_1, \ldots, c_k) = n - k$ for every $1 \leq k \leq n$. Actually, for the set C that we consider below only a subset of C will be our set of "coordinates". This set of coordinates (denoted C_r below) has a number of useful properties which are listed in the facts below. Among other things, the coordinates can be partitioned into finitely many levels: the first level contains all coordinates of SU-rank 1 over $\emptyset$, the second level consists of all coordinates of SU-rank 1 over the first level and so on. Below, C_k is the union of the first k levels (for technical reasons we have also a level C_0 which is empty). Another property of the coordinates is that all coordinates of an element belong to its algebraic closure. Moreover (as said in Lemma 3.7), for tuples $\bar{a}, \bar{b}, \bar{c}$, whether $\bar{a}$ is independent from $\bar{b}$ over $\bar{c}$ is entirely determined by the coordinates of $\bar{a}, \bar{b}$ and $\bar{c}$. We now continue with the technical notions and results that will be used later.

Fact 3.2. *Let $U \subseteq M^{\text{eq}}$ and suppose that only finitely many sorts are represented in U. Then there are $0 < r < \omega$ and*

$$C_0 \subseteq C_1 \subseteq \ldots \subseteq C_r \subseteq C \subseteq M^{\text{eq}}$$

such that:

(i) *$U \subseteq C$, only finitely many sorts are represented in C, and C is* self-coordinatized *in the sense of* [9, Definition 3.3].
(ii) *C and C_i are $\emptyset$-definable, for every $i = 1, \ldots, r$.*
(iii) *$C_0 = \emptyset$ and, for every $n < r$ and every $c \in C_{n+1} \setminus C_n$, $\text{SU}(c/C_n) = 1$.*

(iv) $C \subseteq \mathrm{acl}(C_r)$.
(v) *For every* $1 < n \leq r$ *and every* $c \in C_n$, $\mathrm{acl}(c) \cap C_{n-1} \neq \emptyset$.

Assumption 3.3. In the rest of this section we suppose the following:
(a) $M \subseteq U \subseteq M^{\mathrm{eq}}$ and only finitely many sorts are represented in U.
(b) C and C_i, for $i = 0, \ldots, r$, are as in Fact 3.2.

We can think of C_r as set *coordinates of* C (and hence of M) and we call r the *height* of the coordinatization.

Definition 3.4. (i) For every $\bar{c} \in C$ and every $0 \leq s \leq r$, let $\mathrm{crd}_s(\bar{c}) = \mathrm{acl}(\bar{c}) \cap C_s$.
(ii) We abbreviate 'crd_r' with 'crd'.

Observe that for every $\bar{c} \in C$, $\mathrm{crd}(\bar{c})$ is finite. We can think of $\mathrm{crd}(\bar{c})$ as the *coordinates* of $\bar{c}$ (with respect to the given coordinatization C_r) and $\mathrm{crd}_s(\bar{c})$ as the *coordinates of* $\bar{c}$ *up to "level"* s.

Fact 3.5. (i) *If* $c \in C_r$, $d_1, \ldots, d_n \in M^{\mathrm{eq}}$ *and* $c \in \mathrm{acl}(d_1, \ldots, d_n)$, *then* $c \in \mathrm{acl}(d_i)$ *for some* $1 \leq i \leq n$.
(ii) *For every* $0 < s \leq r$, $(C_s \setminus C_{s-1}, \mathrm{cl})$, *where* $\mathrm{cl}(A) = \mathrm{acl}(A) \cap (C_s \setminus C_{s-1})$ *for all* $A \subseteq C_s \setminus C_{s-1}$, *is a trivial pregeometry*
(iii) *For every* $\bar{c} \in C$ *and every* $0 \leq s \leq r$, $\mathrm{crd}_s(\bar{c}) = \bigcup_{c \in \mathrm{rng}(\bar{c})} \mathrm{crd}_s(c)$. *Thus the same holds for* 'crd' *in place of* 'crd_s'.
(iv) *For all* $\bar{c} \in C$, $\mathrm{acl}(\bar{c}) = \mathrm{acl}(\mathrm{crd}(\bar{c}))$.
(v) *For all* $\bar{a}, \bar{b} \in C$, $\bar{a}$ *is independent from* $\bar{b}$ *over* $\mathrm{crd}(\bar{a}) \cap \mathrm{crd}(\bar{b})$.

Part (i) above is [9, Lemma 3.16]; part (ii) is an immediate consequence of [9, Lemma 3.18], because $C_s \setminus C_{s-1}$ is a $\emptyset$-definable set and a subset of the ($\emptyset$-definable) set N_s considered there [9, Construction 3.13]; part (iii) is [18, Lemma 5.4]. By definition, $\mathrm{crd}(\bar{c}) \subseteq \mathrm{acl}(\bar{c})$, so to prove (iv) it suffices to show that $\bar{c} \in \mathrm{acl}(\mathrm{crd}(\bar{c}))$. By [9, Lemma 5.1], for every $c \in \bar{c}$, $c \in \mathrm{acl}(\mathrm{crd}(c))$. Thus the conclusion now follows from part (iii). Regarding part (v): Let $\bar{c}$ enumerate $\mathrm{crd}(\bar{a})$ and let $\bar{d}$ enumerate $\mathrm{crd}(\bar{b})$. By part (iv), $\mathrm{acl}(\bar{a}) = \mathrm{acl}(\bar{c})$ and $\mathrm{acl}(\bar{b}) = \mathrm{acl}(\bar{d})$, so $\mathrm{acl}(\mathrm{crd}(\bar{a}) \cap \mathrm{crd}(\bar{b})) = \mathrm{acl}(\mathrm{crd}(\bar{c}) \cap \mathrm{crd}(\bar{d}))$. Therefore it suffices to prove that $\bar{c}$ is independent from $\bar{d}$ over $\mathrm{crd}(\bar{c}) \cap \mathrm{crd}(\bar{d})$. Since $\bar{c}, \bar{d} \in C_r$, this is exactly the content of [18, Lemma 5.16].

We note the following strengthening of part (iii) of Fact 3.2:

Fact 3.6. *Let* $0 \leq n < r$. *For every* $c \in C_{n+1} \setminus C_n$, $\mathrm{SU}(c/\mathrm{crd}_n(c)) = 1$.

Proof. Suppose that $c \in C_{n+1} \setminus C_n$. By Fact 3.2 (iii), $\mathrm{SU}(c/C_n) = 1$. By supersimplicity, there is $\bar{d} \in C_n$ such that $\mathrm{SU}(c/\bar{d}) = 1$. Fact 3.5 (v) implies that c is independent from $\bar{d}$ over $\mathrm{crd}(c) \cap \mathrm{crd}(\bar{d})$, so $\mathrm{SU}(c/\mathrm{crd}(c) \cap \mathrm{crd}(\bar{d})) = 1$. Since $\bar{d} \in C_n$ it follows from Fact 3.2 (iii) that $\mathrm{crd}(\bar{d}) \subseteq C_n$. Therefore $\mathrm{SU}(c/\mathrm{crd}_n(c)) = 1$. □

The following generalization of Fact 3.5 (v) will be convenient to use.

Lemma 3.7. *Suppose that* $\bar{a}, \bar{b}, \bar{c} \in C$. *Then* $\bar{a} \underset{\bar{c}}{\perp} \bar{b}$ *if and only if* $\mathrm{crd}(\bar{a}) \cap \mathrm{crd}(\bar{b}) \subseteq \mathrm{acl}(\bar{c})$.

Proof. Suppose that $\mathrm{crd}(\bar{a}) \cap \mathrm{crd}(\bar{b}) \subseteq \mathrm{acl}(\bar{c})$. By extending the sequence $\bar{c}$ with new elements from $\mathrm{crd}(\bar{a}) \cap \mathrm{crd}(\bar{b})$, if necessary, we may assume that $\mathrm{crd}(\bar{a}) \cap \mathrm{crd}(\bar{b}) \subseteq \bar{c}$. By Fact 3.5 (iii), $\mathrm{crd}(\bar{b}\bar{c}) = \mathrm{crd}(\bar{b}) \cup \mathrm{crd}(\bar{c})$, so by Fact 3.5 (v), $\bar{a}$ is independent from $\bar{b}\bar{c}$ over

$$\mathrm{crd}(\bar{a}) \ \cap \ \big(\mathrm{crd}(\bar{b}) \cup \mathrm{crd}(\bar{c})\big) \ = \ \big(\mathrm{crd}(\bar{a}) \cap \mathrm{crd}(\bar{b})\big) \ \cup \ \big(\mathrm{crd}(\bar{a}) \cap \mathrm{crd}(\bar{c})\big).$$

So by monotonicity and the assumption that $\mathrm{crd}(\bar{a}) \cap \mathrm{crd}(\bar{b}) \subseteq \bar{c}$, it follows that $\bar{a}$ is independent from $\bar{b}\bar{c}$ over $\bar{c}$. Hence $\bar{a}$ is independent from $\bar{b}$ over $\bar{c}$.

Now suppose that $\bar{a} \underset{\bar{c}}{\perp} \bar{b}$. For a contradiction, suppose that $d \in \mathrm{crd}(\bar{a}) \cap \mathrm{crd}(\bar{b})$ and $d \notin \mathrm{acl}(\bar{c})$. Then $\mathrm{SU}(d/\bar{c}) \geq 1$. Using that $\mathrm{acl}(\bar{a}) = \mathrm{acl}(\mathrm{crd}(\bar{a}))$ (by Fact 3.5 (iv), we get, by the Lascar equation,

$$\mathrm{SU}(\bar{a}/\bar{c}) \;=\; \mathrm{SU}(\bar{a}d/\bar{c}) \;=\; \mathrm{SU}(\bar{a}/d\bar{c}) \;+\; \mathrm{SU}(d/\bar{c}).$$

Hence $\mathrm{SU}(\bar{a}/d\bar{c}) < \mathrm{SU}(\bar{a}/\bar{c})$. Therefore $\bar{a} \underset{\bar{c}}{\not\perp} d$, and as $d \in \mathrm{crd}(\bar{b}) \subseteq \mathrm{acl}(\bar{b})$, we get $\bar{a} \underset{\bar{c}}{\not\perp} \bar{b}$. □

Definition 3.8. For every $0 \leq s \leq r$ and all $a, b \in C$, let

$$E_s(a,b) \iff \mathrm{crd}_s(a) = \mathrm{crd}_s(b) \;\text{ and }\; \mathrm{tp}\big(a/\mathrm{acl}(\mathrm{crd}_s(a))\big) = \mathrm{tp}\big(b/\mathrm{acl}(\mathrm{crd}_s(b))\big).$$

From Fact 2.1 (iii) it is straightforward to derive the following:

Fact 3.9. *For every $0 \leq s \leq r$, E_s is a $\emptyset$-definable equivalence relation on C.*

Lemma 3.10. *We may, without loss of generality, assume that C_r has the following property: for all $a, b \in M$ and all $c \in C_r$, if $ac \equiv bc$ then* $\mathrm{tp}(a/\mathrm{acl}(c)) = \mathrm{tp}(b/\mathrm{acl}(c))$.

Proof. Let $c \in C_r$ and $q(x) = \mathrm{tp}(c)$. Suppose that there are $a, b \in M$ such that $\mathrm{tp}(a/\mathrm{acl}(c)) \neq \mathrm{tp}(b/\mathrm{acl}(c))$. By Fact 2.1 (i), only finitely many complete types over $\mathrm{acl}(c)$ are realized in M. By part (iii) of the same fact, each such type is isolated. Let $p_1, \ldots, p_n$ enumerate all complete 1-types over $\mathrm{acl}(c)$ which are realized in M. For each i, choose a formula that isolates p_i and let $\bar{d}_i$ be the parameters (from $\mathrm{acl}(c)$) that occur in that formula. Let $\bar{d} = c\bar{d}_1 \ldots \bar{d}_n$. Then $\mathrm{acl}(\bar{d}) = \mathrm{acl}(c)$. As $\mathcal{M}^{\mathrm{eq}}$ has elimination of imaginaries, there is $d \in M^{\mathrm{eq}}$ such that $\mathrm{dcl}(d) = \mathrm{dcl}(\bar{d})$. Let $q' = \mathrm{tp}(d)$. Now remove from C all $c' \in C$ which realize q and then add to what is left of C all $d' \in M^{\mathrm{eq}}$ which realize q'. Then the modified C has the property that whenever $a, b \in M$, $c \in C$, c realizes q' and $ac \equiv bc$, then $\mathrm{tp}(a/\mathrm{acl}(c)) = \mathrm{tp}(b/\mathrm{acl}(c))$. Since (by Assumption 3.3) only finitely many types over $\emptyset$ are realized in C, it follows that we can continue this procedure in finitely many steps and get (new) C and $C_r \subseteq C$ such that the conclusion of the lemma holds. Since the types q and q' above are isolated and every change of element in this process, say from c to d, is such that $\mathrm{acl}(c) = \mathrm{acl}(d)$, it follows that the new C and $C_0 \subseteq \ldots \subseteq C_r$ that we get have all the properties of the earlier facts and lemmas in this section. □

4. The main technical lemmas

Throughout this section we assume that $\mathcal{M}$ is binary, simple, and homogeneous. By Fact 2.4, $\mathcal{M}$ is supersimple, 1-based, with finite SU-rank and with trivial dependence. We thus adopt Assumption 3.3, as well as Notation 3.1. However, the assumption that $\mathcal{M}$ is binary and homogeneous (as opposed to only ω-categorical) is only used once at the end of the proof of Lemma 4.2 and once at the end of the proof of Lemma 4.6.

The goal of this section is to prove the following:

For all $0 < s \leq r$, $a \in M$ and $c_1, c_2 \in \mathrm{crd}_s(a) \setminus C_{s-1}$, if $c_1 \underset{\mathrm{crd}_{s-1}(a)}{\perp} c_2$ then $ac_1 \not\equiv ac_2$.

This is also the statement of Lemma 4.6. It will be used in the next section where we show that we can choose the coordinates to be imaginaries defined by $\emptyset$-definable equivalence relations on M (rather than on M^n for some $n > 1$), and that dividing is controlled by these equivalence relations.

For the rest of this section we fix (an arbitrary) $0 < s \leq r$.

Remark 4.1. (The intuition behind Lemma 4.2.) Let $\mathcal{C}$ be the structure where $C = \mathbb{N}$ and the vocabulary of $\mathcal{C}$ is empty. Let G be the set of all 2-element subsets of C. Turn G into a graph $\mathcal{G}$ by saying that $a, b \in G$ are adjacent if and only if their intersection is a singleton. Since $\mathcal{C}$ is ω-categeorical and stable and $\mathcal{G}$ is interpretable in $\mathcal{C}$ (without parameters) it follows, for example by [16, Theorem 7.3.8] and [32, Ch. III, Lemma 6.7], that $\mathcal{G}$ is ω-categorical and stable, in fact superstable with SU-rank 2, which follows by a straightforward argument using the definition of dividing. However, $\mathcal{G}$ is not homogeneous, because it is easy to see that the following two triples of elements from G satisfy the same quantifier-free formulas, but not the same formulas with quantifiers: $(\{1,2\},\{2,3\},\{1,3\}), (\{1,2\},\{1,3\},\{1,4\})$. Note that the intersection of the elements in the first triple is empty, but the intersection of the elements in the second triple is nonempty.

The idea of the proof of Lemma 4.2 is as follows, where we let $\overline{\mathrm{crd}}_s(a)$ abbreviate '$\mathrm{crd}_s(a) \setminus C_{s-1}$' : If $a \in M$ and $c_1, c_2 \in \overline{\mathrm{crd}}_s(a)$ satisfy the premisses of the lemma, and $E_{s-1}(c_1, c_2)$, then we can find $a, a', a'', a^* \in M$ such that $aa'' \equiv aa^*$ and $a'a'' \equiv a'a^*$, but $aa'a'' \not\equiv aa'a^*$. This is done by choosing the elements in such a way that $\overline{\mathrm{crd}}_s(a) \cap \overline{\mathrm{crd}}_s(a') \cap \overline{\mathrm{crd}}_s(a'') = \emptyset$ and $\overline{\mathrm{crd}}_s(a) \cap \overline{\mathrm{crd}}_s(a') \cap \overline{\mathrm{crd}}_s(a^*) \neq \emptyset$.

The proof of Lemma 3.9 in [20] builds on the same idea. But in its context, $s = 1$ so all elements of C_s have SU-rank 1. Then, by [2, Theorem 5.1], the "canonically embedded structure" (in $\mathcal{M}^{\mathrm{eq}}$) with universe C_1, is, modulo "dividing out by the relation $\mathrm{acl}(x) = \mathrm{acl}(y)$", a reduct of a binary random structure. This simplified the arguments in the proof of [20, Lemma 3.9]. Here we use only (besides the given coordinatization) properties of forking/dividing and, in particular, the independence theorem for simple structures; but the arguments become more intricate.

