ASYMPTOTIC PROBABILITIES OF EXTENSION PROPERTIES AND RANDOM l-COLOURABLE STRUCTURES

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Abstract. We consider a set $\mathbf{K} = \bigcup_{n \in \mathbb{N}} \mathbf{K}_n$ of *finite* structures such that all members of $\mathbf{K}_n$ have the same universe, the cardinality of which approaches ∞ as $n \to \infty$. Each structure in $\mathbf{K}$ may have a nontrivial underlying pregeometry and on each $\mathbf{K}_n$ we consider a probability measure, either the uniform measure, or what we call the *dimension conditional measure*. The main questions are: What conditions imply that for every extension axiom φ, compatible with the defining properties of $\mathbf{K}$, the probability that φ is true in a member of $\mathbf{K}_n$ approaches 1 as $n \to \infty$? And what conditions imply that this is not the case, possibly in the strong sense that the mentioned probability approaches 0 for some φ?

If each $\mathbf{K}_n$ is the set of structures with universe $\{1, \ldots, n\}$, in a fixed relational language, in which certain "forbidden" structures cannot be weakly embedded and $\mathbf{K}$ has the disjoint amalgamation property, then there is a condition (concerning the set of forbidden structures) which, if we consider the uniform measure, gives a dichotomy; i.e. the condition holds if and only if the answer to the first question is 'yes'. In general, we do not obtain a dichotomy, but we do obtain a condition guaranteeing that the answer is 'yes' for the first question, as well as a condition guaranteeing that the answer is 'no'; and we give examples showing that in the gap between these conditions the answer may be either 'yes' or 'no'. This analysis is made for both the uniform measure and for the dimension conditional measure. The later measure has closer relation to random generation of structures and is more "generous" with respect to satisfiability of extension axioms.

Random l-colour*ed* structures fall naturally into the framework discussed so far, but random l-colour*able* structures need further considerations. It is not the case that every extension axiom compatible with the class of l-colourable structures almost surely holds in an l-colourable structure. But a more restricted set of extension axioms turns out to hold almost surely, which allows us to prove a zero-one law for random l-colourable structures, using a probability measure which is derived from the dimension conditional measure, and, after further combinatorial considerations, also for the uniform probability measure.
Keywords: Model theory, finite structure, asymptotic probability, extension axiom, zero-one law, colouring.

Contents

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1. Introduction

Extension axioms have been used as a technical tool for proving zero-one laws [15, 18, 23, 19, 27], but they also have other implications which will be explained below. Extension axioms, by their definition, express possibilities of extending a structure that are compatible (or "consistent") with the definition of a given class of structures under consideration. So given a structure $\mathcal{M}$ from this class, the set of extension axioms which are satisfied in $\mathcal{M}$ tells which possibilities of extending substructures of $\mathcal{M}$, in ways compatible with the context, are actually realized in the particular structure $\mathcal{M}$. Thus, extension axioms have a combinatorial interest of their own.

If we consider the class of all finite L-structures, where L is a language with finite relational vocabulary, then it follows from the proof of the zero-one law (as presented in [15, 18, 23]) that, for every extension axiom, almost all sufficiently large finite L-structures satisfy it. Hence the interesting case to study is the case when there are some restrictions on the structures under consideration. For example, we could restrict ourselves to the class of finite structures in which some particular structure cannot be (weakly) embedded; for instance, the class of triangle-free graphs. Specific classes of this kind have been studied extensively. An overview with emphasis on graphs and partial orders is found in [34]; see also [27, 32] and recent results [3, 4, 5]. An overview with focus on zero-one laws is found in [35]; it takes up, among other things, the number theoretic approach to zero-one laws which was first developed by K. Compton, and which is the subject of a book by S. Burris [9]. However, none of the previously published research focuses specifically on searching for "dividing lines" for asymptotic probabilities of extension properties in a general model theoretic setting. That is the purpose of this article, as well as deriving consequences such as zero-one laws and, finally, studying random l-colourable structures.

The general framework of this article is the following. For some language L, $\mathbf{K} = \bigcup_{n\in\mathbb{N}} \mathbf{K}_n$ is a set of *finite* L-structures such that all members of $\mathbf{K}_n$ have the same universe; often an initial segment of $\{1, 2, 3, \ldots\}$. In addition, each $\mathcal{M} \in \mathbf{K}$ may have a nontrivial closure operator which makes it into a pregeometry; in this case, the closure operator is uniformly definable on all members of $\mathbf{K}$ in the sense described in Definition 7.1. An important special case is when the closure (and pregeometry) is *trivial*, by which we mean that every subset of any structure from $\mathbf{K}$ is closed. If P is a property, then the expression that '*a member of* $\mathbf{K}$ *almost surely has property* P' is shorthand for saying that, with respect to some probability measure μ_n on $\mathbf{K}_n$, the probability that $\mathcal{M} \in \mathbf{K}_n$ has P approaches 1 as $n \to \infty$. If $\mu_n(\mathcal{M}) = 1/|\mathbf{K}_n|$ for all n and all $\mathcal{M} \in \mathbf{K}_n$ (the uniform probability measure), then we may instead say that '*almost all* $\mathcal{M} \in \mathbf{K}$ *have* P'. By a *zero-one law for* $\mathbf{K}$ we mean that for every L-sentence φ, either it or its negation almost surely holds in $\mathbf{K}$.

Suppose that $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{M} \in \mathbf{K}$ and that A and B are closed subsets of M. For a structure $\mathcal{N}$, the $\mathcal{B}/\mathcal{A}$-*extension axiom* holds for $\mathcal{N}$ if for every embedding τ of $\mathcal{A}$ into $\mathcal{N}$ there is an embedding π of $\mathcal{B}$ into $\mathcal{N}$ which extends τ. If the dimension of B is at most $k + 1$, then we call it a k-*extension axiom* of $\mathbf{K}$. If the closure is trivial then dimension is the same as cardinality.

If L has no constant symbols we allow the universe of $\mathcal{A}$ to be empty, and in this case the $\mathcal{B}/\mathcal{A}$-extension axiom expresses that there exists a copy of $\mathcal{B}$ in the ambient structure. Hence, if $\mathcal{M}$ satisfies all k-extension axioms, then every $\mathcal{B} \in \mathbf{K}$ of dimension at most $k + 1$ can be embedded into $\mathcal{M}$; in this case one may say that $\mathcal{M}$ is '$(k + 1)$-universal for $\mathbf{K}$'. By involving pebble games [25, 31] it follows that if L is relational and $\mathcal{M} \in \mathbf{K}$ satisfies all k-extension axioms of $\mathbf{K}$, then $\mathcal{M}$ has the following 'homogeneity property, up to k-variable expressibility': Whenever $\bar{a}, \bar{a}'$ are tuples of elements and there is an isomorphism from the closure of $\bar{a}$ to the closure of $\bar{a}'$ which sends a_i to a'_i, then $\bar{a}$ and $\bar{a}'$ satisfy exactly the same formulas in which at most k distinct variables occur.

If the class $\mathbf{K}^*$ of all structures which can be embedded into some member of $\mathbf{K}$ has (up to taking isomorphic copies) the joint embedding property and the amalgamation property, then a structure $\mathcal{M}$ exists which satisfies all k-extension axioms of $\mathbf{K}$ for every $k \in \mathbb{N}$; because we can let $\mathcal{M}$ be the so-called Fraïssé limit of $\mathbf{K}^*$. However, if $\mathbf{K}$ contains arbitrarily large (finite) structures, then the Fraïssé limit of $\mathbf{K}^*$ is infinite. The question whether, for every $k \in \mathbb{N}$, there exists a *finite* $\mathcal{M} \in \mathbf{K}$ which satisfies every k-extension axiom of $\mathbf{K}$ may be hard. For instance, the problem [11] whether there is a finite triangle-free graph which satisfies every 4-extension axiom is still open. By using the fact that the proportion of triangle-free graphs with vertices $1, \ldots, n$ which are bipartite approaches 1 as n approaches infinity [17, 27], it is straightforward to derive that the proportion of all triangle-free graphs with vertices $1, \ldots, n$ which satisfy all 3-extension axioms approaches 0 as n approaches infinity. The main results in Sections 3 – 7 are concerned with the question of when, for some k and large enough n, it is usual (or unusual), in senses to be made precise, that structures in $\mathbf{K}_n$ satisfy all k-extension axioms.

For the moment, assume that, for each n, μ_n is a probability measure on $\mathbf{K}_n$. Let $Th_\mu(\mathbf{K})$ be the set of sentences φ such that the μ_n-probability that φ is true in a member of $\mathbf{K}_n$ approaches 1 as n approaches infinity. Also assume that $\mathbf{K}^*$, as defined above, satisfies the joint embedding and amalgamation properties and let $Th_{\mathbb{F}}(\mathbf{K})$ be the complete theory of the Fraïssé limit of $\mathbf{K}^*$. If, moreover, the closure is trivial on all members of $\mathbf{K}$, it is straightforward to see that $Th_\mu(\mathbf{K}) = Th_{\mathbb{F}}(\mathbf{K})$ if and only if $Th_\mu(\mathbf{K})$ contains all extension axioms of $\mathbf{K}$. (We can get rid of the assumption that the closure is trivial if we assume that it is "well-behaved", as in Section 7; and then we argue like in Section 8.2.)

The rest of the introduction is devoted to explaining, roughly, the results of this article. We try to appeal to the reader's intuition rather than giving the full definitions of notions involved; but sometimes references to these definitions are given.

We start, in Sections 3 – 5 , by considering $\mathbf{K}$ such that all $\mathcal{M} \in \mathbf{K}$ have trivial closure, so dimension is the same as cardinality. Also, until Section 6 we consider only the uniform measure. The first result, Theorem 3.4, gives a dichotomy for the special case when, for a fixed language L, with finite relational vocabulary, and set $\mathbf{F}$ of "forbidden" L-structures, $\mathbf{K}_n$ is defined to be the set of all L-structures $\mathcal{M}$ with universe $\{1, \ldots, n\}$ such that no $\mathcal{F} \in \mathbf{F}$ can be *weakly embedded* into $\mathcal{M}$ (see Section 2.1). If every $\mathcal{F} \in \mathbf{F}$ is "simple" in a sense which is made precise in Theorem 3.4, then for every extension axiom φ of $\mathbf{K}$, the proportion of $\mathcal{M} \in \mathbf{K}_n$ which satisfy φ approaches 1 as n approaches infinity; and $\mathbf{K}$ has a zero-one law. On the other hand, if there is at least one "non-simple" $\mathcal{F} \in \mathbf{F}$, then for some $0 \leq c < 1$ and $2|F|$-extension axiom φ, the proportion of $\mathcal{M} \in \mathbf{K}_n$ in which φ is true never exceeds c; if the language has no unary relation symbols, then this proportion approaches 0 as n approaches infinity. It may nevertheless be the case that $\mathbf{K}$ has a zero-one law, as in the example of triangle-free graphs [27].

Theorem 3.4, just described, is proved by using the more general Theorems 3.15 and 3.17. In Theorems 3.15 and 3.17 we have no assumptions about how $\mathbf{K}$ is defined. We will call a structure $\mathcal{A}$ *permitted* if it can be embedded into some structure in $\mathbf{K}$. For the sake of simplifying this introductory description of the results, let's assume that every permitted structure is isomorphic to some structure in $\mathbf{K}$; in other words, we assume that $\mathbf{K}$ is, up to taking isomorphic copies, closed under substructures (the 'hereditary property'). The key concept will be that of *substitutions of permitted structures* in a permitted (super)structure $\mathcal{M}$, that is, the act of replacing, in $\mathcal{M}$, the interpretations (of relation symbols) on the universe of $\mathcal{A} \subseteq \mathcal{M}$ by the interpretations in another permitted structure $\mathcal{A}'$ with the same universe as $\mathcal{A}$. If whenever $\mathcal{A}$, $\mathcal{A}'$, $\mathcal{M}$ are permitted, $\mathcal{A} \subseteq \mathcal{M}$ and $\mathcal{A}$ and $\mathcal{A}'$ have the same universe, the result of "replacing $\mathcal{A}$ by $\mathcal{A}'$ in $\mathcal{M}$", denoted $\mathcal{M}[\mathcal{A} \triangleright \mathcal{A}']$, is a permitted structure, then, for every extension axiom of $\mathbf{K}$, the proportion of structures in $\mathbf{K}_n$ in which it is true approaches 1 as n approaches infinity. This statement is a consequence of Theorem 3.15 which, essentially, is a reformulation, with the terminology used here, of known results – although this may not be obvious at first sight.

If, however, there exist permitted $\mathcal{A}$, $\mathcal{A}'$, $\mathcal{M}$ such that $\mathcal{M}[\mathcal{A} \triangleright \mathcal{A}']$ is not permitted – we say "forbidden" – but the reverse substitution, that is, the replacement of $\mathcal{A}'$ by $\mathcal{A}$, *never* produces a forbidden structure from a permitted one, then one of the following holds: (a) $\mathbf{K}$ fails to satisfy the disjoint amalgamation property, or (b) there is an extension axiom φ of $\mathbf{K}$ and $0 \leq c < 1$ such that the proportion of $\mathcal{M} \in \mathbf{K}_n$ which satisfy φ never exceeds c; and if there are no unary relation symbols, then this proportion approaches 0. Consider the example when $\mathbf{K}$ is the set of triangle-free graphs and $\mathcal{A}$ and $\mathcal{A}'$ are graphs with vertex set $\{i, j\}$ where i and j are adjacent in $\mathcal{A}'$ but not in $\mathcal{A}$. Then we can find $\mathcal{M} \in \mathbf{K}$ such that $\mathcal{M}[\mathcal{A} \triangleright \mathcal{A}']$ is forbidden, but since the removal of an edge from a triangle-free graph never produces a triangle and the class of triangle-free graphs has the disjoint amalgamation property we are in case (b). The statement before this example is a consequence of Theorem 3.17 and its corollary. From these results we also get information, in case (a), about an instance of disjoint amalgamation which fails, and in case (b), about the extension axiom φ. Theorem 3.17 is proved by a counting argument. One proves, under the assumption that $\mathbf{K}$ has the disjoint amalgamation property, that for a properly chosen extension axiom φ it is the case that for every $\mathcal{M} \in \mathbf{K}_n$ which satisfies φ, there are sufficiently many $\mathcal{N} \in \mathbf{K}_n$ which do *not* satisfy φ.

There is a third possibility, other than those considered in the previous two paragraphs. It is possible that there are permitted $\mathcal{A}$ and $\mathcal{A}'$ with the same universe such that the substitution of $\mathcal{A}'$ for $\mathcal{A}$ in some permitted (super)structure $\mathcal{M}$ may produce a forbidden (not permitted) structure, but whenever this happens then the reverse substitution of $\mathcal{A}$ for $\mathcal{A}'$ in some permitted $\mathcal{N}$, say, may also produce a forbidden structure. In this case it is possible that for every extension axiom φ of $\mathbf{K}$, the proportion of structures in $\mathbf{K}_n$ in which φ is true approaches 1 as n approaches infinity. But it is also possible that for some extension axiom φ of $\mathbf{K}$, the proportion of structures in $\mathbf{K}$ in which φ is true approaches 0 as n approaches infinity. Section 4 gives examples showing this. The same section also gives examples for which Theorem 3.17 applies. These examples show how the rather technical Theorem 3.17 and its (less technical) corollary can be used. Some examples in Section 4 also serve the purpose of illustrating differences between the uniform probability measure and conditional probability measures, which are introduced in Section 6; these examples will be re-examined in Section 6. Section 5 is devoted to the proof of Theorem 3.17.

In Section 6 conditional probability measures (on $\mathbf{K}_n$) are introduced, motivated and illustrated with examples (that we have already met in Section 4). One reason for introducing these are that the conditions which, according to Theorem 3.15, guarantee that for every extension axiom of $\mathbf{K} = \bigcup_{n \in \mathbb{N}} \mathbf{K}_n$, the proportion of structures in $\mathbf{K}_n$ which satisfy it approaches 1 as n approaches infinity, are rather restrictive. The conditional measures that we consider – or the *dimension conditional measures*, to be precise – are more permissive with respect to satisfiability of extension axioms. This is made precise by Lemma 7.29 and Example 4.3, for instance. Another motivation for considering conditional measures is that they are more closely related to random generation of finite structures. While the uniform measure is conceptually simple it may, for some $\mathbf{K} = \bigcup_{n \in \mathbb{N}} \mathbf{K}_n$, be unclear what type of random generation procedure will, for any $\mathcal{M} \in \mathbf{K}_n$, generate $\mathcal{M}$ with probability exactly $1/|\mathbf{K}_n|$. Often, as in the case of l-coloured, or l-colourable, structures (graphs, for example), the most obvious generation procedure – first randomly assign colours, then randomly assign relationships (e.g.

edges) so that the colouring is respected – corresponds to conditional measures, in the sense of this article. A third reason for considering conditional measures is simply that they may, in some situations, offer a simpler analysis of asymptotic problems than does the uniform measure, while they are still natural in the sense of being related to random generation of finite structures. Finally we note that in some cases, as that of random l-colourable structures, the conditional measure considered here coincides with the uniform probability measure on properties which are first-order definable. This follows from the proofs of the main theorems in Sections 9 and 10

In Section 7 we start working in a context where the structures that we consider have underlying (possibly nontrivial) pregeometries, and 'dimension' takes over the role of 'cardinality'. By a pregeometry on a structure $\mathcal{M}$ we mean a closure operator $\mathrm{cl}_{\mathcal{M}}$ which operates on subsets of the universe of $\mathcal{M}$ and satisfies certain conditions [2, 23]; moreover we require that $\mathrm{cl}_{\mathcal{M}}$ is uniformly definable in all structures considered (Definition 7.1 and Assumption 7.10). The context considered previously is a special case of the framework of Section 7. The main results of this section, Theorems 7.31, 7.32 and 7.34, apply to the dimension conditional measure, which is a conditional measure that "considers" closed sets of dimension 0 first, then closed subsets of dimension 1, then of dimension 2, and so on. These theorems are related to Theorems 3.15 and 3.17. Theorems 7.31 and 7.32 represent the "positive" side of things, like Theorem 3.15, showing that if certain conditions are satisfied, then for every extension axiom of $\mathbf{K}$ the probability (with the dimension conditional measure) that it holds in a member of $\mathbf{K}_n$ approaches 1 as n approaches infinity; and from this a zero-one law is derived. The conditions in question require, as in Section 3, that whenever $\mathcal{M}$ is permitted, then certain "substitutions", or "replacements", of interpretations can be made in $\mathcal{M}$ without producing a forbidden (not permitted) structure. Also, there is a requirement that the underlying pregeometry, and possibly some other structure which is never changed, is *polynomially k-saturated*. This roughly means that for every $k \in \mathbb{N}$ and all sufficiently large n and every $\mathcal{M} \in \mathbf{K}_n$, the reduct of $\mathcal{M}$ to the sublanguage which defines the pregeometry satisfies every k-extension axiom (with respect to the set of such reducts); and moreover, the truth of a k-extension axiom has many different witnesses compared to the size of the universe.

The last result of Section 7, Theorem 7.34, is a "cousin" of Theorem 3.17 and its corollary, and tells that if there are permitted $\mathcal{A}$ and $\mathcal{A}'$ such that $\mathbf{K}$ *accepts* (Definition 7.20) the substitution $[\mathcal{A} \triangleright \mathcal{A}']$ but *not* the reverse subsitution $[\mathcal{A}' \triangleright \mathcal{A}]$, then either $\mathbf{K}$ fails to have the *independent amalgamation property*, or for some extension axioms φ and ψ, the probability, with the dimension conditional measure, that $\varphi \wedge \psi$ holds in a member of $\mathbf{K}_n$ approaches 0 as n approaches infinity. The analogue of Theorem 3.4 in the setting of underlying pregeometries and the dimension conditional measure is given by the corollary in Example 7.36. The proofs of Theorems 7.31, 7.32 and 7.34 appear in Section 8.

Section 8 gives the proofs of the main theorems of Section 7. The definitions appearing in Sections 5 and 8 are only used within those sections.

Sections 9 and 10 study asymptotic properties of random *l-colourable*, as well as *strongly* l-colourable, structures in a fixed (but arbitrary) relational language in which the arity of each symbol is at least 2. Examples 7.22 and 7.23 show that *l-coloured* structures can be treated within the context developed in Section 7. Theorem 7.32 implies that l-coloured structures satisfy a zero-one law with respect to the dimension conditional measure. Since l-colourable structures can be viewed as reducts of l-coloured structures we will also consider a "reduct version" of the dimension conditional measure. With this probability measure it is *not* true that all extension axioms of l-colourable structures hold almost surely; but we can show that all extension axioms of a certain kind, called the *l-colour compatible extension axioms*, hold almost surely in sufficiently large structures; and this is enough for subsequently deriving a zero-one law for l-colourable structures, when using the probability measure derived from the dimension conditional measure (Theorem 9.1). We also prove a result saying that if almost all l-colourable structures have an l-colouring with sufficiently even distribution of colours, then, with the uniform probability measure, every l-colour compatible extension axiom holds almost surely, almost every l-colourable structure has a unique l-colouring (up to permutation of the colours), and a zero-one law holds with the uniform probability measure as well (Theorem 9.16 and Proposition 9.20).

In Section 10 we prove, by combinatorial arguments, that, indeed, almost all l-colourable structures have an l-colouring with sufficiently even distribution of colours (Theorem 10.5). Thereby we confirm that almost all l-colourable structures have a unique l-colouring and that, also with the uniform probability measure, a zero-one law holds for l-colourable structures (Theorems 10.3 and 10.4).

All results of the article hold also if one restricts attention to structures in which certain relation symbols (of arity at least 2) are always interpreted as irreflexive and symmetric relations. All arguments, except those in Section 10, work out in the same way under this assumption.

The results in Sections 9 and 10 may be useful in contexts which do not directly speak about colourings. Suppose that for some $\mathbf{K} = \bigcup_{n \in \mathbb{N}} \mathbf{K}_n$ and probability measure μ_n on $\mathbf{K}_n$ there is $l \in \mathbb{N}$ such that, for n large enough, $\mathcal{M} \in \mathbf{K}_n$ is almost surely l-colourable. If we know that every l-colourable structure (with universe an initial segment of $\{1, 2, \ldots\}$) belongs to $\mathbf{K}$ and that the set of L-structures which are l-colourable has a zero-one law for the measures μ_n, then also $\mathbf{K}$ has a zero-one law for the same measures. This approach was used in [27] when proving that if $\mathbf{K}_n$ is the set of $(l+1)$-clique-free graphs (or $\mathcal{K}_{l+1}$-free graphs) with universe $\{1, \ldots, n\}$, then $\mathbf{K}$ has a zero-one law for the uniform probability measure. The authors of [27] first proved that almost all $(l+1)$-clique-free graphs are l-colourable, with a relatively even distribution of colours, and then that the l-colourable graphs have a zero-one law.

The notions of 'polynomial k-saturation' and 'acceptance of substitutions' in Section 7 are versions, adapted to the context of this article, of the notions 'polynomial k-saturation' and 'k-independence hypothesis' in [14]. This is sufficiently clear for polynomial k-saturation, but it is perhaps harder to see the relationship between admittance of (k-)substitutions and the k-independence hypothesis. However, in both cases the essential difference between Section 7 of this article and [14] is that in [14] complete types of an infinite structure are considered, while here we consider types with only quantifier-free formulas of tuples enumerating the universe of a closed substructure of some permitted structure. But in this article we avoid speaking about such types since it is equally convenient to speak about (sub)structures and formulas describing them up to isomorphism. Lemmas 8.5 – 8.9, as well as their proofs, are adaptations to the context of this article of Lemmas 2.16 – 2.22 in [14]. The results of this article have their beginnings in considerations from two directions. On the one hand, trying to understand asymptotic satisfiability of extension axioms – conditions implying that they almost surely hold, and conditions implying that some almost surely fail – and on the other hand, trying to understand if some zero-one laws for finite structures were hidden in the probabilistic arguments used in [14].

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2. Preliminaries

2.1. **Languages, structures and embeddings.** For basic notions not explained here the reader is refered to [23, 15]. By a ***language*** L we mean the set of (first-order) formulas that can be built up from a ***vocabulary*** (also called ***signature***) which is a set of relation, constant and/or function symbols. We consider the identity symbol '=' as a logical symbol which we may always use, together with connectives and (first-order) quantifiers, to build formulas; so '=' is never mentioned when we describe the symbols of a vocabulary. If the vocabulary has no constant or function symbols, then we call it ***relational***.

Structures will be denoted by "calligraphic", letters: $\mathcal{A}$, $\mathcal{B}$, …, $\mathcal{M}$, $\mathcal{N}$, …. Their universes will be denoted by the corresponding non-calligraphic letter A, B, …, M, N, …, or with bars around the letter; for instance, $|\mathcal{M}|$ as well as M denote the universe of $\mathcal{M}$. The cardinality of a set X is denoted by $|X|$; and the cardinality of the (universe of) the structure $\mathcal{M}$ is denoted by $\|\mathcal{M}\|$, or by $|M|$. Boldface letters always denote classes, usually sets, of structures. Sequences, or tuples, of elements are denoted by $\bar{a}, \bar{b}, \ldots$; and $|\bar{a}|$ denotes the length of the sequence $\bar{a}$. By '$\bar{a} \in M$' we mean that $\bar{a}$ is a sequence such that all of its elements belong to the set M. Sometimes we write $\bar{a} \in M^n$ to show that $\bar{a}$ has length n. By $\mathrm{rng}(\bar{a})$, the ***range*** (or ***image***) of $\bar{a}$, we denote the set of all elements that occur in $\bar{a}$. In the last section we often use the abbreviation $[n] = \{1, \ldots, n\}$ if n is a positive integer. For $\alpha \in \mathbb{R}$, $\lfloor \alpha \rfloor$ denotes the largest integer m such that $m \leq \alpha$. If $f : A \to B$ and $\bar{a} = (a_1, \ldots, a_r) \in A^r$, then $f(\bar{a})$ denotes the sequence $(f(a_1), \ldots, f(a_r))$. If L has no constant symbols, then we allow an L-structure to have an empty universe.

Suppose that $\mathcal{M}$ and $\mathcal{N}$ are L-structures, where L is, as usual, a language. A function $f : M \to N$ is called a ***weak embedding*** of $\mathcal{M}$ (in)to $\mathcal{N}$ if f is *injective* and:

(1) For every constant symbol c, $f(c^{\mathcal{M}}) = c^{\mathcal{N}}$.
(2) For every function symbol g, of arity r, say, and every $\bar{a} \in M^r$, $f(g^{\mathcal{M}}(\bar{a})) = g^{\mathcal{N}}(f(\bar{a}))$.
(3) for every relation symbol, R, of arity r, say, if $\bar{a} \in R^{\mathcal{M}}$ then $f(\bar{a}) \in R^{\mathcal{N}}$.

We say that f is an ***embedding*** if f is *injective* and (1), (2) and the following hold:

(3') for every relation symbol, R, of arity r, say, $\bar{a} \in R^{\mathcal{M}}$ if and only if $f(\bar{a}) \in R^{\mathcal{N}}$.

Thus, embeddings are injective and a bijective embedding is the same as an isomorphism. We say that $\mathcal{M}$ is ***(weakly) embeddable*** into $\mathcal{N}$ if there exists a (weak) embedding from $\mathcal{M}$ to $\mathcal{N}$. We say that $\mathcal{M}$ is a ***weak substructure*** of $\mathcal{N}$, denoted $\mathcal{M} \subseteq_w \mathcal{N}$, if $M \subseteq N$ and the identity mapping $id : M \to N$ is a weak embedding. We call $\mathcal{M}$ a ***substructure*** of $\mathcal{M}$, denoted $\mathcal{M} \subseteq \mathcal{N}$, if $M \subseteq N$ and the identity mapping $id : M \to N$ is an embedding. $\mathcal{A}$ is a *proper* (weak) substructure of $\mathcal{M}$ if $\mathcal{A}$ is a (weak) substructure of $\mathcal{M}$ and $\mathcal{A} \neq \mathcal{M}$. The symbol '$\subset$' means 'proper subset' or 'proper substructure'.

If $\mathcal{M}$ is a structure and $A \subseteq M$, then $\mathcal{M} \restriction A$ denotes the substructure of $\mathcal{M}$ which is generated by A (the smallest substructure $\mathcal{N}$ of $\mathcal{M}$ such that $A \subseteq N$); so if the vocabulary is relational, then $|\mathcal{M} \restriction A| = A$. If L_0 is a language such that $L_0 \subseteq L$ and $\mathcal{M}$ is an L-structure, then $\mathcal{M} \restriction L_0$ denotes the reduct of $\mathcal{M}$ to L_0. Observe that if all constant and function symbols of L belong to the vocabulary of L_0, then the reduct $\mathcal{M} \restriction L_0$ can also be viewed as an L-structure in which the interpretation of every relation symbol which belongs to the vocabulary of L but not to the vocabulary of L_0 is empty. So provided that the smaller language L_0 contains all constant and function symbols we have $\mathcal{M} \restriction L_0 \subseteq_w \mathcal{M}$, from which it is apparent that the notion of weak substructure generalizes the notion of reduct, as well as the notion of substructure.

Since we will several times speak about graphs, we note that, with graph theoretic terminology, if $\mathcal{M}$ and $\mathcal{N}$ are graphs, then $\mathcal{M}$ is a ***subgraph*** of $\mathcal{N}$ if and only if $\mathcal{M}$ is a weak substructure of $\mathcal{N}$; and $\mathcal{M}$ is an ***induced subgraph*** of $\mathcal{N}$ if and only if $\mathcal{M}$ is a substructure of $\mathcal{N}$.

Suppose that R is a relation symbol from the vocabulary of the language of $\mathcal{M}$. Then a tuple $\bar{a}$ of elements from M is called an ***R-relationship*** of M if $\bar{a} \in R^{\mathcal{M}}$ (or equivalently, if $\mathcal{M} \models R(\bar{a})$). If the symbol '$R$' is clear from the context, or if it does not matter which R we refer to, then we may just call an R-relationship a ***relationship***. Sometimes we consider only structures $\mathcal{M}$ in which certain relation symbols $R_1, \ldots, R_k$ are interpreted as irreflexive and symmetric relations (see Remark 2.1). In this case an *R_i-relationship* of $\mathcal{M}$ (for $i = 1, \ldots, k$) is a set $\mathrm{rng}(\bar{a})$ such that $\bar{a} \in (R_i)^{\mathcal{M}}$. So for graphs in general, a relationship is the same as a directed edge; and if we consider only undirected graphs, a relationship is the same as an (undirected) edge.

Remark 2.1. Suppose that R is an n-ary relation on a set A. Then R is called ***irreflexive*** if $(a_1, \ldots, a_n) \in R$ implies that $a_i \neq a_j$ if $i \neq j$. If $(a_1, \ldots, a_n) \in R$ implies that $(a_{\pi(1)}, \ldots, a_{\pi(n)}) \in R$ for every permutation π of $\{1, \ldots, n\}$, then we say that R is ***symmetric***. All results in this article hold also if we assume that the interpretations of certain relation symbols are always irreflexive and symmetric. The proofs in this case are either the same as, or obvious modifications of, the given proofs.

2.2. **Amalgamation.** Let $\mathbf{K}$ be a class of finitely generated L-structures, where L has a countable vocabulary, and let $\widehat{\mathbf{K}}$ be the class consisting of all L-structures $\mathcal{M}$ such that $\mathcal{M}$ is isomorphic to a member of $\mathbf{K}$; so $\widehat{\mathbf{K}}$ is "closed under isomorphism". See [23] (Chapter 7), for example, for definitions of the following notions: ***hereditary property***, or being ***closed under substructures*** as we sometimes say here, ***amalgamation property*** and ***joint embedding property***. We say that $\mathbf{K}$ has any of these properties if $\widehat{\mathbf{K}}$ has it. If the vocabulary of L has only relation symbols then the amalgamation property implies the joint embedding property; but in general the later property is not implied by the first.

If $\widehat{\mathbf{K}}$ has all three properties, then the so-called ***Fraïssé limit*** $\mathcal{M}_{\mathbf{K}}$ of $\widehat{\mathbf{K}}$ exists [23]. $\mathcal{M}_{\mathbf{K}}$ has the following properties: $\mathcal{M}_{\mathbf{K}}$ is countable, every finitely generated $\mathcal{A} \subseteq \mathcal{M}_{\mathbf{K}}$ belongs to $\widehat{\mathbf{K}}$; every $\mathcal{A} \in \mathbf{K}$ can be embedded into $\mathcal{M}_{\mathbf{K}}$, and if $\mathcal{A} \subseteq \mathcal{M}_{\mathbf{K}}$ is finitely generated and $\mathcal{A} \subseteq \mathcal{B} \in \widehat{\mathbf{K}}$, then there is an embedding $f : \mathcal{B} \to \mathcal{M}_{\mathbf{K}}$ such that $f{\restriction}A$ is the identity function [23]. The Fraïssé limit $\mathcal{M}_{\mathbf{K}}$ of $\widehat{\mathbf{K}}$, if it exists, is also called the Fraïssé limit of $\mathbf{K}$.

We will consider the following (stronger) variant of the amalgamation property: We say that $\widehat{\mathbf{K}}$ (and $\mathbf{K}$) has the ***disjoint amalgamation property*** if whenever $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \widehat{\mathbf{K}}$, $\mathcal{A} \subseteq \mathcal{B}$, $\mathcal{A} \subseteq \mathcal{C}$ and $B \cap C = A$, then there is $\mathcal{D} \in \widehat{\mathbf{K}}$ such that $\mathcal{B} \subseteq \mathcal{D}$, $\mathcal{C} \subseteq \mathcal{D}$.

2.3. **Pregeometries.** The notion of ***(combinatorial) pregeometry***, also called ***matroid***, will play a role in sections 7 and 8. See [23] (Chapter 4.6), or [2] (Chapter II.3), for a definition. We use the following terminology when (A, cl) is a pregeometry, with ***closure operator*** cl which maps every $X \subseteq A$ to some closed $Y \subseteq A$. For $X, Y, Z \subseteq A$, ***X is independent from Y over Z*** if for every $a \in X$, $a \in \mathrm{cl}(Y \cup Z) \iff a \in \mathrm{cl}(Z)$. In the special case that $Z = \emptyset$ we say that ***X is independent from Y***. Because of the 'exchange property' of pregeometries, independence is symmetric with respect to X and Y. We say that $a \in A$ is independent from $Y \subseteq A$ over $Z \subseteq A$ if $\{a\}$ is independent from Y over Z. A set $X \subseteq A$ is called ***independent*** if for every $a \in X$, a is independent from $X - \{a\}$ (over $\emptyset$). The ***dimension*** of $X \subseteq A$ is the supremum of the cardinalities of independent subsets of X. A set $X \subseteq A$ is called ***closed*** if $\mathrm{cl}(X) = X$. If $\mathrm{cl}(X) = X$ for every $X \subseteq A$ then we call (A, cl) the ***trivial pregeometry*** on A.

2.4. **Zero-one laws.** Suppose that, for $n \in \mathbb{N}$, $\mathbf{K}_n$ is a set of L-structures and that μ_n is a probability measure on $\mathbf{K}_n$. If $\mu_n(\mathcal{M}) = 1/|\mathbf{K}_n|$ for all $\mathcal{M} \in \mathbf{K}_n$, then we call μ_n the

uniform probability measure on $\mathbf{K}_n$. We say that $\mathbf{K} = \bigcup_{n\in\mathbb{N}} \mathbf{K}_n$ has a ***zero-one law*** if for every L-sentence φ, $\lim_{n\to\infty} \mu_n\big(\{\mathcal{M} \in \mathbf{K}_n : \mathcal{M} \models \varphi\}\big)$ exists and is 0 or 1. When saying "φ is almost surely true (or false)" we mean that the limit is 1 (or 0). If μ_n is the uniform probability measure for all n we may instead say that "almost all $\mathcal{M} \in \mathbf{K}$ satisfy φ" if the limit is 1. By the ***almost sure theory of*** $\mathbf{K}$ (with respect to the measures μ_n), we mean the set of sentences φ such that the probability that φ is true in $\mathbf{K}_n$ approaches 1 as $n \to \infty$.

3. Permitted structures and substitutions

From this section and until Section 7 we work within the following framework:

Assumptions and terminology 3.1. Fix a first-order language L with finite relational vocabulary. Let $(m_n : n \in \mathbb{N})$ be a sequence of positive integers such that $\lim_{n\to\infty} m_n = \infty$. For every $n \in \mathbb{N}$ let $\mathbf{K}_n$ be a set of L-structures with universe $\{1, \dots, m_n\}$; and let $\mathbf{K} = \bigcup_{n\in\mathbb{N}} \mathbf{K}_n$. A structure $\mathcal{M}$ is called ***represented (with respect to*** $\mathbf{K}$***)*** if it is isomorphic to a structure in $\mathbf{K}$. A structure $\mathcal{M}$ is called ***permitted (with respect to*** $\mathbf{K}$) if it is embeddable into a structure in $\mathbf{K}$. A structure which is not permitted is called ***forbidden***. Since we fix $\mathbf{K}$ for rest of the section we sometimes omit the phrase "with respect to $\mathbf{K}$".

Observe that if $\mathbf{K}$ has the hereditary property, then a structure is permitted if and only if it is represented. In this section and the next, all examples of $\mathbf{K}$ which are considered in some detail have the hereditary property. However, since the results do not depend on this we do not assume it. (One example of $\mathbf{K}$ which is *not* closed under substructures is given by letting $\mathbf{K}_n$ be the set of triangle-free graphs with universe $\{1, \dots, n\}$ and diameter 2.)

Definition 3.2. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are permitted structures and that $\mathcal{A}$ is a proper substructure of $\mathcal{B}$.
(i) The ***$\mathcal{B}/\mathcal{A}$-extension axiom*** (or the ***$\mathcal{B}$-extension axiom over*** $\mathcal{A}$) holds, by definition, in $\mathcal{M}$ if the following is true:

> For every embedding τ of $\mathcal{A}$ into $\mathcal{M}$ there exists an embedding π of $\mathcal{B}$ into $\mathcal{M}$ which extends τ (i.e. $\pi(a) = \tau(a)$ whenever $a \in A$).

The $\mathcal{B}/\mathcal{A}$-extension axiom can be expressed by a first-order sentence of the form

$$\forall x_1, \dots, x_n \exists y_1, \dots, y_m \big(\varphi(x_1, \dots, x_n) \longrightarrow \psi(x_1, \dots, x_n, y_1, \dots, y_m)\big),$$

where φ and ψ are quantifier-free. If the language has no constant symbols, then we allow the possibility that the universe of $\mathcal{A}$ is empty, in which case the $\mathcal{B}/\mathcal{A}$-extension axiom is called the $\mathcal{B}/\emptyset$-extension axiom. It is then expressed by an existential formula

$$\exists y_1, \dots, y_m \psi(y_1, \dots, y_m).$$

(ii) If $|\mathcal{B}| \leq k + 1$, then the $\mathcal{B}/\mathcal{A}$-extension axiom is called a ***k-extension axiom of*** $\mathbf{K}$; or, if we do not care about k, just an ***extension axiom of*** $\mathbf{K}$. If $\mathbf{K}$ is clear from the context we may omit saying "of $\mathbf{K}$".

Remark 3.3. *If there are probability measures μ_n on $\mathbf{K}_n$, for $n \in \mathbb{N}$, such that for every extension axiom φ of $\mathbf{K}$, the μ_n-probability that $\mathcal{M} \in \mathbf{K}_n$ satisfies φ approaches 1 as n approaches ∞, then $\mathbf{K}$ has a zero-one law for the measures μ_n.* The usual proof of this statement does not depend on the measures μ_n. It is proved in [18, 15, 23, 35] (for example) by collecting into a theory $T_{\mathbf{K}}$ all extension axioms, together with sentences expressing the possible isomorphism types of substructures of members of $\mathbf{K}$. The general idea of the argument is as follows. By the assumptions in the above statement and compactness, $T_{\mathbf{K}}$ is consistent. By a back-and-forth argument one then proves that $T_{\mathbf{K}}$ is countably categorical and therefore complete. The completeness of $T_{\mathbf{K}}$ (and compactness) implies that $\mathbf{K}$ has a zero-one law.

If we define $\mathbf{K}$ by forbidding certain weak substructures, and the thus obtained $\mathbf{K}$ has the disjoint amalgamation property, then we have the following "dichotomy".

Theorem 3.4. *Let* $\mathbf{F}$ *be a set of finite* L*-structures and, for every* $n \in \mathbb{N}$*, let* $\mathbf{K}_n$ *consist of exactly those* L*-structures* $\mathcal{M}$ *with universe* $\{1, \ldots, n\}$ *such that* no $\mathcal{F} \in \mathbf{F}$ *is* weakly *embeddable into* $\mathcal{M}$ *(so in particular, every member of* $\mathbf{F}$ *is forbidden). Assume that* $\mathbf{K}_n \neq \emptyset$ *for all sufficiently large* n *and that* $\mathbf{K}$ *has the disjoint amalgamation property. Consider the following condition:*

($*$) *There are* $\mathcal{F} \in \mathbf{F}$*, a relation symbol* R *of arity* r*, say, and* $\bar{a} \in F^r$ *such that* $\mathrm{rng}(\bar{a})$ *is a* proper *subset of* F*,* $\bar{a} \in R^{\mathcal{F}}$*, and if* $\mathcal{P}$ *is constructed by removing the* R*-relationship* $\bar{a}$*, but making no other changes in* $\mathcal{F}$*, then* $\mathcal{P}$ *is permitted.*

If ($*$) is false, then, for every $k \in \mathbb{N}$*, the proportion of* $\mathcal{M} \in \mathbf{K}_n$ *which satisfy all* k*-extension axioms of* $\mathbf{K}$ *approaches 1 as* $n \to \infty$*. If ($*$) is true, then letting* $\mathcal{F} \in \mathbf{F}$*,* R *and* $\bar{a}$ *be any witnesses of property ($*$) and letting* α *be the number of permitted structures with universe* $\{1, \ldots, |\mathrm{rng}(\bar{a})|\}$*, the proportion of* $\mathcal{M} \in \mathbf{K}_n$ *which satisfy all* $(2|F| - |\mathrm{rng}(\bar{a})| - 1)$*-extension axioms of* $\mathbf{K}$ *never exceeds* $1 - 1/(1 + \alpha)$*. Moreover, if* L *has no unary relation symbols, then the proportion of* $\mathcal{M} \in \mathbf{K}_n$ *which satisfy all* $(2|F| - |\mathrm{rng}(\bar{a})| - 1)$*-extension axioms approaches 0 as* $n \to \infty$*.*

Theorem 3.4 is a consequence of Theorems 3.15 and 3.17. Since one may see it as an application of these theorems, we give the proof of Theorem 3.4 as Example 4.1 in Section 4. The argument in Example 4.1 gives some information about what happens if $\mathbf{K}$ does not have the disjoint amalgamation property.

Remark 3.5. (i) Suppose, as in Theorem 3.4, that $\mathbf{F}$ is a set of finite L-structures and, for every $n \in \mathbb{N}$, let $\mathbf{K}_n$ consist of exactly those L-structures $\mathcal{M}$ with universe $\{1, \ldots, n\}$ such that no $\mathcal{F} \in \mathbf{F}$ is weakly embeddable into $\mathcal{M}$. Here is a condition on $\mathbf{F}$ which implies that $\mathbf{K}$ has the disjoint amalgamation property. As in [22], let us call an L-structure $\mathcal{M}$ *decomposable* if there are different L-structures $\mathcal{A}$ and $\mathcal{B}$ such that $M = A \cup B$, $\mathcal{A} \restriction A \cap B = \mathcal{B} \restriction A \cap B$ and for every relation symbol R, $R^{\mathcal{M}} = R^{\mathcal{A}} \cup R^{\mathcal{B}}$. Otherwise we call $\mathcal{M}$ *indecomposable*. It is now straightforward to show that if all structures in $\mathbf{F}$ are indecomposable, then $\mathbf{K}$ has the disjoint amalgamation property. (This statement is analogous to Theorem 1.2 (i) in [22].)
(ii) One may ask if the assumption that there are no unary relation symbols is necessary for the last statement of Theorem 3.4. The author does not have an example showing that this statement fails without the assumption that there are no unary relation symbols, *if* we assume, as in Theorem 3.4, that $\mathbf{K}$ has the disjoint amalgamation property. But Example 4.2 shows that when it is assumed that there are no unary relation symbols in Theorem 3.17, then this assumption is necessary.

Two examples follow, one for which ($*$) in Theorem 3.4 does not hold, and one for which ($*$) holds.

Example 3.6. Suppose that L has only one binary relation symbol R and that $\mathbf{F} = \{\mathcal{A}, \mathcal{B}\}$, where $A = \{1\}$, $R^{\mathcal{A}} = \{(1, 1)\}$, $B = \{1, 2\}$ and $R^{\mathcal{B}} = \{(1, 2), (2, 1)\}$. If $\mathbf{K}_n$ and $\mathbf{K}$ are defined as in Theorem 3.4, then an L-structure is permitted if and only if it is an irreflexive and antisymmetric directed graph. Moreover, the property ($*$) fails for $\mathbf{F}$.

Example 3.7. ($\mathcal{K}_l$**-free graphs**) It is not difficult to define $\mathbf{F}$ for which the property ($*$) holds, but let us mention an example which has been studied in some detail [27]. Let L have only one binary relation symbol R and consider only structures in which R is interpreted as an irreflexive and symmetric relation, that is, an undirected graph without loops. Let $l \geq 3$ and let $\mathcal{K}_l$ be the complete (undirected) graph with vertices $1, \dots, l$. If $\mathbf{F} = \{\mathcal{K}_l\}$ then condition $(*)$ holds, since the removal of one edge from $\mathcal{K}_l$ creates a permitted graph. It is easy to see that $\mathbf{K}$ has the disjoint amalgamation property. By Theorem 3.4, the proportion of $\mathcal{M} \in \mathbf{K}_l$ which satisfy all $(2l-3)$–extension axioms of $\mathbf{K}$ approaches 0 as $n \to \infty$. For $l = 3$ at least, this conclusion is not new. Because the proportion of $\mathcal{K}_3$-free graphs (*triangle-free* graphs) which are bipartite approaches 1 as $n \to \infty$ [17, 27]; and a graph is bipartite if and only if it has no cycle of odd length; moreover, it is easy to see that a 5-cycle or 3-cycle exists in every $\mathcal{K}_3$-free graph which satisfies all 3-extension axioms.

Remark 3.8. Even if, for some extension axiom φ of $\mathbf{K}$, the proportion of $\mathcal{M} \in \mathbf{K}_n$ which satisfy φ does not approach 1, $\mathbf{K}$ may nevertheless have a zero-one law with respect to the uniform probability measure. For example, it has been shown [27] that, for every $l \geq 3$, if $\mathbf{F} = \{\mathcal{K}_l\}$ where $\mathcal{K}_l$ is the complete graph with l vertices, and $\mathbf{K}_n$ and $\mathbf{K}$ are defined as in Example 3.7, then $\mathbf{K}$ has a zero-one law for the uniform probability measure.

Definition 3.9. Let $\mathcal{M}$, $\mathcal{A}$ and $\mathcal{B}$ be structures and suppose that $\mathcal{A}$ is a proper substructure of $\mathcal{B}$.
(i) We say that the ***$\mathcal{B}/\mathcal{A}$-multiplicity of $\mathcal{M}$ is at least*** m (or that the ***$\mathcal{B}$-multiplicity over $\mathcal{A}$ in $\mathcal{M}$ is at least*** m) if the following holds:

> Whenever σ is an embedding of $\mathcal{A}$ into $\mathcal{M}$, then there are embeddings σ_i of $\mathcal{B}$ into $\mathcal{M}$, for $i = 1, \dots, m$, such that each σ_i extends σ and if $i \neq j$ then $\sigma_i(B) \cap \sigma_j(B) = \sigma(A)$.

The ***$\mathcal{B}/\mathcal{A}$-multiplicity is*** m if it is at least m but not at least $m+1$.
(ii) We say that $\mathcal{M}$ ***has (at least)*** n ***copies of*** $\mathcal{A}$ if there are (at least) n different substructures $\mathcal{A}'_1, \dots, \mathcal{A}'_n$ of $\mathcal{M}$ such that each $\mathcal{A}'_i$ is isomorphic to $\mathcal{A}$.