Lemma 4.2. *If $a \in M$, $c_1, c_2 \in \mathrm{crd}_s(a) \setminus C_{s-1}$, $ac_1 \equiv ac_2$ and $c_1 \underset{\mathrm{crd}_{s-1}(c_1)}{\downarrow} c_2$, then*

$$\mathrm{tp}\big(c_1/\mathrm{acl}(\mathrm{crd}_{s-1}(c_1))\big) \neq \mathrm{tp}\big(c_2/\mathrm{acl}(\mathrm{crd}_{s-1}(c_1))\big), \quad \textit{and hence}$$
$$\mathrm{tp}\big(c_1/\mathrm{acl}(\mathrm{crd}_{s-1}(a))\big) \neq \mathrm{tp}\big(c_2/\mathrm{acl}(\mathrm{crd}_{s-1}(a))\big).$$

Proof. For a contradiction suppose that there are $a \in M$ and $c_1, c_2 \in \mathrm{crd}_s(a) \setminus C_{s-1}$ such that

$$ac_1 \equiv ac_2, \quad c_1 \underset{\mathrm{crd}_{s-1}(c_1)}{\downarrow} c_2, \text{ and} \tag{4.1}$$

$$\mathrm{tp}(c_1/\mathrm{acl}(\mathrm{crd}_{s-1}(c_1))) = \mathrm{tp}(c_2/\mathrm{acl}(\mathrm{crd}_{s-1}(c_1))). \tag{4.2}$$

Note that this implies that $\mathrm{crd}_{s-1}(c_1) = \mathrm{crd}_{s-1}(c_2)$, so

$$E_{s-1}(c_1, c_2). \tag{4.3}$$

By (4.1) there is $c_1^* \in C$ such that

$$ac_1c_2 \equiv ac_2c_1^*. \tag{4.4}$$

Then

$$c_1^* \in \mathrm{crd}_s(a) \setminus C_{s-1} \quad \text{and} \quad c_2 \underset{\mathrm{crd}_{s-1}(c_1)}{\downarrow} c_1^*. \tag{4.5}$$

From (4.3), (4.4) and Fact 3.9 we also get

$$E_{s-1}(c_2, c_1^*). \tag{4.6}$$

By (4.1), (4.5), (4.6) and the independence theorem there is $c_2' \in C_s \setminus C_{s-1}$ such that

$$c_1c_2' \equiv c_1c_2 \equiv c_2c_2' \text{ and } c_2' \underset{\mathrm{crd}_{s-1}(c_1)}{\downarrow} c_1, c_2. \tag{4.7}$$

In addition, we may, without loss of generality, assume that

$$c_2' \underset{c_1c_2}{\downarrow} a, \tag{4.8}$$

because if this is not the case then we can replace c_2' by a realization of a nondividing extension of $\mathrm{tp}(c_2'/c_1, c_2)$ to $\{a, c_1, c_2\}$ (and recall that $\mathrm{crd}_{s-1}(c_1) \subseteq \mathrm{acl}(c_1)$).

Since (by Fact 3.2 (iii)) $\mathrm{SU}(c_2'/\mathrm{crd}_{s-1}(c_1)) \geq 1$, it follows from $c_2' \underset{\mathrm{crd}_{s-1}(c_1)}{\downarrow} c_1c_2$ (see (4.7)) that $c_2' \notin \mathrm{acl}(c_1, c_2)$. From this together with (4.8) we get

$$c_2' \notin \mathrm{acl}(a) \ \text{ so } \ c_2' \notin \mathrm{crd}_s(a). \tag{4.9}$$

From (4.7), (4.8) and transitivity, we get

$$c_2' \underset{\mathrm{crd}_{s-1}(c_1)}{\downarrow} a. \tag{4.10}$$

By (4.7) there are $a', a'' \in M$ such that

$$a'c_1c_2' \equiv ac_1c_2 \equiv a''c_2c_2'. \tag{4.11}$$

By considering nondividing extensions if necessary we may assume, without loss of generality, that

$$a' \underset{c_1c_2'}{\downarrow} a \quad \text{and} \quad a'' \underset{c_2c_2'}{\downarrow} aa'. \tag{4.12}$$

Before continuing, observe that for every $c \in C_s$, $\mathrm{crd}(c) = \mathrm{crd}_s(c) \subseteq C_s$, because of Fact 3.2 (iii).

Claim 4.3.

$$\mathrm{crd}(a) \cap \mathrm{crd}(a') \ = \ \mathrm{crd}(c_1), \tag{4.13}$$

$$\mathrm{crd}(a) \cap \mathrm{crd}(a'') \ = \ \mathrm{crd}(c_2), \quad \textit{and} \tag{4.14}$$

$$\mathrm{crd}(a') \cap \mathrm{crd}(a'') \ = \ \mathrm{crd}(c_2'). \tag{4.15}$$

Proof of the claim. First note that by the choice of a, c_1 and c_2, and by (4.11), we get $c_1 \in \mathrm{crd}(a) \cap \mathrm{crd}(a') \cap C_s$. Hence $\mathrm{crd}(c_1) \subseteq \mathrm{crd}(a) \cap \mathrm{crd}(a')$. From (4.12) and Lemma 3.7 we get

$$\mathrm{crd}(a) \cap \mathrm{crd}(a') \ \subseteq \ \mathrm{acl}(c_1, c_2'). \tag{4.16}$$

Regarding (4.13), it remains to prove that $\mathrm{crd}(a) \cap \mathrm{crd}(a') \subseteq \mathrm{crd}(c_1)$. Suppose that $d \in \mathrm{crd}(a) \cap \mathrm{crd}(a')$. By (4.16) and Fact 3.5 (i), $d \in \mathrm{acl}(c_1)$ or $d \in \mathrm{acl}(c_2')$. If $d \in \mathrm{acl}(c_1)$ then we have $d \in \mathrm{crd}(c_1)$.

Suppose that $d \in \mathrm{acl}(c_2')$. Hence $d \in \mathrm{crd}(a) \cap \mathrm{crd}(c_2')$. From (4.10) we have $c_2' \underset{\mathrm{crd}_{s-1}(c_1)}{\downarrow} a$, so by Lemma 3.7 we get $d \in \mathrm{acl}(\mathrm{crd}_{s-1}(c_1))$ and hence (by the definition of crd_{s-1}) $d \in \mathrm{crd}(c_1)$. Thus we have proved (4.13).

Observe that (4.13) and Lemma 3.7 imply that

$$a \underset{c_1}{\downarrow} a'. \tag{4.17}$$

If $c_2 \in \mathrm{acl}(a')$ then, as $c_2 \in \mathrm{acl}(a)$, it follows from (4.17) and Lemma 3.7 that $c_2 \in \mathrm{acl}(c_1)$, but this contradicts (4.1). Hence,

$$c_2 \notin \mathrm{acl}(a') \tag{4.18}$$

Now we prove (4.14). From (4.11) it follows that $c_2 \in \mathrm{crd}(a) \cap \mathrm{crd}(a'')$, so $\mathrm{crd}(c_2) \subseteq \mathrm{crd}(a) \cap \mathrm{crd}(b)$. It remains to prove that if $d \in \mathrm{crd}(a) \cap \mathrm{crd}(b)$ then $d \in \mathrm{acl}(c_2)$. So suppose that $d \in \mathrm{crd}(a) \cap \mathrm{crd}(b)$. By (4.12) and Lemma 3.7, $d \in \mathrm{acl}(c_2, c_2')$, so by Fact 3.5 (i), $d \in \mathrm{acl}(c_2)$ or $d \in \mathrm{acl}(c_2')$. If $d \in \mathrm{acl}(c_2)$ then we are done, so suppose that $d \in \mathrm{acl}(c_2')$.

First assume that $d \in C_s \setminus C_{s-1}$. Recall that, by Fact 3.5 (ii), $(C_s \setminus C_{s-1}, \mathrm{cl})$, where 'cl' is 'acl' restricted to $C_s \setminus C_{s-1}$, is a trival pregeometry. By assumption, $d \in \mathrm{acl}(c_2')$, so (by the "exchange property" of pregeometries) $c_2' \in \mathrm{acl}(d)$ and hence $c_2' \in \mathrm{acl}(a)$, contradicting (4.9).

Hence we must have $d \in C_{s-1}$. By assumption we have $d \in \mathrm{crd}(c_2') \cap \mathrm{crd}(a)$. This together with (4.10) and Lemma 3.7 implies that $d \in \mathrm{crd}_{s-1}(c_1)$. By (4.3), $E_{s-1}(c_1, c_2)$, so $\mathrm{crd}_{s-1}(c_1) = \mathrm{crd}_{s-1}(c_2)$ and therefore $d \in \mathrm{crd}_{s-1}(c_2)$. Thus (4.14) is proved.

It remains to prove (4.15). By (4.11), $c_2' \in \mathrm{crd}(a') \cap \mathrm{crd}(a'')$, so $\mathrm{crd}(c_2') \subseteq \mathrm{crd}(a') \cap \mathrm{crd}(a'')$. It remains to prove that if $d \in \mathrm{crd}(a') \cap \mathrm{crd}(a'')$ then $d \in \mathrm{crd}(c_2')$. Suppose that $d \in \mathrm{crd}(a') \cap \mathrm{crd}(a'')$. Then, from (4.12) and Lemma 3.7, we get $d \in \mathrm{acl}(c_2, c_2')$. By Fact 3.5 (i), $d \in \mathrm{acl}(c_2)$ or $d \in \mathrm{acl}(c_2')$. If $d \in \mathrm{acl}(c_2')$ then we are done, so suppose that $d \in \mathrm{acl}(c_2)$.

First assume that $d \in C_s \setminus C_{s-1}$. As $C_s \setminus C_{s-1}$ is a trivial pregeometry (with 'acl' restricted to $C_s \setminus C_{s-1}$) and $d \in \mathrm{acl}(c_2)$ we get $c_2 \in \mathrm{acl}(d) \subseteq \mathrm{acl}(a')$, which contradicts (4.18).

Hence we have $d \in C_{s-1}$. Then $d \in \mathrm{crd}_{s-1}(c_2)$. By (4.3), $E_{s-1}(c_1, c_2)$ and by (4.7) we get $E_{s-1}(c_2, c_2')$, so $\mathrm{crd}_{s-1}(c_2) = \mathrm{crd}_{s-1}(c_2')$. Therefore $d \in \mathrm{crd}_{s-1}(c_2')$. This concludes the proof of Claim 4.3. □

By (4.1) there is $d \in M$ such that

$$ac_1d \equiv ac_2a'', \tag{4.19}$$

so in particular, $c_1 \in \mathrm{crd}_s(d) \setminus C_{s-1}$. By (4.1) and (4.11) we have $a'c_1 \equiv a'c_2'$, so there is $e \in M$ such that

$$a'c_1e \equiv a'c_2'a'', \tag{4.20}$$

so in particular, $c_1 \in \big(\mathrm{crd}_s(d) \cap \mathrm{crd}_s(e)\big) \setminus C_{s-1}$. By (4.20), (4.11), (4.1), (4.11) and (4.19), in the mentioned order, we have

$$c_1e \;\equiv\; c_2'a'' \;\equiv\; c_2a \;\equiv\; c_1a \;\equiv\; c_2a'' \;\equiv\; c_1d.$$

Hence $c_1e \equiv c_1d$ and by Lemma 3.10 we get

$$\mathrm{tp}(d/\mathrm{acl}(c_1)) = \mathrm{tp}(e/\mathrm{acl}(c_1)). \tag{4.21}$$

From (4.14), (4.15) and Lemma 3.7 we get

$$a \mathop{\smile\!\!\!\!\!\mid}_{c_2} a'' \quad \text{and} \quad a' \mathop{\smile\!\!\!\!\!\mid}_{c_2'} a'',$$

which together with (4.19) and (4.20) gives

$$a \mathop{\smile\!\!\!\!\!\mid}_{c_1} d \quad \text{and} \quad a' \mathop{\smile\!\!\!\!\!\mid}_{c_1} e. \tag{4.22}$$

By (4.17), (4.21), (4.22) and the independence theorem there is $a^* \in M$ such that

$$ac_1a^* \equiv ac_2a'' \quad \text{and} \quad a'c_1a^* \equiv a'c_2'a''. \tag{4.23}$$

This together with (4.14) and (4.15) implies that

$$\begin{aligned} &\mathrm{crd}(a) \cap \mathrm{crd}(a^*) = \mathrm{crd}(c_1) \quad \text{and} \\ &\mathrm{crd}(a') \cap \mathrm{crd}(a^*) = \mathrm{crd}(c_1). \end{aligned} \tag{4.24}$$

Hence

$$c_1 \in \mathrm{crd}(a) \cap \mathrm{crd}(a') \cap \mathrm{crd}(a^*) \cap (C_s \setminus C_{s-1}). \tag{4.25}$$

By (4.1) and (4.7), $\{c_1, c_2, c_2'\}$ is an independent set over $\mathrm{crd}_{s-1}(c_1)$. Hence $\mathrm{acl}(c_1) \cap \mathrm{acl}(c_2) \cap \mathrm{acl}(c_2') \cap (C_s \setminus C_{s-1}) = \emptyset$. Now Claim 4.3 implies that

$$\mathrm{crd}(a) \cap \mathrm{crd}(a') \cap \mathrm{crd}(a'') \cap (C_s \setminus C_{s-1}) = \emptyset. \tag{4.26}$$

Since $a, a', a'', a^* \in M$ and $\mathcal{M}$ is a binary structure with elimination of quantifiers, it follows from (4.23) that

$$aa'a'' \equiv aa'a^*. \tag{4.27}$$

But this contradicts (4.25) and (4.26), because the relation "$\mathrm{crd}(x) \cap \mathrm{crd}(y) \cap \mathrm{crd}(z) \cap (C_s \setminus C_{s-1})$ is nonempty" is $\emptyset$-definable in $\mathcal{M}$. This concludes the proof of Lemma 4.2.

□

Before proving our next main lemma we need the following auxilliary lemma:

Lemma 4.4. *Let $a \in M$, $c \in \mathrm{crd}_s(a) \setminus C_{s-1}$ and $p(x,y) = \mathrm{tp}(a,c)$.*
(i) Suppose that $X_1, \ldots, X_n$ enumerates all E_{s-1}-equivalence classes with which $\mathrm{acl}(c) \cap p(a, \mathcal{M}^{\mathrm{eq}})$ has nonempty intersection. Furthermore, suppose that $a' \in M$ and $E_{s-1}(a,a')$ (so in particular $\mathrm{crd}_{s-1}(a) = \mathrm{crd}_{s-1}(a')$). Then there is $c' \in \mathrm{crd}_s(a')$ such that $p(a',c')$ and $\mathrm{acl}(c') \cap p(a', \mathcal{M}^{\mathrm{eq}})$ has nonempty intersection with all of $X_1, \ldots, X_n$.
(ii) Suppose that $a' \in M$, $c' \in \mathrm{crd}_s(a') \setminus C_{s-1}$ and $p(a',c')$. Then $\mathrm{acl}(c') \cap p(a', \mathcal{M}^{\mathrm{eq}})$ has nonempty intersection with the same number of E_{s-1}-equivalence classes as $\mathrm{acl}(c) \cap p(a, \mathcal{M}^{\mathrm{eq}})$ has.
(iii) Suppose that $X_1, \ldots, X_n$ is an enumeration of all E_{s-1}-equivalence classes with which $\mathrm{acl}(c) \cap p(a, \mathcal{M}^{\mathrm{eq}})$ has nonempty intersection. Furthermore suppose that $a' \in M$ and $E_{s-1}(a,a')$. If $c' \in \mathrm{crd}_s(a') \setminus C_{s-1}$, $p(a',c')$ and $E_{s-1}(c,c')$, then $\mathrm{acl}(c') \cap p(a', \mathcal{M}^{\mathrm{eq}})$ has nonempty intersection with all of $X_1, \ldots, X_n$.

Proof. Let $a \in M$, $c \in \mathrm{crd}_s(a) \setminus C_{s-1}$ and $p(x,y) = \mathrm{tp}(a,c)$. In this proof we abbreviate E_{s-1} by E.

(i) We first note that c_E may, strictly speaking, be an element of $(M^{\mathrm{eq}})^{\mathrm{eq}}$. But since $\mathcal{M}^{\mathrm{eq}}$ has elimination of imaginaries we may identify c_E with an element of M^{eq}. By slight abuse of terminology, we also denote the sort of c_E by E. Let $\mathrm{acl}(c) \cap p(a, \mathcal{M}^{\mathrm{eq}}) = \{c_1, \ldots, c_n\}$ and, for each $i = 1, \ldots, n$, let $X_i = [c_i]_E$. From the definition of E $(= E_{s-1})$ it follows that $(c_i)_E \in \mathrm{acl}(\mathrm{crd}_{s-1}(a))$ for all $i = 1, \ldots, n$. Let $\varphi(x, z_1, \ldots, z_n)$ be a formula in the language of $\mathcal{M}^{\mathrm{eq}}$ which expresses the following condition:

"each one of $z_1, \ldots, z_n$ is of sort E and
$\exists y \Big(p(x,y) \ \wedge \ \forall u \Big(\big(p(x,u) \ \wedge \ u \in \mathrm{acl}(y)\big) \ \rightarrow$
for some $1 \leq i \leq n$, u belongs to the E-class represented by $z_i \Big) \Big)$".

Then $\mathcal{M}^{\mathrm{eq}} \models \varphi(a, (c_1)_E, \ldots, (c_n)_E)$. Let $a' \in M$ be such that $E(a,a')$. Then $(c_i)_E \in \mathrm{acl}(\mathrm{crd}_{s-1}(a)) = \mathrm{acl}(\mathrm{crd}_{s-1}(a'))$ for all i, and

$$\mathrm{tp}\big(a/\mathrm{acl}(\mathrm{crd}_{s-1}(a))\big) \ = \ \mathrm{tp}\big(a'/\mathrm{acl}(\mathrm{crd}_{s-1}(a'))\big).$$

Hence we get $\mathcal{M}^{\mathrm{eq}} \models \varphi(a', (c_1)_E, \ldots, (c_n)_E)$. Thus there is $c' \in \mathrm{crd}_s(a')$ such that $\mathcal{M} \models p(a',c')$ and $\mathrm{acl}(c') \cap p(a, \mathcal{M}^{\mathrm{eq}})$ has nonempty intersection with X_i for each $i = 1, \ldots, n$.

(ii) The assumption that $p(a,c)$ and $p(a',c')$ gives $ac \equiv a'c'$ so there is an automorphism of $\mathcal{M}^{\mathrm{eq}}$ which takes ac to $a'c'$. The conclusion follows from this.

(iii) Let $X_1, \ldots, X_n$ be an enumeration of all E-classes with which $\mathrm{acl}(c) \cap p(a, \mathcal{M}^{\mathrm{eq}})$ has nonempty intersection. Suppose that $a' \in M$, $E(a,a')$, $c' \in \mathrm{crd}_s(a')$, $p(a',c')$ and $E(c,c')$. Using part (ii) we can enumerate all E-classes with which $\mathrm{acl}(c') \cap p(a', \mathcal{M}^{\mathrm{eq}})$ has nonempty intersection as $X'_1, \ldots, X'_n$. Without loss of generality, assume that $X_1 = X'_1$ and $c, c' \in X_1$. By part (i), there is $c'' \in \mathrm{crd}_s(a') \setminus C_{s-1}$ such that $p(a',c'')$ and $\mathrm{acl}(c'') \cap p(a', \mathcal{M}^{\mathrm{eq}})$ has nonempty intersection with all $X_1, \ldots, X_n$. In particular, $\mathrm{acl}(c'') \cap p(a', \mathcal{M}^{\mathrm{eq}})$ has nonempty intersection with X_1. Let $c^* \in \mathrm{acl}(c'') \cap p(a', \mathcal{M}^{\mathrm{eq}}) \cap X_1$ (so in particular $c^* \in C_s \setminus C_{s-1}$). As, by Fact 3.5, $C_s \setminus C_{s-1}$ is a trivial pregeometry, with 'acl' restricted to $C_s \setminus C_{s-1}$, we get $\mathrm{acl}(c^*) = \mathrm{acl}(c'')$. Consequently $\mathrm{acl}(c^*) \cap p(a', \mathcal{M}^{\mathrm{eq}})$ has nonempty intersection with all $X_1, \ldots, X_n$. By the choice of c^* we have $a'c^* \equiv a'c'$ and $E(c^*, c')$. Hence Lemma 4.2 implies that $c^* \underset{\mathrm{crd}_{s-1}(c')}{\not\downarrow} c'$. Since, by Fact 3.6, $\mathrm{SU}(c'/\mathrm{crd}_{s-1}(c')) = 1$, we get $c' \in \mathrm{acl}(\{c^*\} \cup \mathrm{crd}_{s-1}(c'))$. By Fact 3.5 (i), we get $c' \in \mathrm{acl}(c^*)$ or $c' \in \mathrm{acl}(\mathrm{crd}_{s-1}(c'))$. But as $\mathrm{SU}(c'/\mathrm{crd}_{s-1}(c')) = 1$ we must have $c' \in \mathrm{acl}(c^*)$. Since $C_s \setminus C_{s-1}$ is a trivial pregeometry we get $\mathrm{acl}(c') = \mathrm{acl}(c^*)$. Then $\mathrm{acl}(c') \cap p(a', \mathcal{M}^{\mathrm{eq}})$ has nonempty intersection with all $X'_1, \ldots, X'_n, X_1, \ldots, X_n$, which, by part (ii) and the choice of $X_1, \dots, X_n$ and $X'_1, \dots, X'_n$, implies that $\{X_1, \dots, X_n\} = \{X'_1, \dots, X'_n\}$. □

Remark 4.5. (The intuition behind Lemma 4.6.) Let $\mathcal{C} = (\mathbb{N}, E)$, where E is interpreted as an equivalence relation with two infinite equivalence classes. Let us assume that one of the classes contains all even numbers and the other all odd numbers. Let

$$G = \{\{n, m\} : n \in \mathbb{N} \text{ is even and } m \in \mathbb{N} \text{ is odd}\}.$$

Turn G into a graph $\mathcal{G}$ by letting $a, b \in G$ be adjacent if and only if their intersection is a singleton. Since $\mathcal{C}$ is ω-categorical and stable, and $\mathcal{G}$ is interpretable in $\mathcal{C}$ (without parameters) it follows that $\mathcal{G}$ is ω-categorical and stable, in fact superstable of SU-rank 2. Moreover, without going into the details, $C\ \ (= \mathbb{N})$ is a $\emptyset$-definable subset of G^{eq} and the equivalence relation E on C is $\emptyset$-definable in $\mathcal{G}^{\text{eq}}$. Consider the following two quadruples of elements from G:

$$(\{1,2\}, \{1,4\}, \{3,6\}, \{3,8\}), \ (\{1,2\}, \{1,4\}, \{3,6\}, \{5,6\}).$$

Clearly, the two quadruples satisfy the same quantifier-free formulas. Note that $\{1,2\}$ and $\{1,4\}$ have a common element in the E-class of odd numbers, and the same is true for $\{3,6\}$ and $\{3,8\}$. Hence the first quadruple above satisfies the formula $\varphi(x_1, x_2, x_3, x_4)$ which expresses "there are $u, v \in C$ such that $E(u, v)$, $x_1 \cap x_2 = \{u\}$ and $x_3 \cap x_4 = \{v\}$". But the second quadruple does not satisfy this formula. Since all elements in the two quadruples above are "real" elements of G^{eq} (i.e. belong to G), it follows that there is a formula in the (graph) language of $\mathcal{G}$ which is satisfied by the first quadruple, but not by the second. Thus $\mathcal{G}$ is not homogeneous.

The idea of the proof of Lemma 4.6 is the following: If $a \in M$, $c_1, c_2 \in \text{crd}_s(a) \setminus C_{s-1}$, $c_1 \underset{\text{crd}_{s-1}(a)}{\perp} c_2$, and $ac_1 \equiv ac_2$, then we can find $a^*, b^*, a', b', b'' \in M$ such that

$$a^*b^*b' \equiv a^*b^*b'' \ \text{ and } \ a'b' \equiv a'b'', \ \text{ but } \ a^*b^*a'b' \not\equiv a^*b^*a'b''.$$

This is done by choosing the elements so that, with $p = \text{tp}(a, c_1)$, there are $c, d \in C_s \setminus C_{s-1}$ such that $E_{s-1}(c, d)$, $p(a^*, c)$, $p(b^*, c)$, $p(a', d)$ and $p(b', d)$, but no such c and d exist if we replace b' by b''. In finding such elements we use Lemma 4.2, which implies that $\neg E_{s-1}(c_1, c_2)$, where 'E_{s-1}' plays the role of 'E' in $\mathcal{G}^{\text{eq}}$.

The same idea is behind the proof of [21, Proposition 4.4], as becomes apparent in the last page of that proof. However, in the context of [21] one can assume that $s = 1$, and then all $c \in C_s$ have SU-rank 1. Moreover, one can assume (in [21]) that for all $c, d \in C_s$, if $d \in \text{acl}(c)$, then $c = d$, and that the "canonically embedded" structure (in $\mathcal{M}^{\text{eq}}$) with universe C_s is a binary random structure (by [2, Theorem 5.1] and some additional observations in [21, Fact 3.6]). In the present context, the arguments in the more specialized situation of [21] are replaced by dividing/forking arguments.

Lemma 4.6. *For all $a \in M$ and all $c_1, c_2 \in \text{crd}_s(a) \setminus C_{s-1}$, if $c_1 \underset{\text{crd}_{s-1}(a)}{\perp} c_2$ then $ac_1 \not\equiv ac_2$.*

Proof. Towards a contradiction suppose that there are $a \in M$ and $c_1, c_2 \in \text{crd}_s(a) \setminus C_{s-1}$ such that $c_1 \underset{\text{crd}_{s-1}(a)}{\perp} c_2$ and $ac_1 \equiv ac_2$. Let

$$q(x) = \text{tp}(a) \quad \text{and} \quad p(x, y) = \text{tp}(a, c_1).$$

Note that if $p(a', c)$ then $c \in \text{crd}_s(a') \setminus C_{s-1}$. So for every $a' \in M$ which realizes q there are $c, c' \in \text{crd}_s(a') \setminus C_{s-1}$ such that $c \underset{\text{crd}_{s-1}(a')}{\perp} c'$ and both $a'c$ and $a'c'$ realize p. Also, for all a' and c such that $a'c$ realizes p there is c' such that $a'c'$ realizes p and $c \underset{\text{crd}_{s-1}(a')}{\perp} c'$.

Choose any $c \in \mathrm{crd}_s(a) \setminus C_{s-1}$ such that ac realizes p. Let $b \in M$ realize a nondividing extension of $\mathrm{tp}\big(a/\mathrm{acl}(\{c\} \cup \mathrm{crd}_{s-1}(a))\big)$ to $\{a\} \cup \mathrm{acl}\big(\{c\} \cup \mathrm{crd}_{s-1}(a)\big)$. Then

$$a \underset{\substack{\{c\}\cup \\ \mathrm{crd}_{s-1}(a)}}{\mathop{\smile\!\!\!\!\!\!\;|}} b, \quad E_{s-1}(a,b) \quad \text{and} \quad p(a,c) \wedge p(b,c). \tag{4.28}$$

By the choice of p, (4.28) and Lemma 3.7 we get

$$p(a, \mathcal{M}^{\mathrm{eq}}) \cap p(b, \mathcal{M}^{\mathrm{eq}}) \ \subseteq\ \mathrm{crd}(a) \cap \mathrm{crd}(b) \ \subseteq\ \mathrm{acl}\big(\{c\} \cup \mathrm{crd}_{s-1}(a)\big).$$

Let $d \in p(a, \mathcal{M}^{\mathrm{eq}}) \cap p(b, \mathcal{M}^{\mathrm{eq}})$. By Fact 3.5 (i), $d \in \mathrm{acl}(c)$ or $d \in \mathrm{acl}(\mathrm{crd}_{s-1}(a))$. In the later case $d \in C_{s-1}$, because of Fact 3.2 (iii), and this contradicts that $p(a, \mathcal{M}^{\mathrm{eq}}) \subseteq C_s \setminus C_{s-1}$. Hence $d \in \mathrm{acl}(c)$, so we have proved that

$$c \in p(a, \mathcal{M}^{\mathrm{eq}}) \cap p(b, \mathcal{M}^{\mathrm{eq}}) \ \subseteq\ \mathrm{crd}(c). \tag{4.29}$$

Let $a' \in M$ realize a nondividing extension of $\mathrm{tp}\big(a/\mathrm{acl}(\mathrm{crd}_{s-1}(a))\big)$ to $\{a,b\} \cup \mathrm{acl}\big(\mathrm{crd}_{s-1}(a)\big)$. Then

$$E_{s-1}(a, a'), \qquad a' \underset{\mathrm{crd}_{s-1}(a)}{\mathop{\smile\!\!\!\!\!\!\;|}} ab, \tag{4.30}$$

and by Lemma 3.7 and Fact 3.5 (iii),

$$\mathrm{crd}(a') \cap (\mathrm{crd}(a) \cup \mathrm{crd}(b)) = \mathrm{crd}_{s-1}(a). \tag{4.31}$$

By Lemma 4.4 (i) there is $c' \in \mathrm{crd}_s(a') \setminus C_{s-1}$ such that $p(a', c')$ and $E_{s-1}(c, c')$. As explained in the beginning of the proof, there is $c'' \in \mathrm{crd}_s(a) \setminus C_{s-1}$ such that $p(a', c'')$ and $c' \underset{\mathrm{crd}_{s-1}(a')}{\mathop{\smile\!\!\!\!\!\!\;|}} c''$. By (4.30), $\mathrm{crd}_{s-1}(a) = \mathrm{crd}_{s-1}(a')$ and therefore

$$c' \underset{\mathrm{crd}_{s-1}(a)}{\mathop{\smile\!\!\!\!\!\!\;|}} c''. \tag{4.32}$$

Let $b' \in M$ realize a nondividing extension of

$$\mathrm{tp}(a'/\{c'\} \cup \mathrm{acl}(\mathrm{crd}_{s-1}(a))) \ \text{ to } \ \{a', a, b, c'\} \cup \mathrm{acl}(\mathrm{crd}_{s-1}(a)).$$

Then

$$E_{s-1}(a', b'), \qquad a'ab \underset{\substack{\{c'\}\cup \\ \mathrm{crd}_{s-1}(a)}}{\mathop{\smile\!\!\!\!\!\!\;|}} b', \tag{4.33}$$

and, in the same way as we proved (4.29), we get

$$c' \in p(a', \mathcal{M}^{\mathrm{eq}}) \cap p(b', \mathcal{M}^{\mathrm{eq}}) \subseteq \mathrm{crd}(c'). \tag{4.34}$$

From (4.31) and $c' \in \mathrm{crd}_s(a')$ we get $c' \underset{\mathrm{crd}_{s-1}(a)}{\mathop{\smile\!\!\!\!\!\!\;|}} ab$, so by (4.33) and transitivity of dividing we also have

$$ab \underset{\mathrm{crd}_{s-1}(a)}{\mathop{\smile\!\!\!\!\!\!\;|}} b'. \tag{4.35}$$

Since $p(a', c')$, $p(a', c'')$ and $E_{s-1}(a', b')$, there is $b'' \in M$ such that

$$a'c'b' \equiv a'c''b'', \ \text{ so } \ E_{s-1}(a', b'') \ \text{ and hence } \ E_{s-1}(b', b''). \tag{4.36}$$

This together with (4.33) implies that

$$a' \underset{\substack{\{c''\}\cup \\ \mathrm{crd}_{s-1}(a)}}{\mathop{\smile\!\!\!\!\!\!\;|}} b''. \tag{4.37}$$

Note that since $E_{s-1}(a,b)$, $E_{s-1}(a,a')$, $E_{s-1}(a',b')$ and $E_{s-1}(b',b'')$, all the elements a, a', b, b' and b'' have the same type over $\mathrm{acl}(\mathrm{crd}_{s-1}(a))$. By considering a nondividing extension of

$$\mathrm{tp}\big(b''/\{a', c''\} \cup \mathrm{acl}(\mathrm{crd}_{s-1}(a))\big) \ \text{ to } \ \{a', c'', a, b, b'\} \cup \mathrm{acl}(\mathrm{crd}_{s-1}(a)),$$

if necessary, we may, in addition, assume that

$$b'' \mathop{\smile\!\!\!\!\!\!\;|}_{\substack{\{a',c''\}\cup \\ \mathrm{crd}_{s-1}(a)}} abb'. \tag{4.38}$$

This together with (4.37) and transitivity gives $b'' \mathop{\smile\!\!\!\!\!\!\;|}_{\substack{\{c''\}\cup \\ \mathrm{crd}_{s-1}(a)}} abb'$. By the choice of c'', $c'' \in \mathrm{crd}_s(a')$. Hence (4.30) implies that $c'' \mathop{\smile\!\!\!\!\!\!\;|}_{\mathrm{crd}_{s-1}(a)} ab$, so by transitivity

$$b'' \mathop{\smile\!\!\!\!\!\!\;|}_{\mathrm{crd}_{s-1}(a)} ab. \tag{4.39}$$

Claim 4.7. $c'' \mathop{\smile\!\!\!\!\!\!\;|}_{\mathrm{crd}_{s-1}(a)} b'$.

Proof of the claim. By (4.33), Lemma 3.7 and facts 3.2 (iii) and 3.5 (i),

$$\big(\mathrm{crd}_s(a') \cap \mathrm{crd}_s(b')\big) \setminus C_{s-1} \subseteq \mathrm{acl}(c').$$

Recall that we have chosen c'' so that $c' \mathop{\smile\!\!\!\!\!\!\;|}_{\mathrm{crd}_{s-1}(a)} c''$. Hence $c'' \notin \mathrm{acl}(c')$. Since $c'' \in \mathrm{crd}_s(a')$ it follows that $c'' \notin \mathrm{acl}(b')$, and hence

$$c'' \notin \mathrm{crd}_s(b').$$

Suppose, for a contradiction, that there is $d \in \big(\mathrm{crd}_s(c'')\cap\mathrm{crd}_s(b')\big)\setminus C_{s-1}$. Since $C_s \setminus C_{s-1}$ is a trivial pregeometry (by Fact 3.5 (ii)), we get $c'' \in \mathrm{acl}(d)$, and hence $c'' \in \mathrm{crd}_s(b')$, contradicting what we obtained above. It follows that $\mathrm{crd}(c'') \cap \mathrm{crd}(b') \subseteq C_{s-1}$, so

$$\mathrm{crd}(c'') \cap \mathrm{crd}(b') = \mathrm{crd}_{s-1}(c'') \cap \mathrm{crd}_{s-1}(b').$$

Since a' and b' have the same type over $\mathrm{acl}(\mathrm{crd}_{s-1}(a)) = \mathrm{acl}(\mathrm{crd}_{s-1}(a')) = \mathrm{acl}(\mathrm{crd}_{s-1}(b'))$, it follows that $\mathrm{crd}_{s-1}(a') = \mathrm{crd}_{s-1}(b')$. As $c'' \in \mathrm{crd}_s(a')$ we get $\mathrm{crd}_{s-1}(c'') \subseteq \mathrm{crd}_{s-1}(b')$. Consequently, $\mathrm{crd}_{s-1}(c'') \cap \mathrm{crd}_{s-1}(b') = \mathrm{crd}_{s-1}(c'')$. Since we proved that $\mathrm{crd}(c'') \cap \mathrm{crd}(b') = \mathrm{crd}_{s-1}(c'') \cap \mathrm{crd}_{s-1}(b')$ it follows from Lemma 3.7 that $c'' \mathop{\smile\!\!\!\!\!\!\;|}_{\mathrm{crd}_{s-1}(c'')} b'$ and hence $c'' \mathop{\smile\!\!\!\!\!\!\;|}_{\mathrm{crd}_{s-1}(a)} b'$. □

On the line after (4.38) we obtained $b'' \mathop{\smile\!\!\!\!\!\!\;|}_{\substack{\{c''\}\cup \\ \mathrm{crd}_{s-1}(a)}} abb'$, from which we get $b'' \mathop{\smile\!\!\!\!\!\!\;|}_{\substack{\{c''\}\cup \\ \mathrm{crd}_{s-1}(a)}} b'$. This together with Claim 4.7 and transitivity gives

$$b' \mathop{\smile\!\!\!\!\!\!\;|}_{\mathrm{crd}_{s-1}(a)} b''. \tag{4.40}$$

We have $E_{s-1}(b', b'')$ and this implies that $\mathrm{crd}_{s-1}(b') = \mathrm{crd}_{s-1}(b'') = \mathrm{crd}_{s-1}(a)$ and

$$\mathrm{tp}(b'/\mathrm{acl}(\mathrm{crd}_{s-1}(a))) \ = \ \mathrm{tp}(b''/\mathrm{acl}(\mathrm{crd}_{s-1}(a))).$$

It follows (from Fact 2.2) that there are $a^+, b^+ \in M$ such that

$$\mathrm{tp}(a, b, b'/\mathrm{acl}(\mathrm{crd}_{s-1}(a))) \ = \ \mathrm{tp}(a^+, b^+, b''/\mathrm{acl}(\mathrm{crd}_{s-1}(a))), \tag{4.41}$$

which by (4.35) implies that

$$a^+b^+ \mathop{\smile\!\!\!\!\!\!\;|}_{\mathrm{crd}_{s-1}(a)} b''. \tag{4.42}$$