Remark 3.10. Observe the following relationships between extension axioms and multiplicity, where we assume that $\mathcal{A} \subset \mathcal{B}$.
(i) $\mathcal{M}$ satisfies the $\mathcal{B}/\mathcal{A}$-extension axiom if and only if the $\mathcal{B}/\mathcal{A}$-multiplicity of $\mathcal{M}$ is at least 1.
(ii) Suppose that there are a structure $\mathcal{C}$ and embeddings $\sigma_1 : \mathcal{B} \to \mathcal{C}$ and $\sigma_2 : \mathcal{B} \to \mathcal{C}$ such that $\sigma_1 \restriction A = \sigma_2 \restriction A$ and $\sigma_1(B) \cap \sigma_2(B) = \sigma_1(A)$. If $\mathcal{M}$ satisfies the $\mathcal{C}/\mathcal{A}$-extension axiom then the $\mathcal{B}/\mathcal{A}$-multiplicity of $\mathcal{M}$ is at least 2.

Definition 3.11. Suppose that the vocabulary of L does not contain any constant symbol. Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{M}$ be L-structures such that $\mathcal{A} \subseteq \mathcal{M}$ and $|\mathcal{A}| = |\mathcal{B}|$. We define $\mathcal{M}[\mathcal{A} \triangleright \mathcal{B}]$ to be the structure obtained by "replacing $\mathcal{A}$ by $\mathcal{B}$ inside $\mathcal{M}$", or more precisely, $\mathcal{M}[\mathcal{A} \triangleright \mathcal{B}]$ is defined to be the structure with the same universe as $\mathcal{M}$ which satisfies the following conditions: For every n and every relation symbol R of arity n,

(1) if $(a_1, \dots, a_n) \in A^n$, then $(a_1, \dots, a_n) \in R^{\mathcal{M}[\mathcal{A} \triangleright \mathcal{B}]} \iff (a_1, \dots, a_n) \in R^{\mathcal{B}}$, and
(2) if $(a_1, \dots, a_n) \in M^n - A^n$, then $(a_1, \dots, a_n) \in R^{\mathcal{M}[\mathcal{A} \triangleright \mathcal{B}]} \iff (a_1, \dots, a_n) \in R^{\mathcal{M}}$.

The notation $\mathcal{M}[\mathcal{A} \triangleright \mathcal{B}]$ may be read as $\mathcal{M}$ ***with*** $\mathcal{A}$ ***replaced by*** $\mathcal{B}$, or $\mathcal{M}$ ***with*** $\mathcal{B}$ ***substituted for*** $\mathcal{A}$.

Definition 3.12. Let $\mathcal{A}$ and $\mathcal{B}$ be permitted structures (with respect to $\mathbf{K}$) with the same universe.
(i) We say that $\mathbf{K}$ ***admits the substitution*** $[\mathcal{A} \triangleright \mathcal{B}]$ if for every represented $\mathcal{M}$ such that $\mathcal{A} \subseteq \mathcal{M}$, $\mathcal{M}[\mathcal{A} \triangleright \mathcal{B}]$ is a represented structure.
(ii) We say that $\mathbf{K}$ ***weakly admits the substitution*** $[\mathcal{A} \triangleright \mathcal{B}]$ if for every represented $\mathcal{M}$ such that $\mathcal{A} \subseteq \mathcal{M}$, $\mathcal{M}[\mathcal{A} \triangleright \mathcal{B}]$ is a permitted structure.
(iii) If $|\mathcal{A}| = |\mathcal{B}|$ and $\|\mathcal{A}\| \leq k$, then we call $[\mathcal{A} \triangleright \mathcal{B}]$ a ***k-substitution***.

(iv) If $\mathbf{K}$ (weakly) admits every k-substitution $[\mathcal{A} \triangleright \mathcal{B}]$, where $\mathcal{A}$ and $\mathcal{B}$ are permitted structures (with the same universe), then we say that $\mathbf{K}$ ***(weakly) admits k-substitutions***.

When speaking of a substitution $[\mathcal{A} \triangleright \mathcal{B}]$ we always assume that $\mathcal{A}$ and $\mathcal{B}$ have the same universe. Note that if every permitted structure is represented, which is the case if $\mathbf{K}$ has the hereditary property, then $\mathbf{K}$ admits a substitution $[\mathcal{A} \triangleright \mathcal{B}]$ if and only if $\mathbf{K}$ weakly admits $[\mathcal{A} \triangleright \mathcal{B}]$.

Lemma 3.13. *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are permitted structures, with respect to $\mathbf{K}$, with the same universe such that the substitution $[\mathcal{A} \triangleright \mathcal{B}]$ is weakly admitted with respect to $\mathbf{K}$. Then for every permitted $\mathcal{M}$ such that $\mathcal{A} \subseteq \mathcal{M}$, $\mathcal{M}[\mathcal{A} \triangleright \mathcal{B}]$ is permitted with respect to $\mathbf{K}$.*

Proof. Suppose that $\mathcal{A}$, $\mathcal{B}$, $\mathcal{M}$ satisfy the premises of the lemma and that $[\mathcal{A} \triangleright \mathcal{M}]$ is weakly admitted. Since $\mathcal{M}$ is permitted there is a represented structure $\mathcal{N}$ such that $\mathcal{M} \subseteq \mathcal{N}$ (recall that the class of represented structures is closed under isomorphism). Since the substitution $[\mathcal{A} \triangleright \mathcal{B}]$ is weakly admitted, $\mathcal{N}[\mathcal{A} \triangleright \mathcal{B}]$ is permitted, so there is a represented $\mathcal{N}'$ such that $\mathcal{N}[\mathcal{A} \triangleright \mathcal{B}] \subseteq \mathcal{N}'$. By assumption $\mathcal{A} \subseteq \mathcal{M} \subseteq \mathcal{N}$, so $\mathcal{M}[\mathcal{A} \triangleright \mathcal{B}] \subseteq \mathcal{N}[\mathcal{A} \triangleright \mathcal{B}] \subseteq \mathcal{N}'$, which means that $\mathcal{M}[\mathcal{A} \triangleright \mathcal{B}]$ is permitted (because $\mathcal{N}'$ is represented). ☐

Remark 3.14. Suppose that ρ is the supremum of the arities of relation symbols in the vocabulary of L.
(i) It is straighforward to see that if $\mathbf{K}$ admits ρ-substitutions, then $\mathbf{K}$ admits k-substitutions for every $k \in \mathbb{N}$; because every k-substitution can be achieved by performing, in sequence, finitely many ρ-substitutions.
(ii) By using Lemma 3.13, it follows, much as in (i), that if $\mathbf{K}$ weakly admits ρ-substitutions, then $\mathbf{K}$ weakly admits k-substitutions for every k.

Remember that a structure $\mathcal{M}$ satisfies the $\mathcal{B}/\mathcal{A}$-extension axiom if and only if the $\mathcal{B}/\mathcal{A}$-multiplicity of $\mathcal{M}$ is at least 1. The next theorem is essentially a rephrasing, with the terminology of this article, of a result of which has been used to prove that every nontrivial parametric class of L-structures has a labeled zero-one law ([29], [15] Theorem 4.2.3). A class $\mathbf{C}$ of finite L-structures is *nontrivial and parametric*, in the sense of [15, 29], if and only if $\mathbf{C}$ is the class of represented structures with respect to some $\mathbf{K} = \bigcup_{n \in \mathbb{N}} \mathbf{K}_n$ which admits k-substitutions for every k, and $\mathbf{K}_n$ is a nonempty set of L-structures with universe $\{1, \ldots, n\}$. The result refered to in [15, 29] is a generalization of the well-known zero-one law for 'random structures' [18, 19], which in the present context amounts to considering the uniform probability measure on the set $\mathbf{K}_n$ of *all* L-structures with universe $\{1, \ldots, n\}$ (assuming that the arity of at least one relation symbol, besides '$=$', is greater than 1).

Theorem 3.15. [15, 18, 29] *Let $\mathbf{K} = \bigcup_{n \in \mathbb{N}} \mathbf{K}_n$ where each $\mathbf{K}_n$ is a nonempty set of L-structures with universe $\{1, \ldots, m_n\}$ and $\lim_{n \to \infty} m_n = \infty$. Suppose that $\mathbf{K}$ admits k-substitutions and let p be any positive integer. Whenever $\mathcal{A} \subset \mathcal{B}$ are permitted and $|\mathcal{B}| \leq k$, then the proportion of structures $\mathcal{M} \in \mathbf{K}_n$ such that the $\mathcal{B}/\mathcal{A}$-multiplicity of $\mathcal{M}$ is at least p approaches 1 as n approaches ∞.*

(The proof of the zero-one law by Glebski et al. [19] does not use extension axioms, but a form of quantifier elimination, which is why their article is not cited in Theorem 3.15.) Since Theorem 3.15 is not quite the same as similar results refered to [15, 18, 29], we give a sketch of its proof.

Proof sketch. For simplicity, consider the case when $\|\mathcal{B}\| = \|\mathcal{A}\| + 1 \leq k$ and $p = 2$. For every positive $d \in \mathbb{N}$, let α_d denote the number of different permitted structures with universe $\{1, \ldots, d\}$. Let $\mathcal{M} \in \mathbf{K}_n$. Since $\mathbf{K}$ admits k-substitutions it follows that for every $d \leq k$, every permitted structure $\mathcal{P}$ with universe $\{1, \ldots, d\}$, all distinct

$i_1, \ldots, i_d \in \{1, \ldots, m_n\}$, and $\mathcal{M} \in \mathbf{K}_n$, the probability that $j \mapsto i_j$ is an embedding of $\mathcal{P}$ into $\mathcal{M}$ is $1/\alpha_d$, with the uniform probability measure. Suppose that $\mathcal{A}'$ is a copy of $\mathcal{A}$ with universe $A' = \{i_1, \ldots, i_d\} \subset M = \{1, \ldots, m_n\}$, so $d < k$. For every $j \in \{1, \ldots, \lfloor m_n/2 \rfloor\} - A'$, the probability that $\mathcal{M} \restriction \{i_1, \ldots, i_d, j\}$ is a copy of $\mathcal{B}$ is at least $1/\alpha_{d+1}$. Therefore the probability that there is no $j \in \{1, \ldots, \lfloor m_n/2 \rfloor\} - A'$ such that this holds is at most $\big(1 - 1/\alpha_{d+1}\big)^{\lfloor m_n/2 \rfloor - d}$. There are at most $\binom{m_n}{d}$ copies of $\mathcal{A}$ in $\mathcal{M}$ and therefore the probability that some copy $\mathcal{A}' \subseteq \mathcal{M}$ of $\mathcal{A}$ cannot be extended to a copy of $\mathcal{B}$ by adding an element from $\{1, \ldots, \lfloor m_n/2 \rfloor\} - A'$ is at most $\binom{m_n}{d}\big(1 - 1/\alpha_{d+1}\big)^{\lfloor m_n/2 \rfloor - d}$ which approaches 0 as n approaches ∞ (because we assume that $\lim_{n\to\infty} m_n = \infty$). In the same way, the probability that some copy $\mathcal{A}' \subseteq \mathcal{M}$ of $\mathcal{A}$ cannot be extended to a copy of $\mathcal{B}$ by adding an element from $\{\lfloor m_n/2 \rfloor + 1, \ldots, m_n\} - A'$ approaches 0 as $n \to \infty$. It follows that the probability that the $\mathcal{A}/\mathcal{B}$-multiplicity of $\mathcal{M} \in \mathbf{K}_n$ is less than 2 approaches 0 as $n \to \infty$. □

With Theorem 3.15 at hand it remains to study what happens, asymptotically, with extension axioms and multiplicities when there are permitted $\mathcal{A}$ and $\mathcal{B}$ (with the same universe) such that the substitution $[\mathcal{A} \triangleright \mathcal{B}]$ is not admitted with respect to $\mathbf{K}$. The assumption that, for some permitted $\mathcal{A}$ and $\mathcal{B}$, the substitution $[\mathcal{A} \triangleright \mathcal{B}]$ is not admitted is not enough, even if we assume that $\mathbf{K}$ has the hereditary property and disjoint amalgamation property, to produce an extension axiom φ of $\mathbf{K}$ such that the proportion of structures in $\mathbf{K}_n$ satisfying φ does *not* approach 1 as $n \to \infty$. In this context it may, or may not, be the case that for every extension axiom, the proportion of structures in $\mathbf{K}_n$ in which it is true approaches 1. Examples 4.6 and 4.7 show this.

But if $\mathbf{K}$ has the hereditary property and disjoint amalgamation property and there are permitted $\mathcal{A}$ and $\mathcal{B}$ with the same universe such that $[\mathcal{A} \triangleright \mathcal{B}]$ is admitted, and permitted $\mathcal{M}$ such that $\mathcal{M}[\mathcal{B} \triangleright \mathcal{A}]$ is forbidden, then (by Corollary 3.18) the proportion of structures in $\mathbf{K}_n$ which satisfy all $(2|M| - 1)$-extension axioms never exceeds some $c < 1$; and if there are no unary relation symbols, then this proportion approaches 0 as $n \to \infty$. If we do not assume that $\mathbf{K}$ has the hereditary and disjoint amalgamation properties, then we can still obtain a related result (Theorem 3.17) if we add another assumption on $\mathcal{A}$ and $\mathcal{B}$. In the case that $\mathbf{K}$ has the hereditary property and disjoint amalgamation property, Lemma 3.16, below, implies that we can find permitted $\mathcal{A}$ and $\mathcal{B}$ which satisfy this added assumption.

Recall that if $\mathbf{K}$ has the hereditary property, then the notions 'permitted structure' and 'represented structure' coincide, and therefore the notions 'admit' (some substitution) and 'weakly admit' (the same substitution) coincide.

Lemma 3.16. *Suppose that $\mathbf{K}$ has the hereditary property and the disjoint amalgamation property. Assume that $\mathcal{A}$ and $\mathcal{B}$ are permitted structures with the same universe and that the substitution $[\mathcal{A} \triangleright \mathcal{B}]$ is admitted, but $[\mathcal{B} \triangleright \mathcal{A}]$ is not admitted. Then there are permitted $\mathcal{A}'$ and $\mathcal{B}'$ such that*

(1) $A' = B' \subseteq A$,
(2) *the substitution $[\mathcal{A}' \triangleright \mathcal{B}']$ is admitted but $[\mathcal{B}' \triangleright \mathcal{A}']$ is not admitted, and*
(3) *for every proper subset $U \subset A'$, $\mathcal{A}' \restriction U = \mathcal{B}' \restriction U$.*

Moreover, if $\mathcal{M}$ is permitted and $\mathcal{M}[\mathcal{B} \triangleright \mathcal{A}]$ is forbidden, then there is permitted $\mathcal{M}'$ with $M' = M$ such that $\mathcal{M}'[\mathcal{B}' \triangleright \mathcal{A}']$ is forbidden.

Proof. With '$\subset$' we mean 'proper subset' or 'proper substructure'. It suffices to prove that if $\mathcal{A}' = \mathcal{A}$ and $\mathcal{B}' = \mathcal{B}$ do not satisfy (1) – (3), then there is $U \subset A$ such that if $\mathcal{A}' = \mathcal{A} \restriction U$ and $\mathcal{B}' = \mathcal{B} \restriction U$, then $[\mathcal{A}' \triangleright \mathcal{B}']$ is admitted but $[\mathcal{B}' \triangleright \mathcal{A}']$ is not admitted. (Because if $A' = B'$ is a singleton set, then (3) trivially holds.) The last statement of the lemma will follow from the proof that there exist $\mathcal{A}'$ and $\mathcal{B}'$ satisfying (1) – (3).

First we prove the following:

Claim. If $U \subset A$, $\mathcal{U} = \mathcal{A}{\restriction}U$ and $\mathcal{V} = \mathcal{B}{\restriction}U$, then the substitution $[\mathcal{U} \triangleright \mathcal{V}]$ is admitted.

Proof of Claim. Let $\mathcal{M}$ be any permitted structure and suppose that $\mathcal{U} \subseteq \mathcal{M}$. We need to show that $\mathcal{M}[\mathcal{U} \triangleright \mathcal{V}]$ is permitted. By the disjoint amalgamation property there is a permitted $\mathcal{C}$ such that $\mathcal{A} \subseteq \mathcal{C}$, $\mathcal{M} \subseteq \mathcal{C}$ and $A \cap M = U$. Since $[\mathcal{A} \triangleright \mathcal{B}]$ is admitted, $\mathcal{C}[\mathcal{A} \triangleright \mathcal{B}]$ is permitted. From $\mathcal{M} \subseteq \mathcal{C}$ and $A \cap M = U$ we get $\mathcal{M}[\mathcal{U} \triangleright \mathcal{V}] \subseteq \mathcal{C}[\mathcal{A} \triangleright \mathcal{B}]$, so $\mathcal{M}[\mathcal{U} \triangleright \mathcal{V}]$ is permitted (and hence represented). $\square$

Suppose that for some $U \subset A$, $\mathcal{A}{\restriction}U \neq \mathcal{B}{\restriction}U$. (Otherwise $\mathcal{A}' = \mathcal{A}$, $\mathcal{B}' = \mathcal{B}$ satisfy (1) – (3).) Let $U_1, \ldots, U_l$ be an enumeration of all proper subsets $U_i \subset A = B$ such that $\mathcal{A}{\restriction}U_i \neq \mathcal{B}{\restriction}U_i$. By assumption there is a permitted $\mathcal{M}$ such that $\mathcal{B} \subset \mathcal{M}$ and $\mathcal{N} = \mathcal{M}[\mathcal{B} \triangleright \mathcal{A}]$ is forbidden. For $i = 1, \ldots, l$, let $\mathcal{U}_i = \mathcal{A}{\restriction}U_i$ and, by induction, define $\mathcal{N}_0 = \mathcal{M}$, $\mathcal{V}_1 = \mathcal{M}{\restriction}U_1$, $\mathcal{N}_{i+1} = \mathcal{N}_i[\mathcal{V}_{i+1} \triangleright \mathcal{U}_{i+1}]$ and $\mathcal{V}_{i+1} = \mathcal{N}_i{\restriction}U_{i+1}$. Let $\mathcal{A}' = \mathcal{N}_l{\restriction}A$. Then $\mathcal{N} = \mathcal{M}[\mathcal{B} \triangleright \mathcal{A}] = \mathcal{N}_l[\mathcal{A}' \triangleright \mathcal{A}]$.

If every one of the substitutions $[\mathcal{V}_1 \triangleright \mathcal{U}_1], \ldots, [\mathcal{V}_l \triangleright \mathcal{U}_l]$ and $[\mathcal{A}' \triangleright \mathcal{A}]$ is admitted, then $\mathcal{N}$ is permitted, which contradicts the assumption about $\mathcal{N}$. First suppose that for some i, the substitution $[\mathcal{V}_i \triangleright \mathcal{U}_i]$ is not admitted. By the claim, $[\mathcal{U}_i \triangleright \mathcal{V}_i]$ is admitted, so we are done (remember the first paragraph of the proof).

Now suppose that for every i, the substitution $[\mathcal{V}_i \triangleright \mathcal{U}_i]$ is admitted, and consequently $[\mathcal{A}' \triangleright \mathcal{A}]$ is not admitted. By the definition of $\mathcal{A}'$, we have $\mathcal{A}'{\restriction}U = \mathcal{A}{\restriction}U$ for every $U \subset A = A'$. Hence, we are done if we can show that the substitution $[\mathcal{A} \triangleright \mathcal{A}']$ is admitted. By the definition of $\mathcal{U}_i$, $\mathcal{V}_i$, $i = 1, \ldots, l$ and $\mathcal{A}'$, the result of the substitution $[\mathcal{B} \triangleright \mathcal{A}']$, in any permitted structure, can be achieved by performing the substitutions

$$[\mathcal{V}_1 \triangleright \mathcal{U}_1], \ldots, [\mathcal{V}_l \triangleright \mathcal{U}_l]$$

sequentially in the order from left to right. By assumption, every substitution $[\mathcal{V}_i \triangleright \mathcal{U}_i]$ is admitted, and hence $[\mathcal{B} \triangleright \mathcal{A}']$ is admitted. Since the result of the substitution $[\mathcal{A} \triangleright \mathcal{A}']$ can be achieved by first performing the substitution $[\mathcal{A} \triangleright \mathcal{B}]$, which is admitted by assumption, and then $[\mathcal{B} \triangleright \mathcal{A}']$, it follows that $[\mathcal{A} \triangleright \mathcal{A}']$ is admitted.

Now we verify the last statement of the lemma. When starting with $\mathcal{M}$ and then performing the substitutions $[\mathcal{V}_1 \triangleright \mathcal{U}_1]$, …, $[\mathcal{V}_l \triangleright \mathcal{U}_l]$ and $[\mathcal{A}' \triangleright \mathcal{A}]$ in this order, then, since $\mathcal{N}$ is forbidden, there is a *first* structure during this process which is forbidden. For $\mathcal{M}'$ we take the last structure during the process such that it and every structure before it is permitted. $\square$

Theorem 3.17. *Assume that $\mathcal{P}$, $\mathcal{S}_\mathcal{P}$ and $\mathcal{S}_\mathcal{F}$ are permitted structures such that $\mathcal{S}_\mathcal{P} \subseteq \mathcal{P}$, $|\mathcal{S}_\mathcal{P}| = |\mathcal{S}_\mathcal{F}|$, $\|\mathcal{S}_\mathcal{P}\| = k$, $\mathcal{F} = \mathcal{P}[\mathcal{S}_\mathcal{P} \triangleright \mathcal{S}_\mathcal{F}]$ is forbidden, but the substitution $[\mathcal{S}_\mathcal{F} \triangleright \mathcal{S}_\mathcal{P}]$ is admitted. Moreover, assume that for every proper substructure $\mathcal{U} \subset \mathcal{S}_\mathcal{P}$, $\mathcal{S}_\mathcal{P}{\restriction}|\mathcal{U}| = \mathcal{S}_\mathcal{F}{\restriction}|\mathcal{U}|$. Let α be the number of different permitted structures with universe $\{1, \ldots, k\}$ (so $\alpha \geq 2$).*

(i) For every n, the proportion of $\mathcal{M} \in \mathbf{K}_n$ such that

(a) *$\mathcal{M}$ contains a copy of $\mathcal{S}_\mathcal{F}$, and*

(b) *the $\mathcal{P}/\mathcal{S}_\mathcal{P}$-multiplicity of $\mathcal{M}$ is at least 2*

never exceeds $1 - 1/(1 + \alpha)$.

(ii) Suppose that there exist a permitted structure $\mathcal{C}$ and embeddings $\sigma_1 : \mathcal{P} \to \mathcal{C}$ and $\sigma_2 : \mathcal{P} \to \mathcal{C}$ such that $\sigma_1(|\mathcal{P}|) \cap \sigma_2(|\mathcal{P}|) = \sigma_1(|\mathcal{S}_\mathcal{P}|)$ and $\sigma_1{\restriction}|\mathcal{S}_\mathcal{P}| = \sigma_2{\restriction}|\mathcal{S}_\mathcal{P}|$. Then, for every n, the proportion of $\mathcal{M} \in \mathbf{K}_n$ that satisfy all $(2\,\|\mathcal{P}\| - k - 1)$-extension axioms never exceeds $1 - 1/(1 + \alpha)$.

(iii) Suppose that L has no unary relation symbols. The proportion of $\mathcal{M} \in \mathbf{K}_n$ such that

(c) *$\mathcal{M}$ satisfies the $\mathcal{S}_\mathcal{F}/\mathcal{U}$-extension axiom, where $\mathcal{U} \subseteq \mathcal{S}_\mathcal{F}$ and $\|\mathcal{U}\| = 1$, and*

(d) *the $\mathcal{P}/\mathcal{S}_\mathcal{P}$-multiplicity of $\mathcal{M}$ is at least 2* *approaches 0 as n approaches ∞.*

Corollary 3.18. *Suppose that $\mathbf{K}$ has the hereditary property and the disjoint amalgamation property. Also assume that there are permitted structures $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{M}$ such that $A = B$ and the substitution $[\mathcal{A} \triangleright \mathcal{B}]$ is admitted, but $\mathcal{M}[\mathcal{B} \triangleright \mathcal{A}]$ is forbidden.*
Then the proportion of structures in $\mathbf{K}_n$ which satisfy all $(2|M| - 1)$-extension axioms never exceeds $1 - 1/(1 + \alpha)$, where α is the number of permitted structures with universe A. If the language has no unary relation symbols then this proportion approaches 0 as $n \to \infty$.

Proof of Corollary 3.18. Assume that $\mathbf{K}$ has the hereditary property and disjoint amalgamation property, and let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{M}$ satisfy the assumptions of the corollary. From Lemma 3.16 it follows that there are permitted structures $\mathcal{P}$, $\mathcal{S}_{\mathcal{P}}$ and $\mathcal{S}_{\mathcal{F}}$ which satisfy the assumptions of Theorem 3.17 and $|\mathcal{S}_{\mathcal{P}}| \subseteq |\mathcal{A}|$ and $|\mathcal{P}| = |\mathcal{M}|$. Since $\mathbf{K}$ has the disjoint amalgamation property, part (ii) of Theorem 3.17 implies that the proportion of structures in $\mathbf{K}_n$ which satisfy all $(2\,\|\mathcal{P}\| - \|\mathcal{S}_{\mathcal{P}}\| - 1)$-extension axioms never exceeds $1 - 1/(1 + \alpha')$, where α' is the number of permitted structures with universe $|\mathcal{S}_{\mathcal{P}}|$. Note that if α is the number of permitted structures with universe A, then, since $\|\mathcal{S}_{\mathcal{P}}\| \leq |A|$, we have $1 - 1/(1 + \alpha') \leq 1 - 1(1 + \alpha)$.

Every structure in $\mathbf{K}$ which satisfies all $(2\,\|\mathcal{P}\| - \|\mathcal{S}_{\mathcal{P}}\| - 1)$-extension axioms satisfies both (c) and (d) in part (iii) of Theorem 3.17. So if the language has no unary relation symbols the proportion of structures in $\mathbf{K}_n$ which satisfy all $(2\,\|\mathcal{P}\| - \|\mathcal{S}_{\mathcal{P}}\| - 1)$-extension axioms must approach 0 as $n \to \infty$. Since $2|M| - 1 \geq 2|P| - 1 \geq 2\,\|\mathcal{P}\| - \|\mathcal{S}_{\mathcal{P}}\| - 1$ we are done. □

4. Examples

In all examples, $\mathbf{K} = \bigcup_{n \in \mathbb{N}} \mathbf{K}_n$ has the hereditary property.

Example 4.1. (Forbidden weak substructures and proof of Theorem 3.4 from Theorem 3.17.) Let L have a finite relational vocabulary, and let $\mathbf{F}$ be a set of finite L-structures. For $n \in \mathbb{N}$, let $\mathbf{K}_n$ be the set of all L-structures $\mathcal{M}$ with universe $\{1, \ldots, n\}$ such that *no* $\mathcal{F} \in \mathbf{F}$ can be weakly embedded into $\mathcal{M}$. Then a structure $\mathcal{A}$ is forbidden if and only if some $\mathcal{F} \in \mathbf{F}$ can be weakly embedded into $\mathcal{A}$. It follows that there exists (at least) one ***minimal forbidden*** structure $\mathcal{F}_{min} \in \mathbf{F}$ in the sense that every *proper* weak substructure of $\mathcal{F}_{min}$ is permitted.

If $\mathcal{F}_{min}$ does not have any relationship at all, that is, if $\mathcal{F}_{min}$ is just a finite set of cardinality m, say, then $\mathbf{K}_n = \emptyset$ for every $n \geq m$. Since we are only interested in the case when $\mathbf{K}_n \neq \emptyset$ for arbitrarily large $n \in \mathbb{N}$, we now assume that every minimal forbidden structure has at least one relationship. From this it follows that $\mathbf{K}_n \neq \emptyset$ for every $n \in \mathbb{N}$, because the assumption ensures that the set $\{1, \ldots, n\}$ without any structure belongs to $\mathbf{K}_n$.

Let $\mathcal{F}_{min}$ be any minimal forbidden structure. By assumption, for some relation symbol R, $R^{\mathcal{F}_{min}}$ is nonempty, so we can remove a relationship $\bar{a}$ from $R^{\mathcal{F}_{min}}$ and call the resulting structure $\mathcal{P}$. Note that $\mathcal{P}$ is permitted (since $\mathcal{F}_{min}$ is minimal forbidden), and that $\mathcal{F}_{min}$ and $\mathcal{P}$ have the same universe which includes $\mathrm{rng}(\bar{a})$. Let $\mathcal{S}_{\mathcal{F}} = \mathcal{F}_{min} \restriction \mathrm{rng}(\bar{a})$ and $\mathcal{S}_{\mathcal{P}} = \mathcal{P} \restriction \mathrm{rng}(\bar{a})$. Then $\mathcal{S}_{\mathcal{P}}$ is permitted, because it is a substructure of $\mathcal{P}$, and $\mathcal{P}$ is permitted. If $\mathrm{rng}(\bar{a}) = |\mathcal{F}_{min}|$ then $\mathcal{S}_{\mathcal{F}} = \mathcal{F}_{min}$ which is forbidden. If this holds for *every* choice of minimal forbidden $\mathcal{F}_{min}$ and $\mathcal{S}_{\mathcal{F}}$ as defined above, then $(*)$ in Theorem 3.4 does not hold, and it is straightforward to verify that $\mathbf{K}$ admits k-substitutions for every k. In this case, Theorem 3.15 implies that, for every extension axiom φ of $\mathbf{K}$, the proportion of $\mathcal{M} \in \mathbf{K}_n$ which satisfy φ approaches 1 as $n \to \infty$; and hence $\mathbf{K}$ has a zero-one law for the uniform measure, by Remark 3.3.

Now suppose that there is a minimal forbidden $\mathcal{F}_{min}$ and R such that for some $\bar{a} \in R^{\mathcal{F}_{min}}$, $\text{rng}(\bar{a})$ is a proper subset of $|\mathcal{F}_{min}|$. Then $(*)$ in Theorem 3.4 holds and $\mathcal{S}_{\mathcal{F}} = \mathcal{F}_{min} \restriction \text{rng}(\bar{a})$ is a *proper* substructure of $\mathcal{F}_{min}$, and since the latter is minimal forbidden, $\mathcal{S}_{\mathcal{F}}$ is permitted. Hence $\mathcal{P}$, $\mathcal{S}_{\mathcal{P}}$ and $\mathcal{S}_{\mathcal{F}}$ are permitted, but $\mathcal{F}_{min} = \mathcal{P}[\mathcal{S}_{\mathcal{P}} \triangleright \mathcal{S}_{\mathcal{F}}]$ is forbidden. (The notions 'admitted' and 'weakly admitted' coincide here because the notions 'permitted' and 'represented' coincide in this example.) But since the removal of a relationship from a permitted structure will never (in the present context) produce a forbidden structure, the substitution $[\mathcal{S}_{\mathcal{F}} \triangleright \mathcal{S}_{\mathcal{P}}]$ is admitted. Moreover, by the definition of $\mathcal{S}_{\mathcal{F}}$ and $\mathcal{S}_{\mathcal{P}}$, they agree on all proper subsets of their common universe. Thus, Theorem 3.17 is applicable. By part (i) of Theorem 3.17, the proportion of $\mathcal{M} \in \mathbf{K}_n$ such that

(a) $\mathcal{M}$ contains a copy of $\mathcal{S}_{\mathcal{F}}$, and
(b) the $\mathcal{P}/\mathcal{S}_{\mathcal{P}}$-multiplicity of $\mathcal{M}$ is at least 2

never exceeds $1 - 1(1+\alpha)$, where α is the number of permitted structures with universe $\{1, \ldots, |\text{rng}(\bar{a})|\}$. If $\mathbf{K}$ has the disjoint amalgamation property (which is assumed in Theorem 3.4), then part (ii) of Theorem 3.17 is applicable, and it follows that the proportion of structures in $\mathbf{K}$ which satisfy all $(2|P| - |\text{rng}(\bar{a})| - 1)$-extension axioms never exceeds $1 - 1/(1+\alpha)$. And if the language has no unary relation symbols and $\mathbf{K}$ has the disjoint amalgamation property, then this proportion approaches 0 as $n \to \infty$, by part (iii) of Theorem 3.17. Note that $\|\mathcal{F}_{min}\| = |P|$, so Theorem 3.4 is proved.

Example 4.2. This example shows that when, in Theorem 3.17, it is assumed that the language has no unary relation symbols, then this assumption is necessary. (The author does not have a corresponding example if one adds the assumption that $\mathbf{K}$ has the disjoint amalgamation property, as in Theorem 3.4 and Corollary 3.18.)

Let P_1 and P_2 be unary relation symbols and let L be a language the vocabulary of which is finite, relational and contains P_1 and P_2. For $n \in \mathbb{N}$, let $\mathbf{K}_n$ consist of all L-structures $\mathcal{M}$ with universe $\{1, \ldots, n\}$ such that

$$\begin{aligned}
&\text{at most one element in } M \text{ satisfies } P_1(x),\\
&\text{at most one element in } M \text{ satisfies } P_2(x), \text{ and}\\
&\mathcal{M} \models \neg\exists x, y\big(P_1(x) \wedge P_2(y)\big).
\end{aligned}$$

$\mathbf{K}_n$ can also be described in the following way, by forbidden weak substructures. Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ have universe $\{1,2\}$ and the following interpretations: $(P_1)^{\mathcal{A}} = \{1\}$, $(P_2)^{\mathcal{A}} = \{2\}$, $(P_1)^{\mathcal{B}} = \{1,2\}$, $(P_2)^{\mathcal{B}} = \emptyset$, $(P_1)^{\mathcal{C}} = \emptyset$, $(P_2)^{\mathcal{C}} = \{1,2\}$ and $R^{\mathcal{A}} = R^{\mathcal{B}} = R^{\mathcal{C}} = \emptyset$ for every other relation symbol R in the vocabulary. Then $\mathbf{K}_n$ can also be described as the set of all L-structures $\mathcal{M}$ such that no $\mathcal{F} \in \mathbf{F} = \{\mathcal{A}, \mathcal{B}, \mathcal{C}\}$ is weakly embeddable in $\mathcal{M}$. Note that $\mathbf{F}$ satisfies the condition labelled $(*)$ in Theorem 3.4, so if α is the number of permitted structures with universe $\{1\}$, then the proportion of $\mathcal{M} \in \mathbf{K}_n$ which satisfy all 2-extension axioms never exceeds $1-1/(1+\alpha)$. We have $\alpha \geq 3$, and if the only unary relation symbols of L are P_1 and P_2 then $\alpha = 3$.

Next, we show that there exists a 0-extension axiom (i.e. an $\mathcal{N}/\emptyset$-extension axiom with N a singleton set) such that the proportion of $\mathcal{M} \in \mathbf{K}_n$ which satisfy it never exceeds $1/2$. For every n, $\mathbf{K}_n$ can be partitioned into three parts: one part, $\mathbf{X}_n$, consisting of all $\mathcal{M} \in \mathbf{K}_n$ which satisfy $\exists x P_1(x)$; another part, $\mathbf{Y}_n$, consisting of all $\mathcal{M} \in \mathbf{K}_n$ which satisfy $\exists x P_2(x)$; and a third part, $\mathbf{Z}_n$, consisting of all $\mathcal{M} \in \mathbf{K}_n$ which do not satisfy either of $\exists x P_1(x)$ or $\exists x P_2(x)$. The definition of $\mathbf{K}_n$ implies that, for each n, $|\mathbf{X}_n| = |\mathbf{Y}_n| = n|\mathbf{Z}_n|$. Let $\mathcal{A}' = \mathcal{A} \restriction \{1\}$. Then $\mathcal{A}'$ is permitted and $\mathcal{A}' \models P_1(1)$. Moreover, for every n, the $\mathcal{A}'/\emptyset$-extension axiom holds exactly for those $\mathcal{M} \in \mathbf{K}_n$ which belong to $\mathbf{X}_n$, and we have

$$\frac{|\mathbf{X}_n|}{|\mathbf{K}_n|} = \frac{|\mathbf{X}_n|}{|\mathbf{X}_n| + |\mathbf{Y}_n| + |\mathbf{Z}_n|} = \frac{n|\mathbf{Z}_n|}{n|\mathbf{Z}_n| + n|\mathbf{Z}_n| + |\mathbf{Z}_n|} = \frac{1}{2 + 1/n} \;\to\; \frac{1}{2} \quad \text{as } n \to \infty.$$

Examples 4.3 and 4.4 show how Theorem 3.17 can be applied. They also provide contrast to Examples 6.3 and 6.4, where a different probability measure is considered.

Example 4.3. (**Graph with a restricted unary predicate**) Let the vocabulary of L consist of a unary relation symbol Q and a binary relation symbol R. Let $\mathbf{K}_n$ be the set of L-structures $\mathcal{M}$ with universe $\{1,\dots,n\}$ such that $R^{\mathcal{M}}$ is irreflexive and symmetric (i.e. an undirected graph) and

$$\mathcal{M} \models \forall x, y\big(R(x,y) \to (\neg Q(x) \wedge \neg Q(y))\big).$$

We use notation which suggests how Theorem 3.17 will be used. Define $\mathcal{S}_{\mathcal{P}}$, $\mathcal{S}_{\mathcal{F}}$, $\mathcal{P}$ and $\mathcal{F}$ as follows: let $|\mathcal{S}_{\mathcal{P}}| = |\mathcal{S}_{\mathcal{F}}| = \{a\}$; $Q^{\mathcal{S}_{\mathcal{P}}} = R^{\mathcal{S}_{\mathcal{P}}} = \emptyset$; $Q^{\mathcal{S}_{\mathcal{F}}} = \{a\}$, $R^{\mathcal{S}_{\mathcal{F}}} = \emptyset$; $|\mathcal{P}| = \{a,b\}$, $Q^{\mathcal{P}} = \emptyset$, $R^{\mathcal{P}} = \{(a,b),(b,a)\}$; and $\mathcal{F} = \mathcal{P}[\mathcal{S}_{\mathcal{P}} \triangleright \mathcal{S}_{\mathcal{F}}]$. Then $\mathcal{S}_{\mathcal{P}}$, $\mathcal{S}_{\mathcal{F}}$ and $\mathcal{P}$ are permitted, but $\mathcal{F}$ is forbidden, since $\mathcal{F} \models Q(a) \wedge R(a,b)$. Hence, the substitution $[\mathcal{S}_{\mathcal{P}} \triangleright \mathcal{S}_{\mathcal{F}}]$ is not admitted, but the reverse substitution $[\mathcal{S}_{\mathcal{F}} \triangleright \mathcal{S}_{\mathcal{P}}]$ is admitted, because we can always remove a Q-relationship without producing a forbidden structure.

By Theorem 3.17 (i), the proportion of $\mathcal{M} \in \mathbf{K}_n$ which contain a copy of $\mathcal{S}_{\mathcal{F}}$ and whose $\mathcal{P}/\mathcal{S}_{\mathcal{P}}$-multiplicity is at least two is not larger than $1 - 1/(1+2) = 2/3$. In this example we can do much better, asymptotically speaking, and show that the proportion of $\mathcal{M} \in \mathbf{K}_n$ which contain a copy of $\mathcal{S}_{\mathcal{F}}$, or equivalently, which satisfy $\exists x Q(x)$, approaches 0 as $n \to \infty$. We can argue as follows to see this. First let

$$\mathbf{X}_n = \big\{\mathcal{M} \in \mathbf{K}_n : \ Q(x) \text{ is satisfied by at least two elements in } |\mathcal{M}|\big\},$$
$$\mathbf{Y}_n = \big\{\mathcal{M} \in \mathbf{K}_n : \ Q(x) \text{ is satisfied by a unique element in } |\mathcal{M}|\big\}.$$

Since

$$\frac{|\mathbf{Y}_n|}{|\mathbf{K}_n|} \leq \frac{n 2^{\binom{n-1}{2}}}{2^{\binom{n}{2}}} = \frac{n}{2^{n-1}} \to 0,$$

as $n \to \infty$, it is sufficient to show that $|\mathbf{X}_n| \leq |\mathbf{Y}_n|$. For $\mathcal{M} \in \mathbf{X}_n$, let $a \in |\mathcal{M}| = \{1,\dots,n\}$ be minimal such that $\mathcal{M} \models Q(a)$, and let $\mathcal{M}'$ be defined as follows: $|\mathcal{M}'| = \{1,\dots,n\}$, $Q^{\mathcal{M}'} = \{a\}$ and let $R^{\mathcal{M}'}$ be the symmetric closure of

$$R^{\mathcal{M}} \cup \big\{(b,c) : b \in Q^{\mathcal{M}} - \{a\}, \quad c \in \{1,\dots,n\} - Q^{\mathcal{M}}\big\}.$$

Note that $\mathcal{M}' \in \mathbf{Y}_n$. It is now easy to verify that the map $\mathcal{M} \mapsto \mathcal{M}'$ from $\mathbf{X}_n$ to $\mathbf{Y}_n$ is injective; thus $|\mathbf{X}_n| \leq |\mathbf{Y}_n|$.

Since $\{\mathcal{M} \in \mathbf{K}_n : \mathcal{M} \models \neg\exists x Q(x)\}$ is the set of all (undirected) graphs with vertices $1,\dots,n$ it follows that, with the uniform probability measure, the almost sure theory of $\mathbf{K}$ is identical to the almost sure theory of all undirected graphs, and consequently $\mathbf{K}$ has a zero-one law for the uniform probability measure. Since the complete theory of the Fraïssé-limit of $\mathbf{K}$ contains the sentence $\exists x Q(x)$ it is different from the almost sure theory of $\mathbf{K}$, for the uniform measure. As we will see later, for the 'dimension conditional probability measure' (where dimension equals cardinality in this example), the almost sure theory of $\mathbf{K}$ is identical to the complete theory of the Fraïssé-limit of $\mathbf{K}$.

Example 4.4. (**Partially coloured binary relation.**) Let the vocabulary of L consist of one binary relation symbol R and two unary relation symbols P_1, P_2. Let $\mathbf{K}_n$ consist of all L-structures $\mathcal{M}$ with universe $\{1,\dots,n\}$ such that

$$\mathcal{M} \models \forall x \neg\big(P_1(x) \wedge P_2(x)\big), \quad \text{and}$$
$$\mathcal{M} \models \forall x, y\big(R(x,y) \ \to \ \big[\neg\big(P_1(x) \wedge P_1(y)\big) \ \wedge \ \neg\big(P_2(x) \wedge P_2(y)\big)\big]\big).$$

We can think of P_i as representing the colour 'i'. Before using Theorem 3.17 to get some information about $\mathbf{K}_n$ we consider the proportion of $\mathcal{M} \in \mathbf{K}_n$ which satisfy $\exists x P_i(x)$. Let

$$\mathbf{X}_n = \big\{\mathcal{M} \in \mathbf{K}_n : (P_1)^{\mathcal{M}} = \emptyset\big\}.$$

For every $\mathcal{M} \in \mathbf{X}_n$, let $\mathbf{Y}_n(\mathcal{M})$ be the set of $\mathcal{N} \in \mathbf{K}_n$ which satisfy that $R^{\mathcal{N}} = R^{\mathcal{M}}$ and either

- $(P_1)^{\mathcal{N}} = \{a\}$ and $(P_2)^{\mathcal{N}} = (P_2)^{\mathcal{M}}$ for some $a \notin (P_2)^{\mathcal{M}}$, or
- $(P_1)^{\mathcal{N}} = \{a\}$ and $(P_2)^{\mathcal{N}} = (P_2)^{\mathcal{M}} - \{a\}$ for some $a \in (P_2)^{\mathcal{M}}$.

It is straightforward to verify that, for every $\mathcal{M} \in \mathbf{X}_n$, $|\mathbf{Y}_n(\mathcal{M})| \geq n$, and, for every $\mathcal{N} \in \mathbf{K}_n$, the number of $\mathcal{M} \in \mathbf{X}_n$ such that $\mathcal{N} \in \mathbf{Y}_n(\mathcal{M})$ is at most 2. It follows that

$$\frac{|\mathbf{X}_n|}{|\mathbf{K}_n|} \leq \frac{|\mathbf{X}_n|}{|\bigcup_{\mathcal{M}\in\mathbf{X}_n} \mathbf{Y}_n(\mathcal{M})|} \leq \frac{|\mathbf{X}_n|}{\frac{1}{2}\sum_{\mathcal{M}\in\mathbf{X}_n} |\mathbf{Y}_n(\mathcal{M})|} \leq \frac{|\mathbf{X}_n|}{\frac{1}{2}|\mathbf{X}_n| n} = \frac{2}{n} \to 0,$$

as $n \to \infty$, so the proportion of $\mathcal{M} \in \mathbf{K}_n$ such that $\mathcal{M} \models \exists x P_1(x)$ approaches 1 as $n \to \infty$. The same argument works for P_2.

For an L-structure $\mathcal{M}$ and $a \in |\mathcal{M}|$, let us say that a is *blank* or *uncoloured* (in $\mathcal{M}$) if $\mathcal{M} \models \neg P_1(a) \wedge \neg P_2(a)$. Let $\mathcal{S}_{\mathcal{P}}$ have universe $\{a\}$ where a is blank in $\mathcal{S}_{\mathcal{P}}$ and $R^{\mathcal{S}_{\mathcal{P}}} = \emptyset$. Let $\mathcal{S}_{\mathcal{F}}$ also have universe $\{a\}$ where a has colour 1 in $\mathcal{S}_{\mathcal{F}}$ (i.e. $\mathcal{S}_{\mathcal{F}} \models P_1(a)$) and $R^{\mathcal{S}_{\mathcal{F}}} = \emptyset$. Then $\mathcal{S}_{\mathcal{P}}$ and $\mathcal{S}_{\mathcal{F}}$ are permitted and it is easily seen that the substitution $[\mathcal{S}_{\mathcal{F}} \triangleright \mathcal{S}_{\mathcal{P}}]$ is admitted, because making a point blank never violates the conditions for being permitted (with respect to $\mathbf{K}$). But if one point in an R-relationship is coloured by i, then colouring the other point in the same R-relationship by the same colour i produces a forbidden structure; so the substitution $[\mathcal{S}_{\mathcal{P}} \triangleright \mathcal{S}_{\mathcal{F}}]$ is not admitted. Now we apply Theorem 3.17. Let $|\mathcal{P}| = \{a, b\}$, $(P_1)^{\mathcal{P}} = \{b\}$ and $R^{\mathcal{P}} = \{(a, b)\}$. Since we know that the proportion of $\mathcal{M} \in \mathbf{K}_n$ which contain a copy of $\mathcal{S}_{\mathcal{F}}$ (i.e satisfy $\exists x P_1(x)$) approaches 1 as $n \to \infty$, it follows that for arbitrarily small $\varepsilon > 0$ and all sufficiently large n, the proportion of $\mathcal{M} \in \mathbf{K}_n$ such that the $\mathcal{P}/\mathcal{S}_{\mathcal{P}}$-multiplicity of $\mathcal{M}$ is at least 2 never exceeds $\big(1 - 1/(1 + 3)\big) + \varepsilon = 3/4 + \varepsilon$. Observe that the $\mathcal{P}/\mathcal{S}_{\mathcal{P}}$-multiplicity of $\mathcal{M}$ is at least 2 if and only if $\mathcal{M}$ satisfies the extension axiom

$$\varphi = \forall x \exists y, z\big([\neg P_1(x) \wedge \neg P_2(x)] \ \to\ [y \neq z \wedge R(x, y) \wedge R(x, z) \wedge P_1(y) \wedge P_1(z)]\big),$$

so the probability, with the uniform probability measure, that this extension axiom is true never exceeds $3/4 + \varepsilon$.

Example 4.5. (**Coloured binary relation.**) Let $\mathbf{K}_n$ be defined as in Example 4.4 *except* that we add the condition that there are *no* blank elements, that is, every $\mathcal{M} \in \mathbf{K}_n$ satisfies $\forall x\big(P_1(x) \vee P_2(x)\big)$. By Theorem 9.16 and Remark 9.17, for every extension axiom φ of $\mathbf{K}$, the proportion of $\mathcal{M} \in \mathbf{K}_n$ which satisfies φ approaches 1 as $n \to \infty$. Since $\mathbf{K}$ has the hereditary property and the disjoint amalgamation property, Lemma 3.16 and Theorem 3.17 (part (ii)) implies that there does *not* exist permitted $\mathcal{S}_{\mathcal{P}}$ and $\mathcal{S}_{\mathcal{F}}$ such that the substitution $[\mathcal{S}_{\mathcal{F}} \triangleright \mathcal{S}_{\mathcal{P}}]$ is admitted and $[\mathcal{S}_{\mathcal{P}} \triangleright \mathcal{S}_{\mathcal{F}}]$ is not admitted. However, since changing one colour to another in a permitted structure may produce a forbidden structure, there are permitted $\mathcal{A}$ and $\mathcal{A}'$ (with singleton universes) such that *none* of the substitutions $[\mathcal{A} \triangleright \mathcal{A}']$ and $[\mathcal{A}' \triangleright \mathcal{A}]$ is admitted.

Examples 4.6 and 4.7 (as well as Example 4.5) show that if $\mathbf{K}$ neither satisfies the conditions of Theorem 3.15, nor the conditions of Theorem 3.17 (or Corollary 3.18), then it may, or may not, be the case that for every extension axiom φ of $\mathbf{K}$ the proportion of $\mathcal{M} \in \mathbf{K}_n$ which satisfy φ approaches 1 as $n \to \infty$. In contrast to Examples 4.2 – 4.5, the last two examples of this section do not have any unary relations.

Example 4.6. (**Complete bipartite graph.**) For all $r, s \in \mathbb{N}$, let $\mathcal{K}_{r,s}$ denote the undirected graph with vertices $a_1, \ldots, a_r, b_1, \ldots, b_s$ and an edge connecting a_i and b_j for all $i \in \{1, \ldots, r\}$ and $j \in \{1, \ldots, s\}$, and no other edges. $\mathcal{K}_{0,s}$ and $\mathcal{K}_{r,0}$ are independent sets (no edges at all) with s and r vertices, respectively.

For every $n \in \mathbb{N}$, let $\mathbf{K}_n$ be the set of all graphs with vertices $1, \ldots, n$ which are isomorphic to $\mathcal{K}_{r,s}$ for some r, s. Clearly, by adding an edge to any represented $\mathcal{M}$ with at least 3 vertices, we create a forbidden graph. Also, by removing an edge from any $\mathcal{K}_{r,s}$ such that $r+s \geq 3$ and $\min(r,s) \geq 1$, we create a forbidden graph.

It is easy to see that if $s, r \geq k+1$, then $\mathcal{K}_{r,s}$ satisfies all k-extension axioms of $\mathbf{K} = \bigcup_{n\in\mathbb{N}} \mathbf{K}_n$. Also, the proportion of $\mathcal{M} \in \mathbf{K}_n$ which are isomorphic to some $\mathcal{K}_{r,s}$ with $r, s \geq k+1$ approaches 1 as $n \to \infty$. It follows that, for every extension axiom φ of $\mathbf{K}$, the proportion of $\mathcal{M} \in \mathbf{K}_n$ which satisfy φ approaches 1 as $n \to \infty$. It is straightforward to verify that the class of represented structures is closed under taking substructures (so 'permitted' is the same as 'represented') and has the disjoint amalgamation property. By Corollary 3.18, there does not exist any permitted $\mathcal{A}$ and $\mathcal{B}$ with $A = B$ such that $[\mathcal{A} \rhd \mathcal{B}]$ is admitted and $[\mathcal{B} \rhd \mathcal{A}]$ is not admitted.

Example 4.7. (**Equivalence relations**) Here we define $\mathbf{K}$ such that (as in the previous example) there are no permitted $\mathcal{A}$ and $\mathcal{B}$ such that $[\mathcal{A} \rhd \mathcal{B}]$ is admitted and $[\mathcal{B} \rhd \mathcal{A}]$ is not admitted. In this example, $\mathbf{K}$ has an extension axiom φ such that the proportion of $\mathcal{M} \in \mathbf{K}_n$ in which φ is true approaches 0 as $n \to \infty$, but nevertheless $\mathbf{K}$ has a zero-one law.

We represent an equivalence relation on a set M as an undirected graph (without loops) with vertex set M such that if a is adjacent to b and b is adjacent to $c \neq a$, then a and c are adjacent. Clearly, such a graph, which we call an *equivalence graph*, is a disjoint union of complete graphs. Let $\mathbf{K}_n$ consist of all equivalence graphs with vertices $1, \ldots, n$. Equivalently, we could have defined $\mathbf{K}_n$ by saying that it consists of all undirected graphs with vertices $1, \ldots, n$ in which $\mathcal{V}$ is not embeddable, where $\mathcal{V}$ denotes the graph with distinct vertices $1, 2, 3$ where 1 is adjacent with 2 and 2 is adjacent with 3, but 1 is not adjacent with 3. It is easily seen that $\mathbf{K}$ has the disjoint amalgamation property. By Lemma 3.16, if there would be $\mathcal{A}$, $\mathcal{B}$ with same universe such that $[\mathcal{A} \rhd \mathcal{B}]$ is admitted, but not $[\mathcal{B} \rhd \mathcal{A}]$, then, because we only have a binary relation symbol, we could assume that the common universe of $\mathcal{A}$ and $\mathcal{B}$ has cardinality 2, and that $[\mathcal{A} \rhd \mathcal{B}]$ means either to remove an edge, or to add an edge. But it is clear that both the removal of an edge, as well as the addition of an edge, may produce a forbidden structure, so *none* of $[\mathcal{A} \rhd \mathcal{B}]$ and $[\mathcal{B} \rhd \mathcal{A}]$ can be admitted, contradicting the assumption. Hence, there does not exist $\mathcal{A}$ and $\mathcal{B}$ such that $[\mathcal{A} \rhd \mathcal{B}]$ is admitted, but not $[\mathcal{B} \rhd \mathcal{A}]$.

We now show that if $\mathcal{A}$ is the graph having only one vertex a and $\mathcal{B}$ has vertex set $\{a, b\}$ where a is adjacent to b in $\mathcal{B}$, then the probability that $\mathcal{M} \in \mathbf{K}_n$ satisfies the $\mathcal{B}/\mathcal{A}$–extension axiom approaches 0 as $n \to \infty$. This contrasts the previous example. Let $\mathbf{X}_n$ be the set of $\mathcal{M} \in \mathbf{K}_n$ which do *not* contain any connected component which is a singleton, and let $\mathbf{X} = \bigcup_{n>1} \mathbf{X}_n$. Note that the class of represented structures with respect to $\mathbf{X}$ is closed under taking disjoint unions and extracting connected components; thus, the class of represented structures with respect to $\mathbf{X}$ is *adequate* in the sense of [10], which we will use. For every $n > 1$, $\mathbf{X}_n$ contains exactly one connected graph (the complete graph with vertices $1, \ldots, n$). Therefore, Theorem 7 in [10] implies that

$$\frac{n|\mathbf{X}_{n-1}|}{|\mathbf{X}_n|} \to \infty \quad \text{as} \quad n \to \infty.$$

Let $\mathbf{Y}_n$ be the set of $\mathcal{M} \in \mathbf{K}_n$ that contain at least one connected component which is a singleton, and let $\mathbf{Y}'_n$ be the set of $\mathcal{M} \in \mathbf{K}_n$ that contain exactly one connected component which is a singleton. Observe that

$$\mathbf{X}_n = \mathbf{K}_n - \mathbf{Y}_n \quad \text{and} \quad |\mathbf{Y}'_n| = n|\mathbf{X}_{n-1}|.$$

It follows that

$$\frac{|\mathbf{X}_n|}{|\mathbf{K}_n|} \leq \frac{|\mathbf{X}_n|}{|\mathbf{Y}'_n|} = \frac{|\mathbf{X}_n|}{n|\mathbf{X}_{n-1}|} \to 0 \quad \text{as} \quad n \to \infty.$$

In other words, the proportion of $\mathcal{M} \in \mathbf{K}_n$ which contain at least one connected component which is a singleton approaches 1 as $n \to \infty$. For every such $\mathcal{M}$, the $\mathcal{B}/\mathcal{A}$–extension axiom fails. Nevertheless, $\mathbf{K}$ has a zero-one law for the uniform probability measure, which follows from Theorem 7 in [10] and the above observed fact that, for every n, there is a unique connected graph in $\mathbf{K}_n$.

5. Proof of Theorem 3.17

Let L have a finite relational vocabulary and let $\mathbf{K} = \bigcup_{n\in\mathbb{N}} \mathbf{K}_n$, where every $\mathbf{K}_n$ is a set of L-structures with universe $\{1, \ldots, m_n\}$ and $\lim_{n\to\infty} m_n = \infty$. Suppose that $\mathcal{P}$, $\mathcal{S}_\mathcal{P}$ and $\mathcal{S}_\mathcal{F}$ are permitted structures such that $\mathcal{S}_\mathcal{P} \subseteq \mathcal{P}$, $|\mathcal{S}_\mathcal{P}| = |\mathcal{S}_\mathcal{F}|$, $\|\mathcal{S}_\mathcal{P}\| = k$, $\mathcal{F} = \mathcal{P}[\mathcal{S}_\mathcal{P} \triangleright \mathcal{S}_\mathcal{F}]$ is forbidden, but the substitution $[\mathcal{S}_\mathcal{F} \triangleright \mathcal{S}_\mathcal{P}]$ is admitted. Morover, assume that for every proper substructure $\mathcal{U} \subset \mathcal{S}_\mathcal{P}$, $\mathcal{S}_\mathcal{P}{\restriction}|\mathcal{U}| = \mathcal{S}_\mathcal{F}{\restriction}|\mathcal{U}|$. Let α be the number of different permitted structures with universe $\{1, \ldots, k\}$ (so $\alpha \geq 2$).

We use the following terminology:

Definition 5.1. (i) A pair of structures $(\mathcal{A}, \mathcal{B})$ is called a ***coexisting pair*** if $\mathcal{A}$ and $\mathcal{B}$ have the same universe.
(ii) We say that two coexisting pairs $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{A}', \mathcal{B}')$ are ***isomorphic*** if there is a bijection $\sigma : |\mathcal{A}| \to |\mathcal{A}'|$ which is an isomorphism from $\mathcal{A}$ to $\mathcal{A}'$ as well as from $\mathcal{B}$ to $\mathcal{B}'$.
(iii) If $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{A}', \mathcal{B}')$ are isomorphic coexisting pairs then we may say that $(\mathcal{A}', \mathcal{B}')$ is a ***copy*** of $(\mathcal{A}, \mathcal{B})$.

Lemma 5.2. *Suppose that $\mathcal{S}_\mathcal{P}$ is a proper substructure of $\mathcal{P}$ and that $\mathcal{M}$ is represented. If $(\mathcal{S}_\mathcal{P}^\mathcal{M}, \mathcal{S}_\mathcal{F}^\mathcal{M})$ is a copy of the coexisting pair $(\mathcal{S}_\mathcal{P}, \mathcal{S}_\mathcal{F})$ and $\mathcal{S}_\mathcal{F}^\mathcal{M} \subseteq \mathcal{M}$, then the $\mathcal{P}/\mathcal{S}_\mathcal{P}$-multiplicity of $\mathcal{M}[\mathcal{S}_\mathcal{F}^\mathcal{M} \triangleright \mathcal{S}_\mathcal{P}^\mathcal{M}]$ is 0.*

Proof. Without loss of generality (by just renaming elements) we may assume that $\mathcal{S}_\mathcal{F} = \mathcal{S}_\mathcal{F}^\mathcal{M} \subseteq \mathcal{M}$ and that $\mathcal{S}_\mathcal{P} = \mathcal{S}_\mathcal{P}^\mathcal{M}$. Then $\mathcal{S}_\mathcal{F}$ $(= \mathcal{S}_\mathcal{F}^\mathcal{M})$ is a substructure of $\mathcal{M}$, and $\mathcal{S}_\mathcal{F}$ and $\mathcal{S}_\mathcal{P}$ $(= \mathcal{S}_\mathcal{P}^\mathcal{M})$ have the same universe which is a subset of $|\mathcal{M}|$. By the assumption that $\mathcal{P}$ is permitted, but $\mathcal{F} = \mathcal{P}[\mathcal{S}_\mathcal{P} \triangleright \mathcal{S}_\mathcal{F}]$ is forbidden (see before Definition 5.1) we have $\mathcal{S}_\mathcal{F} \neq \mathcal{S}_\mathcal{P}$, and, as $\mathcal{S}_\mathcal{F} \subseteq \mathcal{M}$ and $|\mathcal{S}_\mathcal{F}| = |\mathcal{S}_\mathcal{P}|$, $\mathcal{S}_\mathcal{P}$ is not a substructure of $\mathcal{M}$. But $\mathcal{S}_\mathcal{P}$ is a substructure of $\mathcal{M}[\mathcal{S}_\mathcal{F} \triangleright \mathcal{S}_\mathcal{P}]$.

We show that the $\mathcal{P}/\mathcal{S}_\mathcal{P}$-multiplicity of $\mathcal{M}[\mathcal{S}_\mathcal{F} \triangleright \mathcal{S}_\mathcal{P}]$ is 0. Suppose for a contradiction that it is at least 1. Without loss of generality, we may assume that $\mathcal{P} = \mathcal{F}[\mathcal{S}_\mathcal{F} \triangleright \mathcal{S}_\mathcal{P}]$ is a substructure of $\mathcal{M}[\mathcal{S}_\mathcal{F} \triangleright \mathcal{S}_\mathcal{P}]$, so in particular, the common universe of $\mathcal{F}$ and $\mathcal{P} = \mathcal{F}[\mathcal{S}_\mathcal{F} \triangleright \mathcal{S}_\mathcal{P}]$ is a subset of the universe of $\mathcal{M}[\mathcal{S}_\mathcal{F} \triangleright \mathcal{S}_\mathcal{P}]$ and of $\mathcal{M}$. For each relation symbol R, of arity r say, we consider the interpretation of R in $\mathcal{M}{\restriction}|\mathcal{F}|$. If $\bar{a} \in |\mathcal{S}_\mathcal{F}|^r$, then

$$\bar{a} \in R^{\mathcal{M}\restriction F} \iff \bar{a} \in R^{\mathcal{S}_\mathcal{F}} \iff \bar{a} \in R^\mathcal{F} \quad (\text{since } \mathcal{S}_\mathcal{F} \subset \mathcal{F}).$$

If $\bar{a} \in |\mathcal{F}|^r - |\mathcal{S}_\mathcal{F}|^r$, then we use the definition of substitutions (Definition 3.11) and get

$$\begin{aligned} \bar{a} \in R^{\mathcal{M}\restriction F} &\iff \bar{a} \in R^\mathcal{M} \iff \bar{a} \in R^{\mathcal{M}[\mathcal{S}_\mathcal{F} \triangleright \mathcal{S}_\mathcal{P}]} \\ &\iff \bar{a} \in R^{\mathcal{M}[\mathcal{S}_\mathcal{F} \triangleright \mathcal{S}_\mathcal{P}]\restriction F} \iff \bar{a} \in R^{\mathcal{F}[\mathcal{S}_\mathcal{F} \triangleright \mathcal{S}_\mathcal{P}]} \iff \bar{a} \in R^\mathcal{F}. \end{aligned}$$

So whenever $\bar{a} \in |\mathcal{F}|^r$ we have $\bar{a} \in R^\mathcal{M}$ if and only if $\bar{a} \in R^\mathcal{F}$. Since the argument holds for every relation symbol R it follows that the forbidden structure $\mathcal{F}$ is a substructure of $\mathcal{M}$, which contradicts that $\mathcal{M}$ is represented. $\square$

Definition 5.3. Let the expression 'mult$(\mathcal{A}/\mathcal{B}; \mathcal{M}) \geq n$' mean 'the $\mathcal{A}/\mathcal{B}$-multiplicity of $\mathcal{M}$ is at least n'.

Lemma 5.4. *Suppose that $\mathcal{M}, \mathcal{N} \in \mathbf{K}_n$ are different and that* $\mathrm{mult}(\mathcal{P}/\mathcal{S}_\mathcal{P}; \mathcal{M}) \geq 2$ *and* $\mathrm{mult}(\mathcal{P}/\mathcal{S}_\mathcal{P}; \mathcal{N}) \geq 2$. *Let $(\mathcal{S}_\mathcal{P}^\mathcal{M}, \mathcal{S}_\mathcal{F}^\mathcal{M})$ and $(\mathcal{S}_\mathcal{P}^\mathcal{N}, \mathcal{S}_\mathcal{F}^\mathcal{N})$ be copies of the coexisting pair $(\mathcal{S}_\mathcal{P}, \mathcal{S}_\mathcal{F})$ such that $\mathcal{S}_\mathcal{F}^\mathcal{M} \subseteq \mathcal{M}$ and $\mathcal{S}_\mathcal{F}^\mathcal{N} \subseteq \mathcal{N}$. If $\mathcal{M}[\mathcal{S}_\mathcal{F}^\mathcal{M} \triangleright \mathcal{S}_\mathcal{P}^\mathcal{M}] = \mathcal{N}[\mathcal{S}_\mathcal{F}^\mathcal{N} \triangleright \mathcal{S}_\mathcal{P}^\mathcal{N}]$ then $\mathcal{S}_\mathcal{F}^\mathcal{M}$ and $\mathcal{S}_\mathcal{F}^\mathcal{N}$ have the same universe U and $\mathcal{M}$ and $\mathcal{N}$ are different only on U (that is, for every relation symbol R, if $\bar{a}$ belongs to exactly one of the relations $R^\mathcal{M}$ and $R^\mathcal{N}$, then $\bar{a} \in U$.)*

Proof. Let $(\mathcal{S}_\mathcal{P}^\mathcal{M}, \mathcal{S}_\mathcal{F}^\mathcal{M})$ and $(\mathcal{S}_\mathcal{P}^\mathcal{N}, \mathcal{S}_\mathcal{F}^\mathcal{N})$ be copies of the coexisting pair $(\mathcal{S}_\mathcal{P}, \mathcal{S}_\mathcal{F})$ such that $\mathcal{S}_\mathcal{F}^\mathcal{M} \subseteq \mathcal{M}$ and $\mathcal{S}_\mathcal{F}^\mathcal{N} \subseteq \mathcal{N}$. Then there are maps $\sigma_\mathcal{M} : |\mathcal{S}_\mathcal{F}^\mathcal{M}| \to |\mathcal{S}_\mathcal{F}|$ and $\sigma_\mathcal{N} : |\mathcal{S}_\mathcal{F}^\mathcal{N}| \to |\mathcal{S}_\mathcal{F}|$ such that:

- $\sigma_\mathcal{M}$ is an isomorphism from $\mathcal{S}_\mathcal{F}^\mathcal{M}$ to $\mathcal{S}_\mathcal{F}$ and from $\mathcal{S}_\mathcal{P}^\mathcal{M}$ to $\mathcal{S}_\mathcal{P}$, and
- $\sigma_\mathcal{N}$ is an isomorphism from $\mathcal{S}_\mathcal{F}^\mathcal{N}$ to $\mathcal{S}_\mathcal{F}$ and from $\mathcal{S}_\mathcal{P}^\mathcal{N}$ to $\mathcal{S}_\mathcal{P}$.

Let $\{a_1, \ldots, a_k\}$ be the universe of $\mathcal{S}_\mathcal{F}^\mathcal{M}$ (and of $\mathcal{S}_\mathcal{P}^\mathcal{M}$) and let $\{b_1, \ldots, b_k\}$ be the universe of $\mathcal{S}_\mathcal{F}^\mathcal{N}$ (and of $\mathcal{S}_\mathcal{P}^\mathcal{N}$).

Suppose, for a contradiction, that

(I) $\mathcal{M}[\mathcal{S}_\mathcal{F}^\mathcal{M} \triangleright \mathcal{S}_\mathcal{P}^\mathcal{M}] = \mathcal{H} = \mathcal{N}[\mathcal{S}_\mathcal{F}^\mathcal{N} \triangleright \mathcal{S}_\mathcal{P}^\mathcal{N}]$ and that

(II) $\{a_1, \ldots, a_k\} \neq \{b_1, \ldots, b_k\}$.

Then

$$\mathcal{M} = \mathcal{H}[\mathcal{S}_\mathcal{P}^\mathcal{M} \triangleright \mathcal{S}_\mathcal{F}^\mathcal{M}] \text{ and } \mathcal{N} = \mathcal{H}[\mathcal{S}_\mathcal{P}^\mathcal{N} \triangleright \mathcal{S}_\mathcal{F}^\mathcal{N}]. \tag{1}$$

Recall the assumption that for every proper substructure $\mathcal{U} \subset \mathcal{S}_\mathcal{P}$, $\mathcal{S}_\mathcal{P} \restriction |\mathcal{U}| = \mathcal{S}_\mathcal{F} \restriction |\mathcal{U}|$. Since $(\mathcal{S}_\mathcal{P}^\mathcal{M}, \mathcal{S}_\mathcal{F}^\mathcal{M})$ and $(\mathcal{S}_\mathcal{P}^\mathcal{N}, \mathcal{S}_\mathcal{F}^\mathcal{N})$ are copies of $(\mathcal{S}_\mathcal{P}, \mathcal{S}_\mathcal{F})$, it follows that $\mathcal{S}_\mathcal{P}^\mathcal{M}$ and $\mathcal{S}_\mathcal{F}^\mathcal{M}$ agree on all proper subsets of their common universe; and the same with $\mathcal{M}$ replaced by $\mathcal{N}$. From (1) it follows that

$$\text{if } U \subseteq \{1, \ldots, m_n\} \text{ and } |U| < k, \text{ then } \mathcal{M} \restriction U = \mathcal{H} \restriction U = \mathcal{N} \restriction U. \tag{2}$$

Since $\mathcal{H} \restriction \{b_1, \ldots, b_k\} = \mathcal{S}_\mathcal{P}^\mathcal{N}$ and $\mathcal{M}$ is obtained from $\mathcal{H}$ by the substitution $\mathcal{M} = \mathcal{H}[\mathcal{S}_\mathcal{P}^\mathcal{M} \triangleright \mathcal{S}_\mathcal{F}^\mathcal{M}]$, which *only* affects the interpretations of relation symbols on $\{a_1, \ldots, a_k\}$, assumption (II) together with (2) implies that

$$\mathcal{M} \restriction \{b_1, \ldots, b_k\} = \mathcal{S}_\mathcal{P}^\mathcal{N}.$$

Since the $\mathcal{P}/\mathcal{S}_\mathcal{P}$-multiplicity of $\mathcal{M}$ is at least 2, there are $\mathcal{P}_i \subseteq \mathcal{M}$ and isomorphisms $\sigma_i : \mathcal{P}_i \to \mathcal{P}$ such that $\mathcal{S}_\mathcal{P}^\mathcal{N} \subset \mathcal{P}_i$, $\sigma_i \restriction |\mathcal{S}_\mathcal{P}^\mathcal{N}| = \sigma_\mathcal{N}$, for $i = 1, 2$, and $|\mathcal{P}_1| \cap |\mathcal{P}_2| = \big\{b_1, \ldots, b_k\big\} = |\mathcal{S}_\mathcal{P}^\mathcal{N}|$. By assumption (I), $\mathcal{H}$ is obtained from $\mathcal{M}$ by the substitution $\mathcal{H} = \mathcal{M}[\mathcal{S}_\mathcal{F}^\mathcal{M} \triangleright \mathcal{S}_\mathcal{P}^\mathcal{M}]$ which only affects the interpretations of relation symbols on $\{a_1, \ldots, a_k\}$. This together with (II), (2) and the choice of $\mathcal{P}_1$ and $\mathcal{P}_2$ so that $|\mathcal{P}_1| \cap |\mathcal{P}_2| = \big\{b_1, \ldots, b_k\big\}$ implies that for $i = 1$ or $i = 2$, $\mathcal{H} \restriction |\mathcal{P}_i| = \mathcal{P}_i$. Choose i so that

$$\mathcal{H} \restriction |\mathcal{P}_i| = \mathcal{P}_i. \tag{3}$$

Since $\mathcal{S}_\mathcal{P}^\mathcal{N} \subset \mathcal{P}_i$ and $\sigma_i : \mathcal{P}_i \to \mathcal{P}$ is an isomorphism such that $\sigma_i \restriction |\mathcal{S}_\mathcal{P}^\mathcal{N}| = \sigma_\mathcal{N}$, the substitution $[\mathcal{S}_\mathcal{P}^\mathcal{N} \triangleright \mathcal{S}_\mathcal{F}^\mathcal{N}]$ changes $\mathcal{P}_i$ to a structure which is isomorphic with $\mathcal{F}$, that is, $\mathcal{P}_i[\mathcal{S}_\mathcal{P}^\mathcal{N} \triangleright \mathcal{S}_\mathcal{F}^\mathcal{N}] \cong \mathcal{F}$, via the isomorphism σ_i. By applying (1) and (3) we get

$$\mathcal{N} \restriction |\mathcal{P}_i| = (\mathcal{H}[\mathcal{S}_\mathcal{P}^\mathcal{N} \triangleright \mathcal{S}_\mathcal{F}^\mathcal{N}]) \restriction |\mathcal{P}_i| \cong \mathcal{F}.$$

Hence the substructure of $\mathcal{N}$ with universe $|\mathcal{P}_i|$ is isomorphic to the forbidden structure $\mathcal{F}$. Therefore $\mathcal{N}$ is not represented, which contradicts that $\mathcal{N} \in \mathbf{K}_n$.

So if (I) holds then (II) is false and hence all the structures $\mathcal{S}_\mathcal{F}^\mathcal{M}$, $\mathcal{S}_\mathcal{P}^\mathcal{M}$, $\mathcal{S}_\mathcal{F}^\mathcal{N}$ and $\mathcal{S}_\mathcal{P}^\mathcal{N}$ have the same universe, say U. Consequently, from the assumption (I), if R is a relation symbol of arity r, say, and $\bar{a} \in \{1, \ldots, m_n\}^r$ belongs to exactly one of $R^\mathcal{M}$ and $R^\mathcal{N}$, then $\bar{a} \in U$. $\square$

Definition 5.5. (i) For every L-structure $\mathcal{M}$, let $\mathbf{\Sigma}(\mathcal{M}; \mathcal{S}_\mathcal{F} \triangleright \mathcal{S}_\mathcal{P})$ denote the set of all structures of the form $\mathcal{M}[\mathcal{S}_\mathcal{F}^\mathcal{M} \triangleright \mathcal{S}_\mathcal{P}^\mathcal{M}]$ where $(\mathcal{S}_\mathcal{P}^\mathcal{M}, \mathcal{S}_\mathcal{F}^\mathcal{M})$ is a copy of the coexisting pair $(\mathcal{S}_\mathcal{P}, \mathcal{S}_\mathcal{F})$ and $\mathcal{S}_\mathcal{F}^\mathcal{M} \subseteq \mathcal{M}$. (If $\mathcal{M}$ contains no copy of $\mathcal{S}_\mathcal{F}$ then $\mathbf{\Sigma}(\mathcal{M}; \mathcal{S}_\mathcal{F} \triangleright \mathcal{S}_\mathcal{P}) = \emptyset$)
(ii) For every n, let $\mathbf{\Omega}_n$ denote the set of all $\mathcal{M} \in \mathbf{K}_n$ such that $\mathrm{mult}(\mathcal{P}/\mathcal{S}_\mathcal{P}; \mathcal{M}) \geq 2$.
(iii) Recall that α denotes the number of different permitted L-structures with universe $\{1, \ldots, k\}$.

Lemma 5.6. *If $\mathcal{M}_1, \ldots, \mathcal{M}_{\alpha+1} \in \mathbf{\Omega}_n$ and $\mathcal{M}_i \neq \mathcal{M}_j$ whenever $i \neq j$, then*

$$\bigcap_{1 \le i \le \alpha+1} \mathbf{\Sigma}(\mathcal{M}_i; \mathcal{S}_{\mathcal{F}} \rhd \mathcal{S}_{\mathcal{P}}) = \emptyset.$$

In other words, for every structure $\mathcal{N}$, it can belong to $\mathbf{\Sigma}(\mathcal{M}; \mathcal{S}_{\mathcal{F}} \rhd \mathcal{S}_{\mathcal{P}})$ for at most α distinct $\mathcal{M} \in \mathbf{\Omega}_n$.

Proof. Suppose for a contradiction that $\mathcal{M}_1, \ldots, \mathcal{M}_{\alpha+1} \in \Omega_n$ are distinct and that $\mathcal{N} \in \mathbf{\Sigma}(\mathcal{M}_i; \mathcal{S}_{\mathcal{F}} \rhd \mathcal{S}_{\mathcal{P}})$ for every $i \in \{1, \ldots, \alpha+1\}$. Then there are copies $(\mathcal{S}_{\mathcal{P}}^{\mathcal{M}_i}, \mathcal{S}_{\mathcal{F}}^{\mathcal{M}_i})$ of $(\mathcal{S}_{\mathcal{P}}, \mathcal{S}_{\mathcal{F}})$ such that $\mathcal{S}_{\mathcal{F}}^{\mathcal{M}_i} \subseteq \mathcal{M}_i$ and $\mathcal{N} = \mathcal{M}_i[\mathcal{S}_{\mathcal{F}}^{\mathcal{M}_i} \rhd \mathcal{S}_{\mathcal{P}}^{\mathcal{M}_i}]$ for every $i \in \{1, \ldots, \alpha+1\}$. By Lemma 5.4, all $\mathcal{S}_{\mathcal{F}}^{\mathcal{M}_i}$, $i \in \{1, \ldots, \alpha+1\}$, have the same universe, which we denote by U, and for every pair $i, j \in \{1, \ldots, \alpha+1\}$ of distinct numbers, $\mathcal{M}_i$ and $\mathcal{M}_j$ are different only on U. The assumption that $\mathcal{M}_i \neq \mathcal{M}_j$ if $i \neq j$ now implies that for all distinct $i, j \in \{1, \ldots, \alpha+1\}$, $\mathcal{M}_i \restriction U \neq \mathcal{M}_j \restriction U$. Since $|U| = k$, this contradicts the choice of α, being the number of all different permitted L-structures with universe $\{1, \ldots, k\}$. $\square$

Now we have the tools for proving part (i) of Theorem 3.17, and then the other parts of the theorem. Let $\mathbf{\Omega}_n^*$ be the set of all $\mathcal{M} \in \mathbf{K}_n$ such that

(a) $\mathcal{M}$ contains a copy of $\mathcal{S}_{\mathcal{F}}$, and
(b) the $\mathcal{P}/\mathcal{S}_{\mathcal{P}}$-multiplicity of $\mathcal{M}$ is at least 2.

By (b) and the definition of $\mathbf{\Omega}_n$, $\mathbf{\Omega}_n^* \subseteq \mathbf{\Omega}_n$. Since every $\mathcal{M} \in \mathbf{\Omega}_n^*$ contains a copy of $\mathcal{S}_{\mathcal{F}}$, it follows that for every $\mathcal{M} \in \mathbf{\Omega}_n^*$, $\mathbf{\Sigma}(\mathcal{M}; \mathcal{S}_{\mathcal{F}} \rhd \mathcal{S}_{\mathcal{P}}) \neq \emptyset$. Since the substitution $[\mathcal{S}_{\mathcal{F}} \rhd \mathcal{S}_{\mathcal{P}}]$ is admitted, $\mathbf{\Sigma}(\mathcal{M}; \mathcal{S}_{\mathcal{F}} \rhd \mathcal{S}_{\mathcal{P}}) \subseteq \mathbf{K}_n$ for every $\mathcal{M} \in \mathbf{K}_n$. By Lemma 5.2, for every $\mathcal{M} \in \mathbf{\Omega}_n^*$, $\mathbf{\Sigma}(\mathcal{M}; \mathcal{S}_{\mathcal{F}} \rhd \mathcal{S}_{\mathcal{P}}) \subseteq \mathbf{K}_n - \mathbf{\Omega}_n^*$. Lemma 5.6 now implies that

$$\big|\mathbf{K}_n - \mathbf{\Omega}_n^*\big| \ge \Big| \bigcup_{\mathcal{M} \in \mathbf{\Omega}_n^*} \mathbf{\Sigma}(\mathcal{M}; \mathcal{S}_{\mathcal{F}} \rhd \mathcal{S}_{\mathcal{P}}) \Big| \ge \frac{\big|\mathbf{\Omega}_n^*\big|}{\alpha}$$

$$\text{and hence} \quad \alpha \big|\mathbf{K}_n - \mathbf{\Omega}_n^*\big| \ge \big|\mathbf{\Omega}_n^*\big|.$$

From this we get

$$\frac{\big|\mathbf{K}_n - \mathbf{\Omega}_n^*\big|}{\big|\mathbf{K}_n\big|} = \frac{\big|\mathbf{K}_n - \mathbf{\Omega}_n^*\big|}{\big|\mathbf{\Omega}_n^*\big| + \big|\mathbf{K}_n - \mathbf{\Omega}_n^*\big|} \ge \frac{\big|\mathbf{K}_n - \mathbf{\Omega}_n^*\big|}{\alpha\big|\mathbf{K}_n - \mathbf{\Omega}_n^*\big| + \big|\mathbf{K}_n - \mathbf{\Omega}_n^*\big|} = \frac{1}{\alpha+1}.$$

Thus, the proportion of $\mathcal{M} \in \mathbf{K}_n$ *not* satisfying both (a) and (b) is at least $1/(1+\alpha)$. This concludes the proof of part (i) of Theorem 3.17.

Part (ii) of Theorem 3.17 is a straightforward consequence of part (i). For if there exist a permitted structure $\mathcal{C}$ and embeddings $\sigma_1 : \mathcal{P} \to \mathcal{C}$ and $\sigma_2 : \mathcal{P} \to \mathcal{C}$ such that $\sigma_1(|\mathcal{P}|) \cap \sigma_2(|\mathcal{P}|) = \sigma_1(|\mathcal{S}_{\mathcal{P}}|)$, $\sigma_1 \restriction |\mathcal{S}_{\mathcal{P}}| = \sigma_2 \restriction |\mathcal{S}_{\mathcal{P}}|$ and $\mathcal{M} \in \mathbf{K}_n$ satisfies all $(2\,\|\mathcal{P}\| - k - 1)$-extension axioms, then conditions (a) and (b) in part (i) of Theorem 3.17 are satisfied.

Now we prove part (iii) of Theorem 3.17. Here we have added the assumption that L has no unary relation symbols, so there is a unique (up to isomorphism) permitted structure with a singleton universe. (In fact this is sufficient for what we want to prove.) Let $\mathcal{U} \subset \mathcal{S}_{\mathcal{F}}$ be such that $\|\mathcal{U}\| = 1$. Note that since $\mathcal{S}_{\mathcal{F}} \neq \mathcal{S}_{\mathcal{P}}$ (and $|\mathcal{S}_{\mathcal{F}}| = |\mathcal{S}_{\mathcal{P}}|$) we have $\|\mathcal{S}_{\mathcal{F}}\| > 1$. Suppose that $\mathcal{M} \in \mathbf{K}_n$ is such that

(c) $\mathcal{M}$ satisfies the $\mathcal{S}_{\mathcal{F}}/\mathcal{U}$-extension axiom, and
(d) the $\mathcal{P}/\mathcal{S}_{\mathcal{P}}$-multiplicity of $\mathcal{M}$ is at least 2.

Since $\|\mathcal{M}\| = m_n$, there are m_n distinct copies of $\mathcal{U}$ in $\mathcal{M}$. Each one of these copies of $\mathcal{U}$ is, by (c), included in a copy of $\mathcal{S}_{\mathcal{F}}$, so we get at least m_n/k distinct copies of $\mathcal{S}_{\mathcal{F}}$ in $\mathcal{M}$. Recall that our assumptions imply that $\mathcal{S}_{\mathcal{P}} \neq \mathcal{S}_{\mathcal{F}}$ and $\mathcal{S}_{\mathcal{P}} \restriction |\mathcal{V}| = \mathcal{S}_{\mathcal{F}} \restriction |\mathcal{V}|$ for every proper substructure $\mathcal{V} \subset \mathcal{S}_{\mathcal{P}}$. Since $\mathbf{\Sigma}(\mathcal{M}; \mathcal{S}_{\mathcal{F}} \rhd \mathcal{S}_{\mathcal{P}})$ contains all $\mathcal{N}$ which can be obtained from $\mathcal{M}$ by replacing one copy of $\mathcal{S}_{\mathcal{F}}$ by a copy of $\mathcal{S}_{\mathcal{P}}$, we have $\big|\mathbf{\Sigma}(\mathcal{M}; \mathcal{S}_{\mathcal{F}} \rhd \mathcal{S}_{\mathcal{P}})\big| \ge m_n/k$. By

Lemma 5.2, *no* $\mathcal{N} \in \mathbf{\Sigma}(\mathcal{M}; \mathcal{S}_{\mathcal{F}} \triangleright \mathcal{S}_{\mathcal{P}})$ satisfies (d). Hence, if $\mathbf{E}_n$ is the set of all $\mathcal{M} \in \mathbf{K}_n$ which satisfy both (c) and (d), then, by Lemma 5.6,

$$\big|\mathbf{K}_n - \mathbf{E}_n\big| \geq \frac{m_n|\mathbf{E}_n|}{k\alpha} \quad \text{and hence} \quad \frac{\big|\mathbf{E}_n\big|}{\big|\mathbf{K}_n\big|} \leq \frac{|\mathbf{E}_n|}{|\mathbf{K}_n - \mathbf{E}_n|} \leq \frac{k\alpha}{m_n}.$$

As $\lim_{n\to\infty} m_n = \infty$, the proportion of $\mathcal{M} \in \mathbf{K}_n$ which satisfy both (c) and (d) approaches 0 as n approaches ∞. This concludes the proof of Theorem 3.17.

6. Conditional probability measures

In Sections 3 – 5 we saw that a condition that ensures that every extension axiom is true in almost all sufficiently large structures is that every substitution involving (only) permitted structures is admitted. And if this condition does not hold it may happen that some extension axiom is false in almost all sufficiently large structures. In this section we start to develop a theory of conditional probability measures on finite sets of structures. When using this measure we can include more examples of sets of finite structures for which any extension axiom is almost surely true in all sufficiently large structures under consideration. Such examples include Examples 4.3, 4.4 and 4.5, and more generally, coloured structures and partially coloured structures (as in examples 7.22 – 7.24). But there are other examples, such as $\mathcal{K}_l$-free graphs ($l \geq 3$) which are not included; that is, also with the conditional measures considered here there is an extension axiom which almost surely fails for sufficiently large $\mathcal{K}_l$-free graphs.

Although the uniform probability measure is conceptually simple, it does not necessarily correspond to the probability measure associated with a method for randomly generating a structure of some specified kind. The conditional measures to be considered are more closely related to probability measures associated with random generation of structures of a given kind. This is the first point that will be stressed below, after the next two definitions.

Definition 6.1. Let $\mathbf{C}_0$ and $\mathbf{C}_1$ be finite sets of L-structures and let $\mathbb{P}_0$ be a probability measure on $\mathbf{C}_0$. Suppose that

(1) for every $\mathcal{A} \in \mathbf{C}_0$ there is at least one $\mathcal{B} \in \mathbf{C}_1$ such that $\mathcal{A} \subseteq_w \mathcal{B}$, and
(2) for every $\mathcal{B} \in \mathbf{C}_1$ there is a *unique* $\mathcal{A} \in \mathbf{C}_0$ such that $\mathcal{A} \subseteq_w \mathcal{B}$ and whenever $\mathcal{A}' \in \mathbf{C}_0$ and $\mathcal{A}' \subseteq_w \mathcal{B}$, then $\mathcal{A}' \subseteq_w \mathcal{A}$. We denote such $\mathcal{A}$ by $\mathcal{B}{\restriction}0$.

Then we define the ***uniformly*** $\mathbb{P}_0$***-conditional probability measure*** $\mathbb{P}_1$ on $\mathbf{C}_1$ as follows:

For every $\mathcal{B} \in \mathbf{C}_1$, the probability of $\mathcal{B}$ in $\mathbf{C}_1$ is

$$\mathbb{P}_1(\mathcal{B}) = \frac{1}{\big|\{\mathcal{B}' \in \mathbf{C}_1 : \mathcal{B}'{\restriction}0 = \mathcal{B}{\restriction}0\}\big|} \cdot \mathbb{P}_0(\mathcal{B}{\restriction}0),$$

and for $\mathbf{X} = \{\mathcal{B}_1, \ldots, \mathcal{B}_n\} \subseteq \mathbf{C}_1$ (where $\mathbf{X}$ is enumerated without repetition)

$$\mathbb{P}_1(\mathbf{X}) = \sum_{i=1}^{n} \mathbb{P}_1(\mathcal{B}_i).$$

Definition 6.2. More generally, assume that $\mathbf{C}_0, \ldots, \mathbf{C}_r$ are finite sets of L-structures such that, for every $i = 0, \ldots, r-1$, (1) and (2) in Definition 6.1 hold if $\mathbf{C}_0$ and $\mathbf{C}_1$ are replaced by $\mathbf{C}_i$ and $\mathbf{C}_{i+1}$, respectively. Let $\mathbb{P}_0$ denote the uniform probability measure on $\mathbf{C}_0$ (i.e. all elements of $\mathbf{C}_0$ have the same probability $1/|\mathbf{C}_0|$). By induction, define $\mathbb{P}_{i+1}$ to be the uniformly $\mathbb{P}_i$-conditional probability measure, for $i = 0, \ldots, r-1$. We call the probability measure $\mathbb{P}_r$ on $\mathbf{C}_r$, thus obtained, the ***uniformly*** $(\mathbf{C}_0, \ldots, \mathbf{C}_{r-1})$***-conditional probability measure***.

Example 6.3. Let us first illustrate the definitions by considering Example 4.3, where $\mathbf{K}_n$ is the set of undirected graphs with vertices $1, \ldots, n$ (with edge relation represented by R) and a unary relation symbol P subject to the condition: $R(a, b) \implies \neg P(a)$ and $\neg P(b)$. We have proved (see Example 4.3) that with the uniform probability measure, the probability of $\exists x P(x)$ holding in $\mathcal{M} \in \mathbf{K}_n$ approaches 0 as $n \to \infty$. Next we show that with a naturally chosen conditional measure, the probability that $\exists x P(x)$ holds in $\mathcal{M} \in \mathbf{K}_n$ approaches 1 as $n \to \infty$.

Let L denote the language considered in Example 4.3, with one binary relation symbol R and one unary relation symbol P, and let L_0 be the sublanguage of L whose vocabulary contains only P. For every n, let $\mathbf{K}_n \restriction L_0 = \{\mathcal{M} \restriction L_0 : \mathcal{M} \in \mathbf{K}_n\}$. Recall from the definition of weak substructure (Section 2.1), and the discussion after it, that, for every $\mathcal{M} \in \mathbf{K}_n$, $\mathcal{M} \restriction L_0$ may also be viewed as an L-structure (in which the interpretation of R is empty) and it follows that $\mathcal{M} \restriction L_0 \subseteq_w \mathcal{M}$. It is easy to verify that, for every n, if $\mathbf{C}_0 = \mathbf{K}_n \restriction L_0$ and $\mathbf{C}_1 = \mathbf{K}_n$, then conditions (1) and (2) in Definition 6.1 hold. Hence, for every n, the uniformly $(\mathbf{K}_n \restriction L_0)$-conditional probability measure on $\mathbf{K}_n$ is well-defined. Now, the claim that the probability, with this measure, that (the extension axiom) $\exists x P(x)$ holds in $\mathcal{M} \in \mathbf{K}_n$ approaches 1 as $n \to \infty$, is a consequence of Theorem 7.31. But for this simple example it suffices to observe that the probability of $\mathcal{M} \in \mathbf{K}_n$, with the uniformly $(\mathbf{K}_n \restriction L_0)$-conditional measure, is the probability of obtaining $\mathcal{M}$ by the following generating procedure: First go through every $i \in \{1, \ldots, n\}$ and with probability 1/2 let it satisfy $P(x)$; then take the set $\{i_1, \ldots, i_m\}$ of all vertices which do not satisfy $P(x)$, and for each unordered pair $\{i, j\}$ of elements from $\{i_1, \ldots, i_m\}$ assign an edge to it with probability 1/2. So the probability that no $i \in \{1, \ldots, n\}$ satisfies $P(x)$ is $1/2^n$, which approaches 0 as $n \to \infty$.

Example 6.4. Let us now consider Example 4.4 (partially coloured binary relation), where the vocabulary of L is $\{R, P_1, P_2\}$, R is binary and P_i, $i = 1, 2$, are unary, and thought of as "colours". $\mathbf{K}_n$ consists of all structures with universe $\{1, \ldots, n\}$ such that the universe is partially coloured with respect to the relation R, that is, every element has at most one colour (1 or 2), and it may be uncoloured (or "blank"), and whenever $R(a, b)$ holds, then a and b cannot be coloured with the same colour.

How can we, for any given k, design a procedure that generates – by possibly making some random assignments on the way – $\mathcal{M} \in \mathbf{K}_n$ in such a way that the probability of ending up with an $\mathcal{M} \in \mathbf{K}_n$ with exactly k elements with colour 1 is the same as the proportion of $\mathcal{M} \in \mathbf{K}_n$ which have exactly k elements with colour 1? The author does not know, and the point is that, in general, it may not be easy to conceive of a generating procedure, of structures from a given set, such that the probability measure associated with the generating procedure is identical to the uniform probability measure on the given set of structures.

Recall, from Example 4.4, that there is an extension axiom φ such that the probability, with the uniform measure, that φ holds in $\mathcal{M} \in \mathbf{K}_n$ approaches 0 as $n \to \infty$. But if we apply the following generating procedure of $\mathcal{M} \in \mathbf{K}_n$, then, for *every* extension axiom φ, the probability of ending up with an $\mathcal{M} \in \mathbf{K}_n$ which satisfies φ approaches 1 as $n \to \infty$. For every $i \in \{1, \ldots, n\}$, with probability 1/3 let it have colour 1, colour 2 or be blank; then go through all pairs (i, j) such that i and j are not coloured with the same colour and let $(i, j) \in R^{\mathcal{M}}$ with probability 1/2. The probability of obtaining, in this way, a structure $\mathcal{M} \in \mathbf{K}_n$ is the same as the probability of $\mathcal{M}$ with the uniformly $(\mathbf{K}_n \restriction L_0)$-conditional measure on $\mathbf{K}_n$, where L_0 is the sublanguage of L whose vocabulary is $\{P_1, P_2\}$ and $\mathbf{K}_n \restriction L_0 = \{\mathcal{M} \restriction L_0 : \mathcal{M} \in \mathbf{K}_n\}$. By letting the underlying geometry of every structure in $\mathbf{K} = \bigcup_{n \in \mathbb{N}} \mathbf{K}_n$ be trivial (see Remark 7.2) and applying Theorem 7.31 it follows that, for every extension axiom φ of $\mathbf{K}$, the probability, with the uniformly $(\mathbf{K}_n \restriction L_0)$-conditional measure, that φ holds in $\mathbf{K}_n$ approaches 1 as $n \to \infty$; and by

Theorem 7.32, $\mathbf{K}$ has a zero-one law. We have in particular shown that the asymptotic probability, with the uniform probability measure, of a first order definable property in $\mathbf{K}$ may be different from the asymptotic probability of the same property when the $(\mathbf{K}_n \restriction L_0)$-conditional measure is used.

Before taking underlying pregeometries into account, we collect a technical lemma which will be used later.