By (4.35), (4.40), (4.41), (4.42) and the independence theorem there are $a^*, b^* \in M$ such that

$$\begin{aligned} \mathrm{tp}(a^*, b^*, b'/\mathrm{acl}(\mathrm{crd}_{s-1}(a))) \ &= \ \mathrm{tp}(a, b, b'/\mathrm{acl}(\mathrm{crd}_{s-1}(a))), \\ \mathrm{tp}(a^*, b^*, b''/\mathrm{acl}(\mathrm{crd}_{s-1}(a))) \ &= \ \mathrm{tp}(a^+, b^+, b''/\mathrm{acl}(\mathrm{crd}_{s-1}(a))), \ \text{ and} \\ a^*b^* \mathop{\smile\!\!\!\!\!\!\;|}_{\mathrm{crd}_{s-1}(a)} b'b''. \end{aligned} \tag{4.43}$$

By considering a nondividing extension if necessary we may, in addition, assume that

$$a^*b^* \underset{\mathrm{crd}_{s-1}(a)}{\mathop{\smile\!\!\!\!\!\!\mid}} a'b'b''.$$

From (4.41) and (4.43) we get

$$a^*b^*b' \equiv a^*b^*b''.$$

From (4.36) we have $a'b' \equiv a'b''$. Since $a^*, b^*, a', b', b'' \in M$ where $\mathcal{M}$ is binary with elimination of quantifiers it follows that

$$a^*b^*a'b' \equiv a^*b^*a'b''. \tag{4.44}$$

By (4.41) and (4.43) we have $\mathrm{tp}(a^*, b^*/\mathrm{acl}(\mathrm{crd}_{s-1}(a))) = \mathrm{tp}(a, b/\mathrm{acl}(\mathrm{crd}_{s-1}(a)))$. Recall that $c \in p(a, \mathcal{M}^{\mathrm{eq}}) \cap p(b, \mathcal{M}^{\mathrm{eq}})$. Therefore (and by Fact 2.1 (iii)) there is $c^* \in p(a^*, \mathcal{M}^{\mathrm{eq}}) \cap p(b^*, \mathcal{M}^{\mathrm{eq}})$ such that $E_{s-1}(c, c^*)$. We have chosen c' so that, among other things, $E_{s-1}(c, c')$ (see the line after (4.31)). As E_{s-1} is an equivalence relation we get $E_{s-1}(c', c^*)$. These observations and (4.34) imply that the following statement, abbreviated $\varphi(x_1, x_2, x_3, x_4)$, is satisfied by (a^*, b^*, a', b'):

> "There are $y_1, y_2 \in C_s \setminus C_{s-1}$ such that $E_{s-1}(y_1, y_2)$ and $p(x_1, y_1)$, $p(x_2, y_1)$, $p(x_3, y_2)$ and $p(x_4, y_2)$."

Note that $\varphi(x_1, x_2, x_3, x_4)$ can be expressed by a first-order formula in the language of $\mathcal{M}^{\mathrm{eq}}$. The next step is to show that φ is not satisfied by (a^*, b^*, a', b'').

Suppose that $d, e \in C_s \setminus C_{s-1}$ are such that

$$p(a^*, d) \cap p(b^*, d) \ \text{ and } \ p(a', e) \cap p(b'', e).$$

To prove that $\mathcal{M}^{\mathrm{eq}} \not\models \varphi(a^*, b^*, a', b'')$ it suffices to show that $\neg E_{s-1}(d, e)$. By the choice of c^*, (4.28) and (4.43), we have $a^* \underset{\substack{\{c^*\}\cup \\ \mathrm{crd}_{s-1}(a)}}{\mathop{\smile\!\!\!\!\!\!\mid}} b^*$ and therefore

$$\mathrm{crd}(a^*) \cap \mathrm{crd}(b^*) \cap (C_s \setminus C_{s-1}) \subseteq \mathrm{acl}(c^*).$$

Moreover, from (4.37) it follows that

$$\mathrm{crd}(a') \cap \mathrm{crd}(b'') \cap (C_s \setminus C_{s-1}) \subseteq \mathrm{acl}(c'').$$

Therefore the assumptions about d and e imply that

$$d \in \mathrm{acl}(c^*) \cap p(a^*, \mathcal{M}^{\mathrm{eq}}) \ \text{ and } \ e \in \mathrm{acl}(c'') \cap p(a', \mathcal{M}^{\mathrm{eq}}).$$

Since $C_s \setminus C_{s-1}$ is a trivial pregeometry it follows that $c'' \in \mathrm{acl}(e)$, and hence $\mathrm{acl}(e) = \mathrm{acl}(c'')$. Recall that $E_{s-1}(c', c^*)$. By Lemma 4.4 (iii), there is $e' \in \mathrm{acl}(c') \cap p(a', \mathcal{M}^{\mathrm{eq}})$ such that $E_{s-1}(d, e')$. By again using that $C_s \setminus C_{s-1}$ is a trivial pregeometry it follows that $c' \in \mathrm{acl}(e')$, and consequently $\mathrm{acl}(c') = \mathrm{acl}(e')$. Thus we have $\mathrm{acl}(e') = \mathrm{acl}(c')$ and $\mathrm{acl}(e) = \mathrm{acl}(c'')$, and by (4.32) we have $c' \underset{\mathrm{crd}_{s-1}(a)}{\mathop{\smile\!\!\!\!\!\!\mid}} c''$. It follows that $e \underset{\mathrm{crd}_{s-1}(a)}{\mathop{\smile\!\!\!\!\!\!\mid}} e'$. By the choice of e and e' we also have $a'e \equiv a'e'$. Therefore Lemma 4.2 implies that $\neg E_{s-1}(e, e')$. Since $E_{s-1}(d, e')$ we must have $\neg E_{s-1}(d, e)$. Thus we have shown that

$$\mathcal{M}^{\mathrm{eq}} \models \varphi(a^*, b^*, a', b'') \wedge \neg\varphi(a^*, b^*, a', b''),$$

which contradicts (4.44). This concludes the proof of Lemma 4.6. □

5. Coordinatization by equivalence relations

Throughout this section we adopt Notation 3.1. Theorem 5.1, below, is slightly more general than (a) – (c) of the main results in the introduction, because we only assume that $\bar{c} \in C$ here (where $M \subseteq C$). Corollaries 5.3 and 5.4 have more general assumptions than Theorem 5.1 and are derived from its proof.

Theorem 5.1. *Suppose that $\mathcal{M}$ is binary, simple, and homogeneous (hence supersimple with finite SU-rank). Let $\mathbf{R}$ be the (finite) set of all $\emptyset$-definable equivalence relations on M.*

(i) *For every $a \in M$, if $\mathrm{SU}(a) = k$, then there are $R_1, \ldots, R_k \in \mathbf{R}$, depending only on $\mathrm{tp}(a)$, such that $a \in \mathrm{acl}(a_{R_k})$, $\mathrm{SU}(a_{R_1}) = 1$, R_{i+1} refines R_i and $\mathrm{SU}(a_{R_{i+1}}/a_{R_i}) = 1$ for all $1 \leq i < k$ (or equivalently, $\mathrm{SU}(a/a_{R_i}) = k - i$ for all $1 \leq i \leq k$).*

(ii) *Suppose that $a, b \in M$, $\bar{c} \in C$, and $a \not\perp_{\bar{c}} b$ (where we recall that $M \subseteq C \subseteq M^{\mathrm{eq}}$). Then there is $R \in \mathbf{R}$ such that $a \not\perp_{\bar{c}} a_R$ and $a_R \in \mathrm{acl}(b)$ (and hence $a_R \notin \mathrm{acl}(\bar{c})$).*

(iii) *Suppose that all binary $\emptyset$-definable relations on M are symmetric. If $a, b \in M$, $\bar{c} \in C$, and $a \not\perp_{\bar{c}} b$, then there is $R \in \mathbf{R}$ such that $a \not\perp_{\bar{c}} a_R$ and $R(a, b)$ (and therefore $a_R \in \mathrm{acl}(b)$, $a_R \notin \mathrm{acl}(\bar{c})$ and hence $\neg R(a, c)$ for every $c \in \bar{c}$).*

Note that the assumptions of part (iii) imply that $Th(\mathcal{M})$ has only one 1-type over $\emptyset$.

Remark 5.2. (i) Suppose that $a \in M$. The "coordinatization by $R_1, \ldots, R_k$" as in Theorem 5.1 (i) may *not* be unique. In other words, there may also be $\emptyset$-definable equivalence relations $R'_1, \ldots, R'_k$ with the same properties as $R_1, \ldots, R_k$ such that some R'_i is (in a strong sense[2]) not equivalent with R_i. This is shown by the example $\mathcal{M}$ in Section 7.2.
(ii) The conclusion in Theorem 5.1 (ii) *cannot* be strengthened so that it, in addition, says that $R(a, b)$. This is also shown by the example $\mathcal{M}$ in Section 7.2.

The following two corollaries follow from an analysis of the proof of Theorem 5.1, which is given in Section 5.3.

Corollary 5.3. *Suppose that $\mathcal{M}$ is ω-categorical, supersimple with finite SU-rank and with trivial dependence. Also, suppose that part (i) of Theorem 5.1 does not hold for $\mathcal{M}$. Then there are distinct $a_i, b_i \in M$, $i = 1, \ldots, 4$, such that $\mathrm{tp}(a_i, a_j) = \mathrm{tp}(b_i, b_j)$ for all i, j and $\mathrm{tp}(a_1, \ldots, a_4) \neq \mathrm{tp}(b_1, \ldots, b_4)$.*

Corollary 5.4. *Suppose that $\mathcal{M}$ is ω-categorical, supersimple with finite SU-rank and with trivial dependence. Moreover, assume that $\mathcal{M}$ has no $\emptyset$-definable equivalence relation on M with infinitely many infinite equivalence classes. If $\mathrm{SU}(\mathcal{M}) > 1$ then there are distinct $a_i, b_i \in M$, $i = 1, \ldots, 4$, such that $\mathrm{tp}(a_i, a_j) = \mathrm{tp}(b_i, b_j)$, for all i, j, and $\mathrm{tp}(a_1, \ldots, a_4) \neq \mathrm{tp}(b_1, \ldots, b_4)$.*

5.1. **Proof of part (i) of Theorem 5.1.** In this subsection (and the next) we assume that $\mathcal{M}$ is binary, simple and homogeneous. Moreover, we assume that $M \subseteq U \subseteq M^{\mathrm{eq}}$, where U, C and C_i, $i = 1, \ldots, h$, are as in Assumption 3.3. Then we can use all results from sections 2 – 4. The proof is carried out through a sequence of lemmas and is finished by the short argument after Lemma 5.14.

Lemma 5.5. *Suppose that Q is a $\emptyset$-definable equivalence relation on M^n. Let $\bar{a} \in M^n$ and suppose that $b \in \mathrm{acl}(\bar{a}')$ for every $\bar{a}' \in [\bar{a}]_Q$. Then $b \in \mathrm{acl}(\bar{a}_Q)$.*

Proof. If $[\bar{a}]_Q$ is finite the $\mathrm{acl}(\bar{a}) = \mathrm{acl}(\bar{a}_Q)$ and the conclusion is immediate. So suppose that $[\bar{a}]_Q$ is infinite. For a contradiction suppose that $b \notin \mathrm{acl}(\bar{a}_Q)$. The we find $\bar{a}'$ (in some elementary extension of $\mathcal{M}$) realizing a nonforking extension of $\mathrm{tp}(\bar{a}/\bar{a}_Q)$ to $\bar{a}_Q b$, so $\bar{a}' \perp_{\bar{a}_Q} b$. As $\mathcal{M}^{\mathrm{eq}}$ is ω-saturated we may assume that $\bar{a}' \in M^n$. Since $\mathrm{tp}(\bar{a}'/\bar{a}_Q) = \mathrm{tp}(\bar{a}/\bar{a}_Q)$ we have $\bar{a}' \in [\bar{a}]_Q$. As $\bar{a}' \perp_{\bar{a}_Q} b$ and $b \notin \mathrm{acl}(\bar{a}_Q)$, we get $b \notin \mathrm{acl}(\bar{a}')$, contradicting the assumption. □

[2] For example, it can happen, like with $\mathcal{M}$ in Section 7.2, that R_i and R'_i have only infinite classes but $R_i \cap R'_i$ has only singleton classes.

Definition 5.6. Let $a \in M$, $c \in \mathrm{crd}(a)$, $q(x) = \mathrm{tp}(a)$ and $p(x,u) = \mathrm{tp}(a,c)$. Define a relation on M as follows:

$$R_p(x,y) \iff \Big(\neg q(x) \ \wedge \ \neg q(y)\Big) \ \vee \ \exists u,v\Big(p(x,u) \wedge p(y,v) \wedge \mathrm{acl}(u) = \mathrm{acl}(v)\Big).$$

Lemma 5.7. *The relation R_p, as in Definition 5.6, is an equivalence relation and is $\emptyset$-definable.*

Proof. By ω-categoricity, R_p is $\emptyset$-definable. It is straightforward to see that it is reflexive and symmetric, so it remains to show that it is transitive. Suppose that $a, b, c \in M$, $R_p(a,b)$ and $R_p(b,c)$. We assume that $a \neq b$, $a \neq c$, $b \neq c$, $q(a)$, $q(b)$, and $q(c)$, as the other cases are straightforward and only use the definition of R_p. By the definition of R_p, there are i,j,k,l and c_i, c_j, c_k, c_l such that $p(a,c_i)$, $p(b,c_j)$, $p(b,c_k)$, $p(c,c_l)$, $\mathrm{acl}(c_i) = \mathrm{acl}(c_j)$ and $\mathrm{acl}(c_k) = \mathrm{acl}(c_l)$. By the choice of p (in Definition 5.6), it follows that all c_i, c_j, c_k, c_l have the same type over $\emptyset$, and for some $0 < s \leq h$ they all belong to $C_s \setminus C_{s-1}$.

We will prove that $\mathrm{acl}(c_j) = \mathrm{acl}(c_k)$, which implies that $\mathrm{acl}(c_i) = \mathrm{acl}(c_l)$ and from this we immediately get $R_p(a,c)$. By symmetry of the argument, it suffices to show that $c_j \in \mathrm{acl}(c_k)$. By the choice of c_j and c_k we have $p(a,c_j)$ and $p(a,c_k)$ and therefore $bc_j \equiv bc_k$. Then Lemma 4.6 implies that $c_j \underset{\mathrm{crd}_{s-1}(b)}{\not\mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.9em\raise0.6ex\hbox{$|$}}} c_k$. By Facts 3.2 (iii) and 3.6, $\mathrm{SU}(c_j/\mathrm{crd}_{s-1}(b)) = 1$ and therefore $c_j \in \mathrm{acl}\big(\{c_k\} \cup \mathrm{crd}_{s-1}(b)\big)$. By Fact 3.5 (i), $c_j \in \mathrm{acl}(d)$ for some $d \in \{c_k\} \cup \mathrm{crd}_{s-1}(b)$. As $\mathrm{SU}(c_j/\mathrm{crd}_{s-1}(b)) = 1$ we must have $c_j \in \mathrm{acl}(c_k)$. $\square$

Lemma 5.8. *Let $a \in M$, $c \in \mathrm{crd}(a)$, $p = \mathrm{tp}(a,c)$ and let R_p be as in Definition 5.6. Then* $\mathrm{acl}(c) = \mathrm{acl}(a_{R_p})$.

Proof. By the definition of R_p, for every $a' \in [a]_{R_p}$, $c \in \mathrm{acl}(a')$. Hence Lemma 5.5 implies that $c \in \mathrm{acl}(a_{R_p})$. By the definition of R_p, $[a]_{R_p}$ is the unique R_p-class such that for all $a' \in [a]_{R_p}$, there is c' with $\mathrm{acl}(c') = \mathrm{acl}(c)$ and $p(a',c')$. As $\mathcal{M}$ is ω-categorical, the following condition is definable by a formula in the language of $\mathcal{M}^{\mathrm{eq}}$ having only c as a parameter:

> "x is a member of M^{eq} representing an R_p-class such that, for all y in x, there is z such that $\mathrm{tp}(y,z) = p$ and $\mathrm{acl}(z) = \mathrm{acl}(c)$".

Hence $a_{R_p} \in \mathrm{dcl}(c)$. $\square$

Let $a \in M$. Let $h < \omega$ be minimal such that $a \in \mathrm{acl}(C_h)$. It follows (from Fact 3.5 (iv)) that $a \in \mathrm{acl}(\mathrm{crd}_h(a))$.

Definition 5.9. (i) For each $0 < s \leq h$, let ρ_s be maximal so that there are $c_{s,1}, \ldots, c_{s,\rho_s} \in \mathrm{crd}_s(a) \setminus \mathrm{crd}_{s-1}(a)$ such that $\{c_{s,1}, \ldots, c_{s,\rho_s}\}$ is an independent set over $\mathrm{crd}_{s-1}(a)$. (So ρ_s is the "dimension" of $\mathrm{crd}_s(a) \setminus \mathrm{crd}_{s-1}(a)$ over $\mathrm{crd}_{s-1}(a)$.) We now fix such $c_{s,1}, \ldots, c_{s,\rho_s}$.
(ii) For each $0 < s \leq h$ and $1 \leq i \leq \rho_s$, let $p_{s,i} = \mathrm{tp}(a, c_{s,i})$.
(iii) For each $0 < s \leq h$ and $1 \leq i \leq \rho_s$, let $R_{s,i} = R_{p_{s,i}}$ where $R_{p_{s,i}}$ is like R_p in Definition 5.6 with $p = p_{s,i}$.

Observation 5.10. From Lemma 4.6 it follows that, for every $1 \leq s \leq h$ and all $1 \leq i < j \leq \rho_s$, $p_{s,i} \neq p_{s,j}$. And we clearly have $p_{s,i} \neq p_{s',j}$ if $s \neq s'$. It follows that if $(s,i) \neq (s',i')$ then $R_{s,i}$ is different from $R_{s',i'}$.