Lemma 6.5. *Suppose that $\mathbf{C}_0, \ldots, \mathbf{C}_k$ are finite sets of structures such that, for every $i = 0, \ldots, k-1$, (1) and (2) in Definition 6.1 hold if $\mathbf{C}_0$ and $\mathbf{C}_1$ are replaced by $\mathbf{C}_i$ and $\mathbf{C}_{i+1}$, respectively. For $r = 1, \ldots, k$, let $\mathbb{P}_r$ denote the uniformly $(\mathbf{C}_0, \ldots, \mathbf{C}_{r-1})$-conditional probability measure on $\mathbf{C}_r$. If $1 \leq r \leq s \leq k$ and $\mathcal{A} \subseteq \mathbf{C}_r$, then*

$$\mathbb{P}_r(\mathcal{A}) = \mathbb{P}_{r+1}\big(\{\mathcal{B} \in \mathbf{C}_{r+1} : \mathcal{A} \subseteq_w \mathcal{B}\}\big)$$
$$\textit{and } \mathbb{P}_r(\mathcal{A}) = P_s\big(\{\mathcal{B} \in \mathbf{C}_s : \mathcal{A} \subseteq_w \mathcal{B}\}\big).$$

Proof. The second identity follows from the first by induction, and the first identity is a straightforward consequence of Definitions 6.2 and 6.1. □

7. Underlying pregeometries

Definition 7.1. (i) We call an L-structure $\mathcal{A}$ a ***pregeometry*** if

1. there is a closure operation $\mathrm{cl}_\mathcal{A}$ on A such that $(A, \mathrm{cl}_\mathcal{A})$ is a pregeometry,
2. for all $n \in \mathbb{N}$ there is a formula $\theta_n(x_1, \ldots, x_{n+1}) \in L$ such that for all $a_1, \ldots, a_{n+1} \in A$, $a_{n+1} \in \mathrm{cl}_\mathcal{A}(a_1, \ldots, a_n)$ if and only if $\mathcal{A} \models \theta_n(a_1, \ldots, a_{n+1})$, and
3. if $X \subseteq A$ is closed with respect to $\mathrm{cl}_\mathcal{A}$ (i.e. $\mathrm{cl}_\mathcal{A}(X) = X$), then X is closed under interpretations of (eventual) function symbols and constant symbols; so X is the universe of a substructure of $\mathcal{A}$.

(ii) Let $\mathbf{K}$ be a class of L-structures. We call $\mathbf{K}$ a ***pregeometry*** if every $\mathcal{A} \in \mathbf{K}$ is a pregeometry and for every $n \in \mathbb{N}$ there is a formula $\theta_n(x_1, \ldots, x_{n+1}) \in L$ such that for every $\mathcal{A} \in \mathbf{K}$ and all $a_1, \ldots, a_{n+1} \in A$, $a_{n+1} \in \mathrm{cl}_\mathcal{A}(a_1, \ldots, a_n)$ if and only if $\mathcal{A} \models \theta_n(a_1, \ldots, a_{n+1})$.

Remark 7.2. For every structure $\mathcal{A}$, if $\mathrm{cl}_\mathcal{A}(X) = X$ for every $X \subseteq A$, then $\mathcal{A}$ is a pregeometry in the sense of Definition 7.1 (i). This pregeometry is often called ***trivial*** or ***degenerate***. It may happen that for a structure $\mathcal{A}$ there is more than one way to define a pregeometry on A. As noted, we always have a trivial pregeometry on A. But if, for example, $\mathcal{A}$ is a vector space over some finite field (formalized as a first-order structure in a suitable way), then we can also let $\mathrm{cl}_\mathcal{A}(X)$ be the linear span of X, and then $\mathrm{cl}_\mathcal{A}$ becomes a pregeometry on A. When saying that a structure $\mathcal{A}$ is a pregeometry we assume that some particular pregeometry on A (in the sense of Definition 7.1 (i)) is fixed, and if we say that a class of L-structures $\mathbf{K}$ is a pregeometry we assume that, for every $\mathcal{A} \in \mathbf{K}$, some pregeometry $\mathrm{cl}_\mathcal{A}$ is fixed on A and that the condition in Definition 7.1 (ii) holds.

Assumption 7.3. For the rest of this section we assume that $\mathbf{K}$ is a class of L-structures which is a pregeometry, and that the formulas $\theta_n(x_1, \ldots, x_{n+1})$ define the pregeometry in the sense of Definition 7.1 (ii). (Later, in Assumption 7.10, we will add some more assumptions.)

Definition 7.4. (i) As in Sections 3 – 6, we say that structure $\mathcal{A}$ is ***represented (with respect to*** $\mathbf{K}$***)*** if $\mathcal{A}$ is isomorphic to some structure in $\mathbf{K}$. We say that $\mathcal{A}$ is ***permitted (with respect to*** $\mathbf{K}$***)*** if it can be embedded into some structure in $\mathbf{K}$; or equivalently, if it is a substructure of some represented structure. And a structure which is not permitted (with respect to $\mathbf{K}$) is ***forbidden (with respect to*** $\mathbf{K}$***)***. Note that every represented

structure is a pregeometry on which the closure operator is defined by $\theta_n(x_1, \ldots, x_{n+1})$, $n \in \mathbb{N}$. This is what we mean when speaking about a pregeometry and closure on a represented structure.
(ii) If $\mathcal{M}$ is a pregeometry, then the notation $\mathcal{A} \subseteq_{cl} \mathcal{M}$ means that $\mathcal{A}$ is a substructure of $\mathcal{M}$ and $\mathrm{cl}_{\mathcal{M}}(A) = A$. In words, we express '$\mathcal{A} \subseteq_{cl} \mathcal{M}$' by saying that $\mathcal{A}$ ***is a closed substructure of*** $\mathcal{M}$.

Definition 7.5. The notion of $\mathcal{B}/\mathcal{A}$-multiplicity is defined as before (Definition 3.9), *except* that we require that $\mathcal{A}$ and $\mathcal{B}$ are closed in some superstructure. More precisely: Suppose that there is a represented $\mathcal{N}$ such that $\mathcal{A} \subset \mathcal{B} \subseteq \mathcal{N}$ and both A and B are closed in $\mathcal{N}$. We say that the ***$\mathcal{B}/\mathcal{A}$-multiplicity of a (represented) structure $\mathcal{M}$ is at least*** m if the following holds:

> whenever $\mathcal{A}' \subseteq_{cl} \mathcal{M}$ and $\sigma : \mathcal{A}' \to \mathcal{A}$ is an isomorphism, then there are $\mathcal{B}'_i \subseteq_{cl} \mathcal{M}$ and isomorphisms $\sigma_i : \mathcal{B}'_i \to \mathcal{B}$, for $i = 1, \ldots, m$, such that $\mathcal{A}' \subseteq \mathcal{B}'_i$, $\sigma_i \restriction A' = \sigma$ and $B'_i \cap B'_j = A'$ whenever $i \neq j$.

The ***$\mathcal{B}/\mathcal{A}$-multiplicity is*** m if it is at least m but not at least $m + 1$.

Remark 7.6. Observe that we can express, in first-order logic, that sets are closed (or not) in a uniform way. For if $\gamma_n(x_1, \ldots, x_n)$ denotes the formula

$$\neg \exists x_{n+1} \Big(\theta_n(x_1, \ldots, x_n, x_{n+1}) \ \wedge \ \bigwedge_{i=1}^{n} x_i \neq x_{n+1} \Big),$$

then for every $\mathcal{M} \in \mathbf{K}$ and all $a_1, \ldots, a_n \in M$, $\mathcal{M} \models \gamma_n(a_1, \ldots, a_n)$ if and only if $\{a_1, \ldots, a_n\}$ is closed in $\mathcal{M}$. It follows that whenever $\mathcal{M}$ is represented and $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{M}$ are closed substructures of $\mathcal{M}$, then, for every $m \in \mathbb{N}$, there is a sentence φ_m such that for every represented $\mathcal{N}$, $\mathcal{N} \models \varphi_m$ if and only if the $\mathcal{B}/\mathcal{A}$-multiplicity of $\mathcal{N}$ is at least m.

Definition 7.7. For represented $\mathcal{M}$ and closed substructures $\mathcal{A} \subset \mathcal{B} \subseteq \mathcal{M}$, the ***$\mathcal{B}/\mathcal{A}$-extension axiom*** is the statement expressing that the $\mathcal{B}/\mathcal{A}$-multiplicity is at least 1. As noted in Remark 7.6, this statement is expressible with a first-order sentence.

Note that if the closure operator of (structures in) $\mathbf{K}$ is trivial, then the definitions of extension axioms and multiplicity coincide with those given earlier; so the earlier setting is a special case of the current setting.

Definition 7.8. Let $\mathbf{K}$ be a class of L-structures and let $(\mathcal{M}_n : n \in \mathbb{N})$ be a sequence of structures from $\mathbf{K}$.
(i) We say that the sequence $(\mathcal{M}_n : n \in \mathbb{N})$ is ***polynomially k-saturated*** if there are a sequence of numbers $(\lambda_n : n \in \mathbb{N})$ with $\lim_{n \to \infty} \lambda_n = \infty$ and a polynomial $P(x)$ such that for every $n \in \mathbb{N}$:

1. $\lambda_n \leq |M_n| \leq P(\lambda_n)$, and
2. whenever $\mathcal{N}$ is represented and $\mathcal{A} \subset \mathcal{B} \subseteq \mathcal{N}$ are closed (in $\mathcal{N}$) and $\dim_{\mathcal{N}}(A) + 1 = \dim_{\mathcal{N}}(B) \leq k$, then the $\mathcal{B}/\mathcal{A}$-multiplicity of $\mathcal{M}_n$ is at least λ_n.

(ii) We say that $\mathbf{K}$ is ***polynomially k-saturated*** if there are $\mathcal{M}_n \in \mathbf{K}$, for $n \in \mathbb{N}$, such that the sequence $(\mathcal{M}_n : n \in \mathbb{N})$ is ***polynomially k-saturated***.

Example 7.9. While it is possible to construct many different $\mathbf{K}$ which are polynomially k-saturated (by application of Theorem 7.31) the kind of pregeometries that are present in examples that the author can construct are rather limited. So let us look at examples of $\mathbf{K}$ which are polynomially k-saturated for every $k \in \mathbb{N}$ and which do not have any more structure than what is necessary for defining the pregeometry. The cases known are on the one hand the trivial pregeometry and on the other hand (possibly projective or affine variants of) linear spaces over a fixed, but arbitrary, finite field.

If L has empty vocabulary and $\mathcal{E}_n$ is the unique L-structure with universe $\{1, \ldots, n\}$ (with trivial closure operator), then it is straightforward to check that $(\mathcal{E}_n : n \in \mathbb{N})$ is polynomially k-saturated for every $k \in \mathbb{N}$.

Now suppose that $\mathcal{G}_n$ is a vector space with dimension n with universe $\{1, \ldots, p^n\}$ over a finite field F of order p. Let $\mathrm{cl}_{\mathcal{G}_n}$ be linear span. To view $\mathcal{G}_n$ as a first order structure we can let scalar multiplication be represented by unary function symbols (one for every element in F), vector addition by a binary function symbol, and let there be a constant symbol for the zero vector. Then $\{\mathcal{G}_n : n \in \mathbb{N}\}$ is a pregeometry in the sense of Definition 7.1. The proof of Lemma 3.5 in [14] shows that $(\mathcal{G}_n : n \in \mathbb{N})$ is polynomially k-saturated, for every $k \in \mathbb{N}$. In [14] it is explained how one can "transform" $\mathcal{G}_n$ into a first-order structure, which represents a projective space $\mathcal{P}_n$ or affine space $\mathcal{A}_n$ over F of dimension n. By the argument leading to Proposition 3.4 in [14] it follows that $(\mathcal{P}_n : n \in \mathbb{N})$ and $(\mathcal{A}_n : n \in \mathbb{N})$ are polynomially k-saturated, for every $k \in \mathbb{N}$.

There are other "linear geometries" (see [12]) which involve quadratic forms. These may be candidates for other polynomially k-saturated sequences of pregeometries; but for reasons explained in Problem 3.8 in [14], the author has not been able to prove or disprove it.

From now on we work within the following context, in addition to the assumptions already made (see Assumption 7.3).

Assumption 7.10. From now on we assume the following:

(1) $L_0 \subseteq L$ are first-order languages with vocabularies V_0 and V, respectively, such that $V - V_0$ is finite and relational.
(2) $\mathbf{G} = \{\mathcal{G}_n : n \in \mathbb{N}\}$ is a set of L_0-structures which is a pregeometry, in the sense of Definition 7.1. Moreover, assume that the formulas $\theta_n(x_1, \ldots, x_{n+1}) \in L_0$, for $n \in \mathbb{N}$, define the pregeometry in the sense of Definition 7.1.
(3) For $n \in \mathbb{N}$, $\mathbf{K}_n = \mathbf{K}(\mathcal{G}_n)$ is a set of expansions to L of $\mathcal{G}_n$, and $\mathbf{K} = \bigcup_{n \in \mathbb{N}} \mathbf{K}_n$. For each $\mathcal{A} \in \mathbf{K}$, $\mathrm{cl}_{\mathcal{A}}$ is, by definition, the same as $\mathrm{cl}_{\mathcal{A} \restriction L_0}$, where the latter is the same as $\mathrm{cl}_{\mathcal{G}_n}$ for some n, because $\mathcal{A} \restriction L_0 = \mathcal{G}_n$ for some n.
(4) Whenever $\mathcal{M}$ is represented and $\mathcal{A} \subseteq_{cl} \mathcal{M}$, then $\mathcal{A}$ is represented.

Remark 7.11. (i) Note that point (4) in Assumption 7.10 says that the class of represented structures is closed under closed substructures.
(ii) If the closure is trivial, then (4) is equivalent to the hereditary property (for $\mathbf{K}$).
(iii) Analogues of the main theorems of this section can be stated and proved without assumption (4), but then, to get such results, the notion of 'acceptance of substitutions' (Definition 7.20) must be modified, and becomes more complicated. The author opted, in this case, for simplicity rather than some more generality.

Definition 7.12. Let $\mathcal{A} \in \mathbf{K}$ and let d be a natural number.
(i) The ***d-dimensional reduct*** of $\mathcal{A}$, denoted $\mathcal{A} \restriction d$, is the weak substructure of $\mathcal{A}$ which is defined as follows:

(a) $\mathcal{A} \restriction d$ has the same universe as $\mathcal{A}$.
(b) Every symbol in the vocabulary of L_0 is interpreted in the same way in $\mathcal{A} \restriction d$ as in $\mathcal{A}$.
(c) For every relation symbol R which belongs to the vocabulary of L but *not* to the vocabulary of L_0, and for every tuple $\bar{a}$ from the universe of $\mathcal{A}$,

$$\bar{a} \in R^{\mathcal{A} \restriction d} \iff \dim_{\mathcal{A}}(\bar{a}) \leq d \text{ and } \bar{a} \in R^{\mathcal{A}}.$$

(ii) $\mathbf{K} \restriction d = \{\mathcal{A} \restriction d : \mathcal{A} \in \mathbf{K}\}$.
(iii) $\mathbf{K}_n \restriction d = \{\mathcal{A} \restriction d : \mathcal{A} \in \mathbf{K}_n\}$.

Remark 7.13. (i) Observe that if there is no relation symbol whose arity is greater than d, then for every $\mathcal{A} \in \mathbf{K}$, $\mathcal{A} \restriction d = \mathcal{A}$; hence $\mathbf{K} \restriction d = \mathbf{K}$ and $\mathbf{K}_n \restriction d = \mathbf{K}_n$ for every n.
(ii) By Definition 7.12, for every $n \in \mathbb{N}$ and every positive $r \in \mathbb{N}$, the sequence $\mathbf{K}_n \restriction 0, \mathbf{K}_n \restriction 1, \ldots, \mathbf{K}_n \restriction r$ satisfies the conditions for $\mathbf{C}_0, \ldots, \mathbf{C}_r$ in Definition 6.2. Hence, for every $n \in \mathbb{N}$ and every positive $r \in \mathbb{N}$, the uniformly $(\mathbf{K}_n \restriction 0, \ldots, \mathbf{K}_n \restriction r-1)$-conditional measure is well-defined on $\mathbf{K}_n \restriction r$.

Remark 7.14. Note that for an L-structure $\mathcal{M} \in \mathbf{K}$ we have different kinds of "reducts", and the same symbol '$\restriction$' is used in all contexts, but the symbol following '$\restriction$' is a key, besides the context, to what is meant. For a sublanguage $L' \subseteq L$, $\mathcal{M} \restriction L'$ is the reduct of $\mathcal{M}$ to L' in the usual "language wise" sense. For a subset $X \subseteq M$, $\mathcal{M} \restriction X$ denotes the substructure of $\mathcal{M}$ which is generated by X. And for a natural number d, $\mathcal{M} \restriction d$ denotes the d-dimensional reduct of $\mathcal{M}$, which is a *weak* substructure of $\mathcal{M}$, but not necessarily a substructure.

Definition 7.15. (i) Let ρ be equal to the largest arity of a relation symbol in the vocabulary of L. Note that if $r \geq \rho$ then for every $\mathcal{A} \in \mathbf{K}$, $\mathcal{A} \restriction r = \mathcal{A}$; hence $\mathbf{K} \restriction r = \mathbf{K}$ and $\mathbf{K}_n \restriction r = \mathbf{K}_n$ for every n.
(ii) For every $n \in \mathbb{N}$, let $\mathbb{P}_{n,0}$ denote the uniform probability measure on $\mathbf{K}_n \restriction 0$. For every $n \in \mathbb{N}$ and every positive $r \in \mathbb{N}$, let $\mathbb{P}_{n,r}$ denote the uniformly $(\mathbf{K}_n \restriction 0, \ldots, \mathbf{K}_n \restriction r-1)$-conditional measure on $\mathbf{K}_n \restriction r$.
(iii) The uniformly $(\mathbf{K}_n \restriction 0, \ldots, \mathbf{K}_n \restriction \rho - 1)$-conditional measure $\mathbb{P}_{n,\rho}$ on $\mathbf{K}_n = \mathbf{K}_n \restriction \rho$ is also denoted by δ_n and called the ***dimension conditional measure*** on $\mathbf{K}_n$.

Example 7.16. Suppose that L and $\mathbf{K}_n$ are defined as in any of Examples 4.3 – 4.5, let L_0 be the language with empty vocabulary, and let the underlying pregeometry be trivial. If L_0 is defined as in the corresponding example, then the dimension conditional measure on $\mathbf{K}_n$ is, by definition, the same as the uniformly $(\mathbf{K}_n \restriction 0, \mathbf{K}_n \restriction 1)$-conditional measure on $\mathbf{K}_n$, which in turn is identical to the uniformly $(\mathbf{K}_n \restriction L_0)$-conditional measure, considered in the mentioned examples; this follows straightforwardly from the definitions. Examples with nontrivial underlying pregeometry will appear later.

Definition 7.17. We say that the pregeometry $\mathbf{G} = \{\mathcal{G}_n : n \in \mathbb{N}\}$ is ***uniformly bounded*** if there is a function $u : \mathbb{N} \to \mathbb{N}$ such that for every $n \in \mathbb{N}$ and every $X \subseteq |\mathcal{G}_n|$, $\big|\mathrm{cl}_{\mathcal{G}_n}(X)\big| \leq u\big(\dim_{\mathcal{G}_n}(X)\big)$.

Remark 7.18. The trivial pregeometries and the pregeometries obtained from vector spaces over finite fields are uniformly bounded. More examples of uniformly bounded pregeometries can be obtained by applying the variants of the amalgamation construction first developed by E. Hrushovski which produce countably categorical supersimple limit structures with rank 1 [24, 16]. However, the cases of such constructions known to the author do *not* produce pregeometries which are polynomially k-saturated for all k; this can be seen by considering the arguments in Section 2 of [13]. The author does not know an example of a pregeometry (in the sense of this paper) $\mathbf{G} = \{\mathcal{G}_n : n \in \mathbb{N}\}$ which is not uniformly bounded and such that each $\mathcal{G}_n$ is *finite*, as we always assume here.

Terminology 7.19. When saying that two represented structures $\mathcal{A}$ and $\mathcal{A}'$ ***agree on L_0 and on closed proper substructures*** we mean that $\mathcal{A} \restriction L_0 = \mathcal{A}' \restriction L_0$ (so in particular, $\mathrm{cl}_{\mathcal{A}} = \mathrm{cl}_{\mathcal{A}'}$) and whenever $\mathcal{U} \subseteq_{cl} \mathcal{A}$ and $\dim_{\mathcal{A}}(U) < \dim_{\mathcal{A}}(A)$, then $\mathcal{A} \restriction U = \mathcal{A}' \restriction U$.

The next definition generalizes the notion of 'admitting substitutions' from Section 3 to the context of this section.

Definition 7.20. Let $\mathcal{A}$ and $\mathcal{A}'$ be represented structures. Note that, in part (i) and (ii) of this definition, the property defined can only hold if $\mathcal{A}$ and $\mathcal{A}'$ agree on L_0 and on closed proper substructures; so that is the situation which is of interest.

(i) We say that $\mathbf{K}$ ***accepts the substitution*** $[\mathcal{A} \triangleright \mathcal{A}']$ ***over*** L_0 if whenever $\mathcal{M}$ is represented and $\mathcal{A} \subseteq_{cl} \mathcal{M}$, then there is a represented $\mathcal{N}$ such that $\mathcal{N} \restriction L_0 = \mathcal{M} \restriction L_0$, $\mathcal{N} \restriction |\mathcal{A}| = \mathcal{A}'$ and if $\mathcal{U} \subseteq_{cl} \mathcal{N}$, $\dim_{\mathcal{N}}(U) \leq \dim_{\mathcal{N}}(A')$ and $\mathcal{U} \neq \mathcal{A}'$, then $\mathcal{N} \restriction U = \mathcal{M} \restriction U$.
(ii) We say that $\mathbf{K}$ ***accepts*** k***-substitutions over*** L_0 if whenever $\mathcal{A}$ and $\mathcal{A}'$ are represented structures which agree on L_0 and on closed proper substructures, and $\dim_{\mathcal{A}}(A) = \dim_{\mathcal{A}'}(A') \leq k$, then $\mathbf{K}$ accepts the substitution $[\mathcal{A} \triangleright \mathcal{A}']$ over L_0.

Remark 7.21. (i) It is easy to see the following: If there is, up to isomorphism, a unique represented structure with dimension 0, then $\mathbf{K}$ accepts 0-substitutions over L_0.
(ii) Let ρ be the supremum of the arities of all relation symbols that belong to the vocabulary of L but *not* to the vocabulary of L_0. It is straightforward to verify that if $\mathbf{K}$ accepts ρ-substitutions over L_0, then, for every $k \in \mathbb{N}$, $\mathbf{K}$ accepts k-substitutions over L_0.

We now give examples of $\mathbf{K}$ which accept k-substitutions for all $k \in \mathbb{N}$. After Theorem 7.34, which is about $\mathbf{K}$ which do not satisfy this condition, we give more examples, which, for some k, do not not satisfy k-substitutions.

Example 7.22. (**Coloured structures.**) For the sake of having a uniform terminology in this example, and the next, let us have the following convention. For $F = \{1\}$ let L_F be the language with empty vocabulary V_F and let $\mathbf{G}^F = \{\mathcal{G}_n : n \in \mathbb{N}\}$, where $\mathcal{G}_n$ is the unique L_F-structure with universe $\{1, \ldots, n\}$. In this case call $\mathbf{G}^F$ the *vector space pregeometry over* $\{1\}$.

For any finite field F, the *vector space pregeometry over* F refers to the pregeometry $\mathbf{G}^F = \{\mathcal{G}_n : n \in \mathbb{N}\}$ defined in Example 7.9; so $\mathcal{G}_n$ is a vector space over F of dimension n, and L_F and V_F is the language and vocabulary, respectively, of $\mathcal{G}_n$.

Let $\mathbf{G}^F = \{\mathcal{G}_n : n \in \mathbb{N}\}$ be the vector space pregeometry over F, where F is a finite field or $\{1\}$. Then let $l \geq 2$ and assume that $L_{col} \supset L_F$, "the colour language" is the language with vocabulary $V_{col} = V_F \cup \{P_1, \ldots, P_l\}$ where all P_i are unary relation symbols, representing *colours*. Also assume that $L_{rel} \supset L_F$, "the language of relations", has a vocabulary V_{rel} such that $V_{rel} - V_F$ contains only finitely many relation symbols, of any arity. Let L be the language with vocabulary $V = V_{col} \cup V_{rel}$. For every positive $n \in \mathbb{N}$ define $\mathbf{K}_n = \mathbf{K}(\mathcal{G}_n)$ to be set of expansions $\mathcal{M}$ of $\mathcal{G}_n$ to L that satisfy the following three *l-colouring conditions*:

(1) $\mathcal{M} \models \forall x\big(P_1(x) \vee \ldots \vee P_l(x)\big)$.
(2) For all distinct $i, j \in \{1, \ldots, l\}$, and all $a, b \in M - \mathrm{cl}_{\mathcal{M}}(\emptyset)$ such that $a \in \mathrm{cl}_{\mathcal{M}}(b)$, $\mathcal{M} \models \neg\big(P_i(a) \wedge P_j(b)\big)$. (In other words: any two linearly dependent non-zero elements must have the same colour.)
(3) If $R \in V_{rel}$ has arity $m \geq 2$ and $\mathcal{M} \models R(a_1, \ldots, a_m)$, then there are $b, c \in \mathrm{cl}_{\mathcal{M}}(a_1, \ldots, a_m)$ such that for every $k \in \{1, \ldots, l\}$, $\mathcal{M} \models \neg\big(P_k(b) \wedge P_k(c)\big)$; that is, at least two elements in $\mathrm{cl}_{\mathcal{M}}(a_1, \ldots, a_m)$ have different colours.

It is now straightforward to verify that, for every F considered, $\mathbf{K} = \bigcup_{n \in \mathbb{N}} \mathbf{K}_n$ accepts k-substitutions over L_F, for every $k \in \mathbb{N}$. And as mentioned in Example 7.9, $(\mathcal{G}_n : n \in \mathbb{N})$ is polynomially k-saturated for every $k \in \mathbb{N}$. It is also uniformly bounded. Thus, with this setup of $(\mathcal{G}_n : n \in \mathbb{N})$ and $\mathbf{K}$ the premises of Theorems 7.31 and 7.32 (below) are satisfied. This example and the next will be studied more in Sections 9 and 10.

Example 7.23. (**Strongly coloured structures.**) The colourings considered in the previous example are the convention within hypergraph theory [7, 26], but we would also like to consider another sort of colourings, called *strong colourings* in the hypergraph context [1], and we adopt the same terminology. Here, $\mathbf{G}^F$, L_F, L_{col}, L_{rel} and L are defined as in Example 7.22. Let $\mathbf{K} = \bigcup_{n \in \mathbb{N}} \mathbf{K}_n$, where $\mathbf{K}_n$ consists of those L-expansions $\mathcal{M}$ of $\mathcal{G}_n$ which satisfy (1) and (2) from the previous example and the following *strong l-colouring condition*:

(3') If $R \in V_{rel}$ has arity $m \geq 2$, $\mathcal{M} \models R(a_1, \ldots, a_m)$, $b, c \in \mathrm{cl}_{\mathcal{M}}(a_1, \ldots, a_m)$ and b is independent from c (i.e. $b \notin \mathrm{cl}_{\mathcal{M}}(c)$), then for every $k \in \{1, \ldots, l\}$, $\mathcal{M} \models \neg\big(P_k(b) \wedge P_k(c)\big)$; that is, *every* pair of mutually independent elements b and c in the closure of $a_1, \ldots, a_m$ have different colours.

Again, it is straightforward to verify that, for every F considered, $\mathbf{K}$ accepts k-substitutions over L_F, for every $k \in \mathbb{N}$.

Example 7.24. (**Other variations of coloured structures**) In the previous two examples, it is also possible to consider projective or affine spaces over a finite field, instead of a vector space. And, by dropping condition (1), one can consider partial colorings or strong partial colourings. For all these variations, $\mathbf{K}$ accepts k-substitutions for every $k \in \mathbb{N}$.

Example 7.25. (**Random relations on a vector space**) Let F be a finite field and let L_F and $\mathbf{G}^F = \{\mathcal{G}_n : n \in \mathbb{N}\}$ be as in Example 7.9. Let L be the language whose vocabulary consists of the symbols in the vocabulary of L_F and, in addition, relation symbols $R_1, \ldots, R_\rho$, of any arity. For every n, let $\mathbf{K}_n = \mathbf{K}(\mathcal{G}_n)$ be the set of all L-structures $\mathcal{M}$ such that $\mathcal{M}\restriction L_0 = \mathcal{G}_n$. It is straightforward to verify that, for every $k \in \mathbb{N}$, $\mathbf{K} = \bigcup_{n\in\mathbb{N}} \mathbf{K}_n$ accepts k-substitutions over L_F. A similar example can be constructed over projective or affine spaces over F.

Remark 7.26. (**An algebraic approach to adding "pseudo-random" edges**) Here we sketch an algebraic approach to expanding F-vector spaces by a binary irreflexive symmetric relation. The graph structure itself will, in the limit, be the same as the one obtained in the previous example when only one relation symbol $R_1 = R$ is considered and always interpreted as an irreflexive and symmetric relation. But in the algebraic approach it is not sufficiently clear to the author how the vector space structure interacts with the graph structure and therefore the question whether $\mathbf{K}$ defined below accepts 2-substitutions over the vector space language is left open.

Let $F = \mathbb{F}_p$ be the finite field of order p, where p is a prime which is congruent to 1 modulo 4. As in the previous example, let L_F be as in Example 7.9. Every field of order p^n, denoted $\mathbb{F}_{p^n}$, can be viewed as a vector space over $F = \mathbb{F}_p$, and this vector space (of dimension n), formalised as an L_F-structure, is denoted $\mathcal{V}_n$. Let the vocabulary of the "graph language", L_g, contain only one binary relation symbol R, let $\mathbf{K}^g_n$ be the set of undirected graphs (as L_g-structures) with vertices $1, \ldots, n$ and let $\mathbf{K}^g = \bigcup_{n\in\mathbb{N}} \mathbf{K}^g_n$. Then let L be the language whose vocabulary is the union of the vocabularies of L_F and L_g. Every $\mathcal{V}_n$ can be expanded to an L-structure, denoted $\mathcal{V}^g_n$, so that $\mathcal{V}^g_n\restriction L_g$ is an undirected graph, by letting $\mathcal{V}^g_n \models R(a, b)$ if and only if $a - b$ is a square in the field $\mathbb{F}_{p^n}$; so $\mathcal{V}^g_n\restriction L_g$ is a Paley graph. By results about Paley graphs (see Chapter 13 of [8]) it follows that, for every extension axiom φ of $\mathbf{K}^g$, φ is true in $\mathcal{V}^g_n$ for all sufficiently large n. By compactness there is an infinite L-structure $\mathcal{V}$ such that $\mathcal{V}\restriction L_F$ is a vector space over F and $\mathcal{V}\restriction L_g$ is an undirected graph which satisfies every extension axiom of $\mathbf{K}^g$. Now we can let $\mathcal{G}_n$ be an n-dimensional vector space over F, viewed as an L_F-structure, and let $\mathbf{K}_n = \mathbf{K}(\mathcal{G}_n)$ be the set of expansions, $\mathcal{M}$, to L of $\mathcal{G}_n$ such that $\mathcal{M}$ is isomorphic with some substructure of $\mathcal{V}$. We may now ask whether it is true that, for every k, $\mathbf{K} = \bigcup_{n\in\mathbb{N}} \mathbf{K}_n$ accepts k-substitutions over L_F and/or is polynomially k-saturated. Since $\mathcal{V}\restriction L_g$ satisfies every extension axiom of $\mathbf{K}^g$ (and possibly using more information about Paley graphs) one may be tempted to guess that the answers are yes in both cases. However, when dealing with the question of whether $\mathbf{K}$ accepts 2-substitutions over L_F we need to understand (it seems) what graphs can appear as $\mathcal{H} = \mathcal{M}\restriction L_g$ where $\mathcal{M}$ is a substructure of $\mathcal{V}$, so in particular, M is a linearly closed subset of V. This seems to involve deeper understanding of the interaction between the vector space structure of $\mathcal{V}_n$ and the multiplicative structure of $\mathbb{F}_{p^n}$ for all sufficiently large $n \in \mathbb{N}$.

Example 7.27. (**Hypergraph and random graph on a vector space**) Let F be a finite field and let L_F and $\mathbf{G}^F = \{\mathcal{G}_n : n \in \mathbb{N}\}$ be as in Example 7.9. Let L be the language whose vocabulary consists of the symbols in the vocabulary of L_F and, in addition, relation symbols E and R where E is binary and R is ternary. For every $n \in \mathbb{N}$, let $\mathbf{K}_n = \mathbf{K}(\mathcal{G}_n)$ be the set of L-structures $\mathcal{M}$ such that $\mathcal{M} \restriction L_F = \mathcal{G}_n$, E is interpreted as an irreflexive and symmetric relation, so we call E-relationships edges, and, for all $a, b, c \in M$, $(a, b, c) \in R^{\mathcal{M}}$ if and only if the subspace spanned by a, b and c contains an odd number of edges. We show that $\mathbf{K} = \bigcup_{n \in \mathbb{N}} \mathbf{K}_n$ accepts k-substitutions for all $k \in \mathbb{N}$. As was mentioned in Remark 7.21, it suffices to show that $\mathbf{K}$ accepts 3-substitutions. Let L' be the sublanguage of L in which the symbol R has been removed, but all other symbols have been kept. Observe that, for every $\mathcal{M} \in \mathbf{K}$ and for all $a, b, c \in M$, whether $\mathcal{M} \models R(a, b, c)$, or not, is determined by the substructure of $\mathcal{M} \restriction L'$ whose universe is the linear span of a, b and c. This implies that it suffices to show that $\mathbf{K}$ accepts 2-substitutions. Since the only restrictions on E is that it is interpreted as an irreflexive and symmetric relation, it follows that in whichever way we expand $\mathcal{G}_n$ with edges, we get $\mathcal{M} \restriction L'$ for some $\mathcal{M} \in \mathbf{K}_n$. This implies that $\mathbf{K}$ accepts 2-substitutions.

Suppose that L^* is the sublanguage of L where the symbol E has been removed, but all other symbols have been kept, and let $\mathbf{K}^* = \bigcup_{n \in \mathbb{N}} \mathbf{K}_n^*$, where $\mathbf{K}_n^* = \{\mathcal{M} \restriction L^* : \mathcal{M} \in \mathbf{K}_n\}$. It is, when writing this, not clear to the author if $\mathbf{K}^*$ accepts 3-substitutions, or not.

Example 7.28. In this example, pairs of elements as well as elements can be coloured and some restrictions are imposed. Suppose that, for every n, $\mathcal{G}_n$ is a projective space over the 2-element field and let L_0 be the language of $\mathcal{G}_n$. Let $L \supset L_0$ contain, besides the symbols of L_0, three unary relation symbols P_1, P_2, P_3, three binary relation symbols R_1, R_2, R_3 and one ternary relation symbol S. We can think of the P_i as colours of elements, and the R_i as colours of pairs. For every n, $\mathbf{K}_n = \mathbf{K}(\mathcal{G}_n)$ consists of all expansions $\mathcal{M}$ of $\mathcal{G}_n$ to L which satisfy the following conditions:

(a) For every 2-dimensional subspace $X \subseteq M$, if no pair $(a, b) \in X^2$ is coloured, then at least one point in X is coloured.

(b) For every two dimensional subspace $X \subseteq M$, if some pair $(a, b) \in X^2$ is coloured, then there are *not* two different points in X with the same colour (but two different points may be uncoloured).

(c) If $\mathcal{M} \models S(a, b, c)$, then $\{a, b, c\}$ is independent and if $(d_1, d_2), (e_1, e_2) \in \mathrm{cl}_{\mathcal{M}}(a, b, c)$, then (d_1, d_2) and (e_1, e_2) do not have the same colour (but both may be uncoloured).

We show that $\mathbf{K}$ accepts 3-substitutions over L_0. Since no relation symbol has arity greater than 3 it follows (see Remark 7.21) that $\mathbf{K}$ accepts k-substitutions over L_0 for every $k \in \mathbb{N}$.

Let $\mathcal{A}, \mathcal{A}'$ be represented and assume that $\mathcal{A} \restriction L_0 = \mathcal{A}' \restriction L_0$ and that $\mathcal{A}$ and $\mathcal{A}'$ agree on all closed proper substructures. We must show that if $\mathcal{M}$ is represented and $\mathcal{A} \subseteq_{cl} \mathcal{M}$, then there exists a represented $\mathcal{N}$ such that $\mathcal{N} \restriction L_0 = \mathcal{M} \restriction L_0$, $\mathcal{N} \restriction A = \mathcal{A}'$ and whenever $\mathcal{U} \subseteq_{cl} \mathcal{N}$, $\dim_{\mathcal{N}}(U) \leq \dim_{\mathcal{N}}(A')$, and $U \neq A'$, then $\mathcal{N} \restriction U = \mathcal{M} \restriction U$.

First suppose that $\dim_{\mathcal{M}}(A) = 1$. Let $\mathcal{M}' = \mathcal{M}[\mathcal{A} \triangleright \mathcal{A}']$, according to Definition 3.11 (Since $\mathcal{A} \restriction L_0 = \mathcal{A}' \restriction L_0$, the substitution involves only interpretations of relation symbols). Then go through all $\mathcal{B} \subseteq_{cl} \mathcal{M}'$ of dimension 2; whenever we meet such $\mathcal{B}$ which is forbidden we can change some binary relationships (R_i, $i = 1, 2, 3$), but not change any unary relationships (P_i, $i = 1, 2, 3$), and thus get a permitted substructure. When this has been done for all 2-dimensional closed substructures, call the result $\mathcal{M}''$; so all 2-dimensional substructures of $\mathcal{M}''$ are permitted. Then we can just remove all S-relationships from $\mathcal{M}''$ so that in the resulting structure $\mathcal{N}$ the interpretation of S is empty. It now follows from the construction of $\mathcal{N}$ and (a) – (c) that $\mathcal{N}$ is represented. And whenever $U \subseteq N$ is 1-dimensional and different from A', then $\mathcal{N} \restriction U = \mathcal{M} \restriction U$.

Now suppose that $\dim_{\mathcal{M}}(A) = 2$. Let $\mathcal{M}' = \mathcal{M}[\mathcal{A} \triangleright \mathcal{A}']$. Then $\mathcal{M}'$ and $\mathcal{M}$ agree on all closed 1- or 2-dimensional subsets which are different from A'. By removing all S-relationships from $\mathcal{M}'$ we get $\mathcal{N}$ which is represented and such that $\mathcal{N}$ and $\mathcal{M}$ agree on all closed 1- or 2-dimensional subsets which are different from A'. Moreover, $\mathcal{N} \restriction A' = \mathcal{A}'$.

Finally, suppose that $\dim_{\mathcal{A}}(A) = 3$. Both $\mathcal{A}$ and $\mathcal{A}'$ satisfy (a) – (c) (because they are permitted) and $\mathcal{A}$ and $\mathcal{A}'$ agree, by assumption, on substructures of dimension 2. Hence $\mathcal{N} = \mathcal{M}[\mathcal{A} \triangleright \mathcal{A}']$ and $\mathcal{M}$ agree on subsets of dimension 2 and on closed subsets of dimension 3 which are different from A'. Since $\mathcal{A}'$ is represented, and hence satisfies (a) – (c), $\mathcal{N}$ is represented.

The next lemma tells that the notion of 'accepting k-substitutions over L_0' is indeed a generalization of the notion of 'admitting k-substitutions'.

Lemma 7.29. *Let L_0 be the language with empty vocabulary and let $\mathcal{G}_n$ be the unique L_0-structure with universe $\{1, \ldots, m_n\}$ (with the trivial pregeometry) where $\lim_{n\to\infty} m_n = \infty$. Let L be any language with finite relational vocabulary. Suppose that, for every n, $\mathbf{K}_n$ is a set of L-structures with universe $\{1, \ldots, m_n\}$; in other words, $\mathbf{K}_n = \mathbf{K}(\mathcal{G}_n)$ is a set of expansions of $\mathcal{G}_n$ to L; and let $\mathbf{K} = \bigcup_{n\in\mathbb{N}} \mathbf{K}_n$. For every $k \in \mathbb{N}$, if $\mathbf{K}$ admits k-substitutions (in the sense of Definition 3.12), then $\mathbf{K}$ accepts k-substitutions over L_0.*

Proof. One just checks that, under the assumptions, $\mathbf{K}$ does indeed accept k-substitutions over L_0, according to Definition 7.20. ☐

Recall Assumptions 7.10 and Definition 7.15 (iii).

Definition 7.30. For every $n \in \mathbb{N}$ and every L-sentence φ,

$$\text{let } \delta_n(\varphi) \text{ be an abbreviation for } \delta_n\big(\{\mathcal{M} \in \mathbf{K}_n : \mathcal{M} \models \varphi\}\big).$$

Theorem 7.31. *Let $k > 0$. Suppose that $(\mathcal{G}_n : n \in \mathbb{N})$ is uniformly bounded, polynomially k-saturated and that $\mathbf{K} = \bigcup_{n\in\mathbb{N}} \mathbf{K}(\mathcal{G}_n)$ accepts k-substitutions over L_0. Then:*

(i) *For every $(k-1)$-extension axiom φ of $\mathbf{K}$, $\lim_{n\to\infty} \delta_n(\varphi) = 1$.*
(ii) *$\mathbf{K}$ is polynomially k-saturated.*

Theorem 7.32. *Suppose that $(\mathcal{G}_n : n \in \mathbb{N})$ is uniformly bounded and polynomially k-saturated for every $k \in \mathbb{N}$. Also assume that $\mathbf{K} = \bigcup_{n\in\mathbb{N}} \mathbf{K}(\mathcal{G}_n)$ accepts k-substitutions over L_0 for every $k \in \mathbb{N}$. Then, for every L-sentence φ, either $\lim_{n\to\infty} \delta_n(\varphi) = 0$ or $\lim_{n\to\infty} \delta_n(\varphi) = 1$.*

For the last theorem of this section we need a definition.

Definition 7.33. We say that $\mathbf{K}$ has the ***independent amalgamation property*** if the following holds: Whenever $\mathcal{A}$, $\mathcal{B}_1$, $\mathcal{B}_2$ are represented, $\mathcal{A} \subseteq_{cl} \mathcal{B}_i$, for $i = 1, 2$, and $B_1 \cap B_2 = A$, then there is a represented $\mathcal{C}$ such that $\mathcal{B}_i \subseteq_{cl} \mathcal{C}$ for $i = 1, 2$.

Theorem 7.34. *Suppose that $(\mathcal{G}_n : n \in \mathbb{N})$ is uniformly bounded and polynomially k-saturated for every $k \in \mathbb{N}$. Assume that, up to isomorphism, there is a unique represented structure, with respect to $\mathbf{K} = \bigcup_{n\in\mathbb{N}} \mathbf{K}(\mathcal{G}_n)$, with dimension 0 (a particular case of this is when $\mathrm{cl}(\emptyset) = \emptyset$). Let $k \in \mathbb{N}$ be minimal such that $\mathbf{K}$ does not accept k-substitutions over L_0 and suppose that $\mathcal{A}$ and $\mathcal{A}'$ are represented structures (with respect to $\mathbf{K}$) such that $\mathcal{A}$ and $\mathcal{A}'$ have dimension k, agree on L_0 and on closed proper substructures, $\mathbf{K}$ accepts the substitution $[\mathcal{A}' \triangleright \mathcal{A}]$ over L_0, but does* not *accept the substitution $[\mathcal{A} \triangleright \mathcal{A}']$ over L_0. Then at least one of the following holds:*

(i) *$\mathbf{K}$ does not have the independent amalgamation property.*
(ii) *There are $\beta < 1$ and extension axioms φ and ψ such that for all sufficiently large n, $\delta_n(\varphi \wedge \psi) < \beta$. If $k > 1$, then $\lim_{n\to\infty} \delta_n(\varphi \wedge \psi) = 0$.*

Remark 7.35. The proof of Theorem 7.34 shows that if the assumptions of the theorem hold and one particular instance of the independent amalgamation property is satisfied, then case (ii) holds; more information about this instance of independent amalgamation and φ and ψ is given by the proof.

The proofs of Theorems 7.31 – 7.34 are given in the next section.

Example 7.36. (**Forbidden weak substructures**) We will prove a dichotomy, stated by the corollary below, which is analogous to Theorem 3.4, which was proved (using Theorem 3.17) in Example 4.1.

Let $\mathbf{G} = (\mathcal{G}_n : n \in \mathbb{N})$, where all $\mathcal{G}_n$ are L_0-structures, be a pregeometry which satisfies the assumptions of Theorems 7.31 – 7.34. We also assume that $\mathbf{G}$ has the independent amalgamation property (in the same sense as in Definition 7.33 if $\mathbf{K}$ is replaced by $\mathbf{G}$). These assumptions hold for $\mathbf{G} = \mathbf{G}^F$ as in Example 7.9 where the members of $\mathbf{G}^F$ are vector spaces over the finite field F, as well as for projective and affine versions of these spaces. Let L_{rel} be a language with relational vocabulary $\{R_1, \ldots, R_s\}$, and let L be the language whose vocabulary is the union of the vocabularies of L_0 and L_{rel}. Using Henson's terminology in [22], we say that an L_{rel}-structure $\mathcal{M}$ is *decomposable* if there are different L_{rel}-structures $\mathcal{A}$ and $\mathcal{B}$ such that $M = A \cup B$, $\mathcal{A}{\restriction}A \cap B = \mathcal{B}{\restriction}A \cap B$ and for every $i = 1, \ldots, s$, $(R_i)^{\mathcal{M}} = (R_i)^{\mathcal{A}} \cup (R_i)^{\mathcal{B}}$. Otherwise we call $\mathcal{M}$ *indecomposable*. Suppose that $\mathbf{F}$ is a set of finite indecomposable L_{rel}-structures such that if $\mathcal{A}, \mathcal{B} \in \mathbf{F}$ and $\mathcal{A} \neq \mathcal{B}$, then $\mathcal{A}$ is not weakly embeddable into $\mathcal{B}$. Let $\mathbf{K}_n = \mathbf{K}(\mathcal{G}_n)$ be the set of L-structures $\mathcal{M}$ such that $\mathcal{M}{\restriction}L_0 = \mathcal{G}_n$ and no $\mathcal{F} \in \mathbf{F}$ can be weakly embedded into $\mathcal{M}{\restriction}L_{rel}$, and let $\mathbf{K} = \bigcup_{n\in\mathbb{N}} \mathbf{K}_n$. Note that one of the assumptions on $\mathbf{F}$ implies that every $\mathcal{F} \in \mathbf{F}$ is *minimal* in the sense that if $\mathcal{F}'$ is a proper weak substructure of $\mathcal{F}$, then $\mathcal{F}'$ can be weakly embedded into $\mathcal{M}{\restriction}L_{rel}$ for some $\mathcal{M} \in \mathbf{K}$. From the indecomposability of the members of $\mathbf{F}$ it follows, in essentially the same way as the (straightforward) proofs of Lemma 1.1 and Theorem 1.2 (i) in [22], that $\mathbf{K}$ has the independent amalgamation property.

Consider the following statement:

($*$) There are $\mathcal{F} \in \mathbf{F}$, a relation symbol R_i and $\bar{a} \in (R_i)^{\mathcal{F}}$ such that $\mathrm{rng}(\bar{a})$ is a proper subset of F.

Corollary to Theorems 7.31 – 7.34. *(i) If ($*$) holds, then there are $\beta < 1$ and extension axioms φ and ψ of* $\mathbf{K}$ *such that for all sufficiently large n, $\delta_n(\varphi \wedge \psi) < \beta$, and if* $|\mathrm{rng}(\bar{a})| > 1$, *then* $\lim_{n\to\infty} \delta_n(\varphi \wedge \psi) = 0$.
(ii) If ($*$) does not hold, then, for every $k \in \mathbb{N}$, $\mathbf{K}$ *accepts k-substitutions and is polynomially k-saturated, for every extension axiom φ of* $\mathbf{K}$, $\lim_{n\to\infty} \delta_n(\varphi) = 1$, *and* $\mathbf{K}$ *has a zero-one law with respect to the probability measures δ_n.*

Proof. We first prove (ii), so suppose that ($*$) does not hold. We only need to prove that $\mathbf{K}$ accepts k-substitutions for every k, since the other claims then follow from Theorems 7.31 and 7.32. Let $\mathcal{A}$ and $\mathcal{A}'$ be represented structures, with respect to $\mathbf{K}$, that agree on L_0 and on closed proper substructures, and suppose that $\mathcal{A} \subseteq \mathcal{M} \in \mathbf{K}$. Moreover, suppose (for a contradiction) that $\mathcal{N} = \mathcal{M}[\mathcal{A} \triangleright \mathcal{A}']$ is forbidden, so there is $\mathcal{F} \subseteq_w \mathcal{N}{\restriction}L_{rel}$ such that $\mathcal{F}$ is isomorphic to some member of $\mathbf{F}$. We may, without loss of generality, assume that for any i, if any R_i-relationship is removed from $\mathcal{A}'$, giving $\mathcal{A}''$, then $\mathcal{M}[\mathcal{A} \triangleright \mathcal{A}'']$ is represented. Since $\mathcal{M}$ is represented, F must contain some element from $|\mathcal{A}'|$. Since $\mathcal{A}'$ is represenetd, $\mathcal{F}$ is not a weak substructure of $\mathcal{A}'{\restriction}L_{rel}$, so F must also contain some element in $|\mathcal{N}| - |\mathcal{A}'|$. As $\mathcal{F} \subseteq_w \mathcal{N}{\restriction}L_{rel} = \mathcal{M}[\mathcal{A} \triangleright \mathcal{A}']{\restriction}L_{rel}$ and $\mathcal{M} \in \mathbf{K}$, there is some i and R_i-relationship $\bar{a} \in (R_i)^{\mathcal{A}'} \subseteq (R_i)^{\mathcal{N}}$ such that $\mathrm{rng}(\bar{a}) \subseteq |\mathcal{F}|$. But then $\mathrm{rng}(\bar{a})$ is a proper subset of F, which contradicts the assumption that ($*$) is false (since we can, if necessary, remove some relationships whose range includes elements from $|\mathcal{F}| - |\mathcal{A}'|$, to "uncover" $\mathcal{F}$ weakly embedded into $\mathcal{N}\restriction L_{rel}$).

Now we prove (i), so suppose that $(*)$ holds. Let $\mathcal{F} \in \mathbf{F}$ and $\bar{a} \in (R_i)^{\mathcal{F}}$ be such that $\mathrm{rng}(\bar{a})$ is a proper subset of F and such that the removal of the R_i-relationship $\bar{a}$ produces a structure $\mathcal{P}$ which is (weakly) embeddable into $\mathcal{N}\restriction L_{rel}$ for some $\mathcal{N} \in \mathbf{K}$. Let $d = |P|$, let $v_1, \ldots, v_d$ be a basis of $\mathcal{G}_d$, and let $f : P \to \{v_1, \ldots, v_d\}$ be a bijection. Then let $\mathcal{M}$ be the L-structure which is obtained by expanding $\mathcal{G}_d$ in such a way that $f : P \to \mathcal{M}\restriction L_{rel}$ becomes an embedding and if $\bar{b}$ contains an element not in $\{v_1, \ldots, v_d\}$, then $\bar{b}$ is not a R_j-relationship for any j. Then $\mathcal{M} \in \mathbf{K}$. To simplify notation, we may assume that $F = P = \{v_1, \ldots, v_d\}$, so $\mathcal{P} \subseteq \mathcal{M}$. Let $\mathcal{A} = \mathcal{M}\restriction \mathrm{cl}_{\mathcal{M}}(\bar{a})$ and let $\mathcal{A}'$ be the structure obtained from $\mathcal{A}$ by adding the R_i-relationssship $\bar{a}$, but making no other changes. Then the L_{rel}-reduct of $\mathcal{M}[\mathcal{A} \triangleright \mathcal{A}']$ contains a copy of $\mathcal{F}$, so it is forbidden, and hence $\mathbf{K}$ does not accept the substitution $[\mathcal{A} \triangleright \mathcal{A}']$ over L_0. But $\mathbf{K}$ accepts the substitution $[\mathcal{A}' \triangleright \mathcal{A}]$ over L_0, because its effect is only to remove a relationship and this can never create a forbidden structure.

As mentioned before the corollary, $\mathbf{K}$ has the independent amalgamation property, so by Theorem 7.34, there are $\beta < 1$ and extension axioms φ and ψ of $\mathbf{K}$ such that for all sufficiently large n, $\delta_n(\varphi \wedge \psi) < \beta$. Moreover, if $|\mathrm{rng}(\bar{a})| > 1$ then, as $\mathrm{rng}(\bar{a}) \subseteq P$ and $P = \{v_1, \ldots, v_d\}$ is a basis of $\mathcal{M}$, it follows that $\dim(\mathcal{A}) = \dim_{\mathcal{M}}(\bar{a}) > 1$, and hence (by Theorem 7.34) $\lim_{n\to\infty} \delta_n(\varphi \wedge \psi) = 0$. □

Example 7.37. (**l-Colourable structures, and strongly l-colourable structures**) Let F, $\mathbf{K}_n$, L_{rel} and L be as in Example 7.22 (or as in Example 7.23) and let $\mathbf{C}_n = \{\mathcal{M}\restriction L_{rel} : \mathcal{M} \in \mathbf{K}_n\}$ and $\mathbf{C} = \bigcup_{n\in\mathbb{N}} \mathbf{C}_n$. Suppose that all relation symbols of L_{rel} have arity at least 2 and let R be one which has minimal arity, which we denote by k. Assume that $l \geq k$ (or $l \geq$ maximal arity if we consider *strongly* l-colourable structures). Since one can not add arbitrarily many new R-relationships to a sufficiently large independent subset of a structure $\mathcal{M} \in \mathbf{C}$ without finally getting forbidden structure, i.e. one that can not be (strongly) l-coloured, one can show (but we omit the details) that $\mathbf{C}$ does not accept k-substitutions over L_F. On the other hand, we can always remove an R-relationship from a represented structure without producing a forbidden one. It follows that there are represented structures $\mathcal{A}$ and $\mathcal{A}'$ with dimension k which agree on L_F and on closed proper substructures, the substitution $[\mathcal{A}' \triangleright \mathcal{A}]$ over L_F is accepted, but not the substitution $[\mathcal{A} \triangleright \mathcal{A}']$ over L_F. It follows from Theorem 7.34 and since $k > 1$ that either $\mathbf{C}$ does not have the independent amalgamation property *or* that there are extension axioms φ and ψ of $\mathbf{C}$ such that $\lim_{n\to\infty} \delta_n(\varphi \wedge \psi) = 0$, where δ_n is the dimension conditional measure on $\mathbf{C}_n$.

If the only symbol of the vocabulary of L_{rel} which does not belong to the vocabulary of L_F is a binary relation symbol R and $l = 2$, then, by considering a 5-cycle (which cannot be 2-coloured), it is easy to see that $\mathbf{C}$ does not have the independent amalgamation property, since that would force a 5-cycle into some member of $\mathbf{C}$. It is also straightforward to see, by considering 5-cycles and 3-cycles, that if an L_{rel}-structure $\mathcal{M}$ satisfies all 3-extension axioms of $\mathbf{C}$, then it is not 2-colourable. In Sections 9–10 we will see that, nevertheless, for $F = \{1\}$, i.e. the trivial underlying pregeometry, and any L_{rel} as in the beginning of the example, $\mathbf{C}$ has a zero-one law for δ_n, as well as for the uniform probability measure. (The corresponding statement for a finite field F, giving a nontrivial underlying pregeometry, remains open.)

8. Proofs of Theorems 7.31, 7.32 and 7.34

Remember that Theorems 7.31 – 7.34 take place within the setting of Assumptions 7.3 and 7.10. Therefore Assumptions 7.3 and 7.10 are active throughout this section.

8.1. **Proof of Theorem 7.31.** We are assuming that $\mathbf{G} = \{\mathcal{G}_n : n \in \mathbb{N}\}$ is a set of L_0-structures and that $\mathbf{G}$ is a pregeometry. Let $k > 0$. Suppose that $(\mathcal{G}_n : n \in \mathbb{N})$ is polynomially k-saturated and that $\mathbf{K} = \bigcup_{n\in\mathbb{N}} \mathbf{K}_n$, where $\mathbf{K}_n = \mathbf{K}(\mathcal{G}_n)$, accepts k-substitutions over L_0. This means that there exists a sequence of numbers $(\lambda_n : n \in \mathbb{N})$ such that $\lim_{n\to\infty} \lambda_n = \infty$ and a polynomial $P(x)$ such that for every $n \in \mathbb{N}$:

(a) $\lambda_n \leq |G_n| \leq P(\lambda_n)$, and
(b) whenever $\mathcal{A}$ and $\mathcal{B}$ are represented, $\mathcal{A} \subset_{cl} \mathcal{B}$ and $\dim_{\mathcal{B}}(A) + 1 = \dim_{\mathcal{B}}(B) \leq k$, then the $\mathcal{B}/\mathcal{A}$-multiplicity of $\mathcal{G}_n$ is at least λ_n.

We must prove the following:

(i) For every $(k-1)$-extension axiom φ of $\mathbf{K}$, $\lim_{n\to\infty} \delta_n(\varphi) = 1$.
(ii) $\mathbf{K}$ polynomially k-saturated.

Part (i) will be reduced to the problem of proving that the δ_n-probability that $\mathcal{M} \in \mathbf{K}_n$ is sufficiently saturated, in the sense of Definition 8.1 below, tends to 1 as n tends to infinity.

Recall, from Definition 7.15 (i), that ρ is the supremum of the arities of all relation symbols that belong to the vocabulary of L, but not to the vocabulary of L_0. From Assumptions 7.3 and 7.10, Definition 7.12 and Remark 7.13 it follows that whenever $d, n \in \mathbb{N}$ and $\mathcal{M} \in \mathbf{K}_n{\restriction}d$, then $\mathrm{cl}_{\mathcal{M}}$ coincides with $\mathrm{cl}_{\mathcal{G}_n}$ which is the same as $\mathrm{cl}_{\mathcal{M}\restriction L_0}$ since $\mathcal{M}{\restriction}L_0 = \mathcal{G}_n$. Also, if $d \geq \rho$, then for every $\mathcal{M} \in \mathbf{K}$, $\mathcal{M}{\restriction}d = \mathcal{M}$.

In this proof, and the proofs of Theorems 7.32 and 7.34, we often work with $\mathbf{K} \restriction d$, for some $d \in \mathbb{N}$, and consider structures which are represented, permitted, or forbidden, *with respect to* $\mathbf{K}{\restriction}d$. Recall, from Definition 7.15 (iii), that δ_n is an abbreviation for $\mathbb{P}_{n,\rho}$. Essentially, the next definition just repeats point (2) from Definition 7.8 in the case of $\mathbf{K}{\restriction}d$ (instead of $\mathbf{K}$), but it will be convenient to use the terminology defined below.

Definition 8.1. (i) Let $d, m \in \mathbb{N}$ and $\mathcal{M} \in \mathbf{K}{\restriction}d$. We say that $\mathcal{M}$ is (m,k)***-saturated with respect to*** $\mathbf{K}{\restriction}d$ if the following holds:

> Whenever $\mathcal{A}$ and $\mathcal{B}$ are represented *with respect to* $\mathbf{K}{\restriction}d$, $\mathcal{A} \subset_{cl} \mathcal{B}$ and $\dim_{\mathcal{B}}(A) + 1 = \dim_{\mathcal{B}}(B) \leq k$, then the $\mathcal{B}/\mathcal{A}$-multiplicity of $\mathcal{M}$ is at least m.

(i) Since $\mathcal{M}{\restriction}\rho = \mathcal{M}$ for every $\mathcal{M} \in \mathbf{K}$, we say that $\mathcal{M} \in \mathbf{K}$ is (m,k)***-saturated with respect to*** $\mathbf{K}$ if $\mathcal{M}$ is (m,k)-saturated with respect to $\mathbf{K}{\restriction}\rho$.

Definition 8.2. For $r \in \mathbb{N}$ we inductively we define functions $\sigma^r : \mathbb{N} \to \mathbb{N}$. Let $\sigma^0(x) = x$ for all $x \in \mathbb{N}$. Let $\sigma^{r+1}(x) = \lfloor \sqrt{\sigma^r(x)} \rfloor$ for all $x \in \mathbb{N}$.

Note that for every $r \in \mathbb{N}$, $\lim_{n\to\infty} \sigma^r(n) = \infty$. By assumption, $\lim_{n\to\infty} \lambda_n = \infty$, so for every $r \in \mathbb{N}$, $\lim_{n\to\infty} \sigma^r(\lambda_n) = \infty$; this will be used later.

Let φ be a $(k-1)$-extension axiom. In order to prove (i) we need to show that

$$\lim_{n\to\infty} \delta_n\big(\{\mathcal{M} \in \mathbf{K}_n : \mathcal{M} \models \varphi\}\big) = 1. \tag{1}$$

By assumption, φ is the $\mathcal{B}/\mathcal{A}$-extension axiom for some $\mathcal{A} \subset \mathcal{B} \subseteq \mathcal{M}$ such that $\mathcal{M}$ is represented with respect to $\mathbf{K} = \mathbf{K}{\restriction}\rho$, both A and B are closed in $\mathcal{M}$ and $\dim_{\mathcal{B}}(B) \leq k$; in particular $\dim_{\mathcal{B}}(A) < \dim_{\mathcal{B}}(B)$. Then, letting $l = \dim_{\mathcal{B}}(B) - \dim_{\mathcal{B}}(A)$, there are closed substructures $\mathcal{B}_0, \ldots, \mathcal{B}_l$ of $\mathcal{M}$ such that $\mathcal{A} = \mathcal{B}_0 \subset \mathcal{B}_1 \subset \ldots \subset \mathcal{B}_l = \mathcal{B}$ and $\dim_{\mathcal{B}}(B_i) + 1 = \dim_{\mathcal{B}}(B_{i+1})$ for $i = 0, \ldots, l-1$. By Assumption 7.10 (4), every $\mathcal{B}_i$ is represented. As noted above, $\lim_{n\to\infty} \sigma^k(\lambda_n) = \infty$. We now show that if $\mathcal{N}$ is represented with respect to $\mathbf{K}$ and $(\sigma^k(\lambda_n), k)$-saturated, then $\mathcal{N} \models \varphi$. Suppose that $\mathcal{N}$ has these properties. It follows (from Definition 8.1) that, for every $i = 0, \ldots, l-1$, the $\mathcal{B}_{i+1}/\mathcal{B}_i$-multiplicity of $\mathcal{N}$ is at least $\sigma^k(\lambda_n)$ where $\sigma^k(\lambda_n) \geq 1$ for all large enough n. So if $\mathcal{B}'_0 \cong \mathcal{A}$ and $\mathcal{B}'_0 \subseteq_{\mathrm{cl}} \mathcal{N}$, then there are $\mathcal{B}'_i \subseteq_{\mathrm{cl}} \mathcal{N}$ such that $\mathcal{B}'_i \cong \mathcal{B}_i$ and $\mathcal{B}'_{i-1} \subseteq \mathcal{B}'_i$ for $i = 1, \ldots, l$. In particular, $\mathcal{B}'_0 \subseteq \mathcal{B}'_l \cong \mathcal{B}$ and since $\mathcal{B}'_0$ was an arbitrary closed copy of $\mathcal{A}$ in $\mathcal{N}$ it follows that $\mathcal{N}$ satisfies the $\mathcal{B}/\mathcal{A}$-extension axiom, i.e. $\mathcal{N} \models \varphi$. Thus we have shown that in order to prove (1) it is sufficient to show that

$$\lim_{n\to\infty} \delta_n\big(\{\mathcal{M} \in \mathbf{K}_n : \mathcal{M} \text{ is } (\sigma^k(\lambda_n), k)\text{-saturated with respect to } \mathbf{K}\}\big) = 1. \tag{2}$$

For $n \in \mathbb{N}$, let

$$\mathbf{X}_n = \{\mathcal{M} \in \mathbf{K}_n : \mathcal{M} \text{ is } (\sigma^k(\lambda_n), k)\text{-saturated with respect to } \mathbf{K}\},$$

and for $n, r \in \mathbb{N}$ let

$$\mathbf{X}_{n,r} = \{\mathcal{M} \in \mathbf{K}_n \restriction r : \ \mathcal{M} \text{ is } (\sigma^r(\lambda_n), k)\text{-saturated with respect to } \mathbf{K}\restriction r\}.$$

By Lemma 8.3 below, in order to prove (2) it is sufficient to prove that

$$\lim_{n\to\infty} \mathbb{P}_{n,k}(\mathbf{X}_{n,k}) = 1, \tag{3}$$

Lemma 8.3. *For every $n \in \mathbb{N}$, $\delta_n(\mathbf{X}_n) = \mathbb{P}_{n,\rho}(\mathbf{X}_n) = \mathbb{P}_{n,k}(\mathbf{X}_{n,k})$.*

For the proof of Lemma 8.3 we need the following:

Lemma 8.4. *Let $i \in \mathbb{N}$. For every $\mathcal{M} \in \mathbf{K}$, $\mathcal{M}$ is (i, k)-saturated with respect to $\mathbf{K}$ if and only if $\mathcal{M}\restriction k$ is (i, k)-saturated with respect to $\mathbf{K}\restriction k$.*

Proof. Observe that for every $\mathcal{M} \in \mathbf{K}$ and every $A \subseteq M$ with $\dim_{\mathcal{M}}(A) \leq k$ the following holds: for any relation symbol R, of arity r, say, and every $\bar{b} \in A^r$,

$$\bar{b} \in R^{\mathcal{M}} \iff \bar{b} \in R^{\mathcal{M}\restriction k}.$$

In other words, $\mathcal{M}$ and $\mathcal{M}\restriction k$ agree on all subsets A of dimension at most k. It follows, in particular, that for every L-structure $\mathcal{A}$ such that $\mathcal{A}\restriction L_0 \in \mathbf{G}$ and $\mathcal{A}\restriction L_0$ has dimension at most k, $\mathcal{A}$ is represented with respect to $\mathbf{K}$ if and only if $\mathcal{A}$ is represented with respect to $\mathbf{K}\restriction k$. The lemma is now an immediate consequence of the definition of (i, k)-saturation. □

Proof of Lemma 8.3. Recall that ρ is the supremum of the arities of relation symbols which belong to the vocabulary of L but not to the vocabulary of L_0. First suppose that $\rho \leq k$. Let

$$\mathbf{Y}_n = \{\mathcal{N} \in \mathbf{K}_n\restriction k : \mathcal{M} \subseteq_w \mathcal{N} \text{ for some } \mathcal{M} \in \mathbf{X}_n\}.$$

By Lemma 6.5, $\mathbb{P}_{n,\rho}(\mathbf{X}_n) = \mathbb{P}_{n,k}(\mathbf{Y}_n)$. But $\rho \leq k$ implies that, for every $\mathcal{M} \in \mathbf{K}$, $\mathcal{M}\restriction k = \mathcal{M}\restriction\rho = \mathcal{M}$. Hence, $\mathbf{X}_{n,k} = \mathbf{X}_n = \mathbf{Y}_n$, so $\delta_n(\mathbf{X}_n) = \mathbb{P}_{n,\rho}(\mathbf{X}_n) = \mathbb{P}_{n,k}(\mathbf{X}_{n,k})$.

Now suppose that $k < \rho$. From Lemma 8.4 it follows that

$$\mathbf{X}_n = \{\mathcal{N} \in \mathbf{K}_n\restriction\rho : \mathcal{M} \subseteq_w \mathcal{N} \text{ for some } \mathcal{M} \in \mathbf{X}_{n,k}\}$$

By Lemma 6.5, $\mathbb{P}_{n,k}(\mathbf{X}_{n,k}) = \mathbb{P}_{n,\rho}(\mathbf{X}_n) = \delta_n(\mathbf{X}_n)$. □

Thus, it remains to prove (3), i.e. that $\lim_{n\to\infty} \mathbb{P}_{n,k}(\mathbf{X}_{n,k}) = 1$. This will be done by proving, by induction on r, that for every $r = 0, \ldots, k$, $\lim_{n\to\infty} \mathbb{P}_{n,r}(\mathbf{X}_{n,r}) = 1$. In Definition 3.11 the notion of a substitution $\mathcal{M}[\mathcal{A} \triangleright \mathcal{B}]$ of $\mathcal{A}$ for $\mathcal{B}$ inside $\mathcal{M}$ was defined. There it was assumed that the vocabulary of L is relational. However, eventual function or constant symbols in the vocabulary of L already belong to the vocabulary of $L_0 \subseteq L$, and, in what follows, we only consider substitutions when $\mathcal{A}$ and $\mathcal{B}$ agree on L_0 and on proper closed substructures (in the sense of Terminology 7.19). So in this context, substitutions $\mathcal{M}[\mathcal{A} \triangleright \mathcal{B}]$, according to Definition 3.11, make sense; and we will use them.

Lemma 8.5. *Let $0 \leq r < k$, $\mathcal{M} \in \mathbf{K}_n\restriction r+1$ and suppose that $\mathcal{A} \subseteq_{cl} \mathcal{M}$ and $\dim_{\mathcal{M}}(A) = r+1$. Also assume that $\mathcal{B}$ is a represented structure with respect to $\mathbf{K}\restriction r+1$ such that $\mathcal{B}$ and $\mathcal{A}$ agree on L_0 and on closed proper substructures. Then $\mathcal{M}[\mathcal{A} \triangleright \mathcal{B}] \in \mathbf{K}_n\restriction r+1$.*

Proof. Let r, $\mathcal{M}$, $\mathcal{A}$ and $\mathcal{B}$ satisfy the assumptions of the lemma, so in particular $\mathcal{A}{\restriction}L_0 = \mathcal{B}{\restriction}L_0$. Note that since $\mathcal{A}$ and $\mathcal{B}$ have dimension $r+1$ it follows that $\mathcal{A}, \mathcal{B} \in \mathbf{K}$, because for every $\mathcal{C} \in \mathbf{K}$ with dimension at most $r+1$ we have $\mathcal{C}{\restriction}r+1 = \mathcal{C}$. By assumption, $\mathcal{A}$ and $\mathcal{B}$ agree on L_0 and on closed proper substructures. The assumption that $\mathbf{K}$ accepts k-substitutions over L_0 implies that there exists $\mathcal{N} \in \mathbf{K}_n$ such that $\mathcal{N}{\restriction}L_0 = \mathcal{M}{\restriction}L_0$, $\mathcal{N}{\restriction}B = \mathcal{B}$ and for every $\mathcal{U} \subseteq_{cl} \mathcal{N}$ such that $\dim_{\mathcal{N}}(U) \leq r+1$ and $U \neq B$, we have $\mathcal{N}{\restriction}U = \mathcal{M}{\restriction}U$. In particular, $\mathcal{N}{\restriction}U = \mathcal{M}{\restriction}U$ for every U with dimension at most r.

Since $\mathcal{N}{\restriction}r+1 \in \mathbf{K}_n{\restriction}r+1$ it suffices to show that $\mathcal{M}[\mathcal{A} \triangleright \mathcal{B}] = \mathcal{N}{\restriction}r+1$. For this it is enough to show that for every closed substructure $\mathcal{C} \subseteq_{cl} \mathcal{M}[\mathcal{A} \triangleright \mathcal{B}]$ with dimension $r+1$,

$$(*) \qquad \mathcal{N}{\restriction}C = \mathcal{C}.$$

Suppose that $\mathcal{C} \subseteq_{cl} \mathcal{M}[\mathcal{A} \triangleright \mathcal{B}]$. If $\mathcal{C} = \mathcal{B}$ then, by the choice of $\mathcal{N}$, we have $\mathcal{N}{\restriction}C = \mathcal{N}{\restriction}B = \mathcal{B}$. If $\mathcal{C} \neq \mathcal{B}$ then, by the choice of $\mathcal{N}$, we have $\mathcal{N}{\restriction}C = \mathcal{M}{\restriction}C = \mathcal{C}$, where the last identity follows because $\mathcal{M} = \mathcal{M}{\restriction}r+1$ and $\mathcal{C}$ has dimension $r+1$; thus $(*)$ also holds in case when $\mathcal{C} \neq \mathcal{B}$. □

Lemma 8.6. *Let $0 \leq r < k$, $\mathcal{M} \in \mathbf{K}_n{\restriction}r+1$ and suppose that $\mathcal{A} \subseteq_{cl} \mathcal{M}$ and $r < \dim_{\mathcal{M}}(A) \leq k$. Also assume that $\mathcal{B}$ is a represented structure with respect to $\mathbf{K}{\restriction}r+1$ such that $\mathcal{B}{\restriction}L_0 = \mathcal{A}{\restriction}L_0$ and for every closed $U \subseteq A = B$ with dimension r, $\mathcal{A}{\restriction}U = \mathcal{B}{\restriction}U$. Then $\mathcal{M}[\mathcal{A} \triangleright \mathcal{B}] \in \mathbf{K}_n{\restriction}r+1$.*

Proof. Let r, $\mathcal{M}$, $\mathcal{A}$ and $\mathcal{B}$ satisfy the assumptions of the lemma. By definition of $\mathbf{K}{\restriction}r+1$, for every $\mathcal{N} \in \mathbf{K}{\restriction}r+1$ and every relation symbol R which does not belong to the vocabulary of L_0, there is *no* R-relationship $\bar{a} \in R^{\mathcal{N}}$ with dimension greater than $r+1$. Consequently, the structure $\mathcal{M}[\mathcal{A} \triangleright \mathcal{B}]$ can be created by a finite number of substitutions of the kind considered in Lemma 8.5. More precisely: There are $\mathcal{N}_0, \ldots, \mathcal{N}_s \in \mathbf{K}_n{\restriction}r+1$ and $\mathcal{C}_0, \ldots, \mathcal{C}_{2s}$ which dimension $r+1$ such that

$$\mathcal{M} = \mathcal{N}_0, \ \mathcal{M}[\mathcal{A} \triangleright \mathcal{B}] = \mathcal{N}_s,$$
$$\mathcal{N}_{i+1} = \mathcal{N}_i[\mathcal{C}_{2i} \triangleright \mathcal{C}_{2i+1}], \text{ for } i = 1, \ldots, s, \text{ and}$$
$$\mathcal{C}_{2i} \text{ and } \mathcal{C}_{2i+1} \text{ agree on } L_0 \text{ and on closed proper substructures.}$$

By Lemma 8.5, $\mathcal{N}_i \in \mathbf{K}_n{\restriction}r+1$, for $i = 0, \ldots, s$, so we are done. □

Lemma 8.7. *If $0 \leq r < k$ then for every $\mathcal{M} \in \mathbf{K}_n{\restriction}r$ there is $\mathcal{M}' \in \mathbf{K}_n{\restriction}r+1$ such that $\mathcal{M}'{\restriction}r = \mathcal{M}$.*

Proof. If $\mathcal{M} \in \mathbf{K}_n{\restriction}r$ then $\mathcal{M} = \mathcal{N}{\restriction}r$ for some $\mathcal{N} \in \mathbf{K}_n$. Take $\mathcal{M}' = \mathcal{N}{\restriction}r+1$. Then $\mathcal{M}' \in \mathbf{K}_n{\restriction}r+1$ and $\mathcal{M}'{\restriction}r = \mathcal{N}{\restriction}r = \mathcal{M}$. □

Lemma 8.8. *For every n and every $\mathcal{M} \in \mathbf{K}_n{\restriction}0$, $\mathcal{M}$ is (λ_n, k)-saturated.*

Proof. First observe that from Definition 7.12 it follows that whenever $\mathcal{M}$ is permitted (or, equivalently, in the present context, represented) with respect to $\mathbf{K}{\restriction}0$, then $\mathcal{M}$ is an expansion of $\mathcal{M}{\restriction}L_0$ ($\cong \mathcal{G}_n$ for some n) obtained by possibly adding some new relationship(s) involving *only* elements in $\mathrm{cl}_{\mathcal{M}}(\emptyset)$; and whenever $\mathcal{A} \subseteq_{\mathrm{cl}} \mathcal{M}$ then $\mathrm{cl}_{\mathcal{M}}(\emptyset) \subseteq A$

Let $\mathcal{M} \in \mathbf{K}_n{\restriction}0$ and let $\mathcal{A} \subseteq_{cl} \mathcal{B}$ be permitted structures with respect to $\mathbf{K}{\restriction}0$ such that $\dim_{\mathcal{B}}(A) + 1 = \dim_{\mathcal{B}}(B) \leq k$. Suppose that $\mathcal{A}' \subseteq_{cl} \mathcal{M}$ is a copy of $\mathcal{A}$ and that $\tau : \mathcal{A}' \to \mathcal{A}$ is an isomorphism. We must show that there are $\mathcal{B}'_i \subseteq_{cl} \mathcal{M}$ and isomorphisms $\tau_i : \mathcal{B}'_i \to \mathcal{B}$, for $i = 1, \ldots, \lambda_n$, such that $\mathcal{A}' \subseteq_{cl} \mathcal{B}'_i$, $\tau_i{\restriction}A' = \tau$ and $B'_i \cap B'_j = A'$ whenever $i \neq j$. As noted in the beginning of the proof, every relationship of $\mathcal{B}$ (or of $\mathcal{M}$) which involves some element(s) from $B - A$ (or from $M - A'$) is an R-relationship for some relation symbol R of L_0. Observe that $\tau : A' \to A$ can also be viewed as an isomorphism from $\mathcal{A}'{\restriction}L_0$ to $\mathcal{A}{\restriction}L_0$. By (b) in the beginning of the proof of Theorem 7.31, there are

$\mathcal{B}_i \subseteq_{cl} \mathcal{M}\restriction L_0 = \mathcal{G}_n$ and isomorphisms $\tau_i : \mathcal{B}_i \to \mathcal{B}\restriction L_0$, for $i = 1, \ldots, \lambda_n$, such that $\mathcal{A}'\restriction L_0 \subseteq_{cl} \mathcal{B}_i$, $\tau_i\restriction A' = \tau$ and $B_i \cap B_j = A'$ whenever $i \neq j$. For $i = 1, \ldots, \lambda_n$, let $\mathcal{B}'_i \subseteq_{\rm cl} \mathcal{M}$ be such that $\mathcal{B}'_i\restriction L_0 = \mathcal{B}_i$. Then $\mathcal{A}' \subseteq_{\rm cl} \mathcal{B}'_i$ for each i, and since, as observed above, every relationship which involves some element(s) from $M - A'$, or from $B - A$, is an R-relationship for some relation symbol R of L_0, it follows that every τ_i is in fact an isomorphism from $\mathcal{B}'_i$ to $\mathcal{B}$. □

Lemma 8.9. *Suppose that $0 \leq r < k$. For every real $\varepsilon > 0$ there is $n_\varepsilon \in \mathbb{N}$ such that if $n \geq n_\varepsilon$, $\mathcal{M} \in \mathbf{K}_n\restriction r$ is $\big(\sigma^r(\lambda_n), k\big)$-saturated and*

$$\mathbf{E}_{r+1}(\mathcal{M}) = \big\{\mathcal{N} \in \mathbf{K}_n\restriction r+1 : \mathcal{N}\restriction r = \mathcal{M}\big\},$$

then the proportion of $\mathcal{N} \in \mathbf{E}_{r+1}(\mathcal{M})$ which are $\big(\sigma^{r+1}(\lambda_n), k\big)$-saturated is at least $1 - \varepsilon$.

Proof. Let $0 \leq r < k$. We are assuming that $\{\mathcal{G}_n : n \in \mathbb{N}\}$ is a uniformly bounded pregeometry. Hence there is $\alpha \in \mathbb{N}$ such that if $\mathcal{A}$ is permitted with respect to $\mathbf{K}\restriction r+1$ and has dimension at most k, then $|A| \leq \alpha$. Suppose that $\mathcal{M} \in \mathbf{K}_n\restriction r$ is $\big(\sigma^r(\lambda_n), k\big)$-saturated and let $\mathbf{E}_{r+1}(\mathcal{M}) = \big\{\mathcal{N} \in \mathbf{K}_n\restriction r+1 : \mathcal{N}\restriction r = \mathcal{M}\big\}$. We start by proving that, with the uniform probability measure on $\mathbf{E}_{r+1}(\mathcal{M})$, the probability that a randomly chosen $\mathcal{N} \in \mathbf{E}_{r+1}(\mathcal{M})$ is $\big(\sigma^{r+1}(\lambda_n), k\big)$-saturated approaches 1 as n tends to ∞. We do this by finding an upper bound (depending on n) for the probability that a randomly chosen $\mathcal{N} \in \mathbf{E}_{r+1}(\mathcal{M})$ is *not* $\big(\sigma^{r+1}(\lambda_n), k\big)$-saturated; and then observe that this upper bound approaches 0 as n tends to infinity. Finally we note that the argument does not depend on which $\big(\sigma^r(\lambda_n), k\big)$-saturated $\mathcal{M} \in \mathbf{K}_n\restriction r$ we consider; so given $\varepsilon > 0$ there is n_ε which such that for every $n \geq n_\varepsilon$ and every $\big(\sigma^r(\lambda_n), k\big)$-saturated $\mathcal{M} \in \mathbf{K}_n\restriction r$, the proportion of $\mathcal{N} \in \mathbf{E}_{r+1}(\mathcal{M})$ which are *not* $\big(\sigma^{r+1}(\lambda_n), k\big)$-saturated is at most ε.

Let $\mathcal{N} \in \mathbf{E}_{r+1}(\mathcal{M})$ and let $\mathcal{A} \subset_{cl} \mathcal{B}$ be represented structures with respect to $\mathbf{K}\restriction r+1$ such that $\dim_{\mathcal{B}}(A) + 1 = \dim_{\mathcal{B}}(B) \leq k$. Suppose that $\mathcal{A}' \subseteq_{cl} \mathcal{N}$ is a copy of $\mathcal{A}$ and that $\tau : \mathcal{A}' \to \mathcal{A}$ is an isomorphism. Let $l_n = \lfloor\sqrt{\sigma^r(\lambda_n)}\rfloor = \sigma^{r+1}(\lambda_n)$. First we find an upper bound for the probability that there does not exist $\mathcal{B}_i \subseteq_{cl} \mathcal{N}$ and isomorphisms $\tau_i : \mathcal{B}_i \to \mathcal{B}$, for $i = 1, \ldots, l_n$, such that $\mathcal{A}' \subseteq_{cl} \mathcal{B}_i$, $\tau_i\restriction A' = \tau$, and $B_i \cap B_j = A'$ whenever $i \neq j$.

Let $l'_n = \sigma^r(\lambda_n)$. Since $\mathcal{M}$ is $\big(\sigma^r(\lambda_n), k\big)$-saturated there are $\mathcal{B}^-_i \subseteq_{cl} \mathcal{M}$, $i = 1, \ldots, l'_n$ and isomorphisms $\tau_i : \mathcal{B}^-_i \to \mathcal{B}\restriction r$, such that $\mathcal{A}'\restriction r \subseteq_{cl} \mathcal{B}^-_i$, $\tau_i\restriction A' = \tau$ and $B^-_i \cap B^-_j = A'$ whenever $i \neq j$. Let β be the number of represented structures with respect $\mathbf{K}\restriction r+1$ with universe included in $\{1, \ldots, \alpha\}$. Lemma 8.6 implies that the probability that the map $\tau_i : B^-_i \to B$ is an isomorphism from $\mathcal{N}\restriction B^-_i$ to $\mathcal{B}$ is at least $1/\beta$, independently of whether this holds for $j \neq i$. Let s be a natural number such that $0 \leq s < l_n$. The probability that for *every* $i \in \{sl_n + i, \ldots, (s+1)l_n\}$, $\tau_i : B^-_i \to B$ is *not* an isomorphism from $\mathcal{N}\restriction B^-_i$ to $\mathcal{B}$ is at most

$$\big(1 - 1/\beta\big)^{l_n}.$$

Let $m_n = |G_n| = |N|$. By (a) $\lambda_n \leq m_n \leq P(\lambda_n)$ for all $n \in \mathbb{N}$, where P is a polynomial. Since, by assumption, $\lim_{n\to\infty} \lambda_n = \infty$, we have $\lim_{n\to\infty} m_n = \infty$. From the definition of l_n as $l_n = \sigma^{r+1}(\lambda_n)$ and the definition of σ^{r+1} it follows that there is a polynomial Q such that $m_n \leq Q(l_n)$. The number of ways in which we can choose $\mathcal{A}$, $\mathcal{B}$, $\mathcal{A}'$ and s as above is not larger than

$$\beta^2 \cdot (m_n)^\alpha \cdot l_n \quad \leq \quad \beta^2 \cdot (Q(l_n))^\alpha \cdot l_n.$$

Moreover, for every choice of such $\mathcal{A}$, $\mathcal{B}$, $\mathcal{A}'$ and s, there exist, for $i = 1, \ldots, l'_n$, $\mathcal{B}^-_i \subseteq_{cl} \mathcal{M}$ and isomorphisms $\tau_i : \mathcal{B}^-_i \to \mathcal{B}$, with the properties described above. So if $\mathcal{N}$ is not $\big(\sigma^{r+1}(\lambda_n), k\big)$-saturated, then there exist $\mathcal{A}$, $\mathcal{B}$, $\mathcal{A}'$, $\mathcal{B}^-_i$, τ_i, for $i = 1, \ldots, l'_n$, and s as above such that for every $i \in \{sl_n + 1, \ldots, (s+1)l_n\}$, τ_i is not an isomorphism from

$\mathcal{N}\restriction B_i^-$ to $\mathcal{B}$. Hence, the probability that a randomly chosen $\mathcal{N} \in \mathbf{E}_{r+1}(\mathcal{M})$ is not $\big(\sigma^{r+1}(\lambda_n), k\big)$-saturated does *not* exceed

$$f_n = \beta^2 \cdot (Q(l_n))^\alpha \cdot l_n \cdot \big(1 - 1/\beta\big)^{l_n}.$$

Since $\lambda_n \to \infty$ as $n \to \infty$ we also have $l_n \to \infty$ as $n \to \infty$. Because $\beta^2 \cdot (Q(l_n))^\alpha \cdot l_n$ is a polynomial in l_n it follows that $f_n \to 0$ as $n \to \infty$.

Observe that the same expression for f_n works for every $\big(\sigma^r(\lambda_n), k\big)$-saturated $\mathcal{M} \in \mathbf{K}_n\restriction r$. So for every $\varepsilon > 0$ there is n_ε such that for every $n \geq n_\varepsilon$ and every $\big(\sigma^r(\lambda_n), k\big)$-saturated $\mathcal{M} \in \mathbf{K}_n\restriction r$, the proportion of $\mathcal{N} \in \mathbf{E}_{r+1}(\mathcal{M})$ which are $\big(\sigma^{r+1}(\lambda_n), k\big)$-saturated is at least $1 - \varepsilon$. □

Recall that, for $r = 0, 1, \dots, k$,

$$\mathbf{X}_{n,r} = \{\mathcal{M} \in \mathbf{K}_n\restriction r : \mathcal{M} \text{ is } (\sigma^r(\lambda_n), k)\text{-saturated}\}.$$

From Lemma 8.9 we can easily derive the following:

Lemma 8.10. *For every $r = 0, 1, \dots, k-1$ and all sufficiently large n (take $0 < \varepsilon < 1/2$, n_ε and $n > n_\varepsilon$ so that the conclusion of Lemma 8.9 holds),*

$$\mathbf{X}_{n,r} \subseteq \{\mathcal{N}\restriction r : \mathcal{N} \in \mathbf{X}_{n,r+1}\}.$$

Proof. Suppose that $\mathcal{M} \in \mathbf{X}_{n,r}$, so $\mathcal{M}$ is $\big(\sigma^r(\lambda_n), k\big)$-saturated. By Lemma 8.9, for all sufficiently large n, $\mathbf{E}_{r+1}(\mathcal{M})$ will contain a stucture $\mathcal{N}$ which is $\big(\sigma^{r+1}(\lambda_n), k\big)$-saturated; hence $\mathcal{N} \in \mathbf{X}_{n,r+1}$ and $\mathcal{N}\restriction r = \mathcal{M}$. □

Now we can finish the proof of part (i) of Theorem 7.31 by proving (3), in other words, that $\lim_{n\to\infty} \mathbb{P}_{n,k}(\mathbf{X}_{n,k}) = 1$. Let $\varepsilon > 0$. Choose $\varepsilon' > 0$ so that $(1 - \varepsilon')^k \geq 1 - \varepsilon$. By Lemma 8.9, we can choose $n_{\varepsilon'}$ such that if $0 \leq r < k$, $n > n_{\varepsilon'}$ and $\mathcal{M} \in \mathbf{K}_n\restriction r$ is $\big(\sigma^r(\lambda_n), k\big)$-saturated, then the proportion of $\mathcal{N} \in \mathbf{E}_{r+1}(\mathcal{M})$ which are $\big(\sigma^{r+1}(\lambda_n), k\big)$-saturated is at least $1 - \varepsilon'$. By induction we show that, for $r = 0, 1, \dots, k$ and $n > n_{\varepsilon'}$,

$$\mathbb{P}_{n,r}(\mathbf{X}_{n,r}) \geq (1 - \varepsilon')^r \geq 1 - \varepsilon.$$

The base case $r = 0$ is given by Lemma 8.8, so assume that $0 < r \leq k$ and that $\mathbb{P}_{n,r-1}(\mathbf{X}_{n,r-1}) \geq (1 - \varepsilon')^{r-1}$. Let $\mathcal{M}_1, \dots, \mathcal{M}_s$ be an enumeration, without repetition, of $\mathbf{X}_{n,r}$. Then let $\mathcal{M}'_1, \dots, \mathcal{M}'_t$ be an enumeration, without repetition, of the set $\{\mathcal{M}_1\restriction$

$r-1,\dots,\mathcal{M}_s{\restriction}r-1\}$. By the definition of $\mathbb{P}_{n,r}$, the following holds for every $n > n_{\varepsilon'}$:

$$\begin{aligned}
\mathbb{P}_{n,r}(\mathbf{X}_{n,r}) = \mathbb{P}_{n,r}\big(\{\mathcal{M}_1,\dots,\mathcal{M}_s\}\big) &= \sum_{i=1}^{s}\mathbb{P}_{n,r}(\mathcal{M}_i)\\
&= \sum_{i=1}^{s}\frac{1}{\big|\{\mathcal{N}\in\mathbf{K}_n{\restriction}r : \mathcal{N}{\restriction}r-1 = \mathcal{M}_i{\restriction}r-1\}\big|}\cdot\mathbb{P}_{n,r-1}(\mathcal{M}_i{\restriction}r-1)\\
&= \sum_{i=1}^{t}\frac{\big|\{\mathcal{N}\in\mathbf{X}_{n,r} : \mathcal{N}{\restriction}r-1 = \mathcal{M}'_i\}\big|}{\big|\{\mathcal{N}\in\mathbf{K}_n{\restriction}r : \mathcal{N}{\restriction}r-1 = \mathcal{M}'_i\}\big|}\cdot\mathbb{P}_{n,r-1}(\mathcal{M}'_i)\\
&= \sum_{i=1}^{t}\frac{\big|\{\mathcal{N}\in\mathbf{X}_{n,r} : \mathcal{N}{\restriction}r-1 = \mathcal{M}'_i\}\big|}{\big|\mathbf{E}_r(\mathcal{M}'_i)\big|}\cdot\mathbb{P}_{n,r-1}(\mathcal{M}'_i)\\
&\geq (1-\varepsilon')\sum_{i=1}^{t}\mathbb{P}_{n,r-1}(\mathcal{M}'_i) \qquad \text{(by the choice of } n_{\varepsilon'})\\
&= (1-\varepsilon')\mathbb{P}_{n,r-1}\big(\{\mathcal{M}'_1,\dots,\mathcal{M}'_t\}\big)\\
&\geq (1-\varepsilon')\mathbb{P}_{n,r-1}(\mathbf{X}_{n,r-1}) \qquad \text{(by Lemma 8.10)}\\
&\geq (1-\varepsilon')(1-\varepsilon')^{r-1} = (1-\varepsilon')^r \qquad \text{(by the induction hypothesis).}
\end{aligned}$$

Thus (3) is proved, and hence also part (i) of Theorem 7.31.

Now we prove part (ii) of Theorem 7.31. Note that we have proved (2) above, because (3) together with Lemma 8.3 implies (2). By (2), there are, for all $n\in\mathbb{N}$, $\mathcal{M}_n\in\mathbf{K}_n$ such that $\mathcal{M}_n$ is $(\sigma^k(\lambda_n),k)$-saturated. Let $\mu_n = \sigma^k(\lambda_n)$, so $\mathcal{M}_n$ is (μ_n,k)-saturated, where $\lim_{n\to\infty}\mu_n = \infty$. From (a) and the definition of σ^k it follows that there is a polynomial Q such that $\mu_n \leq |M_n| \leq Q(\mu_n)$ for all n. Since $\mathcal{M}_n$ is (μ_n,k)-saturated, the following holds: If $\mathcal{A}\subset_{cl}\mathcal{B}$ are represented structures such that $\dim_{\mathcal{B}}(B)\leq k$, then the $\mathcal{B}/\mathcal{A}$-multiplicity of $\mathcal{M}_n$ is at least μ_n. From Assumption 7.10 (4), it follows that the sequence $(\mathcal{M}_n : n\in\mathbb{N})$ is polynomially k-saturated; and hence $\mathbf{K}$ is polynomially k-saturated. This concludes the proof of part (ii), and hence of Theorem 7.31.

8.2. **Proof of Theorem 7.32.** We still assume that, for every $k>0$, $(\mathcal{G}_n : n\in\mathbb{N})$ is polynomially k-saturated and $\mathbf{K} = \bigcup_{n\in\mathbb{N}}\mathbf{K}_n$, where $\mathbf{K}_n = \mathbf{K}(\mathcal{G}_n)$, accepts k-substitutions over L_0. We want to prove that for every L-sentence φ, either $\lim_{n\to\infty}\delta_n(\varphi) = 0$ or $\lim_{n\to\infty}\delta_n(\varphi)=1$. The general idea of the proof follows a well-known pattern: we collect into a theory $T_{\mathbf{K}}$ all extension axioms of $\mathbf{K}$ together with sentences which express the pregeometry conditions and describe the possible isomorphism types of closed substructures of members of $\mathbf{K}$. By part (i) of Theorem 7.31, $T_{\mathbf{K}}$ is consistent. Then we show that $T_{\mathbf{K}}$ is complete by showing that it is countably categorical. From the completeness, it follows that for every L-sentence φ, either $T_{\mathbf{K}}\models\varphi$ or $T_{\mathbf{K}}\models\neg\varphi$. In the first case there is finite $\Delta\subset T_{\mathbf{K}}$ such that $\Delta\models\varphi$ and in the second case there is finite $\Delta'\subseteq T_{\mathbf{K}}$ such that $\Delta'\models\neg\varphi$. In the first case part (i) of Theorem 7.31 implies that

$$\lim_{n\to\infty}\delta_n(\{\mathcal{M}\in\mathbf{K}_n : \mathcal{M}\models\Delta\}) = 1,$$

and therefore $\lim_{n\to\infty}\delta_n(\varphi) = 1$. In the second case we get, in a similar way, that $\lim_{n\to\infty}\delta_n(\neg\varphi)=1$, so $\lim_{n\to\infty}\delta_n(\varphi)=0$.

Now to the details. We are assuming that $\mathbf{G} = \{\mathcal{G}_n : n\in\mathbb{N}\}$ is a pregeometry where the closure operator of every member of $\mathbf{G}$ is defined by the L_0-formulas $\theta_n(x_1,\dots,x_{n+1})$, $n\in\mathbb{N}$, according to Definition 7.1 and Assumption 7.10. In other words, for all m,n and all $a_1,\dots,a_{n+1}\subseteq G_m$,

$$a_{n+1}\in\mathrm{cl}_{\mathcal{G}_m}(a_1,\dots,a_n) \text{ if and only if } \mathcal{G}_m\models\theta(a_1,\dots,a_{n+1}). \tag{4}$$

Also (by Assumption 7.10), for every m and every $\mathcal{M} \in \mathbf{K}_m = \mathbf{K}(\mathcal{G}_m)$, $\mathrm{cl}_{\mathcal{M}}$ coincides with $\mathrm{cl}_{\mathcal{G}_m}$. Moreover, the pregeometry $\mathbf{G}$ is assumed to be *uniformly* locally finite, so there is $u : \mathbb{N} \to \mathbb{N}$ such that for every $\mathcal{M} \in \mathbf{K}$ and every $X \subseteq M$, $|\mathrm{cl}_{\mathcal{M}}(X)| \leq u(\dim_{\mathcal{M}}(X))$. We may also assume that for every $k \in \mathbb{N}$ the value $u(k)$ is minimal so that this holds.