Definition 5.11. Let $I = \{(s,i) : 1 \leq s \leq h,\ 1 \leq i \leq \rho_s\}$ and let '$\preccurlyeq$' be the lexicographic order on I, in other words, $(s,i) \preccurlyeq (s',i')$ if and only if $s < s'$, or $s = s'$ and $i \leq i'$.

Note that while the ordering in the first coordinate of (s,i) is natural, since s is the "height" of $c_{s,i}$, the order in the second coordinate is arbitrary, since it is given by the arbitrary enumeration $c_{s,1}, \ldots, c_{s,\rho_s}$ of the same elements.

Definition 5.12. For every $(s,i) \in I$, let

$$Q_{s,i} = \bigcap_{(s',i') \preccurlyeq (s,i)} R_{s',i'}.$$

Since intersections/conjuctions of equivalence relations are still equivalence relations it follows from Lemma 5.7 that $Q_{s,i}$ is a $\emptyset$-definable equivalence relation for each (s,i).

Lemma 5.13. *For every $(s,i) \in I$,*

$$\mathrm{acl}(a_{Q_{s,i}}) = \mathrm{acl}\big(\{c_{s',i'} : (s',i') \preccurlyeq (s,i)\}\big).$$

Proof. Let $(s,i) \in I$. We have $c_{s',i'} \in \mathrm{acl}(a)$ for all $(s',i') \preccurlyeq (s,i)$. From the definitions of $Q_{s,i}$ and $R_{s',i'}$ it follows that for every $a' \in a_{Q_{s,i}}$, $c_{s',i'} \in \mathrm{acl}(a')$ for all $(s',i') \preccurlyeq (s,i)$. Lemma 5.5 now implies that $c_{s',i'} \in \mathrm{acl}(a_{Q_{s,i}})$ for all $(s',i') \preccurlyeq (s,i)$.

By Lemma 5.8, for every $(s',i') \preccurlyeq (s,i)$, $\mathrm{acl}(a_{R_{s',i'}}) = \mathrm{acl}(c_{s',i'})$. From the definition of $Q_{i,s}$ it follows that, for any $a', b' \in M$, $a' \in [b']_{Q_{i,s}}$ if and only if $a' \in [b']_{R_{s',i'}}$ for all $(s',i') \preccurlyeq (s,i)$. Consequently,

$$a_{Q_{s,i}} \in \mathrm{acl}\big(\{a_{R_{s',i'}} : (s',i') \preccurlyeq (s,i)\}\big) = \mathrm{acl}\big(\{c_{s',i'} : (s',i') \preccurlyeq (s,i)\}\big). \qquad \square$$

Lemma 5.14. *Suppose that $(s,i) \in I$ is not maximal and that (s',i') is the least element in I which is strictly larger (with respect to '$\preccurlyeq$') than (s,i). Then $\mathrm{SU}(a_{Q_{s,i}}/a_{Q_{s',i'}}) = 1$.*

Proof. Let $(s,i), (s',i') \in I$ satisfy the assumptions of the lemma. By Lemma 5.13, it suffices to show that $\mathrm{SU}(\bar{c}^+/\bar{c}) = 1$, where

$$\bar{c} = \big(c_{t,j} : (t,j) \preccurlyeq (s,i)\big) \quad \text{and} \quad \bar{c}^+ = \big(c_{t,j} : (t,j) \preccurlyeq (s',i')\big).$$

To show this we only need to show that $\mathrm{SU}(c_{s',i'}/\bar{c}) = 1$.

We have two cases. First, suppose that $s = s'$. Then $i' = i+1$. By the choice of the elements $c_{t,j}$ and Facts 3.2 (iii) and 3.6, we get $\mathrm{SU}\big(c_{s,i+1}/\{c_{t,j} : (t,j) \preccurlyeq (s-1, \rho_{s-1})\}\big) = \mathrm{SU}\big(c_{s,i+1}/\mathrm{crd}_{s-1}(a)\big) = 1$. And we also have that $\{c_{s,1}, \ldots, c_{s,i+1}\}$ is independent over $\mathrm{crd}_{s-1}(a)$. Therefore, $\mathrm{SU}(c_{s,i+1}/\bar{c}) = 1$.

Now suppose that $s' = s+1$, so $i = \rho_s$ and $i' = 1$. As in the previous case we get $\mathrm{SU}\big(c_{s+1,1}/\bar{c}) = \mathrm{SU}\big(c_{s+1,1}/\{c_{t,j} : (t,j) \preccurlyeq (s,\rho_s)\}\big) = \mathrm{SU}\big(c_{s+1,1}/\mathrm{crd}_s(a)\big) = 1$ and we are done. $\square$

Now we can finish the proof of part (i) of Theorem 5.1. Recall that (by Fact 3.5 (iv)) $\mathrm{acl}(a) = \mathrm{acl}(\mathrm{crd}_h(a))$ and therefore (using Lemma 5.13)

$$a \in \mathrm{acl}\big(\{c_{s,i} : (s,i) \in I\}\big) = \mathrm{acl}\big(\{a_{Q_{s,i}} : (s,i) \in I\}\big).$$

Since $Q_{s,i}$ refines $Q_{s',i'}$ if $(s',i') \preccurlyeq (s,i)$ we get $a \in \mathrm{acl}(a_{Q_{h,\rho_h}})$. Since $c_{1,1} \in C_1$ we have (using Lemma 5.13 and Fact 3.2 (iii)) $\mathrm{SU}(a_{Q_{1,1}}) = \mathrm{SU}(c_{1,1}) = 1$. From this and Lemma 5.14 it follows, via the Lascar equation, that $\mathrm{SU}(a) = |I|$. Thus the sequence of $\emptyset$-definable equivalence relations that we are looking for is, again using Lemma 5.14,

$$(Q_{s,i} : (s,i) \in I),$$

ordered by '$\preccurlyeq$'.

5.2. **Proof of parts (ii) and (iii) of Theorem 5.1.** The assumptions and framework in this subsection are the same as in the previous (i.e. the proof of part (i)).

Suppose that $a, b \in M$, $\bar{c} \in C$ and $a \not\mathop{\smile\!\!\!\!\!\mid}_{\bar{c}} b$. By Lemma 3.7, there is $d \in \mathrm{crd}(a) \cap \mathrm{crd}(b)$ such that $d \notin \mathrm{acl}(\bar{c})$. Let $p = \mathrm{tp}(a,d)$ and let $R = R_p$ be as in Definition 5.6. By Lemma 5.7, R is a $\emptyset$-definable equivalence relation. By Lemma 5.8, $\mathrm{acl}(d) = \mathrm{acl}(a_R)$. Since $d \in \mathrm{crd}(b) \subseteq \mathrm{acl}(b)$ we get $a_R \in \mathrm{acl}(b)$. By assumption, $d \notin \mathrm{acl}(\bar{c})$ and hence $a_R \notin \mathrm{acl}(\bar{c})$. Then there are distinct $a'_i \in M^{\mathrm{eq}}$, for $i < \omega$, such that $\mathrm{tp}(a'_i/\bar{c}) = \mathrm{tp}(a_R/\bar{c})$, for all $i < \omega$. The type $\mathrm{tp}(a/a_R)$ contains a formula, $\varphi(x, a_R)$ which expresses that

"x belongs to the equivalence class (represented by) a_R".

Since $\{\varphi(x, a'_i) : i < \omega\}$ is clearly 2-inconsistent it follows that $a \not\mathop{\smile}\limits_{\bar{c}} a_R$. This concludes the proof of Theorem 5.1 (ii).

Now assume, in addition, that every binary $\emptyset$-definable relation on M is symmetric. Suppose that $a, b, \bar{c} \in M$ and $a \not\mathop{\smile}\limits_{\bar{c}} b$. Just as in the proof of part (ii), we get $d \in \mathrm{crd}(a) \cap \mathrm{crd}(b)$ such that $d \notin \mathrm{acl}(\bar{c})$. By letting $p = \mathrm{tp}(a, d)$ and $R = R_p$ be just as in the proof of part (ii), we conclude (just as in part (ii)) that R is a $\emptyset$-definable equivalence relation and $\mathrm{acl}(d) = \mathrm{acl}(a_R)$.

Let $0 < s \leq r$ be such that $d \in C_s \setminus C_{s-1}$. By Fact 3.5, $C_s \setminus C_{s-1}$ is a trivial pregeometry. So if there is $e \in C_s \setminus C_{s-1}$ such that $\mathrm{acl}(d) = \mathrm{acl}(e)$ and $p(b, e)$, then $R_p(a, b)$ (by Definition 5.6) so $R(a, b)$ and hence $a_R = b_R$. Since $\mathrm{acl}(d) = \mathrm{acl}(a_R)$ and $d \notin \mathrm{acl}(\bar{c})$ we must have $a_R \notin \mathrm{acl}(\bar{c})$.

Now suppose (towards a contradiciton) that, for every $e \in C_s \setminus C_{s-1}$ such that $\mathrm{acl}(d) = \mathrm{acl}(e)$, we have $\mathrm{tp}(b, e) \neq p$.

Let a' realize a nondividing extension of

$$\mathrm{tp}\big(a/(\mathrm{crd}_{s-1}(a) \cap \mathrm{crd}_{s-1}(b)) \cup \{d\}\big) \quad \text{to} \quad (\mathrm{crd}_{s-1}(a) \cap \mathrm{crd}_{s-1}(b)) \cup \{d, b\}.$$

Then a' is independent from b over $\big(\mathrm{crd}_{s-1}(a') \cap \mathrm{crd}_{s-1}(b)\big) \cup \{d\}$. As $C_s \setminus C_{s-1}$ is a trivial pregeometry, it follows that if $e \in \big(\mathrm{crd}_s(a') \cap \mathrm{crd}_s(b)\big) \setminus C_{s-1}$, then $\mathrm{acl}(e) = \mathrm{acl}(d)$. By assumption, for every $e \in \big(\mathrm{crd}_s(a') \cap \mathrm{crd}_s(b)\big) \setminus C_{s-1}$, $\mathrm{tp}(b, e) \neq p$. But then $\mathrm{tp}(a', b) \neq \mathrm{tp}(b, a')$. Since $a', b \in M$ we get $\mathrm{tp}_{\mathcal{M}}(a', b) \neq \mathrm{tp}_{\mathcal{M}}(b, a')$. As every complete type over $\emptyset$ is isolated it follows that there is a binary $\emptyset$-definable relation which is not symmetric, which contradicts an assumption of part (iii). Thus the proof of part (iii) is finished.

5.3. **Proof of Corollaries 5.3 and 5.4.** Suppose that $\mathcal{M}$ is ω-categorical, supersimple with finite SU-rank and with trivial dependence. Moreover, suppose that part (i) of Theorem 5.1 does not hold for $\mathcal{M}$. The proof of part (i) of Theorem 5.1 only uses

- results from Section 3 all of which hold for all ω-categorical, supersimple structures with finite SU-rank and with trivial dependence,
- Lemma 4.6, and
- results from Section 5.1 which, besides Lemma 4.6 only depend on the assumption that $\mathcal{M}$ is ω-categorical, supersimple with finite SU-rank and with trivial dependence.

So, assuming that part (i) of Theorem 5.1 fails for $\mathcal{M}$, it must be because Lemma 4.6 fails for $\mathcal{M}$. But the proof of Lemma 4.6 is a proof by contradiction. It assumes that Lemma 4.2 holds (and consequently Lemma 4.4 holds) and that Lemma 4.6 fails, and then finds $a^*, b^*, a', b', b'' \in M$ such that

$$a^*b^*b' \equiv a^*b^*b'' \quad \text{and} \quad a'b' \equiv a'b'', \quad \text{but} \quad a^*b^*a'b' \not\equiv a^*b^*a'b''.$$

This finishes the proof of Corollary 5.3 unless Lemma 4.2 fails for $\mathcal{M}$. But if Lemma 4.2 fails, then (by its proof) there are $a, a', a'', a^* \in M$ such that $aa'' \equiv aa^*$ and $a'a'' \equiv a'a^*$, but $aa'a'' \not\equiv aa'a^*$. This finishes the proof of Corollary 5.3.

Now we prove Corollary 5.4. Suppose that $\mathcal{M}$ is ω-categorical, supersimple with finite SU-rank and with trivial dependence. Moreover, assume that $\mathcal{M}$ has no $\emptyset$-definable equivalence relation on M with infinitely many infinite equivalence classes. By the proof of [20, Lemma 3.3], $M \subseteq \mathrm{acl}(C_1)$.[3] Furthermore, assume that $\mathrm{SU}(\mathcal{M}) > 1$, so $\mathrm{SU}(a) > 1$ for some $a \in M$.

[3] Lemma 3.3 in [20] assumes that $\mathcal{M}$ is primitive, but its proof only needs the assumption that that there is no $\emptyset$-definable equivalence relation on M which has infinitely many infinite equivalence classes.

Suppose that $\mathrm{SU}(a) = \rho_1 > 1$. By Fact 3.5 (iv), $\mathrm{acl}(a) = \mathrm{acl}(\mathrm{crd}_1(a))$, and hence there are $c_{1,1}, \ldots, c_{1,\rho_1} \in \mathrm{crd}_1(a)$ such that $\{c_{1,1}, \ldots, c_{1,\rho_1}\}$ is an independent set over $\emptyset$ and $\mathrm{acl}(a) = \mathrm{acl}(c_{1,1}, \ldots, c_{1,\rho_1})$.

Suppose that Lemma 4.6 holds for $\mathcal{M}$. Then $ac_{1,i} \not\equiv ac_{1,j}$ whenever $i \neq j$. Let $p = \mathrm{tp}(a, c_{1,1})$. By Lemma 5.7, R_p (as in Definition 5.6) is a $\emptyset$-definable equivalence relation on M. Since $\mathrm{SU}(c_{1,1}) = 1$, it follows from Lemma 5.14 that R_p has infinitely many infinite equivalence classes. This contradicts the assumptions of Corollary 5.4.

Hence Lemma 4.6 fails for $\mathcal{M}$. Then, in the same way as in the proof of Corollary 5.3, we find $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4 \in M$ such that $a_i a_j \equiv b_i b_j$ for all i and j, but $a_1 a_2 a_3 a_4 \not\equiv b_1 b_2 b_3 b_4$. This completes the proof of Corollary 5.4.

6. Extension properties

We are interested in knowing under what conditions two or more types are subtypes of a single type. More precisely, if $\bar{a}_i, \bar{b}_i \in M$, for $i = 1, \ldots, n$, under what circumstances is there $\bar{a} \in M$ such that $\mathrm{tp}(\bar{a}, \bar{b}_i) = \mathrm{tp}(\bar{a}_i, \bar{b}_i)$ for all $i = 1, \ldots, n$? Under rather general conditions, the answer is yes for the Rado graph, the "generic bipartite graph", as well as a number of other structures that can be constructed by procedures that involve a "high degree of randomness". (The most up to date study of extension problems in the context of binary ω-categorical structures is probably [1], by Ahlman, where more references can be found.) Therefore, the idea here is that if the answer is 'yes' under fairly general conditions, then this is a manifestation of a "high degree of randomness".

Definition 6.1. Here we call the following an *extension problem* of $\mathcal{M}$:

> Suppose that $\bar{a}_1, \ldots, \bar{a}_n, \bar{b}_1, \ldots, \bar{b}_n \in M$. We ask: is there $\bar{e} \in M$ such that $\mathrm{tp}(\bar{e}, \bar{b}_i) = \mathrm{tp}(\bar{a}_i, \bar{b}_i)$ for all $i = 1, \ldots, n$? If such $\bar{e} \in M$ exists then we say that *the extension problem of* $\mathrm{tp}(\bar{a}_i, \bar{b}_i)$*,* $i = 1, \ldots, n$*, has a solution* and call $\bar{e}$ a *solution* to this extension problem.

Observe that since we will assume that $\mathcal{M}$ is homogeneous (hence ω-saturated) it follows that if an extension problem has a solution $\bar{e}$ in some elementary extension of $\mathcal{M}$, then it also has a solution in $\mathcal{M}$.

Note also that if we have $\bar{a}_i$ and $\bar{b}_i$ as above and, for every $i = 1, \ldots, n$, there is $\bar{a}'_i$ such that for every $i < n$, $\mathrm{tp}(\bar{a}'_{i+1}, \bar{b}_1, \ldots, \bar{b}_i) = \mathrm{tp}(\bar{a}'_i, \bar{b}_1, \ldots, \bar{b}_i)$ and $\mathrm{tp}(\bar{a}'_{i+1}, \bar{b}_{i+1}) = \mathrm{tp}(\bar{a}_{i+1}, \bar{b}_{i+1})$, then we have $\mathrm{tp}(\bar{a}'_n, \bar{b}_i) = \mathrm{tp}(\bar{a}_i, \bar{b}_i)$ for every $i = 1, \ldots, n$. Therefore we will only consider the problem of extending two types.

Here we study binary relational structures with elimination of quantifiers. Under this assumption, if $\bar{c} = (c_1, \ldots, c_k)$, and $\bar{e}$ is a solution to the extension problem of the types $\mathrm{tp}(\bar{a}, c_1), \ldots, \mathrm{tp}(\bar{a}, c_k), \mathrm{tp}(\bar{b}, \bar{d})$, then $\bar{e}$ is also a solution to the extension problem of the types $\mathrm{tp}(\bar{a}, \bar{c})$ and $\mathrm{tp}(\bar{b}, \bar{d})$. And as pointed out above, the extension problem of the types $\mathrm{tp}(\bar{a}, c_1), \ldots, \mathrm{tp}(\bar{a}, c_k), \mathrm{tp}(\bar{b}, \bar{d})$ can be reduced to a sequence of k extension problems of two types of the form $\mathrm{tp}(\bar{a}', c')$ and $\mathrm{tp}(\bar{b}', \bar{d}')$, where c' is a single element.

By considering one coordinate at a time in the sequences $\bar{a}_1, \ldots, \bar{a}_n$, and using our observations above, it follows that the extension problem of the types $\mathrm{tp}(\bar{a}_1, \bar{b}_1), \ldots, \mathrm{tp}(\bar{a}_n, \bar{b}_n)$ can be reduced to a sequence of extension problems of two types of the form $\mathrm{tp}(a', c')$ and $\mathrm{tp}(b', \bar{d}')$, where a', b' and c' are single elements. Therefore we will only consider the extension problem of two types $\mathrm{tp}(a, c)$ and $\mathrm{tp}(b, \bar{d})$, where a, b and c are a single elements. Recall that notation 3.1 is in effect in this section.