By the ***finiteness property***, for a pregeometry (A, cl), we mean the property that for all $a \in A$ and $X \subseteq A$, $a \in \mathrm{cl}(X)$ if and only if $a \in \mathrm{cl}(Y)$ for some *finite* $Y \subseteq X$. Besides the finiteness property, all other properties of a pregeometry can, when (4) holds, be expressed for finite subsets of A by using the formulas $\theta_n(x_1, \ldots, x_{n+1})$, $n \in \mathbb{N}$. Let T_{preg} be the set of sentences which express all properties of a pregeometry (for finite subsets) except the finiteness property. Then every $\mathcal{M} \in \mathbf{K}$ is a model of T_{preg}.

Note that, for every $\mathcal{M} \in \mathbf{K}$ and all $a_1, \ldots, a_n \in M$, the statement "$\{a_1, \ldots, a_n\}$ is a closed set (in $\mathcal{M}$)" is uniformly expressed by the first-order formula

$$\neg \exists x_{n+1} \Big(\bigwedge_{i=1}^{n} x_{n+1} \neq x_i \ \wedge \ \theta_n(x_1, \ldots, x_{n+1}) \Big),$$

which we denote by $\gamma_n(x_1, \ldots, x_n)$. For every positive $m \in \mathbb{N}$, let $s(m)$ be the the number of nonisomorphic structures of cardinality at most m which occur as closed substructures of members of $\mathbf{K}$, and let $\mathcal{M}_{m,1}, \ldots, \mathcal{M}_{m,s(m)}$ be an enumeration of all isomorphism types of such structures. For $1 \leq i \leq s(m)$, let $\chi_{m,i}(x_1, \ldots, x_m)$ describe the isomorphism type of $\mathcal{M}_{m,i}$ in such a way that we require that all variables $x_1, \ldots, x_m$ actually occur in $\chi_{m,i}$. It means that if $\|\mathcal{M}_{m,i}\| < m$, then $\chi_{m,i}(x_1, \ldots, x_m)$ must express that some variables refer to the same element, by saying '$x_k = x_l$' for some $k \neq l$. For every $k \in \mathbb{N}$ let ψ_k denote the sentence

$$\forall x_1, \ldots, x_k \exists x_{k+1}, \ldots, x_{u(k)} \Big(\gamma_{u(k)}(x_1, \ldots, x_{u(k)}) \ \wedge \ \bigvee_{i=1}^{s(u(k))} \bigvee_{\pi} \chi_{u(k),i}(x_{\pi(1)}, \ldots, x_{\pi(u(k))}) \Big),$$

where the second disjunction ranges over all permutations π of $\{1, \ldots, u(k)\}$. If $k = 0$ and $u(k) > 0$, then the universal quantifiers do not occur so ψ_0 is an existential formula. If $u(0) = 0$, then, by convention, ψ_0 is $\forall x(x = x)$. If $u(k) = k$, then the existential quantifiers do not occur and ψ_k is a universal formula. Note that for every $k \in \mathbb{N}$ and every $\mathcal{M} \in \mathbf{K}$, $\mathcal{M} \models \psi_k$. Let $T_{iso} = \{\psi_k : k \in \mathbb{N}\}$ so every $\mathcal{M} \in \mathbf{K}$ is a model of T_{iso}.

Finally, let T_{ext} consist (exactly) of all extension axioms of $\mathbf{K}$ and let

$$T_{\mathbf{K}} = T_{preg} \cup T_{iso} \cup T_{ext}.$$

By Theorem 7.31 and compactness, $T_{\mathbf{K}}$ is consistent. Note that every model of $T_{\mathbf{K}}$ is infinite, because we assume that $(\mathcal{G}_n : n \in \mathbb{N})$ is polynomially k-saturated (for every $k > 0$), which implies that for some sequence $(\lambda_n : n \in \mathbb{N})$ which tends to infinity as $n \to \infty$, $\mathcal{G}_n$ contains at least λ_n different elements.

Lemma 8.11. *Suppose that $\mathcal{M} \models T_{\mathbf{K}}$ and define $\mathrm{cl}_{\mathcal{M}}$ as follows:*

(a) *for all $n \in \mathbb{N}$ and all $a_1, \ldots, a_{n+1} \in M$, $a_{n+1} \in \mathrm{cl}_{\mathcal{M}}(a_1, \ldots, a_n) \iff \mathcal{M} \models \theta_n(a_1, \ldots, a_{n+1})$.*

(b) *for all $X \subseteq M$ and all $a \in M$, $a \in \mathrm{cl}_{\mathcal{M}}(X) \iff$ for some finite $Y \subseteq X$, $a \in \mathrm{cl}_{\mathcal{M}}(Y)$.*

Then $(M, \mathrm{cl}_{\mathcal{M}})$ is a pregeometry such that for every finite $X \subseteq M$, $|\mathrm{cl}_{\mathcal{M}}(X)| \leq u(\dim_{\mathcal{M}}(X))$.

Proof. Suppose that $\mathcal{M} \models T_{\mathbf{K}}$. Since $T_{preg} \subseteq T_{\mathbf{K}}$, it follows from part (a) that $\mathrm{cl}_{\mathcal{M}}$ satisfies all properties of a pregeometry on finite subsets of M. But (b) guarantees that $\mathrm{cl}_{\mathcal{M}}$ has the finiteness property, and then all other properties follow for all subsets of M. So $(M, \mathrm{cl}_{\mathcal{M}})$ is a pregeometry. Since $T_{iso} \subset T_{\mathbf{K}}$ it follows that, for every $X \subseteq M$, $|\mathrm{cl}_{\mathcal{M}}(X)| \leq u(\dim_{\mathcal{M}}(X))$. □

To complete the proof of Theorem 7.32 we only need to prove:

Lemma 8.12. *$T_{\mathbf{K}}$ is countably categorical and hence complete.*

Proof. Let $\mathcal{M}$ and $\mathcal{N}$ be countable models of $T_{\mathbf{K}}$. We show that $\mathcal{M} \cong \mathcal{N}$, by a back-and-forth argument. By symmetry it is sufficient to show the following:

> Suppose that $\mathcal{A}$ is a closed finite substructure of $\mathcal{M}$ (or $A = \emptyset$), that $\mathcal{B}$ is a closed finite substructure of $\mathcal{N}$ (or $B = \emptyset$), that $f : \mathcal{A} \to \mathcal{B}$ is an isomorphism (if A and B are nonempty) and that $a \in M - A$. Then there are a closed $\mathcal{B}' \subseteq M$ such that $B \subset B'$ and an isomorphism $g : \mathrm{cl}_{\mathcal{M}}(A \cup \{a\}) \to \mathcal{B}'$ which extends f.

So suppose that $\mathcal{A}$ is a closed finite substructure of $\mathcal{M}$, that $\mathcal{B}$ is a closed finite substructure of $\mathcal{N}$, that $f : \mathcal{A} \to \mathcal{B}$ is an isomorphism and that $a \in M - A$. Since $\mathcal{M} \models T_{\mathbf{K}} \supset T_{iso}$, $\mathcal{A}$, $\mathcal{B}$ and $\mathrm{cl}_{\mathcal{M}}(A \cup \{a\})$ are isomorphic with closed substructures of members of $\mathbf{K}$. Since $\mathcal{N} \models T \supset T_{ext}$, it follows that $\mathcal{N}$ satisfies the $\mathrm{cl}_{\mathcal{M}}(A \cup \{a\})/\mathcal{A}$-extension axiom, and as $\mathcal{B} \cong \mathcal{A}$ there is a closed $\mathcal{B}' \subset \mathcal{N}$ such that $\mathcal{B} \subset \mathcal{B}'$ and an isomorphism $g : \mathrm{cl}_{\mathcal{M}}(A \cup \{a\}) \to \mathcal{B}'$ which extends f. Recall the convention that for every structure $\mathcal{P}$ which is isomorphic with a closed substructure of a member of $\mathbf{K}$, the statement "there exists a closed copy of $\mathcal{P}$" is an extension axiom, called the $\mathcal{P}/\emptyset$-extension axiom; this takes care of the case $A = B = \emptyset$. □

8.3. **Proof of Theorem 7.34.** Let $\mathbf{G} = \{\mathcal{G}_n : n \in \mathbb{N}\}$ be a set of L_0-structures which form a uniformly bounded pregeometry, and suppose that $(\mathcal{G}_n : n \in \mathbb{N})$ is polynomially k-saturated for every $k \in \mathbb{N}$. Assume that there is, up to isomorphism, a unique represented structure with dimension 0; hence $\mathbf{K}$ accepts 0-substitutions over L_0. Suppose that k is *minimal* such that $\mathbf{K}$ does *not* accept k-substitutions over L_0; hence $k > 0$ and $\mathbf{K}$ accepts $(k-1)$-substitutions over L_0. Moreover assume that there are represented structures, with respect to $\mathbf{K}$, $\mathcal{A}$ and $\mathcal{A}'$ such that

- $\mathcal{A}$ and $\mathcal{A}'$ have dimension k,
- $\mathcal{A}$ and $\mathcal{A}'$ agree on L_0 and on closed proper substructures,
- $\mathbf{K}$ accepts the substitution $[\mathcal{A}' \triangleright \mathcal{A}]$ over L_0, but
- $\mathbf{K}$ does *not* accept the substitution $[\mathcal{A} \triangleright \mathcal{A}']$ over L_0.

Let ρ be the supremum of the arities of all relation symbols which belong to the vocabulary of L but not to the vocabulary of L_0. By Remark 7.21, $0 < k \leq \rho$.

In order to prove Theorem 7.34, we assume that $\mathbf{K}$ has the independent amalgamation property and show that there are extension axioms φ and ψ such that $\lim_{n\to\infty} \delta_n(\varphi \wedge \psi) = 0$. We start with the following, which is straightforward to verify:

Observation 8.13. *For every L-structure $\mathcal{M}$ and $d \in \mathbb{N}$, $\mathcal{M}$ is represented with respect to $\mathbf{K}{\restriction}d$ if and only if there is $\mathcal{M}'$ such that $\mathcal{M}'$ is represented with respect to $\mathbf{K}$ and $\mathcal{M} = \mathcal{M}'{\restriction}d$.*

Note that the notion of 'acceptance of l-substitutions over L_0', which was defined for $\mathbf{K}$, can equally well be defined for $\mathbf{K}{\restriction}r$ for any r; the only difference is that the notion 'represented' is in this case with respect to $\mathbf{K}{\restriction}r$. By assumption, $\mathbf{K}$ accepts $(k-1)$-substitutions over L_0. From Observation 8.13 it follows that $\mathbf{K}{\restriction}(k-1)$ accepts $(k-1)$-substitutions over L_0. Note that for every $\mathcal{M} \in \mathbf{K}{\restriction}k-1$ and every relation symbol R in the vocabulary of L but not in the vocabulary of L_0, $\mathcal{M}$ does not have any R-relationship with dimension greater than $k-1$. From this and the assumption that $\mathbf{K}$ accepts $(k-1)$-substitutions over L_0 it follows that

(5) $\mathbf{K}{\restriction}(k-1)$ accepts l-substitutions over L_0 for *every* $l \in \mathbb{N}$.

By assumption, $\mathbf{K}$ accepts the substitution $[\mathcal{A}' \triangleright \mathcal{A}]$, and by Observation 8.13 it follows that $\mathbf{K}{\restriction}k$ accepts the substitution $[\mathcal{A}' \triangleright \mathcal{A}]$.

Since $\mathcal{A}$ and $\mathcal{A}'$ agree on L_0 it makes sense to speak about the substitution $\mathcal{M}[\mathcal{A}\triangleright\mathcal{A}']$ if $\mathcal{A} \subseteq_{cl} \mathcal{M}$, or $\mathcal{M}[\mathcal{A}'\triangleright\mathcal{A}]$ if $\mathcal{A}' \subseteq_{cl} \mathcal{M}$, as was explained in the paragraph before Lemma 8.5. Since $\mathcal{A}$ and $\mathcal{A}'$ have dimension k and $\mathbf{K}$ accepts the substitution $[\mathcal{A}' \triangleright \mathcal{A}]$ over L_0, it follows that if $\mathcal{M}$ is represented with respect to $\mathbf{K}{\restriction}k$, and $\mathcal{A}' \subseteq_{cl} \mathcal{M}$, then $\mathcal{M}[\mathcal{A}' \triangleright \mathcal{A}]$ is represented with respect to $\mathbf{K}{\restriction}k$. In other words, $\mathbf{K}{\restriction}k$ *admits* the substitution $[\mathcal{A}' \triangleright \mathcal{A}]$. By assumption, $\mathbf{K}$ does not accept the substitution $[\mathcal{A} \triangleright \mathcal{A}']$. Therefore we can argue similarly as we just did for the substitution $[\mathcal{A}\triangleright\mathcal{A}']$ to conclude that there is $\mathcal{P}$ such that $\mathcal{P}$ is represented with respect to $\mathbf{K}{\restriction}k$, $\mathcal{A} \subset_{cl} \mathcal{P}$ and $\mathcal{P}[\mathcal{A} \triangleright \mathcal{A}']$ is forbidden with respect to $\mathbf{K}{\restriction}k$.

Since the core of the argument (the proof of Lemma 8.16 below) is an adaptation of the proof of Theorem 3.17 to the present context, we introduce the same notation as in Section 5. We rename $\mathcal{A}$ and $\mathcal{A}'$ with $\mathcal{S}_\mathcal{P}$ and $\mathcal{S}_\mathcal{F}$, so in particular $\mathcal{S}_\mathcal{P}$ and $\mathcal{S}_\mathcal{F}$ have dimension k. As concluded above, $\mathbf{K}{\restriction}k$ *admits* the substitution $[\mathcal{S}_\mathcal{F} \triangleright \mathcal{S}_\mathcal{P}]$, in the sense that whenever $\mathcal{M}$ is represented with respect to $\mathbf{K}{\restriction}k$, then $\mathcal{M}[\mathcal{S}_\mathcal{F} \triangleright \mathcal{S}_\mathcal{P}]$ is represented with respect to $\mathbf{K}{\restriction}k$. Moreover, there is $\mathcal{P}$ such that $\mathcal{P}$ is represented with respect to $\mathbf{K}{\restriction}k$, $\mathcal{S}_\mathcal{P} \subseteq_{cl} \mathcal{P}$ and $\mathcal{F} = \mathcal{P}[\mathcal{S}_\mathcal{P} \triangleright \mathcal{S}_\mathcal{F}]$ is *forbidden* with respect to $\mathbf{K}{\restriction}k$. This implies that the dimension of $\mathcal{P}$ is strictly larger than the dimension of $\mathcal{S}_\mathcal{P}$ which is k.

By Observation 8.13, there is $\widehat{\mathcal{P}}$ which is represented with respect to $\mathbf{K}$ and such that $\widehat{\mathcal{P}}{\restriction}k = \mathcal{P}$. We are assuming that $\mathbf{K}$ has the independent amalgamation property. Hence, there are a represented $\mathcal{C}$, with respect to $\mathbf{K}$, and embeddings $\tau_i : \widehat{\mathcal{P}} \to \mathcal{C}$, for $i = 1, 2$, such that $\tau_1{\restriction}|\mathcal{S}_\mathcal{P}| = \tau_2{\restriction}|\mathcal{S}_\mathcal{P}|$ and $|\mathcal{S}_\mathcal{P}| = \tau_1(|\mathcal{P}|) \cap \tau_2(|\mathcal{P}|)$; so in particular $\mathcal{S}_\mathcal{P} \subset_{cl} \mathcal{C}$. By replacing $\mathcal{C}$ with the closure of $\tau_1(|\widehat{\mathcal{P}}|) \cup \tau_2(|\widehat{\mathcal{P}}|)$ in $\mathcal{C}$, we may assume that $\dim_\mathcal{C}(|\mathcal{C}|) = 2\dim_\mathcal{P}(|\mathcal{P}|) - \dim_{\mathcal{S}_\mathcal{P}}(|\mathcal{S}_\mathcal{P}|) = 2\dim_\mathcal{P}(|\mathcal{P}|) - k$. Let $c = \dim_\mathcal{C}(|\mathcal{C}|)$. Since $\dim_\mathcal{P}(|\mathcal{P}|) > k > 0$ (as noted above), we have $c > \dim_\mathcal{P}(|\mathcal{P}|) > k > 0$, so $c \geq 3$.

If $k = 1$ then let $\mathcal{U}$ be the unique closed proper substructure of $\mathcal{S}_\mathcal{F}$ with dimension 0. If $k > 1$ then let $\mathcal{U}$ be any closed proper substructure of $\mathcal{S}_\mathcal{F}$ with dimension 1. In both cases $\mathcal{U}$ is represented with respect to $\mathbf{K}$, with respect to $\mathbf{K}{\restriction}k$, and with respect to $\mathbf{K}{\restriction}k-1$.

Let φ denote the $\mathcal{S}_\mathcal{F}/\mathcal{U}$-extension axiom and let ψ denote the $\mathcal{C}/\mathcal{S}_\mathcal{P}$-extension axiom. We prove that $\lim_{n\to\infty} \delta_n(\varphi \wedge \psi) = 0$. Let $\mathcal{C}' = \mathcal{C}{\restriction}k$, so $\mathcal{C}'$ is represented with respect to $\mathbf{K}{\restriction}k$, and note that since the dimension of $\mathcal{S}_\mathcal{P}$ and of $\mathcal{S}_\mathcal{F}$ is k and $\mathcal{U} \subset_{cl} \mathcal{S}_\mathcal{F}$ we have $\mathcal{U}{\restriction}k = \mathcal{U}$, $\mathcal{S}_\mathcal{P}{\restriction}k = \mathcal{S}_\mathcal{P}$ and $\mathcal{S}_\mathcal{F}{\restriction}k = \mathcal{S}_\mathcal{F}$. The next lemma shows that instead of working with $\mathbf{K}$, φ and ψ we can work with $\mathbf{K}{\restriction}k$, the $\mathcal{S}_\mathcal{F}/\mathcal{U}$-extension axiom and the $\mathcal{C}'/\mathcal{S}_\mathcal{P}$-extension axiom.

Lemma 8.14. *Let p be the probability, with the measure δ_n, that a structure in $\mathbf{K}_n$ satisfies both the $\mathcal{S}_\mathcal{F}/\mathcal{U}$-extension axiom ($=\varphi$) and the $\mathcal{C}/\mathcal{S}_\mathcal{P}$-extension axiom ($=\psi$). Let q be the probability, with the measure $\mathbb{P}_{n,k}$, that a structure in $\mathbf{K}_n{\restriction}k$ satisfies both the $\mathcal{S}_\mathcal{F}/\mathcal{U}$-extension axiom and the $\mathcal{C}'/\mathcal{S}_\mathcal{P}$-extension axiom. Then $p \leq q$.*

Proof. Recall that $k \leq \rho$. By the definitions of $\mathbb{P}_{n,k}$ and δ_n, for every $\mathcal{M} \in \mathbf{K}_n{\restriction}k$,

$$\mathbb{P}_{n,k}(\mathcal{M}) = \delta_n\big(\{\mathcal{N} \in \mathbf{K}_n : \mathcal{N}{\restriction}k = \mathcal{M}\}\big).$$

As mentioned above, $\mathcal{S}_\mathcal{P}{\restriction}k = \mathcal{S}_\mathcal{P}$ and $\mathcal{S}_\mathcal{F}{\restriction}k = \mathcal{S}_\mathcal{F}$. So whenever $\mathcal{N} \in \mathbf{K}_n$ satisfies the $\mathcal{S}_\mathcal{F}/\mathcal{U}$-extension axiom, then $\mathcal{N}{\restriction}k$ satisfies the $\mathcal{S}_\mathcal{F}/\mathcal{U}$-extension axiom. And whenever $\mathcal{N} \in \mathbf{K}_n$ satisfies the $\mathcal{C}/\mathcal{S}_\mathcal{P}$-extension axiom, then $\mathcal{N}{\restriction}k$ satisfies the $\mathcal{C}'/\mathcal{S}_\mathcal{P}$-extension axiom. Therefore p cannot exceed q. □

By Lemma 8.14 it suffices to prove that

(6) there is $\beta < 1$ such that for all sufficiently large n the probability, with the measure $\mathbb{P}_{n,k}$, that a structure in $\mathbf{K}_n{\restriction}k$ satisfies both the $\mathcal{S}_\mathcal{F}/\mathcal{U}$-extension axiom and

the $\mathcal{C}'/\mathcal{S}_\mathcal{P}$-extension axiom does not exceed β; and if $k > 1$, then this probability tends to 0 as $n \to \infty$.

The claim (6) follows from the next two lemmas and the definition of the measures $\mathbb{P}_{n,r}$, $r \in \mathbb{N}$. Remember that c is the dimension of $\mathcal{C}$ (and of $\mathcal{C}'$).

Lemma 8.15. *The probability, with the measure $\mathbb{P}_{n,k-1}$, that a structure in $\mathbf{K}_n \restriction k-1$ is $(\sigma^c(\lambda_n), c)$-saturated, with respect to $\mathbf{K}\restriction k-1$, tends to 1 as $n \to \infty$.*

Lemma 8.16. *Let α be the number of represented structures with universe $|\mathcal{S}_\mathcal{F}|$. Suppose that $\mathcal{M} \in \mathbf{K}_n\restriction k-1$ is $(\sigma^c(\lambda_n), c)$-saturated with respect to $\mathbf{K}\restriction k-1$ and let*

$$\mathbf{E}_k(\mathcal{M}) = \big\{\mathcal{N} \in \mathbf{K}\restriction k : \mathcal{N}\restriction k-1 = \mathcal{M}\big\}.$$

(i) The proportion of structures in $\mathbf{E}_k(\mathcal{M})$ which satisfy both the $\mathcal{S}_\mathcal{F}/\mathcal{U}$-extension axiom and the $\mathcal{C}'/\mathcal{S}_\mathcal{P}$-extension axiom never exceeds $1 - 1/(1+\alpha)$.
(ii) If $k > 1$ then the proportion of structures in $\mathbf{E}_k(\mathcal{M})$ which satisfy both the $\mathcal{S}_\mathcal{F}/\mathcal{U}$-extension axiom and the $\mathcal{C}'/\mathcal{S}_\mathcal{P}$-extension axiom never exceeds $\alpha\,\|\mathcal{S}_\mathcal{F}\|\,/\sigma^c(\lambda_n)$. Note that this expression does not depend on $\mathcal{M}$ and approaches 0 as $n \to \infty$.

Proof of Lemma 8.15 Note that when saying that $\mathbf{K}\restriction k-1$ accepts r-substitutions over L_0 we only consider substitutions of the form $[\mathcal{A} \triangleright \mathcal{A}']$ where $\mathcal{A}$ and $\mathcal{A}'$ are represented with respect to $\mathbf{K}\restriction k-1$.

Let $\mathbf{K}'_n = \mathbf{K}_n\restriction k-1$ and $\mathbf{K}' = \mathbf{K}\restriction k-1$. Let $\mathbb{P}'_{n,0}$ be the uniform measure on $\mathbf{K}'_n\restriction 0$ ($= \mathbf{K}_n\restriction 0$) and for positive $r \in \mathbb{N}$, let $\mathbb{P}'_{n,r}$ be the $(\mathbf{K}'_n\restriction 0, \ldots, \mathbf{K}'_n\restriction r-1)$-conditional measure on $\mathbf{K}'_n\restriction r$. Observe that we have the following:

For $r \leq k-1$, $\mathbf{K}'_n\restriction r = \mathbf{K}_n\restriction r$ and $\mathbb{P}'_{n,r}$ coincides with $\mathbb{P}_{n,r}$

For $r \geq k-1$, $\mathbf{K}'_n\restriction r = \mathbf{K}'_n = \mathbf{K}_n\restriction k-1$ and $\mathbb{P}'_{n,r}$ coincides with $\mathbb{P}'_{n,k-1}$

As $c > k-1$, we in particular have

$\mathbf{K}'_n\restriction c = \mathbf{K}'_n = \mathbf{K}_n\restriction k-1$

and $\mathbb{P}'_{n,c}$ coincides with $\mathbb{P}'_{n,k-1}$ which in turn coincides with $\mathbb{P}_{n,k-1}$.

So $\mathbb{P}'_{n,c}$ and $\mathbb{P}_{n,k-1}$ are the same measure on $\mathbf{K}'_n\restriction c = \mathbf{K}_n\restriction k-1$. Thus, in order to prove Lemma 8.15 it suffices to show that the probability, with the measure $\mathbb{P}'_{n,c}$, that a structure in $\mathbf{K}'_n\restriction c$ is $(\sigma^c(\lambda_n), c)$-saturated, with respect to $\mathbf{K}'\restriction c$, tends to 1 as $n \to \infty$. If, for $n, r \in \mathbb{N}$, we let

$$\mathbf{X}'_{n,r} = \{\mathcal{M} \in \mathbf{K}'_n\restriction r : \mathcal{M} \text{ is } (\sigma^r(\lambda_n), c)\text{-saturated}\},$$

then the claim of Lemma 8.15 is that

$$\lim_{n\to\infty} \mathbb{P}'_{n,c}(\mathbf{X}'_{n,c}) = 1. \tag{7}$$

By assumption, $(\mathcal{G}_n : n \in \mathbb{N})$ is polynomially c-saturated, and, as mentioned in the beginning of the proof, $\mathbf{K}'$ ($= \mathbf{K}\restriction k-1$) accepts r-substitutions over L_0 for every $r \in \mathbb{N}$, so in particular for $r = c$. In other words, $\mathbf{K}'$ satisfies the same assumptions, with respect to $(\mathcal{G}_n : n \in \mathbb{N})$ and L_0, as $\mathbf{K}$ did in the proof of Theorem 7.31, and $\mathbf{P}'_{n,c}$ is the $(\mathbf{K}'_n\restriction 0, \ldots, \mathbf{K}'_n\restriction r-1)$-conditional measure on $\mathbf{K}'_n\restriction r$, where $\mathbf{K}'_n\restriction 0 = \mathbf{K}_n\restriction 0$. Therefore, the statement of (7) (and its underlying assumptions) is the same as the statement of (3) (and its underlying assumptions) if we replace $\mathbf{K}$, $\mathbb{P}_{n,k}$ and $\mathbf{X}_{n,k}$ by $\mathbf{K}'$, $\mathbb{P}'_{n,c}$ and $\mathbf{X}'_{n,c}$, respectively. Hence, (7) is proved in exactly the same way as (3), by just replacing $\mathbf{K}$, $\mathbb{P}_{n,r}$ and $\mathbf{X}_{n,r}$ with $\mathbf{K}'$, $\mathbb{P}'_{n,r}$ and $\mathbf{X}'_{n,r}$, respectively, for $n, r \in \mathbb{N}$. □

Proof of Lemma 8.16. Suppose that $\mathcal{M} \in \mathbf{K}_n\restriction k-1$ is $(\sigma^c(\lambda_n), c)$-saturated, with respect to $\mathbf{K}\restriction k-1$, and let

$$\mathbf{E}_k(\mathcal{M}) = \big\{\mathcal{N} \in \mathbf{K}\restriction k : \mathcal{N}\restriction k-1 = \mathcal{M}\big\}.$$

Let α be the number of represented structures, with respect to $\mathbf{K}{\restriction}k$, with universe $|\mathcal{S}_{\mathcal{P}}|$. It suffices to show that the proportion of structures in $\mathbf{E}_k(\mathcal{M})$ which satisfy both the $\mathcal{S}_{\mathcal{F}}/\mathcal{U}$-extension axiom and the $\mathcal{C}'/\mathcal{S}_{\mathcal{P}}$-extension axiom does not exceed $1 - 1/(1+\alpha)$; and if $k > 1$ then this proportion approaches 0 as $n \to \infty$. We will consider the cases $k = 0$ and $k > 0$ one by one.

First assume that $k = 1$. Then, by the choice of $\mathcal{U}$, $\mathcal{U}$ has dimension 0 and is represented, since it is a closed substructure of a represented structure. By assumption there is a unique, up to isomorphism, represented structure of dimension 0. Hence, every represented structure (with respect to $\mathbf{K}$, $\mathbf{K}{\restriction}k$ or $\mathbf{K}{\restriction}k-1$) contains a copy of $\mathcal{U}$. Therefore every $\mathcal{M} \in \mathbf{K}{\restriction}k$ which satisfies the $\mathcal{S}_{\mathcal{F}}/\mathcal{U}$-extension axiom contains a copy of $\mathcal{S}_{\mathcal{F}}$. Note that if $\mathcal{N} \in \mathbf{K}{\restriction}k$ satisfies the $\mathcal{C}'/\mathcal{S}_{\mathcal{P}}$-extension axiom, then the $\mathcal{P}/\mathcal{S}_{\mathcal{P}}$-multiplicity of $\mathcal{N}$ is at least 2. Now we can argue as in Section 5. More precisely, the proofs of Lemmas 5.2, 5.4 and 5.6 as well as the proof of part (i) of Theorem 3.17 carry over to the present context if we have the following in mind: The structures $\mathcal{S}_{\mathcal{P}}$, $\mathcal{S}_{\mathcal{F}}$, $\mathcal{P}$ and $\mathcal{F}$ play the same roles in the present context as in Section 5; in the present context 'closed substructures' play the role of 'substructures' in Section 5; dimension plays the role here that cardinality had in that section; and $\mathbf{E}_k(\mathcal{M})$ plays the role here that '$\mathbf{K}_n$' had in that section. In this way we can conclude that the proportion of $\mathcal{N} \in \mathbf{E}_k(\mathcal{M})$ which contain a copy of $\mathcal{S}_{\mathcal{F}}$ and satisfy the $\mathcal{C}'/\mathcal{S}_{\mathcal{P}}$-extension axiom never exceeds $1 - 1/(1+\alpha)$.

Now suppose that $k > 1$. Again, the reasoning from Section 5 carries over to the present context. Since we assume $k > 1$, $\mathcal{U}$ has dimension 1 and $\mathcal{U} \subset_{cl} \mathcal{S}_{\mathcal{F}}$. As noted earlier, $c > k > 1$. Since $\mathcal{M}$ is $(\sigma^c(\lambda_n), c)$-saturated, with respect to $\mathbf{K}{\restriction}k-1$, $\mathcal{M}$ contains at least $\sigma^c(\lambda_n)$ distinct copies of $\mathcal{U}$. Since $\mathcal{M}$ and every $\mathcal{N} \in \mathbf{E}_k(\mathcal{M})$ agree on all substructures of dimension at most $k-1 \geq 1$, it follows that every $\mathcal{N} \in \mathbf{E}_k(\mathcal{M})$ contains at least $\sigma^c(\lambda_n)$ distinct copies of $\mathcal{U}$. Suppose that $\mathcal{N} \in \mathbf{E}_k(\mathcal{M})$ satisfies both the $\mathcal{S}_{\mathcal{F}}/\mathcal{U}$-extension axiom and the $\mathcal{C}'/\mathcal{S}_{\mathcal{P}}$-extension axiom. First we notice that the satisfaction of the $\mathcal{S}_{\mathcal{F}}/\mathcal{U}$-extension axiom implies that $\mathcal{N}$ contains at least $\sigma^c(\lambda_n)/\|\mathcal{S}_{\mathcal{F}}\|$ distinct copies of $\mathcal{S}_{\mathcal{F}}$ (the copies may *partially* overlap, but this poses no problem). Secondly, the satisfaction of the $\mathcal{C}'/\mathcal{S}_{\mathcal{P}}$-extension axiom implies that the $\mathcal{P}/\mathcal{S}_{\mathcal{P}}$-multiplicity of $\mathcal{N}$ is at least 2.

As in the previous case (when $k = 1$) the proofs of lemmas 5.2, 5.4 and 5.6 carry over – with the already mentioned provisos – to this context. But we are now able to continue the argument similarly as in the proof of part (iii) of Theorem 3.17. The number $\sigma^c(\lambda_n)$ plays the same role here as the number 'm_n' did in the proof of part (iii) of Theorem 3.17. In a similar way as in that proof we can now derive that '$\alpha \|\mathcal{S}_{\mathcal{F}}\| / \sigma^c(\lambda_n)$' (instead of '$k\alpha/m_n$' as in the proof of part (iii) of Theorem 3.17) is an upper bound for the proportion of $\mathcal{N} \in \mathbf{E}_k(\mathcal{M})$ such that $\mathcal{N}$ satisfies the $\mathcal{S}_{\mathcal{F}}/\mathcal{U}$-extension axiom and the $\mathcal{P}/\mathcal{S}_{\mathcal{P}}$-multiplicity of $\mathcal{N}$ is at least 2. □

9. Random l-colourable structures

In this section and the next we consider l-colourable, as well as *strongly* l-colourable, relational structures and zero-one laws for these, with the uniform probability measure and with a measure which is derived from the dimension conditional measure with trivial underlying pregeometry. In all cases we have a zero-one law, and we get the same almost sure theory whether we work with the uniform probability measure or with the probability measure derived from the dimension conditional measure. (The notions 'zero-one law' and 'almost sure theory' are explained in Section 2.4.) In the case when one considers the probability measure derived from the dimension conditional measure the proof only uses methods of formal logic, while in the case when one considers the uniform probability measure the proof uses, in addition, results about the typical distribution of colours, which are proved by combinatorial arguments. Therefore, we start, in this section, by considering the probability measure derived from the dimension conditional measure. In Section 10 we state the corresponding results for the uniform probability measure and complete their proofs.

In this section, $l \geq 2$ is a fixed integer and for each $n \in \mathbb{N}$, $\mathbf{K}_n$ is defined as in Example 7.22 for $F = \{1\}$, and $\mathbf{SK}_n$ is defined as $\mathbf{K}_n$ in Example 7.23 for $F = \{1\}$. Note that '$F = \{1\}$' means that the universe of every $\mathcal{M} \in \mathbf{K}_n$ is $\{1, \ldots, n\}$ and that the pregeometry is trivial (i.e. $\mathrm{cl}_{\mathcal{M}}(X) = X$ for every $\mathcal{M} \in \mathbf{K}_n$ and every $X \subseteq M$). As usual, let $\mathbf{K} = \bigcup_{n\in\mathbb{N}} \mathbf{K}_n$ and $\mathbf{SK} = \bigcup_{n\in\mathbb{N}} \mathbf{SK}_n$. The notation L_{col} (the language of the l colours), L_{rel} (the language of relations) and L mean the same as in the mentioned examples. *But we add the assumption that all relation symbols of the vocabulary of L_{rel} have arity at least 2.* (Colouring unary relations is not so interesting.) *When working with strong l-colourings, that is, with $\mathbf{SK}$, we also assume that l is at least as great as the arity of every relation symbol in the vocabulary of L_{rel}*; for otherwise the interpretations of some relation symbol(s) will be empty for all l-coloured structures, and then there is no point in having this (or these) relation symbol(s). Observe that if there are no relation symbols of arity greater than 2, then $\mathbf{K} = \mathbf{SK}$, as the pregeometry is trivial. A structure which is isomorphic with one in $\mathbf{K}$ is called ***l-coloured***. A structure which is isomorphic with one in $\mathbf{SK}$ is called ***strongly l-coloured***. Note that being l-coloured (strongly l-coloured) is equivalent to being represented with respect to $\mathbf{K}$ ($\mathbf{SK}$).

For each n, let

$$\mathbf{C}_n = \big\{\mathcal{M}{\restriction}L_{rel} : \mathcal{M} \in \mathbf{K}_n\big\}, \qquad \mathbf{C} = \bigcup_{n\in\mathbb{N}} \mathbf{C}_n,$$

$$\mathbf{S}_n = \big\{\mathcal{M}{\restriction}L_{rel} : \mathcal{M} \in \mathbf{SK}_n\big\} \quad \text{and} \quad \mathbf{S} = \bigcup_{n\in\mathbb{N}} \mathbf{S}_n.$$

A structure which is isomorphic to one in $\mathbf{C}$ (i.e. represented with respect to $\mathbf{C}$) will be called ***l-colourable***. A structures which is isomorphic to one in $\mathbf{S}$ will be called ***strongly l-colourable***. It is clear that an L_{rel}-structure $\mathcal{M}$ is (strongly) l-colourable if and only if there is a function $f : M \to \{1, \ldots, l\}$, called an ***(strong) l-colouring***, such that the expansion $\mathcal{M}'$ of $\mathcal{M}$ to L, defined by $\mathcal{M}' \models P_i(a)$ if and only if $f(a) = i$, is isomorphic with a member of $\mathbf{K}$ ($\mathbf{SK}$). Therefore we can, when convenient, use (strong) l-colouring functions instead of the relation symbols $P_1, \ldots, P_l$ to represent (strong) l-colourings.

In this section, $\delta_n^{\mathbf{K}}$ denotes the dimension conditional measure on $\mathbf{K}_n$ and $\delta_n^{\mathbf{SK}}$ denotes the dimension conditional measure on $\mathbf{SK}_n$ (see Definition 7.15). For each n, we consider the measures, $\delta_n^{\mathbf{C}}$ on $\mathbf{C}_n$ and $\delta_n^{\mathbf{S}}$ on $\mathbf{S}_n$ which are inherited from $\mathbf{K}_n$ and $\mathbf{SK}_n$, respectively, in the following sense:

$$\text{For every } \mathbf{X} \subseteq \mathbf{C}_n, \quad \delta_n^{\mathbf{C}}(\mathbf{X}) = \delta_n^{\mathbf{K}}\big(\{\mathcal{M} \in \mathbf{K}_n : \mathcal{M}{\restriction}L_{rel} \in \mathbf{X}\}\big).$$
$$\text{For every } \mathbf{X} \subseteq \mathbf{S}_n, \quad \delta_n^{\mathbf{S}}(\mathbf{X}) = \delta_n^{\mathbf{SK}}\big(\{\mathcal{M} \in \mathbf{SK}_n : \mathcal{M}{\restriction}L_{rel} \in \mathbf{X}\}\big).$$

For every L_{rel}-sentence φ, let $\delta_n^{\mathbf{C}}(\varphi) = \delta_n^{\mathbf{C}}\big(\{\mathcal{M} \in \mathbf{C}_n : \mathcal{M} \models \varphi\}\big)$ and
$\delta_n^{\mathbf{S}}(\varphi) = \delta_n^{\mathbf{S}}\big(\{\mathcal{M} \in \mathbf{S}_n : \mathcal{M} \models \varphi\}\big)$.

Theorem 9.1. *For every sentence $\varphi \in L_{rel}$,*

(i) $\lim_{n\to\infty} \delta_n^{\mathbf{C}}(\varphi) = 0$ *or* $\lim_{n\to\infty} \delta_n^{\mathbf{C}}(\varphi) = 1$, *and*
(ii) $\lim_{n\to\infty} \delta_n^{\mathbf{S}}(\varphi) = 0$ *or* $\lim_{n\to\infty} \delta_n^{\mathbf{S}}(\varphi) = 1$.

Theorem 9.1 will be proved in Section 9.1. We also state the corresponding theorem for the uniform probability measure, although it will be restated, with more detail as Theorems 10.3 and 10.4, in Section 10 where its proof will be completed.

Theorem 9.2. *For every sentence $\varphi \in L_{rel}$ the following holds:*

(i) *The proportion of $\mathcal{M} \in \mathbf{C}_n$ in which φ is true approaches either 0 or 1, as n approaches infinity.*

(ii) *The proportion of $\mathcal{M} \in \mathbf{S}_n$ in which φ is true approaches either 0 or 1, as n approaches infinity.*

Remark 9.3. (i) Let the relation symbols of L_{rel} be $R_1, \ldots, R_\rho$ and let $I \subseteq \{1, \ldots, \rho\}$. If we add the restriction that for every $i \in I$, R_i is always interpreted as an irreflexive and symmetric relation (see Remark 2.1), then Theorems 9.1, 9.2 and Proposition 9.20 still hold. The proofs in this section are exactly the same even if we add this extra assumption. But the combinatorial arguments in Section 10, needed to complete the proof of Theorem 9.2 are sensitive to whether a relation symbol is always interpreted as an irreflexive and symmetric relation, or not. For this reason the notation in Section 10 (but not in this section) specifies which relation symbols are always interpreted as irreflexive and symmetric relations.

(ii) It is open whether Theorems 9.1 and 9.2 still hold if F is allowed to be a (fixed) finite field, thus giving a nontrivial underlying pregeometry, and $\mathbf{K}_n$, $\mathbf{SK}_n$, $\mathbf{C}_n$ and $\mathbf{S}_n$ are, apart from this difference, defined as before.

9.1. **Proof of Theorem 9.1.** The proof depends on Theorem 7.31 which is used in the proof of Lemma 9.7 below. Apart from Lemmas 9.5 and 9.9 below, the proof is the same, except for obvious changes of notation, in the case of $\mathbf{S}$ (strongly l-colourable structures) as in the case of $\mathbf{C}$ (l-colourable structures). For this reason, and to avoid cluttering notation and language, we prove Theorem 9.1 by speaking of $\mathbf{K}_n$, $\mathbf{C}_n$, l-coloured structures and l-colourable structures. Only when proving Lemmas 9.5 and 9.9 will we separate the two cases explicitly.

The general pattern of the proof is a familiar one. We collect into a theory $T_{\mathbf{C}}$ a certain type of extension axioms (to be called 'l-colour compatible extension axioms') together with sentences which describe all possible isomorphism types of structures in $\mathbf{C}$. Then we show that for every $\psi \in T_{\mathbf{C}}$, $\lim_{n\to\infty} \delta_n^{\mathbf{C}}(\psi) = 1$, which implies (via compactness) that $T_{\mathbf{C}}$ is consistent. After this we show that $T_{\mathbf{C}}$ is complete by showing that it is countably categorical. The zero-one law is now a straightforward consequence of the previously proven facts, together with compactness.

Remark 9.4. We can *not* expect that for every extension axiom φ of $\mathbf{C}$, $\lim_{n\to\infty} \delta_n^{\mathbf{C}}(\varphi) = 1$. For example, suppose that the vocabulary of L_{rel} contains only one relation symbol which is binary (which implies $\mathbf{S} = \mathbf{C}$), and that $l = 2$. Then there is *no* 2-colourable L_{rel}-structure which satisfies all 3-extension axioms of $\mathbf{C}$. For if $\mathcal{M}$ would be such a structure, then it is easy to see that $\mathcal{M}$ would contain a 3-cycle or a 5-cycle (it does not matter if it is directed or not) which contradicts that $\mathcal{M}$ is 2-colourable.

In order to define the type of extension axioms that are useful in this context, we need to find a way of expressing, with an L_{rel}-formula, that two elements in an L-structure have the same colour. In fact, it suffices to find an L_{rel}-formula $\xi(y, z)$ such that with $\delta_n^{\mathbf{K}}$-probability approaching 1 as $n \to \infty$: if $\mathcal{M} \in \mathbf{K}_n$ and $a, b \in M$, then $\mathcal{M} \models \xi(a, b)$ if and only if a and b have the same colour in $\mathcal{M}$. The following lemma is a first step in that direction:

Lemma 9.5. *There is an (strongly) l-colourable structure $\mathcal{S}$ and distinct $a, b \in S$ such that the following hold:*

(a) *Whenever $\gamma : S \to \{1, \ldots, l\}$ is an (strong) l-colouring of $\mathcal{S}$, then $\gamma(a) = \gamma(b)$; in other words, whenever $\mathcal{S}$ is (strongly) l-coloured then a and b get the same colour.*

(b) *For every $i \in \{1, \ldots, l\}$, there is an (strong) l-colouring $\gamma_i : S \to \{1, \ldots, l\}$ of $\mathcal{S}$ such that $\gamma(a) = \gamma(b) = i$.*

Proof. We must treat the case of $\mathbf{C}$, i.e. l-colourable structures, and the case of $\mathbf{S}$, i.e. strongly l-colourable structures, separately. We start with the case of $\mathbf{C}$. By assumption all relation symbols in the vocabulary of L_{rel} have arity at least 2. Let r be the minimum of the arities of relation symbols in the vocabulary of L_{rel}, so $r \geq 2$, and let R be a relation symbol in the vocabulary of L_{rel} which has arity r. Let $S = \{0, 1, \ldots, (r-1)l\}$ and let $R^{\mathcal{S}}$ consist exactly of all tuples $(s_1, \ldots, s_r)$ of *distinct* elements from S such that

$$\{s_1, \ldots, s_r\} \subseteq S - \{0\} \quad \text{or} \quad \{s_1, \ldots, s_r\} \subseteq S - \{1\}.$$

For all other relation symbols Q of the vocabulary of L_{rel}, let $Q^{\mathcal{S}} = \emptyset$. Note that there is no relationship in $\mathcal{S}$ which contains both 0 and 1.

We first show that there is a colouring $\gamma : S \to \{1, \ldots, l\}$ of $\mathcal{S}$ such that $\gamma(0) = \gamma(1) = 1$. This will prove (b), because any permutation of the colours of an l-colouring gives a new l-colouring. Let both 0 and 1 be assigned the colour 1. Then assign the colour 1 to exactly $r - 2$ elements $s_1, \ldots, s_{r-2} \in S - \{0, 1\}$. So exactly r elements of $S = \{0, 1, \ldots, (r-1)l\}$ have been assigned the colour 1; and these elements are $0, 1, s_1, \ldots, s_{r-2}$. Hence

$$\big|\{S - \{0, 1, s_1, \ldots, s_{r-2}\}\big| = (r-1)l + 1 - r = (r-1)(l-1),$$

so $S - \{0, 1, s_1, \ldots, s_{r-2}\}$ can be partitioned into $l-1$ parts each of which contains exactly $r-1$ elements. Consequently, we can, for each colour $i \in \{2, \ldots, l\}$, assign the colour i to exactly $r-1$ elements in $S - \{0, 1, s_1, \ldots, s_{r-2}\}$. Since no colour other than 1 has been assigned to more that $r-1$ elements, the result is an l-colouring of $\mathcal{S}$.

We now prove (a). Assume that $\gamma : S \to \{1, \ldots, l\}$ is a colouring of $\mathcal{S}$. Note that $\big|S - \{0\}\big| = \big|S - \{1\}\big| = (r-1)l$. By the definition of $\mathcal{S}$, every r-tuple of distinct elements $(s_1, \ldots, s_r) \in (S - \{0\})^r$ is an R-relationship. Hence, for every colour $i \in \{1, \ldots, l\}$, we must have $\big|\gamma^{-1}(i) \cap (S - \{0\})\big| = r - 1$. Suppose that $\gamma(1) = 1$. (If $\gamma(1) \in \{2, \ldots, l\}$ the argument is analogous.) Assume, for a contradiction, that $\gamma(0) = i \neq 1$. Above we concluded that $\big|\gamma^{-1}(i) \cap (S - \{0\})\big| = r - 1$. Since $\gamma(0) = i$ we get $\big|\gamma^{-1}(i)\big| = r$, and as $\gamma(1) \neq i$, we get $\gamma^{-1}(i) \subseteq S - \{1\}$. Hence, there are distinct $s_1, \ldots, s_r \in \gamma^{-1}(i) \subseteq S - \{1\}$. By the definition of $\mathcal{S}$, $(s_1, \ldots, s_r) \in R^{\mathcal{S}}$. Since γ assigns all elements $s_1, \ldots, s_r$ the colour i, this contradicts that γ is a colouring of $\mathcal{S}$. So if we take $a = 0$ and $b = 1$, then the lemma holds for this $\mathcal{S}$ in the case of (not necessarily strong) l-colourings.