Theorem 6.2. *Suppose that $\mathcal{M}$ is binary, simple and homogeneous. Let $a, b, c, \bar{d} \in M$.*

(i) *There is a $\emptyset$-definable equivalence relation R on M such that $c \mathop{\smile\!\!\!\!\!\!\;|}_{c_R} \bar{d}$.*

(ii) *If for some R as in part (i),*
$$a \mathop{\smile\!\!\!\!\!\!\;|}_{c_R} c, \quad b \mathop{\smile\!\!\!\!\!\!\;|}_{c_R} \bar{d} \quad \text{and} \quad \mathrm{tp}(a/\mathrm{acl}(c_R)) = \mathrm{tp}(b/\mathrm{acl}(c_R)),$$

then the extension problem of $\mathrm{tp}(a,c)$ *and* $\mathrm{tp}(b,\bar{d})$ *has a solution. Otherwise it may not have a solution, not even when* $\bar{d}$ *is a single element.*

Proof. This follows from Lemmas 6.3 – 6.6 (and the examples in Sections 7.1 – 7.3). □

In the rest of this section we assume that $\mathcal{M}$ ***is binary, simple and homogeneous, so Theorem 5.1 applies.***

Lemma 6.3. *For all* $c, \bar{d} \in M$ *there is a* $\emptyset$*-definable equivalence relation* R *such that* $c \mathop{\smile\!\!\!\!\!|}_{c_R} \bar{d}$.

Proof. Recall that, by Assumption 3.3, $M \subseteq U \subseteq M^{\mathrm{eq}}$ and only finitely many sorts are represented in U. The only assumption on U that is necessary for Fact 3.2 to hold is that only finitely many sorts are represented in U . Since there are only finitely many $\emptyset$-definable equivalence relations on M, we may, without loss of generality, assume that for every $\emptyset$-definable equivalence relation E on M and every $a \in M$, $a_E \in U$ and hence $a_E \in C$.

Now we prove (i). Let $c, \bar{d} \in M$. If $c \mathop{\smile\!\!\!\!\!|} \bar{d}$ then we can take R to be the equivalence relation with only one equivalence class. So suppose that $c \not\mathop{\smile\!\!\!\!\!|} \bar{d}$. Then $c \not\mathop{\smile\!\!\!\!\!|} d$ for some $d \in \bar{d}$. By Theorem 5.1 (ii), there is a $\emptyset$-definable equivalence relation R_1 such that $c \not\mathop{\smile\!\!\!\!\!|} c_{R_1}$ and $c_{R_1} \in \mathrm{acl}(d) \subseteq \mathrm{acl}(\bar{d})$. If $c \mathop{\smile\!\!\!\!\!|}_{c_{R_1}} \bar{d}$ then we are done with $R = R_1$. If not, then $c \not\mathop{\smile\!\!\!\!\!|}_{c_{R_1}} d$ for some $d \in \bar{d}$ and by Theorem 5.1 (ii) again (where we use that $c_{R_1} \in C$ which is why we need the argument in the first paragraph of the proof), there is a $\emptyset$-definable equivalence relation R_2 such that $c \not\mathop{\smile\!\!\!\!\!|}_{c_{R_1}} c_{R_2}$ and $c_{R_2} \in \mathrm{acl}(d) \subseteq \mathrm{acl}(\bar{d})$. If $c \mathop{\smile\!\!\!\!\!|}_{c_{R_2}} \bar{d}$ then we are done with $R = R_2$. If not, we continue in the same way. Since $\mathcal{M}$ has finite SU-rank we will, after finitely many iterations of this procedure, find a $\emptyset$-definable equivalence relation R_k such that $a \mathop{\smile\!\!\!\!\!|}_{c_{R_k}} \bar{d}$. (Or alternatively, one could appeal to the fact that there are only finitely many $\emptyset$-definable equivalence relations on M.) □

Lemma 6.4. *Suppose that* $a, b, c, \bar{d} \in M$ *and that* R *is a* $\emptyset$*-definable equivalence relation on* M *such that* $c \mathop{\smile\!\!\!\!\!|}_{c_R} \bar{d}$. *If* $a \not\mathop{\smile\!\!\!\!\!|}_{c_R} c$ *or* $b \not\mathop{\smile\!\!\!\!\!|}_{c_R} \bar{d}$, *then the extension problem of* $\mathrm{tp}(a,c)$ *and* $\mathrm{tp}(b,\bar{d})$ *may* not *have a solution.*

Proof. This is shown by the examples in Sections 7.1 and 7.2. □

Lemma 6.5. *Suppose that* $a, b, c, \bar{d} \in M$, *that* R *is a* $\emptyset$*-definable equivalence relation on* M *such that* $c \mathop{\smile\!\!\!\!\!|}_{c_R} \bar{d}$ *and that* $a \mathop{\smile\!\!\!\!\!|}_{c_R} c$ *and* $b \mathop{\smile\!\!\!\!\!|}_{c_R} \bar{d}$. *If* $\mathrm{tp}(a/\mathrm{acl}(c_R)) \neq \mathrm{tp}(b/\mathrm{acl}(c_R))$ *then the extension problem of* $\mathrm{tp}(a,c)$ *and* $\mathrm{tp}(b,\bar{d})$ *may not have a solution.*

Proof. This is shown by the example in Section 7.3. □

Lemma 6.6. *Suppose that* $a, b, c, \bar{d} \in M$, R *is a* $\emptyset$*-definable equivalence relation on* M *such that* $c \mathop{\smile\!\!\!\!\!|}_{c_R} \bar{d}$, $a \mathop{\smile\!\!\!\!\!|}_{c_R} c$, $b \mathop{\smile\!\!\!\!\!|}_{c_R} d$ *and* $\mathrm{tp}(a/\mathrm{acl}(c_R)) = \mathrm{tp}(b/\mathrm{acl}(c_R))$. *Then the extension problem of* $\mathrm{tp}(a,c)$ *and* $\mathrm{tp}(b,\bar{d})$ *has a solution.*

Proof. If the premisses of the lemma are satisfied, then all premisses of the independence theorem of simple theories are satisfied, and hence a solution exists in some elementary extension of $\mathcal{M}$. Since $\mathcal{M}$ is ω-saturated we find a solution in M. □

7. Examples

In sections 7.1 – 7.3 we give examples that prove the claims made in Remark 5.2 and in lemmas 6.4 and 6.5. Section 7.4 tells how certain metric spaces fit nicely into the context of this article when viewed as binary structures (namely, $\mathcal{R}$-Urysohn spaces for finite distance monoids $\mathcal{R}$).

7.1. **Cross cutting equivalence relations.** In this subsection we prove Lemma 6.4. This is also done, in a stronger sense, in Section 7.2, but the example of this section may nevertheless be instructive because of its simplicity.

Let $\mathcal{M} = (M, P^{\mathcal{M}}, Q^{\mathcal{M}})$, where M is a countably infinite set and $P^{\mathcal{M}}$ and $Q^{\mathcal{M}}$ are equivalence relations such that the equivalence relation $P^{\mathcal{M}} \cap Q^{\mathcal{M}}$

- partitions every equivalence class of $P^{\mathcal{M}}$ into infinitely many parts, all of which are infinite, and
- partitions every equivalence class of $Q^{\mathcal{M}}$ into infinitely many parts, all of which are infinite.

It is a basic exercise to show that $\mathcal{M}$ is homogeneous and superstable with SU-rank 2. Let X_1 and X_2 be two distinct equivalence classes of $P^{\mathcal{M}}$ and let Y_1 and Y_2 be two distinct equivalence classes of $Q^{\mathcal{M}}$. Pick $a \in X_1 \cap Y_1$, $b \in X_2 \cap Y_1$, $c \in X_1 \cap Y_2$ and $d \in X_2 \cap Y_2$. Then it is straightforward to verify that $c \underset{c_Q}{\perp} d$, $a \underset{c_Q}{\not\perp} c$ and $b \underset{c_Q}{\not\perp} d$, where '$c_Q$' is shorthand for '$c_{Q^{\mathcal{M}}}$'. Moreover, the extension problem of $\text{tp}_{\mathcal{M}}(a, c)$ and $\text{tp}_{\mathcal{M}}(b, d)$ does not have a solution, because if e would be a solution then $\mathcal{M} \models P(e, c) \wedge P(e, d)$, so $\mathcal{M} \models P(c, d)$, contradicting the choice of c and d.

7.2. **Bipedes with bicoloured legs.** In this subsection we prove the claims made in Remark 5.2 and Lemma 6.4. For any set A, let $[A]^2 = \{X \subseteq A : |X| = 2\}$. Let

$$\mathcal{N}^- = (\mathbb{N} \cup [\mathbb{N}]^2, F^{\mathcal{N}^-}, L^{\mathcal{N}^-}),$$

where

$$F^{\mathcal{N}^-} = \mathbb{N} \text{ and } L^{\mathcal{N}^-} = \big\{(\{m, n\}, k) : \{m, n\} \in [N]^2 \text{ and } k \in \{m, n\}\big\}.$$

We can think of the elements of $F^{\mathcal{N}^-} = \mathbb{N}$ as "feet" and elements of $[\mathbb{N}]^2$ as "bodies". Each body $\{m, n\} \in [\mathbb{N}]^2$ has two feet, namely m and n. Clearly, some different bodies, like $\{1, 2\}$ and $\{2, 3\}$, share a foot, while others do not. We can also imagine any given pair $(\{m, n\}, n) \in L^{\mathcal{N}^-}$ as a "leg" which joins the body $\{m, n\}$ to the foot n. We further imagine that for every body, one of its legs is coloured "blue" and the other is coloured "red". Moreover, the decision regarding which one is blue and which one is red is taken randomly and independently of the colouring of the legs of other "bodies". Note that only legs are coloured. A given foot may be the end of a blue leg and also the end of red leg, in which case the later leg belongs to another body than the first leg.

More formally, we construct such a structure as follows. Let B and R (for "blue" and "red") be new binary relation symbols and let Ω be the set of expansions

$$\mathcal{N} = (\mathbb{N} \cup [\mathbb{N}]^2, F^{\mathcal{N}}, L^{\mathcal{N}}, B^{\mathcal{N}}, R^{\mathcal{N}})$$

of $\mathcal{N}^-$ which satisfy the following sentences:

$$\forall x, y\Big(\big[B(x, y) \vee R(x, y)\big] \to L(x, y)\Big),$$

$$\forall x, y\Big(L(x, y) \to \big[(B(x, y) \wedge \neg R(x, y)) \vee (R(x, y) \wedge \neg B(x, y))\big]\Big), \quad \text{and}$$

$$\forall x\Big(\neg F(x) \to \exists y, z\big[B(x, y) \wedge R(x, z)\big]\Big).$$

For any set X let 2^X denote the set of functions from X to $\{0, 1\}$ For every finite $A \subseteq \mathbb{N}$ and every $f \in 2^A$, let $\langle A, f \rangle = \{g \in 2^{\mathbb{N}} : g(n) = f(n) \text{ for all } n \in A\}$. If $|A| = m$ then we let $\mu_0(\langle A, f \rangle) = 2^{-m}$. By standard notions and results in measure theory, there is a σ-algebra $\Sigma \subseteq 2^{\mathbb{N}}$, containing all $\langle A, f \rangle$ for finite A and $f \in 2^A$, and a countably subadditive probability measure $\mu : \Sigma \to \mathbb{R}$ which extends μ_0.[4] Let $\lambda : [\mathbb{N}]^2 \to \mathbb{N}$ be a bijection. For every $f \in 2^{\mathbb{N}}$ we get an expansion $\mathcal{N}_f \in \Omega$ of $\mathcal{N}$ that satisfies:

[4] These notions and results can be found in, for example, [11, chapters 1.1–1.4].

For every $\{m,n\} \in [\mathbb{N}]^2$ with $m < n$, if $f(\lambda(\{m,n\})) = 0$, then $\mathcal{N}_f \models B(\{m,n\},m) \wedge R(\{m,n\},n)$, and otherwise $\mathcal{N}_f \models R(\{m,n\},m) \wedge B(\{m,n\},n)$.

Moreover, it is clear that for every $\mathcal{N} \in \Omega$ there is a unique $f \in 2^{\mathbb{N}}$ such that $\mathcal{N} = \mathcal{N}_f$. Via this bijection between $2^{\mathbb{N}}$ and Ω we may also view Ω as a probability space.

Lemma 7.1. *There is $\mathcal{N} \in \Omega$ with the following property. Let $0 < n < \omega$, $a_1, \ldots, a_n \in \mathbb{N}$ and $f : \{1, \ldots, n\} \to \{0, 1\}$. Then there are distinct $b_i \in \mathbb{N} \setminus \{a_1, \ldots, a_n\}$, for all $i < \omega$, such that, for every $i < \omega$ and every $1 \leq k \leq n$, the following holds:*

- *If $f(k) = 0$ then $\mathcal{N} \models B(\{a_k, b_i\}, b_i) \wedge R(\{a_k, b_i\}, a_k)$.*
- *If $f(k) = 1$ then $\mathcal{N} \models R(\{a_k, b_i\}, b_i) \wedge B(\{a_k, b_i\}, a_k)$.*

Proof. We will prove that with probability 1 a structure in Ω has the stated property. By countable subadditivity of μ, it suffices to show the following:

For any choice of $0 < n < \omega$, $a_1, \ldots, a_n \in \mathbb{N}$, $f : \{1, \ldots, n\} \to \{0, 1\}$ and distinct $b^i_j \in \mathbb{N} \setminus \{a_1, \ldots, a_n\}$ for $i, j < \omega$,

$$\mu(X_i) = 0, \text{ for every } i < \omega, \text{ where}$$
$$X_i = \Big\{ g \in 2^{\mathbb{N}} : \text{ for all } j < \omega \text{ there is } 1 \leq k \leq n \text{ such that } g(\lambda(\{a_k, b^i_j\})) \neq f(\lambda(\{a_k, b^i_j\})) \Big\}.$$

By using the definition of μ_0 and the fact that μ extends μ_0 we get

$$\mu(X_i) \ \leq \ \big(1 - 2^{-(n+1)}\big)^j$$

for every $i < \omega$ and every $j < \omega$. Hence $\mu(X_i) = 0$ for every $i < \omega$ and the proof is finished. □

For the rest of this subsection we assume that $\mathcal{N}$ is like in Lemma 7.1.

Definition 7.2. (i) For every $A \subseteq \mathbb{N} \cup [\mathbb{N}]^2$, $\mathrm{cl}'(A) = A \cup \{b \in \mathbb{N} : \exists a \in A \cap [\mathbb{N}]^2, b \in a\}$.
(ii) For every $A \subseteq \mathbb{N} \cup [\mathbb{N}]^2$, $\mathrm{cl}''(A) = A \cup \{b \in [\mathbb{N}]^2 : \exists m, n \in A \cap \mathbb{N}, b = \{m, n\}\}$.
(iii) For every $A \subseteq \mathbb{N} \cup [\mathbb{N}]^2$, $\mathrm{cl}(A) = \mathrm{cl}''(\mathrm{cl}'(A))$.
(iv) We say that $A \subseteq \mathbb{N} \cup [\mathbb{N}]^2$ is *closed* if $\mathrm{cl}(A) = A$.

Lemma 7.3. *Suppose that $A \subseteq \mathbb{N} \cup [\mathbb{N}]^2$ and $a \in \mathrm{cl}(A)$. Then there is $B \subseteq A$ such that $|B| \leq 2$ and $a \in \mathrm{dcl}_{\mathcal{N}}(B)$. If $a \in \mathbb{N}$, then there is $b \in A$ such that $a \in \mathrm{dcl}_{\mathcal{N}}(b)$.*

Proof. This is because,

(a) for any two (different) feet there is a unique body which has precisely these two feet, and
(b) every body has a unique foot on the other end of its blue leg and a unique foot on the other end of its red leg. □

Lemma 7.4. *Suppose that $\{a_1, \ldots, a_n\}, \{b_1, \ldots, b_n\} \in \mathbb{N} \cup [\mathbb{N}]^2$ are two closed sets such that $(a_1, \ldots, a_n) \equiv^{at}_{\mathcal{N}} (b_1, \ldots, b_n)$.*
(i) For every $a_{n+1} \in \mathbb{N} \cup [\mathbb{N}]^2$ there is $b_{n+1} \in \mathbb{N} \cup [\mathbb{N}]^2$ such that $\mathrm{cl}(a_1, \ldots, a_{n+1}) \setminus \{a_1, \ldots, a_{n+1}\}$ and $\mathrm{cl}(b_1, \ldots, b_{n+1}) \setminus \{b_1, \ldots, b_{n+1}\}$ can be enumerated as $a'_1, \ldots, a'_m$ and $b'_1, \ldots, b'_m$, respectively, so that

$$(a_1, \ldots, a_{n+1}, a'_1, \ldots, a'_m) \equiv^{at}_{\mathcal{N}} (b_1, \ldots, b_{n+1}, b'_1, \ldots, b'_m).$$

(ii) There is an automorphism σ of $\mathcal{N}$ such that $\sigma(a_i) = b_i$ for every $1 \leq i \leq n$.
(iii) $\mathcal{N}$ is ω-categorical.
(iv) If $a, b \in \mathbb{N}$ or if $a, b \in [\mathbb{N}]^2$, then $\mathrm{tp}_{\mathcal{N}}(a) = \mathrm{tp}_{\mathcal{N}}(b)$.

Proof. (i) We consider two cases. First assume that $a_{n+1} \in \mathbb{N} \setminus \{a_1, \ldots, a_n\}$. Without loss of generality, assume that $\{a_1, \ldots, a_n\} \cap \mathbb{N} = \{a_1, \ldots, a_k\}$ for some $k \leq n$. Then $\{b_1, \ldots, b_n\} \cap \mathbb{N} = \{b_1, \ldots, b_k\}$ Since $(a_1, \ldots, a_n) \equiv^{at}_{\mathcal{N}} (b_1, \ldots, b_n)$ it suffices to find $b_{n+1} \in \mathbb{N}$ such that for every $1 \leq i \leq k$:

- If $B(\{a_{n+1}, a_i\}, a_{n+1})$ then $B(\{b_{n+1}, b_i), b_{n+1})$.
- If $R(\{a_{n+1}, a_i\}, a_{n+1})$ then $R(\{b_{n+1}, b_i), b_{n+1})$.

But Lemma 7.1 guarantees that such $b_{n+1} \in \mathbb{N}$ exists.

Now suppose that $a_{n+1} = \{i, j\} \in [\mathbb{N}]^2 \setminus \{a_1, \ldots, a_n\}$. Then at least one of i or j does not belong to $\{a_1, \ldots, a_n\}$, because this set is, by assumption, closed. First, suppose that $i \in \{a_1, \ldots, a_n\}$ and $j \notin \{a_1, \ldots, a_n\}$. Without loss of generality, assume that $i = a_1$. Then, by the previous case, we find $j' \in \mathbb{N}$ such that $\mathrm{cl}(a_1, \ldots, a_n, j) \setminus \{a_1, \ldots, a_n, j\}$ and $\mathrm{cl}(b_1, \ldots, b_n, j') \setminus \{b_1, \ldots, b_n, j'\}$ can be enumerated as $a'_1, \ldots, a'_m$ and $b'_1, \ldots, b'_m$, respectively, so that

$$(a_1, \ldots, a_n, j, a'_1, \ldots, a'_m) \equiv^{at}_{\mathcal{N}} (b_1, \ldots, b_n, j', b'_1, \ldots, b'_m).$$

Moreover, since these sequences are closed, there is $1 \leq l \leq m$ such that $a_{n+1} = \{i, j\} = \{a_1, j\} = a'_l$ and hence $\{b_1, j'\} = b'_l$, so we are done by taking $b_{n+1} = b'_l$.

Now suppose that $i, j \notin \{a_1, \ldots, a_n\}$. Then we apply what we have already proved twice. First we find we find $i' \in \mathbb{N}$ such that $\mathrm{cl}(a_1, \ldots, a_n, i) \setminus \{a_1, \ldots, a_n, i\}$ and $\mathrm{cl}(b_1, \ldots, b_n, i') \setminus \{b_1, \ldots, b_n, i'\}$ can be enumerated as $a'_1, \ldots, a'_m$ and $b'_1, \ldots, b'_m$, respectively, so that

$$(a_1, \ldots, a_n, i, a'_1, \ldots, a'_m) \equiv^{at}_{\mathcal{N}} (b_1, \ldots, b_n, i', b'_1, \ldots, b'_m).$$

Then we find $j' \in \mathbb{N}$ such that $\mathrm{cl}(a_1, \ldots, a_n, i, a'_1, \ldots, a'_m, j) \setminus \{a_1, \ldots, a_n, i, a'_1, \ldots, a'_m, j\}$ and $\mathrm{cl}(b_1, \ldots, b_n, i', b'_1, \ldots, b'_m, j') \setminus \{b_1, \ldots, b_n, i', b'_1, \ldots, b'_m, j'\}$ can be enumerated as $a''_1, \ldots, a''_s$ and $b''_1, \ldots, b''_s$, respectively, so that

$$(a_1, \ldots, a_n, i, a'_1, \ldots, a'_m, j, a''_1, \ldots, a''_s) \equiv^{at}_{\mathcal{N}} (b_1, \ldots, b_n, i', b'_1, \ldots, b'_m, j', b''_1, \ldots, b''_s).$$

Then $a_{n+1} = \{i, j\} = a''_l$ for some l, and we take $b_{n+1} = \{i', j'\} = b''_l$.

(ii) By part (i), we can carry out a standard back and forth argument to produce an automorphism f such that $f(a_i) = b_i$ for all i.

(iii) By the definition of 'cl' it is clear that, for every finite $A \subseteq \mathbb{N} \cup [\mathbb{N}]^2$, $|\mathrm{cl}(A)| \leq 3|A| + \binom{3|A|}{2}$. Together with part (ii) this implies that there are, up to equivalence in $Th(\mathcal{N})$, only finitely many formulas with free variables $x_1, \ldots, x_n$, for every $n < \omega$. Hence $\mathcal{N}$ is ω-categorical.

(iv) If $a, b \in \mathbb{N}$, then $\{a\}$ and $\{b\}$ are closed and $a \equiv^{at}_{\mathcal{N}} b$, so part (ii) gives $\mathrm{tp}_{\mathcal{N}}(a) = \mathrm{tp}_{\mathcal{N}}(b)$. If $a, b \in [\mathbb{N}]^2$, then it is clear from the definition of 'cl' that $\mathrm{cl}(a)$ and $\mathrm{cl}(b)$ can be ordered as a, a', a'' and b, b', b'', respectively, so that $(a, a', a'') \equiv^{at}_{\mathcal{N}} (b, b', b'')$ and again we use part (ii) to get $\mathrm{tp}_{\mathcal{N}}(a, a', a'') = \mathrm{tp}_{\mathcal{N}}(b, b', b'')$. □

Lemma 7.5. *For every* $A \subseteq \mathbb{N} \cup [\mathbb{N}]^2$, $\mathrm{acl}_{\mathcal{N}}(A) = \mathrm{cl}(A) = \mathrm{dcl}_{\mathcal{N}}(A)$.

Proof. By the definition of 'cl' it suffices to prove the lemma for finite A. By Lemma 7.3, we have $\mathrm{cl}(A) \subseteq \mathrm{dcl}_{\mathcal{N}}(A) \subseteq \mathrm{acl}_{\mathcal{N}}(A)$. Hence it suffices to show that if $b \notin \mathrm{cl}(A)$ then $b \notin \mathrm{acl}_{\mathcal{N}}(A)$.

Suppose that $b \notin \mathrm{cl}(A)$. Let $\mathrm{cl}(A) = \{a_1, \ldots, a_n\}$ and let $b'_1, \ldots, b'_m$ enumerate $\mathrm{cl}(A \cup \{b\}) \setminus (\mathrm{cl}(A) \cup \{b\})$. By Lemma 7.4 (ii) it is enough to find distinct b_i, for $i < \omega$, such that, for each $i < \omega$, $\mathrm{cl}(A \cup \{b_i\}) \setminus (\mathrm{cl}(A) \cup \{b_i\})$ can be enumerated as $b'_{i,1}, \ldots, b'_{i,m}$ so that

$$(a_1, \ldots, a_n, b, b'_1, \ldots, b'_m) \equiv^{at}_{\mathcal{N}} (a_1, \ldots, a_n, b_i, b'_{i,1}, \ldots, b'_{i,m}).$$

To show this one can argue similarly as in the proof of part (i) of Lemma 7.4 (hence using Lemma 7.1). The details are left for the reader. □

Lemma 7.6. *Suppose that* $a_1, \ldots, a_n, b_1, \ldots, b_n \in [\mathbb{N}]^2$ *and* $(a_i, a_j) \equiv_{\mathcal{N}} (b_i, b_j)$ *for all* $1 \leq i, j \leq n$. *Then* $\mathrm{cl}(a_1, \ldots, a_n) \setminus \{a_1, \ldots, a_n\}$ *and* $\mathrm{cl}(b_1, \ldots, b_n) \setminus \{b_1, \ldots, b_n\}$ *can be ordered as* $a'_1, \ldots, a'_m$ *and* $b'_1, \ldots, b'_m$, *respectively, so that*

$$(a_1, \ldots, a_n, a'_1, \ldots, a'_m) \equiv^{at}_{\mathcal{N}} (b_1, \ldots, b_n, b'_1, \ldots, b'_m).$$

Proof. This is a straightforward consequence of Lemma 7.3. □

Lemma 7.7. *Suppose that* $a_1, \ldots, a_n, b_1, \ldots, b_n \in [\mathbb{N}]^2$ *and* $(a_i, a_j) \equiv_{\mathcal{N}} (b_i, b_j)$ *for all* $1 \leq i, j \leq n$. *Then* $(a_1, \ldots, a_n) \equiv_{\mathcal{N}} (b_1, \ldots, b_n)$.

Proof. Immediate consequence of Lemmas 7.4 (ii) and 7.6. □

Definition 7.8. Let $\mathcal{M}$ be a structure with universe $[\mathbb{N}]^2$ and such that, for every $p = \mathrm{tp}_{\mathcal{N}}(a, b)$ where $a, b \in [\mathbb{N}]^2$ are distinct, $\mathcal{M}$ has a relation symbol R_p which is interpreted as the set of realizations of p in $\mathcal{N}$. The vocabulary of $\mathcal{M}$ has no other relation symbols.

Lemma 7.9. *(i) For all* $\bar{a}, \bar{b} \in [\mathbb{N}]^2$ *of the same length,* $\bar{a} \equiv_{\mathcal{N}} \bar{b}$ *if and only if* $\bar{a} \equiv_{\mathcal{M}} \bar{b}$.
(ii) $\mathcal{M}$ *is homogeneous and has only one complete 1-type over* $\emptyset$.
(iii) For every $A \subseteq [\mathbb{N}]^2$, $\mathrm{acl}_{\mathcal{M}}(A) = \mathrm{cl}(A) \cap [\mathbb{N}]^2 = \mathrm{dcl}_{\mathcal{M}}(A)$.

Proof. (i) Let $a_1, \ldots, a_n, b_1, \ldots, b_n \in [\mathbb{N}]^2$, $\bar{a} = (a_1, \ldots, a_n)$ and $\bar{b} = (b_1, \ldots, b_n)$. If $\bar{a} \equiv_{\mathcal{M}} \bar{b}$, then in particular $(a_i, a_j) \equiv_{\mathcal{M}} (b_i, b_j)$ for all i, j. By the definition of $\mathcal{M}$ we get $(a_i, a_j) \equiv_{\mathcal{N}} (b_i, b_j)$ for all i, j, and then Lemma 7.7 gives $\bar{a} \equiv_{\mathcal{N}} \bar{b}$. If $\bar{a} \equiv_{\mathcal{N}} \bar{b}$, then, as $\mathcal{N}$ is ω-categorical and countable, there is an automorphism σ of $\mathcal{N}$ such that $\sigma(\bar{a}) = \bar{b}$. Since $[\mathbb{N}]^2$ is $\emptyset$-definable in $\mathcal{N}$ (by $\neg F(x)$), σ fixes $[\mathbb{N}]^2$ setwise. From the definition of $\mathcal{M}$ it now follows that the restriction of σ to $[\mathbb{N}]^2$ is an automorphism of $\mathcal{M}$ and hence $\bar{a} \equiv_{\mathcal{M}} \bar{b}$.

Part (ii) follows from (i) and lemmas 7.7 and 7.4. Part (iii) follows from (i) and Lemma 7.5 □

Lemma 7.10. *For all tuples* $\bar{a}, \bar{b}, \bar{c}$ *of elements from* $[\mathbb{N}]^2$, *the following holds regardless of whether dividing is considered in* $\mathcal{M}$ *or in* $\mathcal{N}$: $\bar{a} \not\downarrow_{\bar{c}} \bar{b}$ *if and only if*

(a) *there is* $a \in \bar{a}$ *such that* $a \in \mathrm{cl}(\bar{b}) \setminus \mathrm{cl}(\bar{c})$, *or*
(b) *there are* $a \in \bar{a}$ *and* $b \in \bar{b}$ *such that* $a \cap b \neq \emptyset$, *but* $a \cap c = \emptyset$ *for all* $c \in \bar{c}$.

Proof sketch. If (a) or (b) holds, then it is straightforward to show that $\bar{a} \not\downarrow_{\bar{c}} \bar{b}$ (regardless of whether dividing is considered in $\mathcal{M}$ or in $\mathcal{N}$). If neither (a) nor (b) holds, then one can use Lemma 7.1 similarly as in the proof of Lemma 7.4 to show that $\bar{a} \downarrow_{\bar{c}} \bar{b}$ (again regardless of whether dividing is in $\mathcal{M}$ or in $\mathcal{N}$). We leave the details to the reader. □

Lemma 7.11. $\mathcal{M}$ *is supersimple with SU-rank 2, but not stable.*

Proof. To prove that $\mathcal{M}$ is supersimple it suffices to prove (by [34, Theorem 2.4.7 and Definition 2.8.12]) that if $\mathcal{M}' \equiv \mathcal{M}$, $\bar{a} \in M'$ and $B \subseteq M'$, then there is a finite $C \subseteq B$ such that $\mathrm{tp}_{\mathcal{M}'}(\bar{a}/B)$ does not divide over C.

Let $\varphi(x, y)$ be a formula in the language of $\mathcal{M}$ such that for all $a, b \in M (= [\mathbb{N}]^2)$, $\mathcal{M} \models \varphi(a, b)$ if and only if $a \neq b$ and $a \cap b \neq \emptyset$. Recall that $\mathrm{cl}(A) = \mathrm{dcl}_{\mathcal{M}}(A)$ for every $A \subseteq M$. From Lemma 7.10 it now follows that for any $\mathcal{M}' \models Th(\mathcal{M})$ and any $\bar{a}, \bar{b}, \bar{c} \in M'$, $\bar{a} \not\downarrow_{\bar{c}} \bar{b}$ if and only if either there is some $a \in \bar{a}$ such that $a \in \mathrm{dcl}_{\mathcal{M}'}(\bar{b}) \setminus \mathrm{dcl}_{\mathcal{M}'}(\bar{c})$, or there is $b \in \bar{b}$ such that $\mathcal{M}' \models \varphi(a, b)$, but $\mathcal{M}' \models \neg\varphi(a, c)$ for all $c \in \bar{c}$.

Now suppose that $\mathcal{M}' \models Th(\mathcal{M})$, $\bar{a} \in M'$ and $B \subseteq M'$. Let $C' = \bar{a} \cap \mathrm{dcl}_{\mathcal{M}'}(B)$. For every $a \in \bar{a} \setminus B$ such that there is $b \in B$ such that $\mathcal{M}' \models \varphi(a, b)$, choose exactly one such b and call it b_a. Let $C = C' \cup \{b_a : a \in \bar{a}\}$. Then, for every finite $B' \subseteq B$, $\mathrm{tp}_{\mathcal{M}'}(\bar{a}/B'C)$

does not divide over C. By the finite character of dividing, $\text{tp}_{\mathcal{M}'}(\bar{a}/B)$ does not divide over C.

We leave the verification that $\mathcal{M}$ has SU rank 2 to the reader. By using Lemma 7.1, it is straightforward to see that $\mathcal{N}$ has the independence property. From this one can derive that also $\mathcal{M}$ has the independence property, from which it follows that it is unstable. $\square$

Consider the following equivalence relation on $[\mathbb{N}]^2$:

$$E_B(a,b) \iff \text{there is } m \in a \cap b \text{ such that } \mathcal{N} \models B(a,m) \wedge B(b,m).$$

It is clearly $\emptyset$-definable in $\mathcal{N}$ and hence it is $\emptyset$-definable in $\mathcal{M}$. By replacing 'B' with 'R' we get a similar $\emptyset$-definable equivalence relation E_R. The equivalence classes of E_B and E_R correspond to elements of $\mathcal{M}^{\text{eq}}$. It follows from the definitions of E_B, E_R and choice of $\mathcal{N}$, that for all $a, b \in [\mathbb{N}]^2$, $E_B(a,b) \wedge E_R(a,b)$ if and only if $a = b$.

Let $a \in [\mathbb{N}]^2$. By using Lemma 7.10 and basic "forking/dividing calculus" one can now show that, for every $a \in [\mathbb{N}]^2$, $\text{SU}(a_{E_B}) = \text{SU}(a_{E_R}) = 1$ and $\text{SU}(a_=/a_{E_B}) = \text{SU}(a_=/a_{E_R}) = 1$ (where clearly $a \in \text{acl}_{\mathcal{M}^{\text{eq}}}(a_=)$). This proves the claim made in Remark 5.2 (i), namely that the "coordinatization sequence" of equivalence relations, called $R_1, \ldots, R_k$ in Theorem 5.1 (i), need not be unique.

Lemma 7.12. *E_B and E_R are the only nontrivial $\emptyset$-definable (in $\mathcal{M}$) equivalence relations on $M = [\mathbb{N}]^2$.*

Proof. Suppose that E is a nontrivial $\emptyset$-definable (in $\mathcal{M}$) equivalence relation on $[\mathbb{N}]^2$ and that $E \neq E_B$ and $E \neq E_R$. Suppose that $a, b \in [\mathbb{N}]^2$ are such that $a \cap b = \emptyset$ and $E(a,b)$. Then one can prove, using Lemma 7.1, that $E(a', b')$ for all $a', b' \in [\mathbb{N}]^2$, contradicting that E is nontrivial. We do not give the details, but the idea is that, for any $a', b' \in [\mathbb{N}]^2$, one can (by Lemma 7.1) find $c \in [\mathbb{N}]^2$ such that $(a', c) \equiv_{\mathcal{N}} (b', c) \equiv_{\mathcal{N}} (a, b)$, and consequently $E(a', c)$ and $E(b', c)$, and thus $E(a', b')$. Hence, we conclude that, for all $a, b \in [\mathbb{N}]^2$, $E(a,b)$ implies that $a \cap b \neq \emptyset$.

Using the construction of $\mathcal{M}$, one can show that there are exactly two binary nontrivial $\emptyset$-definable relations which properly refine E_B, and none of these two relations is symmetric, hence none of them is an equivalence relation. In the same way one can show that there is no nontrivial $\emptyset$-definable equivalence relation which properly refines E_R. From this (and since $E \neq E_B$ and $E \neq E_R$) it follows that E does not refine E_B or E_R.

Suppose that for all $a, b \in [\mathbb{N}]^2$, $E(a,b)$ implies $E_B(a,b) \vee E_R(a,b)$. Since E does not refine E_B or E_R, and since $\mathcal{M}$ has a unique 1-type over $\emptyset$, it follows that there are distinct $a, b, c \in M$ such that $E(a,b), E(b,c)$, $E_B(a,b)$ and $E_R(b,c)$. Then $E(a,c)$, so by assumption, $E_B(a,c)$ or $E_R(a,c)$. But neither case is possible because $E_B(a,b)$ and $E_R(b,c)$.