Now we prove the lemma in the case of strong l-colourings. Let $S = \{0, 1, \ldots, l\}$. Let R be any symbol from the vocabulary of L_{rel}, so the arity r of R is at least 2. By assumption, since we work with strong l-colourings now, $2 \leq r \leq l$. Let $R^{\mathcal{S}}$ consist exactly of all tuples $(s_1, \ldots, s_r)$ of *distinct* elements from S such that

$$\{s_1, \ldots, s_r\} \subseteq S - \{0\} \quad \text{or} \quad \{s_1, \ldots, s_r\} \subseteq S - \{1\}.$$

For all other relation symbols Q of the vocabulary of L_{rel}, let $Q^{\mathcal{S}} = \emptyset$. Note that there is no relationship in $\mathcal{S}$ which contains both 0 and 1. Therefore any assignment of the same colour $i \in \{1, \ldots, l\}$ to 0 and 1 can be extended to a strong l-colouring of $\mathcal{S}$. Also note that every strong l-colouring of $\mathcal{S}$ must give all elements in $S - \{0\}$ different colours; and it must give all elements in $S - \{1\}$ different colours. Since $|S| = l - 1$ there is no other choice but giving 0 and 1 the same colour. Hence the lemma, in the case of strong l-colourings, holds for this $\mathcal{S}$ with $a = 0$ and $b = 1$. $\square$

Notation 9.6. (i) Let $\mathcal{S}$ be an l-colourable structure and $a, b \in S$ distinct elements such that Lemma 9.5 is satisfied. Note that we must have $|S| \geq 3$. Without loss of generality we assume that $|\mathcal{S}| = S = \{1, \ldots, s\}$ for some $s \geq 3$ and that $a = s - 1$ and $b = s$. Hence every assignment of the same colour to $s - 1$ and s can be extended to an l-colouring of $\mathcal{S}$, and every l-colouring of $\mathcal{S}$ gives $s - 1$ and s the same colour.
(ii) Let $\chi_{\mathcal{S}}(x_1, \ldots, x_s)$ be a quantifier-free L_{rel}-formula which expresses the L_{rel}-isomorphism type of $\mathcal{S}$; more precisely, for every L_{rel}-structure $\mathcal{M}$ and all $a_1, \ldots, a_s \in M$,

$\mathcal{M} \models \chi_{\mathcal{S}}(a_1, \ldots, a_s)$ if and only if the map $a_i \mapsto i$ is an isomorphism from $\mathcal{M} \restriction \{a_1, \ldots, a_s\}$ to $\mathcal{S}$.
(iii) Let $\xi(y, z)$ be the formula

$$y = z \ \vee \ \exists u_1, \ldots, u_{s-2} \chi_{\mathcal{S}}(u_1, \ldots, u_{s-2}, y, z).$$

(iv) For $n, k \in \mathbb{N}$ let $\mathbf{X}_{n,k} \subseteq \mathbf{K}_n$ be the set of all $\mathcal{M} \in \mathbf{K}_n$ which satisfy all k-extension axioms with respect to $\mathbf{K}$.

Lemma 9.7. *For every $k \in \mathbb{N}$, $\lim_{n\to\infty} \delta_n^{\mathbf{K}}(\mathbf{X}_{n,k}) = 1$.*

Proof. As mentioned in Examples 7.9 and 7.22, for every $k \in \mathbb{N}$, the trivial pregeometry is polynomially k-saturated and $\mathbf{K}$ accepts k-substitutions over the language with empty vocabulary. By Theorem 7.31 (i), for every extension axiom φ of $\mathbf{K}$, $\lim_{n\to\infty} \delta_n^{\mathbf{K}}(\varphi) = 1$. The lemma follows since there are only finitely many k-extension axioms. (In the case of strongly l-colourable structures we look back at Example 7.23 instead of Example 7.22.) $\square$

Lemma 9.8. *Let $\mathcal{M} \in \mathbf{K}$ and $a, b \in M$.*
(i) If $\mathcal{M} \models \xi(a, b)$ then a and b have the same colour in $\mathcal{M}$, i.e. for some $i \in \{1, \ldots, l\}$, $\mathcal{M} \models P_i(a) \wedge P_i(b)$.
(ii) If $k \geq \|\mathcal{S}\|$ and $\mathcal{M} \in \mathbf{X}_{n,k}$, then $\mathcal{M} \models \xi(a, b)$ if and only if a and b have the same colour in $\mathcal{M}$.

Proof. (i) Suppose that $\mathcal{M} \in \mathbf{K}$ and $\mathcal{M} \models \xi(a, b)$. If $a = b$ then a and b have the same colour, so suppose that $a \neq b$. Then there are $m_1, \ldots, m_{s-2} \in M$ such that

$$\mathcal{M} \models \chi_{\mathcal{S}}(m_1, \ldots, m_{s-2}, a, b).$$

It follows that the L_{rel}-reduct of $\mathcal{M} \restriction \{m_1, \ldots, m_{s-2}, a, b\}$ is isomorphic with $\mathcal{S}$ via the L_{rel}-isomorphism $m_i \mapsto i$, for $i = 1, \ldots, s-2$, $a \mapsto s-1$ and $b \mapsto s$. Then we get an l-colouring of $\mathcal{S}$ by letting i get the same colour as m_i, for $i = 1, \ldots, s-2$, letting $s-1$ get the same colour as a, and letting s get the same colour as b. From Lemma 9.5 it follows that $s-1$ and s must have the same colour in $\mathcal{S}$; hence a and b must have the same colour in $\mathcal{M}$.

(ii) Let $k \geq \|\mathcal{S}\|$ and $\mathcal{M} \in \mathbf{X}_{n,k}$. If $a = b$ then immediately from the definition of $\xi(y, z)$ we get $\mathcal{M} \models \xi(a, b)$. Suppose that $a, b \in M$ are distinct elements which have the same colour in $\mathcal{M}$, that is, for some colour $i \in \{1, \ldots, l\}$, $\mathcal{M} \models P_i(a) \wedge P_i(b)$. By Lemma 9.5 and Notation 9.6, there is an l-coloured structure $\mathcal{S}_i$ such that $\mathcal{S}_i \restriction L_{rel} = \mathcal{S}$ and $\mathcal{S}_i \models P_i(s-1) \wedge P_i(s)$. Let $\mathcal{S}'_i = \mathcal{S}_i \restriction \{s-1, s\}$. Since a and b have the same colour in $\mathcal{M}$, there is no binary relationship of $\mathcal{M}$ which includes both a and b. Hence $\mathcal{M} \restriction \{a, b\}$ has no other relationships than the colour of a and of b which is i in both cases. By the properties of $\mathcal{S}$ (given by Lemma 9.5 and Notation 9.6), $\mathcal{S}'_i$ has no other relationships than the colour of $s-1$ and of s which is i in both cases. Hence, any bijection between $\{s-1, s\}$ and $\{a, b\}$ is an isomorphism between $\mathcal{S}'_i$ and $\mathcal{M} \restriction \{a, b\}$. Since $\mathcal{M} \in \mathbf{X}_{n,k}$ and $k \geq \|\mathcal{S}\|$, it follows that $\mathcal{M}$ satisfies the $\mathcal{S}_i/\mathcal{S}'_i$-extension axiom. This implies that there are $m_1, \ldots, m_{s-2} \in M$ such that the map $m_i \mapsto i$, for $i = 1, \ldots, s-2$, $a \mapsto s-1$ and $b \mapsto s$, is an isomorphism from $\mathcal{M} \restriction \{m_1, \ldots, m_{s-2}, a, b\}$ to $\mathcal{S}_i$. Since $\mathcal{S}_i \restriction L_{rel} = \mathcal{S}$ we get $\mathcal{M} \models \chi_{\mathcal{S}}(m_1, \ldots, m_{s-2}, a, b)$, so $\mathcal{M} \models \xi(a, b)$. $\square$

Besides being able to express (with high probability) with the L_{rel}-formula $\xi(y, z)$ that two elements have the same colour, we also need to be able to represent colours by elements (having those colours) in a structure, and we must be able to define such elements with an L_{rel}-formula. This is taken care of by Lemma 9.9, Notation 9.10 and Lemma 9.11, below. In some more detail, the structure $\mathcal{U}$ in the next lemma will help us to define an L_{rel}-formula $\zeta(x_1, \ldots, x_u)$, in Notation 9.10, where $u \geq l$, such that if

$\mathcal{M} \in \mathbf{X}_{n,k}$, then $\mathcal{M} \models \exists x_1, \ldots, x_u \zeta(x_1, \ldots, x_u)$ and if $\mathcal{M} \models \zeta(a_1, \ldots, a_u)$, then the first l elements $a_1, \ldots, a_l$ have different colours in $\mathcal{M}$. The formula ζ will be used (before Lemma 9.12) when we define a restricted version of extension axioms for $\mathbf{C}$, the 'l-colour compatible extension axioms'.

Lemma 9.9. *There is an (strongly) l-colourable structure $\mathcal{U}$ which is not (strongly) $(l-1)$-colourable and such that $\|\mathcal{U}\|$ is divisible by l and every partition of $|\mathcal{U}|$ into l parts of equal size gives rise to an (strong) l-colouring of $\mathcal{U}$.*

Proof. We deal with the cases of l-colourings and strong l-colourings separately and begin with the case of l-colourings. Let R be a relation symbol from the vocabulary of L_{rel}, so the arity, call it r, of R is at least 2. Recall that $l \geq 2$. Let $\mathcal{U}$ be the L_{rel}-structure with universe $U = \{1, \ldots, l(r-1)\}$, where

$$R^{\mathcal{U}} = \big\{(u_1, \ldots, u_r) \in U^r : i \neq j \Rightarrow u_i \neq u_j\big\},$$

and the interpretation of every other relation symbol is empty. Then U can be partitioned into l parts, each part with exactly $r-1$ elements. Hence every tuple $(u_1, \ldots, u_r) \in U^r$ of distinct elements must contain u_i and u_j from different parts of the partition. Consequently, $\mathcal{U}$ is l-colourable. However, if U is partitioned into $l-1$ parts, then at least one part must contain r distinct elements $u_1, \ldots, u_r$, and since $(u_1, \ldots, u_r) \in R^{\mathcal{U}}$, the partition does not represent an $(l-1)$-colouring of $\mathcal{U}$. Thus, $\mathcal{U}$ is not $(l-1)$-colourable.

The case of strong l-colourings is even simpler. Again we take any relation symbol R from the vocabulary of L_{rel}. Its arity, say r, is by assumption at least 2. By the extra assumption when dealing with strongly l-colourable structures we in fact have $2 \leq r \leq l$. We then let $U = \{1, \ldots, l\}$ and define the interpretations in $\mathcal{U}$ as above. It is clear that $\mathcal{U}$ is strongly l-colourable, but not strongly $(l-1)$-colourable. □

Notation 9.10. (i) Let, according to Lemma 9.9, $\mathcal{U}$ be an l-colourable, but not $(l-1)$-colourable, structure such that $\|\mathcal{U}\|$ is divisible by l and every partition of $|\mathcal{U}|$ into l parts of equal size gives rise to an l-colouring of $\mathcal{U}$. Let the universe of $\mathcal{U}$ be $U = \{1, \ldots, u\}$, so $u \geq l$.
(ii) Let $\chi_{\mathcal{U}}(x_1, \ldots, x_u)$ be a quantifier-free L_{rel}-formula which expresses the isomorphism type of $\mathcal{U}$.
(iii) Let $\widehat{\mathcal{U}} \in \mathbf{K}$ be an expansion of $\mathcal{U}$, that is, $\widehat{\mathcal{U}}$ is an l-colouring of $\mathcal{U}$. Without loss of generality we may assume that the elements $1, \ldots, l \in U$ have different colours in $\widehat{\mathcal{U}}$.
(iv) Let I be the set of all unordered pairs $\{i, j\} \subseteq U$ such that i and j have the same colour in $\widehat{\mathcal{U}}$, and let $\zeta(x_1, \ldots, x_u)$ denote the formula

$$\chi_{\mathcal{U}}(x_1, \ldots, x_u) \ \wedge \bigwedge_{\{i,j\} \in I} \xi(x_i, x_j) \ \wedge \bigwedge_{\{i,j\} \notin I} \neg\, \xi(x_i, x_j).$$

Lemma 9.11. *(i) Suppose that $k \geq \max(\|\mathcal{S}\|, \|\mathcal{U}\|)$ and $\mathcal{M} \in \mathbf{X}_{n,k}$. Then*

$$\mathcal{M} \models \exists x_1, \ldots, x_u \zeta(x_1, \ldots, x_u),$$

$$\mathcal{M} \models \forall x_1, \ldots, x_u \Big(\zeta(x_1, \ldots, x_u) \to \bigwedge_{i<j\leq l} \neg\, \xi(x_i, x_j)\Big),$$

$$\mathcal{M} \models \forall y, x_1, \ldots, x_u \Big(\zeta(x_1, \ldots, x_u) \to \bigvee_{i=1}^{l} \xi(x_i, y)\Big),$$

$$\mathcal{M} \models \forall y \xi(y, y) \ \wedge \ \forall y_1, y_2 \big(\xi(y_1, y_2) \ \to \ \xi(y_2, y_1)\big), \ \textit{and}$$

$$\mathcal{M} \models \forall y_1, y_2, y_3 \big([\xi(y_1, y_2) \ \wedge \ \xi(y_2, y_3)] \ \to \ \xi(y_1, y_3)\big).$$

(ii) If ψ is any one of the sentences in part (i), then $\lim_{n\to\infty} \delta_n^{\mathbf{C}}(\psi) = 1$.

Proof. (i) Recall Notation 9.10 (iii). Since $k \geq \|\mathcal{U}\| = \left\|\widehat{\mathcal{U}}\right\|$ and $\mathcal{M} \in \mathbf{X}_{n,k}$, the $\widehat{\mathcal{U}}/\emptyset$-extension axiom is satisfied in $\mathcal{M}$, so there are $m_1, \ldots, m_u \in M$ such that the map $m_i \mapsto i$ is an isomorphism from $\mathcal{M}\restriction\{m_1, \ldots, m_u\}$ to $\widehat{\mathcal{U}}$, so in particular it preserves the colours. As $\mathcal{M} \in \mathbf{X}_{n,k}$ and $k \geq \|\mathcal{S}\|$, Lemma 9.8 implies that $\mathcal{M} \models \zeta(m_1, \ldots, m_u)$.

Now suppose that $b, a_1, \ldots, a_u \in M$ and $\mathcal{M} \models \zeta(a_1, \ldots, a_u)$, that is,

$$\mathcal{M} \models \chi_{\mathcal{U}}(a_1, \ldots, a_u) \ \wedge \bigwedge_{\{i,j\} \in I} \xi(a_i, a_j) \ \wedge \bigwedge_{\{i,j\} \notin I} \neg\, \xi(a_i, a_j).$$

Together with the definition of $\mathcal{U}$ and I (Notation 9.10), this implies that if $i < j \leq l$, then $\mathcal{M} \models \neg\, \xi(a_i, a_j)$. Since $\mathcal{M} \in \mathbf{X}_{n,k}$ and $k \geq \|\mathcal{S}\|$, Lemma 9.8 implies that if $i < j \leq l$, then a_i and a_j have different colours. Since there are only l colours, there is $i \leq l$ such that b has the same colour as a_i in $\mathcal{M}$. By Lemma 9.8 again, $\mathcal{M} \models \xi(a_i, b)$. So we have proved that

$$\mathcal{M} \models \forall x_1, \ldots, x_u \Big(\zeta(x_1, \ldots, x_u) \rightarrow \bigwedge_{i<j\leq l} \neg\, \xi(x_i, x_j) \Big), \quad \text{and}$$

$$\mathcal{M} \models \forall y, x_1, \ldots, x_u \Big(\zeta(x_1, \ldots, x_u) \rightarrow \bigvee_{i=1}^{l} \xi(x_i, y) \Big).$$

The relation 'y has the same colour as z' is an equivalence relation which under the given conditions is defined by $\xi(y, z)$ (by Lemma 9.8). This immediately implies the rest of part (i).

(ii) Let ψ be any one of the sentences in part (i). Since $\psi \in L_{rel}$ we have

$$\{\mathcal{M} \in \mathbf{C}_n : \mathcal{M} \models \psi\} = \{\mathcal{N}\restriction L_{rel} : \mathcal{N} \in \mathbf{K}_n \text{ and } \mathcal{N} \models \psi\},$$

so by the definition of $\delta_n^{\mathbf{C}}$ we get $\delta_n^{\mathbf{C}}(\psi) = \delta_n^{\mathbf{K}}(\psi)$, for every n. Therefore it suffices to show that $\lim_{n\to\infty} \delta_n^{\mathbf{K}}(\psi) = 1$. Take $k \geq \max(\|\mathcal{S}\|, \|\mathcal{U}\|)$. By Lemma 9.7, $\lim_{n\to\infty} \delta_n^{\mathbf{K}}(\mathbf{X}_{n,k}) = 1$. By part (i), for every n and every $\mathcal{M} \in \mathbf{X}_{n,k}$, $\mathcal{M}$ satisfies ψ, so $\delta_n^{\mathbf{K}}(\psi) \geq \delta_n^{\mathbf{K}}(\mathbf{X}_{n,k}) \to 1$, as $n \to \infty$. $\square$

Next, we define 'l-colour compatible extension axioms'. Suppose that $\mathcal{B}$ is l-colourable (and finite) and let $\mathcal{A} \subset \mathcal{B}$. Without loss of generality we assume that $A = \{1, \ldots, \alpha\}$ and $B = \{1, \ldots, \beta\}$, so $\alpha < \beta$. Let $\chi_{\mathcal{A}}(x_1, \ldots, x_\alpha)$ and $\chi_{\mathcal{B}}(x_1, \ldots, x_\beta)$ be quantifier-free L_{rel}-formulas which express the isomorphism types of $\mathcal{A}$ and $\mathcal{B}$, respectively; so for any L_{rel}-structure $\mathcal{M}$, $\mathcal{M} \models \chi_{\mathcal{A}}(m_1, \ldots, m_\alpha)$ if and only if the map $m_i \mapsto i$ is an isomorphism from $\mathcal{M}\restriction\{m_1, \ldots, m_\alpha\}$ to $\mathcal{A}$; and similarly for $\chi_{\mathcal{B}}$. Let us say that an l-colouring $\gamma : \{1, \ldots, \alpha\} \to \{1, \ldots, l\}$ of $\mathcal{A}$ is a ***$\mathcal{B}$-good colouring*** if it can be extended to an l-colouring $\gamma' : \{1, \ldots, \beta\} \to \{1, \ldots, l\}$ of $\mathcal{B}$ (i.e. $\gamma'\restriction A = \gamma$). Let $\gamma : \{1, \ldots, \alpha\} \to \{1, \ldots, l\}$ be a $\mathcal{B}$-good colouring of $\mathcal{A}$ and let $\gamma' : \{1, \ldots, \beta\} \to \{1, \ldots, l\}$ be any colouring of $\mathcal{B}$ that extends γ. Let τ be any permutation of $\{1, \ldots, l\}$. The idea in what follows is that, for $j \in \{1, \ldots, \alpha\}$, the colour of j is associated with the colour of the element which will be substituted for the variable $x_{\tau\gamma(j)}$, where $\tau\gamma(j) = \tau(\gamma(j))$. Let $\theta_{\gamma,\tau}(x_1, \ldots, x_l, y_1, \ldots, y_\alpha)$ be the conjunction of all $\xi(x_{\tau\gamma(j)}, y_j)$ where $j \in \{1, \ldots, \alpha\}$. Similarly, let $\theta_{\gamma',\tau}(x_1, \ldots, x_l, y_1, \ldots, y_\beta)$ be the conjunction of all $\xi(x_{\tau\gamma'(j)}, y_j)$ where $j \in \{1, \ldots, \beta\}$. We call the following sentence an ***instance of the l-colour compatible $\mathcal{B}/\mathcal{A}$-extension axiom***:

$$\begin{aligned} &\forall x_1, \ldots, x_u, y_1, \ldots, y_\alpha \exists y_{\alpha+1}, \ldots, y_\beta \big(\\ &\quad \big[\zeta(x_1, \ldots, x_u) \wedge \chi_{\mathcal{A}}(y_1, \ldots, y_\alpha) \ \wedge \ \theta_{\gamma,\tau}(x_1, \ldots, x_l, y_1, \ldots, y_\alpha)\big] \ \longrightarrow \\ &\qquad\qquad \big[\chi_{\mathcal{B}}(y_1, \ldots, y_\beta) \ \wedge \ \theta_{\gamma',\tau}(x_1, \ldots, x_l, y_1, \ldots, y_\beta)\big]\big). \end{aligned}$$

In the special case that $A = \emptyset$ and γ' is an arbitrary l-colouring of $\mathcal{B}$, the above formula should be interpreted as

$$\forall x_1, \ldots, x_u \exists y_1, \ldots, y_\beta \big(\zeta(x_1, \ldots, x_u) \ \longrightarrow \ \chi_{\mathcal{B}}(y_1, \ldots, y_\beta) \wedge \theta_{\gamma',\tau}(x_1, \ldots, x_l, y_1, \ldots, y_\beta)\big).$$

Since there are only finitely many l-colourings of any finite structure, there are only finitely many instances of the l-colour compatible $\mathcal{B}/\mathcal{A}$-extension axiom. The ***l-colour compatible $\mathcal{B}/\mathcal{A}$-extension axiom*** is, by definition, the conjunction of all instances of the l-colour compatible $\mathcal{B}/\mathcal{A}$-extension axiom. If $|\mathcal{B}| \leq k+1$ then the l-colour compatible $\mathcal{B}/\mathcal{A}$-extension axiom is also called an ***l-colour compatible k-extension axiom***.

Lemma 9.12. *Suppose that $\mathcal{B}$ is l-colourable (and finite) and let $\mathcal{A} \subset \mathcal{B}$. Let φ denote the l-colour compatible $\mathcal{B}/\mathcal{A}$-extension axiom. If $k \geq \max(\|\mathcal{S}\|, \|\mathcal{B}\|)$ and $\mathcal{M} \in \mathbf{X}_{n,k}$, then $\mathcal{M} \models \varphi$.*

Proof. Let $\mathcal{A}$, $\mathcal{B}$, φ, k and $\mathcal{M}$ satisfy the premisses of the lemma, so in particular $\mathcal{M} \in \mathbf{X}_{n,k} \subseteq \mathbf{K}_n$. We consider only the case when $\|\mathcal{A}\| \geq 1$, since the case when $\|\mathcal{A}\| = 0$ is analogous. Without loss of generality we assume that $A = \{1, \ldots, \alpha\}$ and $B = \{1, \ldots, \beta\}$ where $\alpha < \beta$. It suffices to prove that every instance of the l-colour compatible $\mathcal{B}/\mathcal{A}$-extension axiom is true in $\mathcal{M}$.

Let $\gamma : \{1, \ldots, \alpha\} \to \{1, \ldots, l\}$ be a $\mathcal{B}$-good l-colouring of $\mathcal{A}$ and let $\gamma' : \{1, \ldots, \beta\} \to \{1, \ldots, l\}$ be an l-colouring of $\mathcal{B}$ which extends γ. Also, let τ be a permutation of $\{1, \ldots, l\}$. We prove that $\mathcal{M}$ satisfies the following instance of the l-colour compatible $\mathcal{B}/\mathcal{A}$-extension axiom, where $\theta_{\gamma,\tau}$ is the conjunction of all $\xi(x_{\tau\gamma(j)}, y_j)$ where $j \in \{1, \ldots, \alpha\}$, and $\theta_{\gamma',\tau}$ is the conjunction of all $\xi(x_{\tau\gamma(j)}, y_j)$ where $j \in \{1, \ldots, \beta\}$:

$$\begin{aligned}
&\forall x_1, \ldots, x_u, y_1, \ldots, y_\alpha \exists y_{\alpha+1}, \ldots, y_\beta \big(\\
&\quad \big[\zeta(x_1, \ldots, x_u) \wedge \chi_{\mathcal{A}}(y_1, \ldots, y_\alpha) \ \wedge \ \theta_{\gamma,\tau}(x_1, \ldots, x_l, y_1, \ldots, y_\alpha)\big] \ \longrightarrow \\
&\qquad\qquad \big[\chi_{\mathcal{B}}(y_1, \ldots, y_\beta) \ \wedge \ \theta_{\gamma',\tau}(x_1, \ldots, x_l, y_1, \ldots, y_\beta)\big]\big).
\end{aligned}$$

Note that since $\mathcal{M} \in \mathbf{X}_{n,k}$ and $k \geq \max(\|\mathcal{S}\|, \|\mathcal{B}\|)$, we can, and will repeatedly, use Lemma 9.8 which implies that for all $a, b \in M$, $\mathcal{M} \models \xi(a, b)$ if and only if a and b have the same colour in $\mathcal{M}$.

Suppose that

$$\mathcal{M} \models \zeta(m_1, \ldots, m_u) \ \wedge \ \chi_{\mathcal{A}}(a_1, \ldots, a_\alpha) \ \wedge \ \theta_{\gamma,\tau}(m_1, \ldots, m_l, a_1, \ldots, a_\alpha).$$

By the definition of ζ (Notation 9.10 (iii), (iv)), if $i, j \leq l$ and $i \neq j$, then m_i and m_j have different colours. Hence, there is a permutation π of $\{1, \ldots, l\}$ such that, for every $i \in \{1, \ldots, l\}$, m_i has colour $\pi(i)$, i.e. $\mathcal{M} \models P_{\pi(i)}(m_i)$. Let $\widehat{\mathcal{B}} \in \mathbf{K}$ be the expansion of $\mathcal{B}$ such that,

$$\text{for every } j \in \{1, \ldots, \beta\}, \ \widehat{\mathcal{B}} \models P_{\pi\tau\gamma'(j)}(j).$$

In other words, $j \in \{1, \ldots, \beta\}$ gets the same colour in $\widehat{\mathcal{B}}$ as $m_{\tau\gamma'(j)}$ in $\mathcal{M}$, and this colour is $\pi\tau\gamma'(j)$. In particular, this holds whenever $j \leq \alpha$ and γ' is replaced by γ. Since we assume that

$$\mathcal{M} \models \chi_{\mathcal{A}}(a_1, \ldots, a_\alpha) \ \wedge \ \theta_{\gamma,\tau}(m_1, \ldots, m_l, a_1, \ldots, a_\alpha)$$

it follows that the map $j \mapsto a_j$, for $j \in \{1, \ldots, \alpha\}$, is an isomorphism from $\widehat{\mathcal{A}} = \widehat{\mathcal{B}} \restriction \{1, \ldots, \alpha\}$ to $\mathcal{M} \restriction \{a_1, \ldots, a_\alpha\}$. Since $\mathcal{M} \in \mathbf{X}_{n,k}$ and k is sufficiently large, $\mathcal{M}$ satisfies the $\widehat{\mathcal{B}}/\widehat{\mathcal{A}}$-extension axiom. Hence, there are $a_{\alpha+1}, \ldots, a_\beta \in M$ such that the map $j \mapsto a_j$, for $j \in \{1, \ldots, \beta\}$, is an isomorphism from $\widehat{\mathcal{B}}$ to $\mathcal{M} \restriction \{a_1, \ldots, a_\beta\}$. This implies that $\mathcal{M} \models \chi_{\mathcal{B}}(a_1, \ldots, a_\beta)$ and that, for all $j \in \{1, \ldots, \beta\}$, $\mathcal{M} \models P_{\pi\tau\gamma'(a_j)}(a_j)$, which means that a_j has the same colour as $m_{\tau\gamma'(j)}$, so $\mathcal{M} \models \xi(m_{\tau\gamma'(j)}, a_j)$. Hence

$$\mathcal{M} \models \theta_{\gamma',\tau}(m_1, \ldots, m_l, a_1, \ldots, a_\beta),$$

and we are done. $\square$

Corollary 9.13. *For every l-colour compatible extension axiom φ, $\lim_{n\to\infty}\delta_n^{\mathbf{C}}(\varphi) = 1$.*

Proof. Let φ be an l-colour compatible extension axiom. Since $\varphi \in L_{rel}$ we have

$$\{\mathcal{M} \in \mathbf{C}_n : \mathcal{M} \models \varphi\} = \{\mathcal{N}\restriction L_{rel} : \mathcal{N} \in \mathbf{K}_n \text{ and } \mathcal{N} \models \varphi\},$$

so by the definition of $\delta_n^{\mathcal{C}}$ we get $\delta_n^{\mathbf{C}}(\varphi) = \delta_n^{\mathbf{K}}(\varphi)$, for every n. Therefore it suffices to show that $\lim_{n\to\infty}\delta_n^{\mathbf{K}}(\varphi) = 1$. For some l-colourable L_{rel}-structures $\mathcal{A} \subset \mathcal{B}$, φ is the l-colour compatible $\mathcal{B}/\mathcal{A}$-extension axiom. Take $k \geq \max(\|\mathcal{S}\|, \|\mathcal{B}\|)$. By Lemma 9.7, $\lim_{n\to\infty}\delta_n^{\mathbf{K}}(\mathbf{X}_{n,k}) = 1$. By Lemma 9.12, for every n and every $\mathcal{M} \in \mathbf{X}_{n,k}$, $\mathcal{M}$ satisfies φ, so $\delta_n^{\mathbf{K}}(\varphi) \geq \delta_n^{\mathbf{K}}(\mathbf{X}_{n,k}) \to 1$, as $n \to \infty$. $\square$

For every integer $n > 0$ let $\mathcal{M}_{(n,1)}, \ldots, \mathcal{M}_{(n,m_n)}$ be an enumeration of all isomorphism types of l-colourable structures of cardinality at most n. Let $\chi_i^n(x_1, \ldots, x_n)$ describe the isomorphism type of $\mathcal{M}_{(n,i)}$ in such a way that we require that all variables $x_1, \ldots, x_n$ actually occur in χ_i^n. It means that if $\|\mathcal{M}_{(n,i)}\| < n$, then $\chi_i^n(x_1, \ldots, x_n)$ must express that some variables refer to the same element, by saying '$x_k = x_l$' for some $k \neq l$. For every $n \in \mathbb{N}$ let ψ_n denote the sentence

$$\forall x_1, \ldots, x_n \bigvee_{i=1}^{m_n} \bigvee_{\pi} \chi_i^n(x_{\pi(1)}, \ldots, x_{\pi(n)}),$$

where the second disjunction ranges over all permutations π of $\{1, \ldots, n\}$. Then let $T_{iso} = \{\psi_n : n \in \mathbb{N},\ n > 0\}$ and note that every ψ_n is true in every l-colourable structure. Let T_{ext} consist of all l-colour compatible extension axioms and let T_{col} consist of the sentences appearing in part (i) of Lemma 9.11. Finally, let $T_{\mathbf{C}} = T_{iso} \cup T_{ext} \cup T_{col}$. By part (ii) of Lemma 9.11, Corollary 9.13 and compactness, $T_{\mathbf{C}}$ is consistent. Since $T_{ext} \subset T_{\mathbf{C}}$, every model of $T_{\mathbf{C}}$ is infinite. In order to prove Theorem 9.1 it is enough to prove that $T_{\mathbf{C}}$ is complete.

Lemma 9.14. *$T_{\mathbf{C}}$ is countably categorical and therefore complete.*

Proof. Suppose that the L_{rel}-structures $\mathcal{M}$ and $\mathcal{M}'$ are countable models of $T_{\mathbf{C}}$. We will prove that $\mathcal{M} \cong \mathcal{M}'$ by a back and forth argument, but first we need some preparation. Recall that $T_{col} \subset T_{\mathbf{C}}$ and that T_{col} contains the formulas that appear in part (i) of Lemma 9.11. Therefore there are $m_1, \ldots, m_u \in M$ and $m'_1, \ldots, m'_u \in M'$ such that $\mathcal{M} \models \zeta(m_1, \ldots, m_u)$ and $\mathcal{M}' \models \zeta(m'_1, \ldots, m_u)$. Moreover, because of the sentences in T_{col}, the following hold:

- $\xi(y, z)$ defines an equivalence relation R_M on M and an equivalence relation $R_{M'}$ on M'.
- The elements $m_1, \ldots, m_l$ belong to different equivalence classes; the elements $m'_1, \ldots, m'_l$ belong to different equivalence classes.
- Every element in M is equivalent to one of $m_1, \ldots, m_l$, so R_M has exactly l equivalence classes; and the same is true for $m'_1, \ldots, m'_l$ and $R_{M'}$.

We prove that $\mathcal{M} \cong \mathcal{M}'$ by a back and forth argument in which partial isomorphisms between $\mathcal{M}$ and $\mathcal{M}'$ are extended step by step. It suffices to prove the following:

Claim. *Suppose that $\mathcal{A}$ and $\mathcal{A}'$ are finite substructures of $\mathcal{M}$ and $\mathcal{M}'$, respectively, and that f is an isomorphism from $\mathcal{A}$ to $\mathcal{A}'$ such that for all $a \in A$ and all $i \in \{1, \ldots, l\}$, $\mathcal{M} \models \xi(m_i, a) \iff \mathcal{M}' \models \xi(m'_i, f(a))$. For every $b \in M - A$ (or $b' \in M' - A'$), there are $b' \in M' - A'$ (or $b \in M - A$) and an isomorphism $g : \mathcal{M}\restriction A \cup \{b\} \to \mathcal{M}'\restriction A' \cup \{b'\}$ such that g extends f (so $g(b) = b'$) and, for every $i \in \{1, \ldots, l\}$, $\mathcal{M} \models \xi(m_i, b) \iff$*

$\mathcal{M}' \models \xi(m'_i, b')$.

Suppose that $\mathcal{A}$ and $\mathcal{A}'$ are finite substructures of $\mathcal{M}$ and $\mathcal{M}'$, respectively, and that f is an isomorphism from $\mathcal{A}$ to $\mathcal{A}'$ such that for all $a \in A$ and all $i \in \{1, \ldots, l\}$, $\mathcal{M} \models \xi(m_i, a) \iff \mathcal{M}' \models \xi(m'_i, f(a))$. Let $A = \{a_1, \ldots, a_\alpha\}$, let $b = a_{\alpha+1} \in M - A$ and let $\mathcal{B} = \mathcal{M}{\restriction}\{a_1, \ldots, a_{\alpha+1}\}$. We will find the required $b' \in M' - A'$ by defining a suitable instance of the l-colour compatible $\mathcal{B}/\mathcal{A}$-extension axiom and then use the assumption that $\mathcal{M}' \models T_{\mathbf{C}} \supset T_{ext}$.

Let

$$X = \{m_1, \ldots, m_u, a_1, \ldots, a_{\alpha+1}\}.$$

Since $\xi(y, z)$ is an existential formula (see Notation 9.6 (iii)), there is a *finite* substructure $\mathcal{N} \subset \mathcal{M}$ such that

$$X \subseteq N, \quad \text{and whenever} \quad c, d \in X, \quad \text{then} \quad \mathcal{M} \models \xi(c, d) \iff \mathcal{N} \models \xi(c, d).$$

Since N is finite and $\mathcal{N} \subset \mathcal{M} \models T_{\mathbf{C}} \supset T_{iso}$, $\mathcal{N}$ is l-colourable. Let $\gamma^* : N \to \{1, \ldots, l\}$ be an l-colouring of $\mathcal{N}$, and define an equivalence relation $\sim^*$ on N by:

$$c \sim^* d \iff \gamma^*(c) = \gamma^*(d).$$

By the choice of $\mathcal{N}$ and Lemma 9.8 (i), for all $c, d \in X$,

$$R_M(c, d) \iff \mathcal{M} \models \xi(c, d) \iff \mathcal{N} \models \xi(c, d) \implies \gamma^*(c) = \gamma^*(d) \iff c \sim^* d.$$

This means that the restriction of R_M to X is a refinement of the restriction of $\sim^*$ to X. We have already observed that the restriction of R_M to X has exactly l equivalence classes, because all $m_1, \ldots, m_l$ belong to different classes. Moreover, since $\mathcal{M} \models \zeta(m_1, \ldots, m_u)$ we get, by the definition of ζ, $\mathcal{M} \models \chi_{\mathcal{U}}(m_1, \ldots, m_u)$, and since $\mathcal{U}$ is l-colourable, but not $(l-1)$-colourable, it follows that $\sim^*$ has exactly l equivalence classes. It follows that the restriction of R_M to X is the *same* relation as the restriction of $\sim^*$ to X. Hence,

$$\text{for all} \quad c, d \in X, \ \mathcal{M} \models \xi(c, d) \iff \gamma^*(c) = \gamma^*(d).$$

Therefore, there is a permutation τ of $\{1, \ldots, l\}$ such that,

$$\text{for every} \quad j \in \{1, \ldots, \alpha + 1\}, \ \mathcal{M} \models \xi(m_{\tau\gamma^*(a_j)}, a_j).$$

Let $\gamma' = \gamma^*{\restriction}\{a_1, \ldots, a_{\alpha+1}\}$ and $\gamma = \gamma^*{\restriction}\{a_1, \ldots, a_\alpha\}$. Then let

$$\theta_{\gamma',\tau}(x_1, \ldots, x_l, y_1, \ldots, y_{\alpha+1})$$

be the conjunction of all $\xi(x_{\tau\gamma'(a_j)}, y_j)$ where $j \in \{1, \ldots, \alpha + 1\}$, and let

$$\theta_{\gamma,\tau}(x_1, \ldots, x_l, y_1, \ldots, y_\alpha)$$

be the conjunction of all $\xi(x_{\tau\gamma(a_j)}, y_j)$ where $j \in \{1, \ldots, \alpha\}$. Let $\chi_{\mathcal{A}}(y_1, \ldots, y_\alpha)$ and $\chi_{\mathcal{B}}(y_1, \ldots, y_{\alpha+1})$ be quantifier-free formulas which describe the isomorphism types of $\mathcal{A}$ and $\mathcal{B}$, respectively. Now the following is an instance of the l-colour compatible $\mathcal{B}/\mathcal{A}$-extension axiom:

$$\begin{aligned}
&\forall x_1, \ldots, x_u, y_1, \ldots, y_\alpha \exists y_{\alpha+1} \big(\\
&\quad \big[\zeta(x_1, \ldots, x_u) \wedge \chi_{\mathcal{A}}(y_1, \ldots, y_\alpha) \ \wedge \ \theta_{\gamma,\tau}(x_1, \ldots, x_l, y_1, \ldots, y_\alpha)\big] \ \longrightarrow \\
&\qquad\qquad \big[\chi_{\mathcal{B}}(y_1, \ldots, y_{\alpha+1}) \ \wedge \ \theta_{\gamma',\tau}(x_1, \ldots, x_l, y_1, \ldots, y_{\alpha+1})\big]\big).
\end{aligned}$$

Since

$$\mathcal{M} \models \zeta(m_1, \ldots, m_u) \ \wedge \ \chi_{\mathcal{A}}(a_1, \ldots, a_\alpha) \ \wedge \ \theta_{\gamma,\tau}(m_1, \ldots, m_l, a_1, \ldots, a_\alpha),$$

it follows from the assumptions that

$$\mathcal{M}' \models \zeta(m'_1, \ldots, m'_u) \ \wedge \ \chi_{\mathcal{A}}(f(a_1), \ldots, f(a_\alpha)) \ \wedge \ \theta_{\gamma,\tau}(m'_1, \ldots, m'_l, f(a_1), \ldots, f(a_\alpha)).$$

Since $\mathcal{M}'$ satisfies all l-colour compatible extension axioms it follows that there is $b' \in M' - A'$ such that if $g(a_{\alpha+1}) = b'$ and $g(a) = f(a)$ for all $a \in A$, then

$$\mathcal{M}' \models \chi_{\mathcal{B}}(g(a_1), \ldots, g(a_{\alpha+1})) \ \wedge \ \theta_{\gamma',\tau}(m'_1, \ldots, m'_l, g(a_1), \ldots, g(a_{\alpha+1})).$$

It follows that g is an isomorphism from $\mathcal{M}{\restriction}A \cup \{b\}$ to $\mathcal{M}'{\restriction}A' \cup \{b'\}$. Since $\mathcal{M} \models \xi(m'_i, b')$ for a unique $i \in \{1, \ldots, l\}$ (by the conclusions in the beginning of the proof), it also follows that, for every $i \in \{1, \ldots, l\}$, $\mathcal{M}' \models \xi(m'_i, b') \iff \mathcal{M} \models \xi(m_i, b)$, where $b = a_{\alpha+1}$.

Note that the argument also works for the 'base case' when $A = A' = \emptyset$ and f is the empty map; the difference is merely notational. If we start out with $b' \in M' - A'$, then we argue symmetrically. Thus the claim, and hence the lemma, is proved. □

By the preceeding lemmas, $T_{\mathbf{C}}$ is a complete theory such that whenever $m \in \mathbb{N}$ and $\psi_1, \ldots, \psi_m \in T_{\mathbf{C}}$, then $\lim_{n\to\infty} \delta_n^{\mathbf{C}}\Big(\bigwedge_{i=1}^m \psi_i\Big) = 1$. By compactness and completeness it follows that if $T_{\mathbf{C}} \models \varphi$, then $\lim_{n\to\infty} \delta_n^{\mathbf{C}}(\varphi) = 1$, and if $T_{\mathbf{C}} \not\models \varphi$, then $\lim_{n\to\infty} \delta_n^{\mathbf{C}}(\varphi) = 0$.

9.2. **Relationship between the dimension conditional measure and the uniform measure.** In this section we prove that the results that we have seen for the probability measures $\delta_n^{\mathbf{C}}$ and $\delta_n^{\mathbf{S}}$ transfer to the uniform probability measures on $\mathbf{C}_n$ and $\mathbf{S}_n$, respectively, if one condition about (strongly) l-colourable structures holds. In Section 10 we prove that this condition does indeed hold.

Definition 9.15. Let $m \in \mathbb{R}$.
(i) Suppose that $\gamma : S \to \{1, \ldots, l\}$ is a function. We say that γ is ***m-rich*** if, for every $i \in \{1, \ldots, l\}$, $|\gamma^{-1}(i)| \geq m$, that is, at least m members of S are mapped to i.
(ii) We call $\mathcal{M} \in \mathbf{K}$ (or $\mathcal{M}{\restriction}L_{col}$) ***m-richly l-coloured*** if for every $i \in \{1, \ldots, l\}$, $\big|\{a \in M : \mathcal{M} \models P_i(a)\}\big| \geq m$.
(iii) We also call $\mathcal{M} \in \mathbf{SK}$ (or $\mathcal{M}{\restriction}L_{col}$) ***m-richly l-coloured*** if for every $i \in \{1, \ldots, l\}$, $\big|\{a \in M : \mathcal{M} \models P_i(a)\}\big| \geq m$. (If $\mathcal{M} \in \mathbf{SK}$ then it is understood that we are dealing with strong l-colourings, although it was not explicitly reflected in the terminology defined.)

Recall the notion of an *l-colour compatible extension axiom*, defined in Section 9.1, before Lemma 9.13. These axioms are essentially the same whether we consider (not necessarily strong) l-colourings, or strong l-colourings. The only difference is that the structures $\mathcal{S}$ and $\mathcal{U}$ which are implicitly refered to (via the formulas ζ and ξ) are different in the two cases.

Theorem 9.16. *Let $f : \mathbb{N} \to \mathbb{R}$ be such that $f(n)/\ln n \to \infty$ as $n \to \infty$.*
(i) Suppose that the proportion of $\mathcal{M} \in \mathbf{K}_n$ which are $f(n)$-richly l-coloured approaches 1 as $n \to \infty$. Then, for every extension axiom φ of $\mathbf{K}$, the proportion of $\mathcal{M} \in \mathbf{K}_n$ which satisfy φ approaches 1 as $n \to \infty$. Consequently, $\mathbf{K}$ has a 0-1 law for the uniform probability measure.
(ii) Suppose that the proportion of $\mathcal{M} \in \mathbf{C}_n$ which have an $f(n)$-rich l-colouring approaches 1 as $n \to \infty$. Then, for every l-colour compatible extension axiom φ of $\mathbf{C}$, the proportion of $\mathcal{M} \in \mathbf{C}_n$ which satisfy φ approaches 1 as $n \to \infty$. Moreover, $\mathbf{C}$ has a 0-1 law for the uniform probability measure.
(iii) Parts (i) and (ii) hold if $\mathbf{K}_n$ and $\mathbf{C}_n$ are replaced by $\mathbf{SK}_n$ and $\mathbf{S}_n$, repectively, and 'strong' is added before 'l-colouring'.

Remark 9.17. In Section 10 we will prove (Theorem 10.5) that there is a constant $\mu > 0$ such that the proportion of $\mathcal{M} \in \mathbf{C}_n$ (or $\mathcal{M} \in \mathbf{S}_n$) which have a μn-rich (strong) l-colouring approaches 1 as $n \to \infty$. It follows (Remark 10.6) that the proportion of $\mathcal{M} \in \mathbf{K}_n$ (or $\mathcal{M} \in \mathbf{SK}_n$) which are μn-richly l-coloured approaches 1 as $n \to \infty$.

9.3. **Proof of Theorem 9.16.** The proof is exactly the same whether we consider (not necessarily strongly) l-colourable (or l-coloured) structures or *strongly* l-colourable (or l-coloured) structures. This is because we only need to use (in Lemma 9.19 below) the properties of the structure $\mathcal{U}$ and the formulas ζ and ξ, from Lemmas 9.9 and 9.8, and not their precise definitions in the respective case. Therefore we will speak only about $\mathbf{K}_n$, $\mathbf{C}_n$, l-colourings and l-colourable (or l-coloured) structures; the proof in the case of strong l-colourings is obtained by making the obvious changes of notation and terminology. Throughout the proof, $l \geq 2$ is fixed so we may occasionally say 'colouring' instead of 'l-colouring'.

Suppose that $f : \mathbb{N} \to \mathbb{R}$ is such that $f(n)/\ln n \to \infty$ as $n \to \infty$. A straightforward consequence is that for every $k \in \mathbb{N}$ and every $0 < \alpha < 1$, $\lim_{n\to\infty} n^k \cdot \alpha^{f(n)} = 0$. For if $\beta = 1/\alpha$, then $\beta > 1$, $\ln \beta > 0$ and $\ln \frac{\beta^{f(n)}}{n^k} = \big(f(n)\ln\beta - k\ln n\big) \to \infty$ as $n \to \infty$; which gives $\lim_{n\to\infty} \frac{\beta^{f(n)}}{n^k} = \infty$, and $n^k \cdot \alpha^{f(n)} = \frac{n^k}{\beta^{f(n)}} \to 0$ as $n \to \infty$.

We first prove (i). As said in Remark 3.3, the zero-one law for $\mathbf{K}$, with the uniform measure, follows if we can show that for every extension axiom of $\mathbf{K}$, the proportion of structures in $\mathbf{K}_n$ which satisfy it approaches 1 as $n \to \infty$. Suppose that the proportion of $\mathcal{M} \in \mathbf{K}_n$ which are $f(n)$-richly coloured approaches 1 as $n \to \infty$.

Let φ be an extension axiom of $\mathbf{K}$. It suffices to consider the case when φ has only one existential quantifier, so let φ have the form

$$\forall x_1, \ldots, x_k \exists x_{k+1} \big(\psi(x_1, \ldots, x_k) \to \psi'(x_1, \ldots, x_k, x_{k+1})\big),$$

where ψ and ψ' are quantifier-free.

For every L_{col}-structure $\mathcal{A}$ with universe $\{1, \ldots, n\}$ for some n (or equivalently, for every $\mathcal{A} \in \mathbf{K}{\restriction}1$), let

$$\mathbf{E}_L(\mathcal{A}) = \{\mathcal{M} \in \mathbf{K} : \mathcal{M}{\restriction}L_{col} = \mathcal{A}\}.$$

Since we assume that the proportion of $\mathcal{M} \in \mathbf{K}_n$ which are $f(n)$-richly coloured approaches 1 as $n \to \infty$, it is sufficient to prove that

(a) for every $\varepsilon > 0$ there is n_ε such that for every $n > n_\varepsilon$, if $\mathcal{A} \in \mathbf{K}_n{\restriction}1$ is an $f(n)$-rich colouring, then the proportion of $\mathcal{M} \in \mathbf{E}_L(\mathcal{A})$ which satisfy φ is at least $1 - \varepsilon$.

The proof of (a) is a slight variant of the well known proof that, with the uniform measure, the probability that an extension axiom is true in a randomly picked structure (without any restrictions on its relations, and with at least one relation with arity > 1) with universe $\{1, \ldots, n\}$ approaches 1 as n tends to infinity (see [18, 15, 23]).

Suppose that $\mathcal{A} \in \mathbf{K}_n{\restriction}1$ is an $f(n)$-rich colouring. Let α be the number of nonequivalent quantifier-free L-formulas with free variables (exactly) $x_1, \ldots, x_{k+1}$. We show that, with the uniform measure, the probability that $\mathcal{M} \in \mathbf{E}_L(\mathcal{A})$ does *not* satisfy φ approaches 0 as $n \to \infty$; moreover, the convergence is uniform in the sense that it depends only on $n = \|\mathcal{A}\|$. From this (a) follows.