Hence, there are $a, b \in [\mathbb{N}]^2$ such that $E(a,b)$, $\neg E_B(a,b)$ and $\neg E_R(a,b)$ (so $a \neq b$). Then there is $m \in a \cap b$ such that $\mathcal{N} \models B(a,m) \wedge R(b,m)$ or vice versa. Without loss of generality, suppose that $\mathcal{N} \models B(a,m) \wedge R(b,m)$. Then all a', b' such that $\text{tp}_{\mathcal{M}}(a', b') = \text{tp}_{\mathcal{M}}(a,b)$ or $\text{tp}_{\mathcal{M}}(b', a') = \text{tp}_{\mathcal{M}}(a,b)$ satisfy $E(a', b')$. Since (by Lemma 7.9) $\text{tp}_{\mathcal{M}}(a) = \text{tp}_{\mathcal{M}}(b)$, there is $c \in [\mathbb{N}]^2$ such that $\text{tp}_{\mathcal{M}}(a,b) = \text{tp}_{\mathcal{M}}(b,c)$, so in particular, $E(b,c)$. Then $\text{tp}_{\mathcal{N}}(a,b) = \text{tp}_{\mathcal{N}}(b,c)$ so there is $n \in b \cap c$ such that $\mathcal{N} \models B(b,n) \wedge R(c,n)$. Since $\mathcal{N} \models R(b,m)$ we have $n \neq m$. Since $\text{acl}_{\mathcal{M}}(b) = \text{cl}(b) \cap [\mathbb{N}]^2 = \{b\}$ and $b \neq c$ (because $a \neq b$) we can assume that $c \notin \text{acl}_{\mathcal{M}}(a,b)$, from which it follows (together with $n \in b \cap c$) that $a \cap c = \emptyset$. But then $\neg E(a,c)$, contradicting the transitivity of E. $\square$

Now we prove the claim made in Remark 5.2 (ii). Suppose that $a, b \in [\mathbb{N}]^2$, $a \neq b$, $m \in a \cap b$, $\mathcal{N} \models B(a,m) \wedge R(b,m)$. Then (by Lemma 7.10) $a \not\mathop{\smile\!\!\!\!\!\!|}\, b$, $a_{E_B} \in \text{acl}_{\mathcal{M}}(b)$, and (by some standard forking calculus) $a \not\perp a_{E_B}$. However, by Lemma 7.12, there is *no* $\emptyset$-definable equivalence relation E such that $E(a,b)$ and $a \not\perp a_E$.

Now we prove Lemma 6.4 again, this time giving a "stronger" example than in Section 7.1 in the sense that, with the notation of Lemma 6.4, $a \underset{c_R}{\not\perp} b$ but $b \underset{c_R}{\perp} d$. By the choice of $\mathcal{N}$ and Lemma 7.1, there are distinct $i,j,k,l,m \in \mathbb{N}$ such that, with $a = \{i,j\}$, $b = \{k,l\}$, $c = \{j,l\}$ and $d = \{l,m\}$, the following holds in $\mathcal{N}$:

$$B(a,j), \ R(c,j), \ B(b,l), \ R(d,l).$$

Then $c \underset{c_{E_B}}{\perp} d$, $b \underset{c_{E_B}}{\perp} d$, and $a \underset{c_{E_B}}{\not\perp} c$. (The somewhat tedious, but standard, verifications of this are left to the reader.) Suppose, for a contradiction, that the extension problem (in $\mathcal{M}$) of $\mathrm{tp}_{\mathcal{M}}(a,c)$ and $\mathrm{tp}_{\mathcal{M}}(b,d)$ has a solution $e = \{i',j'\}$. Then $i' = j$ or $j' = j$. We can as well assume that $j' = j$. Since $e \neq c$ we get $i' \neq l$. As $b \cap d \neq \emptyset$ we must have $e \cap d \neq \emptyset$, which gives $i' = m$. Hence $e = \{j,m\}$. Since $\mathrm{tp}_{\mathcal{M}}(e,c) = \mathrm{tp}_{\mathcal{M}}(a,c)$ we get $\mathrm{tp}_{\mathcal{N}}(e,c) = \mathrm{tp}_{\mathcal{N}}(a,c)$. Hence $B(e,j)$ and consequently $R(e,m)$. Then

$$\mathcal{N} \models \exists x\big(R(e,x) \wedge B(d,x)\big) \ \wedge \ \neg\exists x\big(R(b,x) \wedge B(d,x)\big).$$

Hence $\mathrm{tp}_{\mathcal{N}}(e,d) \neq \mathrm{tp}_{\mathcal{N}}(b,d)$ and therefore $\mathrm{tp}_{\mathcal{M}}(e,d) \neq \mathrm{tp}_{\mathcal{M}}(b,d)$, which contradicts that e is a solution to the given extension problem.

7.3. **ω-Pedes.** In this subsection we outline a proof of Lemma 6.5. The constructions and arguments are similar to, but easier than, those in Section 7.2. Therefore the proofs of the lemmas that follow are left out. Let $\mathcal{N} = (\mathbb{N}, F^{\mathcal{N}}, E_0^{\mathcal{N}}, E_1^{\mathcal{N}})$ where:

- F is unary and $F^{\mathcal{N}}$ and $\mathbb{N} \setminus F^{\mathcal{N}}$ are infinite.
- $E_0^{\mathcal{N}}$ and $E_1^{\mathcal{N}}$ are equivalence relations such that $E_1^{\mathcal{N}} \subseteq E_0^{\mathcal{N}}$.
- E_0 partitions $F^{\mathcal{N}}$ into infinitely many infinite equivalence classes and $E_1^{\mathcal{N}}$ partitions each $E_0^{\mathcal{N}}$-class which is included in $F^{\mathcal{N}}$ into exactly two $E_1^{\mathcal{N}}$-classes, both of which are infinite.
- All $a,b \in \mathbb{N} \setminus F^{\mathcal{N}}$ belong to the same $E_1^{\mathcal{N}}$-class (hence to the same $E_0^{\mathcal{N}}$-class).

Let L be a binary relation symbol and let Ω be the set of expansions

$$\mathcal{M} = (\mathbb{N}, F^{\mathcal{M}}, E_0^{\mathcal{M}}, E_1^{\mathcal{M}}, L^{\mathcal{M}})$$

of $\mathcal{N}$ which have the following properties:

- $\mathcal{M} \models \forall x,y\Big(L(x,y) \rightarrow \big(\neg F(x) \wedge F(y)\big)\Big)$.
- For every $a \in \mathbb{N} \setminus F^{\mathcal{N}}$, every $E_0^{\mathcal{N}}$-class $X \subseteq F^{\mathcal{N}}$ and distinct $E_1^{\mathcal{N}}$-classes $Y, Z \subseteq X$, either $\mathcal{M} \models L(a,b)$ for all $b \in Y$ and $\mathcal{M} \models \neg L(a,c)$ for all $c \in Z$, or vice versa.

Let

$$\Psi = \big\{(a,X) : a \in \mathbb{N} \setminus F^{\mathcal{N}} \text{ and } X \subseteq F^{\mathcal{N}} \text{ is an } E_0^{\mathcal{N}}\text{-class.}\big\}$$

Let Σ and μ be precisely as in Section 7.2. Let $\lambda : \Psi \rightarrow \mathbb{N}$ be a bijection and let Y_i, $i < \omega$, be an enumeration of all $E_1^{\mathcal{N}}$-classes which are included in $F^{\mathcal{N}}$. For every $f \in 2^{\mathbb{N}}$, let $\mathcal{M}_f$ be the unique structure in Ω which has the following property:

> For every $(a,X) \in \Psi$ and $Y_i, Y_j \subseteq X$, where $i < j$, if $f(\lambda(a,X)) = 0$ then $\mathcal{M} \models L(a,b)$ for all $b \in Y_i$ and $\mathcal{M} \models \neg L(a,c)$ for all $c \in Y_j$, otherwise $\mathcal{M} \models \neg L(a,b)$ for all $b \in Y_i$ and $\mathcal{M} \models L(a,c)$ for all $c \in Y_j$.

Moreover, for every $\mathcal{M} \in \Omega$ there is a unique $f \in 2^{\mathbb{N}}$ such that $\mathcal{M} = \mathcal{M}_f$. In a similar spirit as in the proof of Lemma 7.4 (but easier), one can now prove the following:

Lemma 7.13. *There is $\mathcal{M} \in \Omega$ with the following properties:*
(i) For all $0 < n < \omega$, all $a_1, \ldots, a_n \in \mathbb{N} \setminus F^{\mathcal{M}}$ and every $f : \{1, \ldots, n\} \rightarrow \{0,1\}$, there is an $E_0^{\mathcal{M}}$-class $X \subseteq F^{\mathcal{M}}$ with $Y_i, Y_j \subseteq X$, where $i < j$, such that

for every $1 \leq k \leq n$*, if* $f(k) = 0$ *then* $\mathcal{M} \models L(a_k, b)$ *for all* $b \in Y_i$ *(and hence* $\mathcal{M} \models \neg L(a, c)$ *for all* $c \in Y_j$*), and otherwise* $\mathcal{M} \models \neg L(a_k, b)$ *for all* $b \in Y_i$ *(and hence* $\mathcal{M} \models L(a, c)$ *for all* $c \in Y_j$*).*

(ii) For all $0 < n < \omega$*, all* $E_0^{\mathcal{M}}$*-classes* $X_1, \ldots, X_n$ *and every* $f : \{1, \ldots, n\} \to \{0, 1\}$*, there is* $a \in \mathbb{N} \setminus F^{\mathcal{M}}$ *such that*

for every $1 \leq k \leq n$ *and* $Y_i, Y_j \subseteq X_k$*, where* $i < j$*, if* $f(k) = 0$ *then* $\mathcal{M} \models L(a, b)$ *for every* $b \in Y_i$*, and otherwise* $\mathcal{M} \models \neg L(a, b)$ *for every* $b \in Y_i$*.*

For the rest of this subsection assume that $\mathcal{M}$ ***is like in Lemma 7.13.*** Using Lemma 7.13, one can prove the following by a standard back-and-forth argument which builds up an automorphism:

Lemma 7.14. $\mathcal{M}$ *is homogeneous.*

It is straightforward to see, using Lemma 7.13, that for every $A \subseteq M$, $\mathrm{acl}_{\mathcal{M}}(A) = A$. With this at hand, it is also straightforward to characterize dividing as follows:

Lemma 7.15. *For all* $\bar{a}, \bar{b}, \bar{c} \in M$*,* $\mathrm{tp}_{\mathcal{M}}(\bar{a}/\bar{b}\bar{c})$ *divides over* $\bar{c}$ *if and only if there is* $a \in \bar{a}$ *such that*

(i) $a \in \bar{b} \setminus \bar{c}$*, or*
(ii) $\mathcal{M} \models F(a)$ *and there is* $b \in \bar{b}$ *such that* $\mathcal{M} \models E_0(a, b)$ *and for all* $c \in \bar{c}$*,* $\mathcal{M} \models \neg E_0(a, c)$*.*

With Lemma 7.15 and standard arguments as in the proof of Lemma 7.11, one can prove:

Lemma 7.16. $\mathcal{M}$ *is supersimple (but not stable). If* $\mathcal{M} \models F(a)$ *then* $\mathrm{SU}(a) = 2$*, otherwise* $\mathrm{SU}(a) = 1$*.*

Now we are ready to prove Lemma 6.5. There are $c, d \in M$ such that

$$\mathcal{M} \models F(c) \wedge F(d) \wedge E_0(c, d) \wedge \neg E_1(c, d).$$

By Lemma 7.13, we can also find $a, b \in M$ such that $\mathcal{M} \models L(a, c) \wedge L(b, d)$, and hence $\mathcal{M} \models \neg F(a) \wedge \neg F(b)$. Since $\mathcal{M}$ is homogeneous there is an automorphism of $\mathcal{M}^{\mathrm{eq}}$ which takes (a, c) to (b, d). This automorphism can be extended to an automorphism of $\mathcal{M}^{\mathrm{eq}}$. Since $E_0(c, d)$ it follows that this automorphism (of $\mathcal{M}^{\mathrm{eq}}$) fixes c_{E_0}. Hence $\mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}(a/c_{E_0}) = \mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}(b/c_{E_0})$. But there is no e such that $\mathrm{tp}_{\mathcal{M}}(e, c) = \mathrm{tp}_{\mathcal{M}}(a, c)$ and $\mathrm{tp}_{\mathcal{M}}(e, d) = \mathrm{tp}_{\mathcal{M}}(b, d)$, because this would give $L(e, c) \wedge L(e, d)$ where $E_0(c, d) \wedge \neg E_1(c, d)$. However note that $\mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}(a/\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(c_{E_0})) \neq \mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}(b/\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(c_{E_0}))$, because $c_{E_1}, d_{E_1} \in \mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(c_{E_0})$.

7.4. **Metric spaces.** Unlike sections 7.1 – 7.3, the examples of this section are not meant to show that things can be more complicated than one might have hoped for. Instead these are examples for which the main results of this article are concretized in nice and natural ways.

In [7], Conant studies the infinite countable homogeneous (in a more general sense that in this article) metric space, denoted $\mathcal{U}_{\mathcal{R}}$ and called *$\mathcal{R}$-Urysohn space*, over a countable *distance monoid* $\mathcal{R} = (R, \oplus, \leq, 0)$ (see [7, Section 2] for a definition of distance monoid). In other words, fix some distance monoid $\mathcal{R}$ and let $\mathcal{K}_{\mathcal{R}}$ be the class of all finite $\mathcal{R}$-metric spaces. Then, for a suitable relational language, $\mathcal{K}_{\mathcal{R}}$ is closed under isomorphism and has the hereditary property and the amalgamation property. Hence the Fraïssé limit of $\mathcal{K}_{\mathcal{R}}$ exists and we denote it by $\mathcal{U}_{\mathcal{R}}$. The language that we use has a binary relation symbol d_r for every $r \in R$, where $d_r(a, b)$ is interpreted as "the distance between a and b is at most r". So a structure $\mathcal{M}$ for this vocabulary is viewed as an *$\mathcal{R}$-metric space* if for all $a, b, c \in M$,

- $d_0(a, b)$ if and only if $a = b$,

- for all $r \in R$, $d_r(a,b)$ if and only if $d_r(b,a)$, and
- (triangle inequality) for all $r,s,t \in R$, if $d_r(a,b)$, $d_s(b,c)$ and $d_t(a,c)$, then $r \oplus s \geq t$.

If $\mathcal{R}$ is finite then the vocabulary of an $\mathcal{R}$-metric space is finite and hence $\mathcal{U}_\mathcal{R}$ is homogeneous in the sense of this article. From now on, assume that $\mathcal{R}$ is a *finite* distance monoid. As examples of finite distance monoids one can take $\mathcal{R} = (R, \oplus, \leq, 0)$, where $R \subseteq \mathbb{R}^{\geq 0}$ is finite and chosen so that $0 \in R$, '$\leq$' is the usual order on $\mathbb{R}$, '$\oplus$' is '$+_R$' where for all $r,s \in R$,

$$r +_R s = \max\{x \in R : x \leq r + s\} \quad \text{ and } \quad +_R \text{ is associative.}$$

For example, this holds if one takes $R = \{0,1,2\}$ or $R = \{0,1,3,4\}$. In the first case, however, $\mathcal{U}_\mathcal{R}$ is essentially the Rado graph, by viewing "$d_1(a,b) \wedge \neg d_0(a,b)$" as saying that there is an edge between a and b, and "$d_2(a,b) \wedge \neg d_1(a,b)$" as saying that there is no edge between a and b (and $a \neq b$). More examples of finite distance sets are analyzed in Appendix A of L. Nguyen van Thé's thesis [33].

By [7, Theorem 4.9], *$\mathcal{U}_\mathcal{R}$ is simple if and only if for all $r,s \in R$ such that $r \leq s$, $r \oplus r \oplus s = r \oplus s$.* One can check that if, for example, $R = \{0,1,3,4\}$ then this condition holds. From now on, suppose that $\mathcal{U}_\mathcal{R}$ is simple. Hence it is (by Fact 2.4) supersimple with finite SU-rank and trivial dependence. An element $r \in R$ is called *idempotent* if $r \oplus r = r$. By [7, Theorem 4.16], *the SU-rank of $\mathcal{U}_\mathcal{R}$ is the number of non-maximal idempotent elements in R.* Moreover, by [7, Corollary 7.9], *the $\emptyset$-definable equivalence relations on the universe of $\mathcal{U}_\mathcal{R}$ are exactly those which are defined by the formulas $d_r(x,y)$ where r is idempotent.* Suppose that $0 < r < s \in R$ are idempotent elements. Using the idempotency one can easily show that the equivalence relation $d_r(x,y)$ partitions every class of the equivalence relation $d_s(x,y)$ into infinitely many parts, all of which are infinite. Thus the sequence of equivalence relations $R_1, \ldots, R_k$ in Theorem 5.1 (i) corresponds, in the case of $\mathcal{U}_\mathcal{R}$, to $d_{r_1}(x,y), \ldots, d_{r_k}(x,y)$, where $r_1 > \ldots > r_k$ is a list of all non-maximal idempotent elements (so $r_k = 0$).

For any $r \in R$, let '$2r$' denote '$r \oplus r$'. From the characterization of $\mathcal{U}_\mathcal{R}$ being simple (given above), it follows that $2r$ is idempotent for every $r \in R$. Let $d(a,b)$ be the least $r \in R$ such that $d_r(a,b)$ holds. From [7, Corollary 4.10] we have for all a, b and $\bar{c}$ from any model of $Th(\mathcal{U}_\mathcal{R})$:

$$a \not\mathop{\smile}_{\bar{c}} b \iff 2d(a,b) \ < \ 2d(a,c) \ \text{ for all } c \in \bar{c}.$$

Since $2r$ is idempotent for every $r \in R$, it follows that, for every $r \in R$, $a \not\mathop{\smile}_{\bar{c}} b$ if and only if there is a $\emptyset$-definable equivalence relation E, defined by $d_r(x,y)$ for some idempotent r, such that $E(a,b)$ but $\neg E(a,c)$ for all $c \in \bar{c}$. This is the specific version of Theorem 5.1 (iii) in the case of $\mathcal{U}_\mathcal{R}$.

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