Note that the only restriction on the interpretations of relation symbols from L_{rel} in structures in $\mathbf{E}_L(\mathcal{A})$ is that the interpretations respect the colouring of $\mathcal{A}$. Suppose that $\bar{a} = (a_1, \ldots, a_k) \in |\mathcal{A}|^k$, $\mathcal{M} \in \mathbf{E}_L(\mathcal{A})$ and $\mathcal{M} \models \psi(\bar{a})$. Let $a_{k+1} \in |\mathcal{A}| - \mathrm{rng}(\bar{a})$ be any of the at least $f(n) - k$ elements not in $\mathrm{rng}(\bar{a})$ which have the colour, say i, which is specified for x_{k+1} by $\psi'(x_1, \ldots, x_{k+1})$. Then the probability, with the uniform measure, that, for such a_{k+1}, $\mathcal{M} \models \psi'(a_1, \ldots, a_k, a_{k+1})$ is at least $1/\alpha$. So the probability that this is not true is at most $1 - 1/\alpha$; and the probability that $\mathcal{M} \not\models \psi'(a_1, \ldots, a_k, a)$ for every one of the at least $f(n) - k$ elements a outside of $\mathrm{rng}(\bar{a})$ with colour i is at most $(1 - 1/\alpha)^{f(n)-k}$. There are n^k choices of $\bar{a} \in |\mathcal{A}|^k$ for which $\exists x_{k+1}\psi'(\bar{a}, x_{k+1})$ could fail to be true in $\mathcal{M}$, so the probability that $\mathcal{M} \not\models \varphi$ is at most $n^k \cdot (1 - 1/\alpha)^{f(n)-k} \to 0$ as $n \to \infty$; by the assumption about $f(n)$. Since we get the same expression '$n^k \cdot (1 - 1/\alpha)^{f(n)-k}$' for every $f(n)$-rich colouring $\mathcal{A} \in \mathbf{K}_n{\restriction}1$ we have proved (a), and hence (i).

Let $\mathcal{S}$ and $\mathcal{U}$ be the L_{rel}-structures from Notation 9.6 and 9.10, and let $m = \max(\|\mathcal{S}\|, \|\mathcal{U}\|)$. Also, let $\xi(y,z)$ be the L_{rel}-formula from Notation 9.6. Fix an arbitrary $k \geq m$ and define

$$\begin{aligned}
\mathbf{X}_n^{\mathbf{K}} &= \{\mathcal{M} \in \mathbf{K}_n : \mathcal{M} \text{ satisfies all } k\text{-extension axioms of } \mathbf{K}\},\\
\mathbf{X}_n^{\mathbf{C}} &= \{\mathcal{M} \in \mathbf{C}_n : \mathcal{M} = \mathcal{N}{\restriction}L_{rel} \text{ for some } \mathcal{N} \in \mathbf{X}_n^{\mathbf{K}}\},\\
\mathbf{Y}_n^{\mathbf{K}} &= \{\mathcal{M} \in \mathbf{K}_n : \mathcal{M} \text{ is } f(n)\text{-richly coloured}\},\\
\mathbf{Y}_n^{\mathbf{C}} &= \{\mathcal{M} \in \mathbf{C}_n : \mathcal{M} \text{ has an } f(n)\text{-rich colouring}\}.
\end{aligned}$$

Lemma 9.18. *Every $\mathcal{M} \in \mathbf{X}_n^{\mathbf{C}}$ satisfies all l-colour compatible k-extension axioms.*

Proof. The notation $\mathbf{X}_n^{\mathbf{K}}$, introduced before the lemma, denotes the same set of structures as the notation $\mathbf{X}_{n,k}$ defined in Notation 9.6 (iv). Therefore Lemma 9.12 tells that every $\mathcal{M} \in \mathbf{X}_n^{\mathbf{K}}$ satisfies all l-colour compatible k-extension axioms. Since all such axioms are L_{rel}-sentences it follows that for every $\mathcal{M} \in \mathbf{X}_n^{\mathbf{K}}$, $\mathcal{M}{\restriction}L_{rel}$ satisfies all l-colour compatible k-extension axioms. The lemma now follows from the definition of $\mathbf{X}_n^{\mathbf{C}}$. $\square$

From Lemma 9.18 it follows that in order to prove that the proportion of $\mathcal{M} \in \mathbf{C}_n$ which satisfy all l-colour compatible k-extension axioms approaches 1 as $n \to \infty$, it suffices to show that $|\mathbf{X}_n^{\mathbf{C}}|/|\mathbf{C}_n| \to 1$ as $n \to \infty$.

Lemma 9.19. *For all $\mathcal{M} \in \mathbf{X}_n^{\mathbf{C}}$ the following hold:*
(i) For every l-colouring $\gamma : M \to \{1,\dots,l\}$ of $\mathcal{M}$, and all $a,b \in M$, $\mathcal{M} \models \xi(a,b) \iff \gamma(a) = \gamma(b)$.
(ii) $\mathcal{M}$ has a unique l-colouring up to permutation of the colours.

Proof. As in the proof of the previous lemma, recall that $\mathbf{X}_n^{\mathbf{K}}$ means the same as $\mathbf{X}_{n,k}$ in Section 9. By Lemma 9.11 and the definition of ζ (in Notation 9.10), for every $\mathcal{M} \in \mathbf{X}_n^{\mathbf{K}}$, the following hold:

- $\mathcal{U}$ is embeddable into $\mathcal{M}$.
- $\xi(y,z)$ defines an equivalence relation on M with exactly l equivalence classes.

Since $\mathcal{U}$ is an L_{rel}-structure and ξ an L_{rel}-formula, it follows from the definition of $\mathbf{X}_n^{\mathbf{C}}$ that the above two points hold for every $\mathcal{M} \in \mathbf{X}_n^{\mathbf{C}}$ as well.

Let $\mathcal{M} \in \mathbf{X}_n^{\mathbf{C}}$ and let $\gamma : M \to \{1,\dots,l\}$ be an l-colouring of $\mathcal{M}$. Since $\mathcal{U}$ is embeddable into $\mathcal{M}$ and $\mathcal{U}$ is not $(l-1)$-colourable, it follows that the equivalence relation $\gamma(a) = \gamma(b)$ has exactly l equivalence classes. Observe that the colouring γ gives rise to a unique expansion of $\mathcal{M}$ that belongs to $\mathbf{K}$. Therefore Lemma 9.8 (i) implies that if $\mathcal{M} \models \xi(a,b)$ then $\gamma(a) = \gamma(b)$. Hence, the equivalence relation defined by $\xi(y,z)$ is a refinement of the equivalence relation $\gamma(y) = \gamma(z)$. Since both equivalence relations have exactly l equivalence classes they must be the same. In other words, for all $a,b \in M$, $\mathcal{M} \models \xi(a,b)$ if and only if $\gamma(a) = \gamma(b)$. Hence (i) is proved. Part (ii) is now immediate, for if γ and γ' are two l-colourings of $\mathcal{M} \in \mathbf{X}_n^{\mathbf{C}}$, then

$$\gamma(a) = \gamma(b) \iff \mathcal{M} \models \xi(a,b) \iff \gamma'(a) = \gamma'(b).$$

$\square$

Now we have the tools to complete the proof of part (ii) of the theorem. Observe that with the notation used in the proof of part (i) we have

$$\mathbf{Y}_n^{\mathbf{K}} = \bigcup \{\mathbf{E}_L(\mathcal{A}) : \mathcal{A} \in \mathbf{K}_n{\restriction}1 \text{ is an } f(n)\text{-rich } l\text{-colouring}\},$$

and (a) implies that

(b) $$\lim_{n\to\infty} \frac{|\mathbf{X}_n^{\mathbf{K}} \cap \mathbf{Y}_n^{\mathbf{K}}|}{|\mathbf{Y}_n^{\mathbf{K}}|} = 1.$$

Note that for every l-colouring of $\mathcal{M} \in \mathbf{C}$, the colours can be permuted in $l!$ ways. Therefore,

$$\text{(c)} \qquad |\mathbf{K}_n| \geq l!|\mathbf{C}_n| \quad \text{and} \quad \big|\mathbf{Y}_n^{\mathbf{K}}\big| \geq l!\big|\mathbf{Y}_n^{\mathbf{C}}\big|.$$

Lemma 9.19 implies that

$$\text{(d)} \qquad \big|\mathbf{X}_n^{\mathbf{K}}\big| = l!\big|\mathbf{X}_n^{\mathbf{C}}\big| \quad \text{and} \quad \big|\mathbf{X}_n^{\mathbf{K}} \cap \mathbf{Y}_n^{\mathbf{K}}\big| = l!\big|\mathbf{X}_n^{\mathbf{C}} \cap \mathbf{Y}_n^{\mathbf{C}}\big|.$$

Assume that the proportion of $\mathcal{M} \in \mathbf{C}_n$ which have an $f(n)$-rich colouring approaches 1 as $n \to \infty$. In other words,

$$\text{(e)} \qquad \lim_{n\to\infty} \frac{\big|\mathbf{Y}_n^{\mathbf{C}}\big|}{|\mathbf{C}_n|} = 1.$$

By (c) and (d),

$$\text{(f)} \qquad \frac{\big|\mathbf{X}_n^{\mathbf{K}} \cap \mathbf{Y}_n^{\mathbf{K}}\big|}{\big|\mathbf{Y}_n^{\mathbf{K}}\big|} \leq \frac{l!\big|\mathbf{X}_n^{\mathbf{C}} \cap \mathbf{Y}_n^{\mathbf{C}}\big|}{l!\big|\mathbf{Y}_n^{\mathbf{C}}\big|} = \frac{\big|\mathbf{X}_n^{\mathbf{C}} \cap \mathbf{Y}_n^{\mathbf{C}}\big|}{|\mathbf{C}_n|} \cdot \frac{|\mathbf{C}_n|}{\big|\mathbf{Y}_n^{\mathbf{C}}\big|} \leq 1.$$

Now (b), (e) and (f) imply that

$$\text{(g)} \qquad \lim_{n\to\infty} \frac{\big|\mathbf{X}_n^{\mathbf{C}} \cap \mathbf{Y}_n^{\mathbf{C}}\big|}{|\mathbf{C}_n|} = 1.$$

By Lemma 9.18 and (g), the proportion of $\mathcal{M} \in \mathbf{C}_n$ which satisfy all l-colour compatible k-extension axioms of $\mathbf{C}$ approaches 1 as n approaches ∞. This has been derived for arbitrary $k \geq m$, under the assumption that the proportion of $\mathcal{M} \in \mathbf{C}_n$ which have an $f(n)$-rich colouring approaches 1, as $n \to \infty$. Since every l-colour compatible extension axiom is an l-colour compatible k-extension axiom for all sufficiently large k, we have proved: If the proportion of $\mathcal{M} \in \mathbf{C}_n$ which have an $f(n)$-rich colouring approaches 1 as $n \to \infty$, then for every l-colour compatible extension axiom φ, the proportion of $\mathcal{M} \in \mathbf{C}_n$ which satisfy φ approaches 1 as $n \to \infty$.

Now suppose that the proportion of $\mathcal{M} \in \mathbf{C}_n$ which have an $f(n)$-rich colouring approaches 1 as $n \to \infty$. Define $T_{\mathbf{C}} = T_{iso} \cup T_{ext} \cup T_{col}$ exactly as in Section 9, just before Lemma 9.14. By the definition of T_{iso}, every $\varphi \in T_{iso}$ is true in every $\mathcal{M} \in \mathbf{C}_n$. By the last statement of the preceeding paragraph, for every $\varphi \in T_{ext}$ the proportion of $\mathcal{M} \in \mathbf{C}_n$ in which φ holds approaches 1 as $n \to \infty$. Recall that the formulas ξ and ζ (defined in Notation 9.6 and 9.10) are L_{rel}-formulas. Lemma 9.11 and the definition of $\mathbf{X}_n^{\mathbf{C}}$ (and of $\mathbf{X}_{n,k}$ in Notation 9.6) imply that for every $\varphi \in T_{col}$, the proportion of $\mathcal{M} \in \mathbf{C}_n$ in which φ is true approaches 1 as $n \to \infty$. Hence, for every finite $\Delta \subset T_{\mathbf{C}}$, the proportion of $\mathcal{M} \in \mathbf{C}_n$ such that $\mathcal{M} \models \Delta$ approaches 1 as $n \to \infty$. By the completeness of $T_{\mathbf{C}}$ (Lemma 9.14) and compactness, $\mathbf{C}$ has a zero-one law for the uniform probability measure. Thus, we have proved part (ii) of Theorem 9.16 and hence the proof of that theorem is completed (since, as explained in the beginning of the proof, the proof of part (iii) is the same except for obvious changes in notation and terminology).

Observe that, by Lemma 9.19 and (g), we have also proved the following:

Proposition 9.20. *Let $f : \mathbb{N} \to \mathbb{R}$ be such that $\lim_{n\to\infty} \frac{f(n)}{\ln n} = \infty$. Suppose that the proportion of $\mathcal{M} \in \mathbf{C}_n$ which have an $f(n)$-rich colouring approaches 1 as $n \to \infty$. Then the proportion of $\mathcal{M} \in \mathbf{C}_n$ such that every l-colouring γ of $\mathcal{M}$ is definable by $\xi(y, z)$, in the sense that $\mathcal{M} \models \xi(a, b) \Leftrightarrow \gamma(a) = \gamma(b)$, approaches 1 as $n \to \infty$. Consequently, the proportion of $\mathcal{M} \in \mathbf{C}_n$ which have a unique l-colouring, up to permutation of colours, approaches 1 as $n \to \infty$. The same statements hold if $\mathbf{C}_n$ is replaced by $\mathbf{S}_n$ (in which case the formula $\xi(y, z)$ may be different).*

10. The uniform probability measure and the typical distribution of colours

In [27], Kolaitis, Prömel and Rothschild proved that almost all l-colourable undirected graphs are uniquely l-colourable (Corollary 1.23 [27]), and the distribution of colours is relatively even (Corollaries 1.20 and 1.21 [27]). They also proved that the class of l-colourable undirected graphs has a zero-one law, with the uniform probability measure, which together with their first main result – that almost all $\mathcal{K}_{l+1}$-free undirected graphs are l-colourable – implies the other main result, that the class of $\mathcal{K}_{l+1}$-free graphs has a zero-one law. In the above context $l \geq 2$ is a fixed integer. A further study of l-colourable graphs was made by Prömel and Steger in [33], where $l = l(n)$ was allowed to grow, and the authors found a threshold function $l = l(n)$ for the property of being uniquely l-colourable. As in the previous section, we will let $l \geq 2$ be a fixed integer and study random (strongly) l-colourable relational structures, but now only for the uniform probability measure.

The main results of this section, Theorems 10.3 and 10.4, generalize the zero-one law and (almost always) uniqueness of l-colouring for random l-colourable graphs in [27] to random (strongly) l-colourable L_{rel}-structures for any relational language L_{rel} subject to some mild assumptions. They also tell that, almost always, the partition of the universe induced by an (strong) l-colouring is L_{rel}-definable without parameters. Because of Theorem 9.16 (ii) and Proposition 9.20, in order to prove these things we only need to show that, for some function $f : \mathbb{N} \to \mathbb{R}$ such that $\lim_{n\to\infty} f(n)/\ln n = \infty$, the proportion of (strongly) l-colourable L_{rel}-structures $\mathcal{M}$ with universe $\{1, \ldots, n\}$ which have an $f(n)$-rich (strong) l-colouring approaches 1 as $n \to \infty$. We will show (Theorem 10.5) that there is a constant $\mu > 0$ (depending on l, L_{rel} and whether we consider l-colourings or *strong* l-colourings) such that the proportion of L_{rel}-structures $\mathcal{M}$ with universe $\{1, \ldots, n\}$ which have *only* μn-rich (strong) l-colourings approaches 1 as $n \to \infty$. The proof involves counting and estimating the number of (strongly) multichromatic m-tuples and m-sets (Definition 10.2) for m ranging from 2 to the maximum arity of the relation symbols.

As in the previous sections we will allow the possibility that certain relation symbols are always interpreted as irreflexive and symmetric relations (see Remark 2.1). As the arguments in this section are sensitive to whether this restriction applies to a given relation symbol, we will (in contrast to previous sections) be careful to let the notation indicate which relation symbols (if any) are always interpreted as irreflexive and symmetric relations. Note that, apart from making this information visible, the notation below agrees with that which was introduced in the beginning of Section 9.

Assumption 10.1. We fix an integer $l \geq 2$ and a relational language L_{rel} with vocabulary $\{R_1, \ldots, R_\rho\}$, where $\rho > 0$ and each R_k has arity $r_k \geq 2$.

Definition 10.2. (i) For positive integers n, we use the abbreviation $[n] = \{1, \ldots, n\}$.
(ii) Let A be a set and let $\gamma : A \to [l]$. An m-tuple $(a_1, \ldots, a_m) \in A^m$ is called ***monochromatic with respect to*** γ if $\gamma(a_1) = \ldots = \gamma(a_m)$. Otherwise we call $(a_1, \ldots, a_m)$ ***multichromatic with respect to*** γ. Note that if $m \geq 3$, then a multichromatic m-tuple may have repetitions of elements ($a_i = a_j$ for some $i \neq j$).
(iii) An m-tuple $(a_1, \ldots, a_m) \in A^m$ is called ***strongly multichromatic with respect to*** γ if $\gamma(a_i) \neq \gamma(a_j)$ whenever $i \neq j$.
(iv) Let $\mathcal{M} = (M, R_1^{\mathcal{M}}, \ldots, R_\rho^{\mathcal{M}})$ be an L_{rel}-structure and let $\gamma : M \to [l]$. We say that γ is an ***(strong) l-colouring of*** $\mathcal{M}$ if, for every $k = 1, \ldots, \rho$, every $(a_1, \ldots, a_{r_k}) \in R_k^{\mathcal{M}}$ is (strongly) multichromatic with respect to γ.
(v) An L_{rel}-structure $\mathcal{M}$ is called ***(strongly) l-colourable*** if there is $\gamma : M \to [l]$ which is an (strong) l-colouring of $\mathcal{M}$.
(vi) We say that an L_{rel}-structure $\mathcal{M}$ is ***uniquely (strongly) l-colourable*** if it is

(strongly) l-colourable and for all (strong) l-colourings γ and γ' of $\mathcal{M}$ and all $a, b \in M$, $\gamma(a) = \gamma(b) \iff \gamma'(a) = \gamma'(b)$.
(vii) For every $I \subseteq [\rho]$, $\mathbf{C}_n^I$ denotes the set of l-colourable L_{rel}-structures $\mathcal{M}$ with universe $[n] = \{1, \ldots, n\}$ such that for every $k \in I$, R_k is interpreted as an irreflexive and symmetric relation in $\mathcal{M}$. Let $\mathbf{C}^I = \bigcup_{n \in \mathbb{N}} \mathbf{C}_n^I$, where $\mathbb{N}$ is the set of positive integers.
(viii) For every $I \subseteq [\rho]$, $\mathbf{S}_n^I$ denotes the set of strongly l-colourable L_{rel}-structures $\mathcal{M}$ with universe $[n]$ such that for every $k \in I$, R_k is interpreted as an irreflexive and symmetric relation in $\mathcal{M}$. Let $\mathbf{S}^I = \bigcup_{n \in \mathbb{N}} \mathbf{S}_n^I$.
(iv) For $\alpha \in \mathbb{R}$, a function $\gamma : [n] \to [l]$ is called ***α-rich*** if $|f^{-1}(i)| \geq \alpha$ for every $i \in [l]$.

As usual when the uniform probability measure is considered, the phrase ***'almost all $\mathcal{M} \in \mathbf{C}^I$ has property P'*** means that the proportion of $\mathcal{M} \in \mathbf{C}_n^I$ which have property P approaches 1 as n approaches infinity. The phrase ***'$\mathbf{C}^I$ has a zero-one law'*** means that for every L_{rel}-sentence φ, either φ or its negation, $\neg\varphi$, is satisfied by almost all $\mathcal{M} \in \mathbf{C}^I$. (And similarly for $\mathbf{S}^I$ in place of $\mathbf{C}^I$.)

Theorem 10.3. *For every $I \subseteq [\rho]$ the following hold:*
(i) There is an L-formula $\xi(x, y)$ such that for almost all $\mathcal{M} \in \mathbf{C}^I$ the following holds: for every l-colouring $\gamma : M \to [l]$ of $\mathcal{M}$ and all $a, b \in M$, $\gamma(a) = \gamma(b)$ if and only if $\mathcal{M} \models \xi(a, b)$.
(ii) Almost all $\mathcal{M} \in \mathbf{C}^I$ are uniquely l-colourable.
(iii) $\mathbf{C}^I$ has a zero-one law.

Theorem 10.4. *Suppose that every relation symbol has arity $\leq l$. For every $I \subseteq [\rho]$, all three parts (i), (ii) and (iii) of Theorem 10.3 hold if $\mathbf{C}^I$ is replaced by $\mathbf{S}^I$ and 'strong' is added before 'l-colouring/colourable'.*

Recall from Remark 9.3 that if $I \subseteq \{1, \ldots, \rho\}$ and $\mathbf{C}_n = \mathbf{C}_n^I$ and $\mathbf{S}_n = \mathbf{S}_n^I$, then Theorem 9.16 and Proposition 9.20 hold. Hence, Theorems 10.3 and 10.4 are immediate consequences of Theorem 10.5 below and Theorem 9.16 and Proposition 9.20.

Theorem 10.5. *(i) For every $I \subseteq [\rho]$ there are constants $\mu, \lambda > 0$ such that, for all sufficiently large n, the proportion of $\mathcal{M} \in \mathbf{C}_n^I$ which have an l-colouring that is* not *μn-rich is at most $2^{-\lambda n^m}$, where m is the maximum arity of the relation symbols (so $m \geq 2$). Consequently, the proportion of $\mathcal{M} \in \mathbf{C}_n^I$ which have only μn-rich l-colourings approaches 1 as $n \to \infty$.*
(ii) If at least one relation symbol has arity $\leq l$, then part (i) also holds if $\mathbf{C}_n^I$ is replaced by $\mathbf{S}_n^I$, 'l-colouring' by 'strong l-colouring', and m is the largest arity $\leq l$ (so $m \geq 2$).

Remark 10.6. (i) In both parts of Theorem 10.5, the proof shows how to compute μ from the number of colours, l, and the arities $r_1, \ldots, r_\rho$ of the relation symbols of the language L_{rel}.
(ii) Theorem 10.5 gives a bit more than what has been said above; namely that the assumption in part (i) of Theorem 9.16 is true, which can be seen as follows. Let $\mathbf{K}_n$ and $\mathbf{SK}_n$ be defined as in the previous section. By Theorem 10.5 (i), there are constants $\mu, \lambda > 0$ such that if $\mathbf{Y}_n^{\mathbf{K}}$ is the set of $\mathcal{M} \in \mathbf{K}_n$ which are μn-richly l-coloured, then, for all sufficiently large n,

$$\big|\{\mathcal{M}{\restriction}L_{rel} : \mathcal{M} \in \mathbf{K}_n - \mathbf{Y}_n^{\mathbf{K}}\}\big| \Big/ |\mathbf{C}_n| \ \leq \ 2^{-\lambda n^2}.$$

Since for each $\mathcal{M} \in \mathbf{K}_n$, $\mathcal{M}{\restriction}L_{rel}$ can be l-coloured, or equivalently, expanded to an L-structure (using the notation of the previous section), in at most $l^n = 2^{\beta n}$ (for some $\beta > 0$) ways, we get $\big|\mathbf{K}_n - \mathbf{Y}_n^{\mathbf{K}}\big| / |\mathbf{K}_n| \leq 2^{\beta n - \lambda n^2} \to 0$ as $n \to \infty$. Therefore the assumption in part (i) of Theorem 9.16 holds, and it follows that, for every $k \in \mathbb{N}$, the proportion of $\mathcal{M} \in \mathbf{K}_n$ which satisfy all k-extension axioms of $\mathbf{K}$ approaches 1 as $n \to \infty$. The same argument can be carried out for $\mathbf{SK}_n$, $\mathbf{S}_n$ and strong l-colourings.

Example 10.7. Here follows applications of Theorems 10.3 and 10.4.

(i) Let $\mathcal{F}$ be the Fano plane as a 3-hypergraph, that is, $\mathcal{F}$ has seven vertices and seven 3-hyperedges (3-subsets of the vertex set) such that every pair of distinct vertices is contained in a unique 3-hyperedge. If $\mathbf{K}_n$ is the set of all 3-hypergraphs with vertices $1, \ldots, n$ in which $\mathcal{F}$ is not embeddable, and $\mathbf{K} = \bigcup_{n\in\mathbb{N}} \mathbf{K}_n$, then almost all members of $\mathbf{K}$ are 2-colourable [30]. Since $\mathcal{F}$ cannot be weakly embedded into any 2-colourable 3-hypergraph, it follows from Theorem 10.3 that $\mathbf{K}$ has a zero-one law for the uniform probability measure.

(ii) Let $\mathcal{G}$ be the 3-hypergraph with vertices $1, 2, 3, 4, 5$ and 3-hyperedges $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{3, 4, 5\}$, and let $\mathbf{K}_n$ be the set of 3-hypergraphs with vertices $1, 2, \ldots, n$ in which $\mathcal{G}$ is not weakly embeddable. Then almost all members of $\mathbf{K} = \bigcup_{n\in\mathbb{N}} \mathbf{K}_n$ are strongly 3-colourable [6]. (Tripartite in [6] means the same as strongly 3-colourable here.) Since $\mathcal{G}$ cannot be weakly embedded into any strongly 3-colourable 3-hypergraph it follows from Theorem 10.4 that $\mathbf{K}$ has a zero-one law for the uniform probability measure.

In Section 10.1 we derive an upper bound on the number of multichromatic m-tuples if the l-colouring $\gamma : [n] \to [l]$ is *not* $\frac{n}{a}$-rich and a is sufficiently large. Then we show that the number of multichromatic m-sets are fairly tightly controlled by the number of multichromatic m-tuples. These results are used in Section 10.2 where we prove part (i) of Theorem 10.5. In Section 10.3 we consider *strongly* multichromatic m-tuples and m-sets and derive similar results as in Section 10.1 which are used in Section 10.4 where part (ii) of Theorem 10.5 is proved.

10.1. **Counting multichromatic tuples and sets.**

Notation 10.8. (i) Let $n, m, l \in \mathbb{N}$ and suppose that $n \geq l \geq 2$ and $m \geq 2$.
(ii) Let $\gamma : [n] \to [l]$.
(iii) Let $\mathrm{mult}(n, \gamma, m)$ denote the number of ordered m-tuples $(a_1, \ldots, a_m) \in [n]^m$ which are multichromatic with respect to γ.
(iv) For every $i \in [l]$, let $p(n, \gamma, i) = \left|\gamma^{-1}(i)\right|$, so $p(n, \gamma, i)$ is the number of elements in $[n]$ which are assigned the colour i by γ.

The number of $(a_1, \ldots, a_m) \in [n]^m$ which are monochromatic with respect to γ is $\sum_{i=1}^{l} p^m(n, \gamma, i)$, from which it follows that

$$\mathrm{mult}(n, \gamma, m) \;=\; n^m \;-\; \sum_{i=1}^{l} p^m(n, \gamma, i). \tag{1}$$

Remark 10.9. *Let $k \in \mathbb{N}$ and $\alpha \in \mathbb{R}^+$. The function $f_{k,m}(x_1, \ldots, x_k) = \sum_{i=1}^{k} x_i^m$ constrained by $x_1 + \ldots + x_k = \alpha$ and $x_i \geq 0$, for $i = 1, \ldots, k$, attains its minimal value in the point $(\alpha/k, \ldots, \alpha/k)$, and hence this value is α^m / k^{m-1}.* This fact is easily proved by using the method of Lagrange multipliers [20]. Alternatively, one can use a variant of Hölder's inequality: In the result with number 16 in [21] (p. 26), take $r = 1$, $s = m$ and $a = (x_1, \ldots, x_k)$ and the claim "$\mathfrak{M}_r(a) < \mathfrak{M}_s(a)$ unless ..." becomes $(x_1 + \ldots + x_k)/k < \left(\frac{1}{k} f_{k,m}(x_1, \ldots, x_k)\right)^{1/m}$ unless all x_i are equal. Since we assume that $x_1 + \ldots + x_k = \alpha$, the claim follows by taking the mth power on both sides.

Lemma 10.10. *Let $a > 0$. If $\gamma : [n] \to [l]$ is* not *$\frac{n}{a}$-rich, then*

$$\mathrm{mult}(n, \gamma, m) \;\leq\; \left(1 \;-\; \left[\frac{a-1}{a}\right]^m \frac{1}{(l-1)^{m-1}}\right) n^m.$$

Proof. Let $a > 0$. Suppose that $\gamma : [n] \to [l]$ is not $\frac{n}{a}$-rich, which means that, for some $i \in [l]$, $p(n, \gamma, i) < \frac{n}{a}$. For simplicity of notation (and without loss of generality) assume that $i = l$. Then

$$n \; - \; p(n, \gamma, l) \; > n \; - \; \frac{n}{a} \; = \; \frac{a-1}{a} n. \tag{2}$$

Now we have

$$\begin{aligned}
&\mathrm{mult}(n, \gamma, m) \; = \\
&= \; n^m \; - \; \sum_{i=1}^{l-1} p^m(n, \gamma, i) \; - \; p^m(n, \gamma, l) \\
&\leq \; n^m \; - \; \frac{\big[n - p(n, \gamma, l)\big]^m}{(l-1)^{m-1}} \; - \; p^m(n, \gamma, l) \\
&\qquad\qquad\qquad \text{by Remark 10.9 with } \alpha = n - p(n, \gamma, l) \text{ and } k = l - 1 \\
&< \; n^m \; - \; \left[\frac{a-1}{a}\right]^m \frac{n^m}{(l-1)^{m-1}} \; - \; p^m(n, \gamma, l) \quad \text{by (2)} \\
&\leq \; \left(1 \; - \; \left[\frac{a-1}{a}\right]^m \frac{1}{(l-1)^{m-1}}\right) n^m. \qquad \square
\end{aligned}$$

Notation 10.11. (i) As usual, by a *k-set* we mean a set of cardinality k.
(ii) For every integer $k \geq 2$, every $n \in \mathbb{N}$ and every $\gamma : [n] \to [l]$, let $\overline{\mathrm{mult}}(n, \gamma, k)$ be the number of k-subsets $\{a_1, \ldots, a_k\} \subseteq [n]$ such that there are $i, j \in [k]$ with $\gamma(a_i) \neq \gamma(a_j)$. We call such a k-set $\{a_1, \ldots, a_k\}$ ***multichromatic***.
(iii) For integers $1 \leq i \leq k$, let $\mathrm{perm}(i, k)$ be the number of (ordered) k-tuples $(a_1, \ldots, a_k)$ of elements of an i-set A such that every $a \in A$ occurs at least once in $(a_1, \ldots, a_k)$.

Lemma 10.12. *Let $m_{\max} \geq 2$ be an integer and suppose that $\sigma_n : [n] \to [l]$ and $\gamma_n : [n] \to [l]$, for $n \in \mathbb{N}$. Moreover, assume that for all $2 \leq m \leq m_{\max}$ there are constants $c_m, d_m > 0$ such that for all sufficiently large n,*

$$c_m n^m \;\leq\; \mathrm{mult}(n, \sigma_n, m) - \mathrm{mult}(n, \gamma_n, m) \;\leq\; d_m n^m.$$

Then, for all $2 \leq m \leq m_{\max}$, there are constants $c'_m, d'_m > 0$ such that for all sufficiently large n,

$$c'_m n^m \;\leq\; \overline{\mathrm{mult}}(n, \sigma_n, m) - \overline{\mathrm{mult}}(n, \gamma_n, m) \;\leq\; d'_m n^m.$$

Proof. Suppose that for all for all $2 \leq m \leq m_{\max}$ there are $c_m, d_m > 0$ such that for all sufficiently large n,

$$c_m n^m \;\leq\; \mathrm{mult}(n, \sigma_n, m) - \mathrm{mult}(n, \gamma_n, m) \;\leq\; d_m n^m. \tag{3}$$

Note that if an m-tuple $(a_1, \ldots, a_m) \in [n]^m$ is multichromatic with respect to $\gamma : [n] \to [l]$, then $\big|\{a_1, \ldots, a_m\}\big| = i$ for some $2 \leq i \leq m$. Therefore (with the notation introduced before the lemma), for every m and every $\gamma : [n] \to [l]$ we have

$$\mathrm{mult}(n, \gamma, m) \;=\; \sum_{i=2}^{m} \overline{\mathrm{mult}}(n, \gamma, i) \cdot \mathrm{perm}(i, m). \tag{4}$$

We use induction on $m = 2, \ldots, m_{\max}$. If $m = 2$ then (3) and (4) give

$$\frac{c_2}{\mathrm{perm}(2,2)} \;\leq\; \overline{\mathrm{mult}}(n, \sigma_n, 2) - \overline{\mathrm{mult}}(n, \gamma_n, 2) \;\leq\; \frac{d_2}{\mathrm{perm}(2,2)},$$

so we can take $c'_2 = c_2 / \mathrm{perm}(2,2)$ and $d'_2 = d_2 / \mathrm{perm}(2,2)$.

As induction hypothesis, suppose that for $m = 2, \dots, k < m_{\max}$ there are $c'_m, d'_m > 0$ such that for all sufficiently large n,

$$c'_m n^m \;\le\; \overline{\text{mult}}(n, \sigma_n, m) - \overline{\text{mult}}(n, \gamma_n, m) \;\le\; d'_m n^m. \tag{5}$$

By assumption, for all sufficiently large n we have

$$c_{k+1} n^{k+1} \;\le\; \text{mult}(n, \sigma_n, k+1) - \text{mult}(n, \gamma_n, k+1) \;\le\; d_{k+1} n^{k+1},$$

and by (4) with $m = k+1$ it follows that for all sufficiently large n,

$$\begin{aligned} c_{k+1} n^{k+1} \;\le\; &\sum_{i=2}^{k} \text{perm}(i, k+1)\Big[\overline{\text{mult}}(n, \sigma_n, i) - \overline{\text{mult}}(n, \gamma_n, i)\Big] \\ &+ \text{perm}(k+1, k+1)\Big[\overline{\text{mult}}(n, \sigma_n, k+1) - \overline{\text{mult}}(n, \gamma_n, k+1)\Big] \;\le\; d_{k+1} n^{k+1}. \end{aligned}$$

By the induction hypothesis, (5) holds for $m = 2, \dots, k$ and all sufficiently large n. Hence, for all sufficiently large n,

$$\begin{aligned} &c_{k+1} n^{k+1} \;\le\; \\ &\text{perm}(k+1, k+1)\Big[\overline{\text{mult}}(n, \sigma_n, k+1) - \overline{\text{mult}}(n, \gamma_n, k+1)\Big] \;+\; O\big(n^k\big) \;\le\; d_{k+1} n^{k+1}, \end{aligned}$$

so there must be $c'_{k+1}, d'_{k+1} > 0$ such that for all sufficiently large n,

$$c'_{k+1} n^{k+1} \;\le\; \overline{\text{mult}}(n, \sigma_n, k+1) - \overline{\text{mult}}(n, \gamma_n, k+1) \;\le\; d'_{k+1} n^{k+1}. \qquad \square$$

10.2. **Proof of the first part of Theorem 10.5.** We continue to use the terminology and notation introduced in Notation 10.8 and 10.11. Recall that all arities $r_1, \dots, r_\rho$ of the relation symbols $R_1, \dots, R_\rho$ are at least 2. Let $m_{\max} = \max(r_1, \dots, r_\rho)$. For all $l, m \ge 2$ we have

$$\frac{1}{(l-1)^{m-1}} \;>\; \frac{1}{l^{m-1}}.$$

Let $a > l$ be large enough so that, whenever $2 \le m \le m_{\max}$,

$$\left[\frac{a-1}{a}\right]^m \frac{1}{(l-1)^{m-1}} \;>\; \frac{1}{l^{m-1}}. \tag{6}$$

For every $n \in \mathbb{N}$ such that $n \ge l$, fix $\sigma_n : [n] \to [l]$ such that, for every $i \in [l]$,

$$\frac{n}{l} - 1 \;\le\; p(n, \sigma_n, i) \;\le\; \frac{n}{l} + 1. \tag{7}$$

Then, for all sufficiently large n, σ_n is $\frac{n}{a}$-rich (because we chose $a > l$). Observe that if $\gamma : [n] \to [l]$, then the number of $\mathcal{M} \in \mathbf{C}_n^I$ for which γ is an l-colouring is

$$2^{\sum_{k \in [\rho] - I} \text{mult}(n, \gamma, r_k) \;+\; \sum_{k \in I} \overline{\text{mult}}(n, \gamma, r_k)}.$$

Therefore,

$$\big|\mathbf{C}_n^I\big| \;\ge\; 2^{\sum_{k \in [\rho] - I} \text{mult}(n, \sigma_n, r_k) \;+\; \sum_{k \in I} \overline{\text{mult}}(n, \sigma_n, r_k)}. \tag{8}$$

A lower bound of $\text{mult}(n, \sigma_n, m)$ is obtained as follows:

$$\begin{aligned} \text{mult}(n, \sigma_n, m) \;=\; n^m \;-\; \sum_{i=1}^{l} p^m(n, \sigma_n, i) \;&\ge\; n^m \;-\; l\left(\frac{n}{l} + 1\right)^m \qquad \text{by (7)} \\ &=\; n^m \;-\; \frac{n^m}{l^{m-1}} \;\pm\; O\big(n^{m-1}\big), \end{aligned}$$

so

$$\text{mult}(n, \sigma_n, m) \;\ge\; \left(1 - \frac{1}{l^{m-1}}\right) n^m \;\pm\; O\big(n^{m-1}\big). \tag{9}$$

For every $n \in \mathbb{N}$, choose $\gamma_n : [n] \to [l]$ such that

$$\gamma_n \text{ is } \textit{not } \frac{n}{a}\text{-rich, and} \tag{10}$$

for every $\gamma : [n] \to [l]$ which is *not* $\frac{n}{a}$-rich,

$$\begin{aligned} \sum_{k \in [\rho] - I} \mathrm{mult}(n, \gamma, r_k) \ + \ \sum_{k \in I} \overline{\mathrm{mult}}(n, \gamma, r_k) \\ \leq \sum_{k \in [\rho] - I} \mathrm{mult}(n, \gamma_n, r_k) \ + \ \sum_{k \in I} \overline{\mathrm{mult}}(n, \gamma_n, r_k). \end{aligned} \tag{11}$$

Let $\mathbf{X}_n \subseteq \mathbf{C}_n^I$ be the set of all $\mathcal{M} \in \mathbf{C}_n^I$ which have an l-colouring which is *not* $\frac{n}{a}$-rich. It suffices to prove that $|\mathbf{X}_n| / |\mathbf{C}_n^I| \to 0$ as $n \to \infty$. If $\mathcal{M} \in \mathbf{X}_n$ then there is an l-colouring $\gamma : [n] \to [l]$ of $\mathcal{M}$ which is not $\frac{n}{a}$-rich, and the number of $\mathcal{N} \in \mathbf{C}_n^I$ for which γ is an l-colouring is at most $2^{\sum_{k \in [\rho] - I} \mathrm{mult}(n, \gamma, r_k) \ + \ \sum_{k \in I} \overline{\mathrm{mult}}(n, \gamma, r_k)}$. Since the number of functions $\gamma : [n] \to [l]$ is $l^n = 2^{\beta n}$, for some $\beta > 0$, it follows from (10) and (11) that

$$\big|\mathbf{X}_n\big| \ \leq \ 2^{\beta n \ + \ \sum_{k \in [\rho] - I} \mathrm{mult}(n, \gamma_n, r_k) \ + \ \sum_{k \in I} \overline{\mathrm{mult}}(n, \gamma_n, r_k)}. \tag{12}$$

From (10) and Lemma 10.10 it follows that

$$\mathrm{mult}(n, \gamma_n, m) \ \leq \ \left(1 \ - \ \left[\frac{a-1}{a} \right]^m \frac{1}{(l-1)^{m-1}} \right) n^m. \tag{13}$$

Note that for all n, m and $\gamma : [n] \to [l]$ we have $\overline{\mathrm{mult}}(n, \gamma, m) \leq \mathrm{mult}(n, \gamma, m) \leq n^m$. Therefore, (9) and (13) imply that

$$\begin{aligned} n^m \ &\geq \ \mathrm{mult}(n, \sigma_n, m) - \mathrm{mult}(n, \gamma_n, m) \\ &\geq \ \left[1 - \frac{1}{l^{m-1}} \ - \ \left(1 \ - \ \left[\frac{a-1}{a} \right]^m \frac{1}{(l-1)^{m-1}} \right) \right] n^m \ \pm \ O\big(n^{m-1}\big) \\ &= \ \left[\left[\frac{a-1}{a} \right]^m \frac{1}{(l-1)^{m-1}} \ - \ \frac{1}{l^{m-1}} \right] n^m \ \pm \ O\big(n^{m-1}\big). \end{aligned}$$

Together with (6) this implies that there is $c > 0$ such that whenever $2 \leq m \leq m_{\max}$ and n is sufficiently large

$$cn^m \ \leq \ \mathrm{mult}(n, \sigma_n, m) \ - \ \mathrm{mult}(n, \gamma_n, m) \ \leq \ n^m. \tag{14}$$

Lemma 10.12 now implies that for all $2 \leq m \leq m_{\max}$ there are $c'_m > 0$ such that for all sufficiently large n,

$$c'_m n^m \ \leq \ \overline{\mathrm{mult}}(n, \sigma_n, m) \ - \ \overline{\mathrm{mult}}(n, \gamma_n, m). \tag{15}$$

By (8) and (12) we have

$$\begin{aligned} &\big|\mathbf{X}_n\big| \big/ \big|\mathbf{C}_n^I\big| \leq \\ &\leq 2^{\beta n \ + \ \sum_{k \in [\rho] - I} \left[\mathrm{mult}(n, \gamma_n, r_k) \ - \ \mathrm{mult}(n, \sigma_n, r_k) \right] \ + \ \sum_{k \in I} \left[\overline{\mathrm{mult}}(n, \gamma_n, r_k) \ - \ \overline{\mathrm{mult}}(n, \sigma_n, r_k) \right]}. \end{aligned} \tag{16}$$

From (14) and (15) it follows that for all sufficiently large n,

$$k \in [\rho] - I \implies \mathrm{mult}(n, \gamma_n, r_k) - \mathrm{mult}(n, \sigma_n, r_k) \ \leq \ -cn^{r_k}, \quad \text{and} \tag{17}$$

$$k \in I \implies \overline{\mathrm{mult}}(n, \gamma_n, r_k) - \overline{\mathrm{mult}}(n, \sigma_n, r_k) \ \leq \ -c'_{r_k} n^{r_k}, \tag{18}$$

where $c, c'_{r_k} > 0$ for all $k \in [\rho]$. Since $r_k \geq 2$ for all $k \in [\rho]$ it follows from (16)–(18) that, for $m = m_{\max}$ and some $\lambda > 0$, we have (for all large enough n)

$$\big|\mathbf{X}_n\big| \big/ \big|\mathbf{C}_n^I\big| \ \leq \ 2^{-\lambda n^m} \ \to \ 0 \quad \text{as} \ \ n \to \infty.$$

In other words, the proportion of $\mathcal{M} \in \mathbf{C}_n^I$ which only have $\frac{n}{a}$-rich l-colourings approaches 1 as $n \to \infty$; in part (i) of Theorem 10.5 we can take $\mu = \frac{1}{a}$.

10.3. **Counting strongly multichromatic tuples and sets.**

Notation 10.13. (i) Let $n, m, l \in \mathbb{N}$ and suppose that $n \geq l \geq m \geq 2$.
(ii) Let $\gamma : [n] \to [l]$.
(iii) Let $\mathrm{smult}(n, \gamma, m)$ denote the number of ordered m-tuples $(a_1, \ldots, a_m) \in [n]^m$ which are strongly multichromatic with respect to γ.
(iv) Let $\overline{\mathrm{smult}}(n, \gamma, m)$ be the number of m-subsets $\{a_1, \ldots, a_m\} \subseteq [n]$ such that $\gamma(a_i) \neq \gamma(a_j)$ whenever $i \neq j$.
(v) For every $i \in [l]$, let $p(n, \gamma, i) = \left|\gamma^{-1}(i)\right|$, so $p(n, \gamma, i)$ is the number of elements in $[n]$ which are assigned the colour i by γ.

Observe that

$$\mathrm{smult}(n, \gamma, m) \ = \ m! \ \overline{\mathrm{smult}}(n, \gamma, m) \tag{19}$$

and

$$\overline{\mathrm{smult}}(n, \gamma, m) \ = \sum_{1 \leq i_1 < \ldots < i_m \leq l} p(n, \gamma, i_1) \ldots p(n, \gamma, i_m). \tag{20}$$

For $k \geq m$, let

$$g_{k,m}(x_1, \ldots, x_k) \ = \sum_{1 \leq i_1 < \ldots < i_m \leq k} x_{i_1} \ldots x_{i_m}.$$

The next lemma is a special case of an inequality found in [21] (p. 52), see Remark 10.15 below, but here we give a proof based on the better known method of Lagrange multipliers [20].

Lemma 10.14. *Let $\alpha > 0$. Subject to the constraints $x_1 + \ldots + x_k = \alpha$ and $x_i \geq 0$ for all $i \in [l]$, $g_{k,m}$ attains its maximum in $(\alpha/k, \ldots, \alpha/k)$, and hence the maximum is*

$$g_{k,m}(\alpha/k, \ldots, \alpha/k) \ = \ \binom{k}{m}\left(\frac{\alpha}{k}\right)^m.$$

Proof. Suppose that, under the given constraints, $g_{k,m}$ attains its maximum in $(a_1, \ldots, a_k)$. Then at least one a_i is non-zero, hence positive. We show that for every j, $a_j = a_i$. By the constraint $a_1 + \ldots + a_k = \alpha$ it follows that $a_j = \alpha/k$ for all $j \in [k]$. For simplicity of notation, and without loss of generality, assume that $i = 1$, so $a_1 > 0$. Since the following argument works out in the same way for all $j = 2, \ldots, k$, let's assume that $j = 2$ (simplifying notation again).

Let $h(x_1, x_2) = g_{k,m}(x_1, x_2, a_3, \ldots, a_k)$. Since we assume that $g_{k,m}$ attains its maximum, under the given constraints, in $(a_1, \ldots, a_k)$, it follows that $h(x_1, x_2)$ attains its maximum, under the constraints $x_1 + x_2 = \alpha - a_3 - \ldots - a_k$ (where $\alpha - a_3 - \ldots - a_k > 0$ since $a_1 > 0$) and $x_1, x_2 \geq 0$, in (a_1, a_2). Observe that

$$\begin{aligned}
h(x_1, x_2) \ &= \sum_{3 \leq i_1 < \ldots < i_{m-2} \leq k} x_1 x_2 a_{i_1} \ldots a_{i_{m-2}} \ + \sum_{3 \leq i_1 < \ldots < i_{m-1} \leq k} x_1 a_{i_1} \ldots a_{i_{m-1}} \\
&+ \sum_{3 \leq i_1 < \ldots < i_{m-1} \leq k} x_2 a_{i_1} \ldots a_{i_{m-1}} \ + \sum_{3 \leq i_1 < \ldots < i_m \leq k} a_{i_1} \ldots a_{i_m} \\
&= \sum_{3 \leq i_1 < \ldots < i_{m-2} \leq k} x_1 x_2 a_{i_1} \ldots a_{i_{m-2}} \ + \sum_{3 \leq i_1 < \ldots < i_{m-1} \leq k} (x_1 + x_2) a_{i_1} \ldots a_{i_{m-1}} \\
&+ \sum_{3 \leq i_1 < \ldots < i_m \leq k} a_{i_1} \ldots a_{i_m}.
\end{aligned}$$

Subject to the constraint $x_1 + x_2 = \alpha - a_3 - \ldots - a_k$, the part

$$\sum_{3 \leq i_1 < \ldots < i_{m-1} \leq k} (x_1 + x_2) a_{i_1} \ldots a_{i_{m-1}} \ + \sum_{3 \leq i_1 < \ldots < i_m \leq k} a_{i_1} \ldots a_{i_m}$$

is constant, and hence $h(x_1, x_2)$ attains its maximum in the same point as $h^*(x_1, x_2) = cx_1x_2$, where $c > 0$ is a constant. The reason that we can assume that $c > 0$ is that if $m > 2$, then there are at least $m - 2$ non-zero a_i's with $i > 2$; because otherwise $g_{k,m}$ would be zero in $(a_1, \ldots, a_k)$ and then this point could not be a maximum, contrary to assumption. Thus it suffices to show that, for any $\beta > 0$, if $h^*(x_1, x_2) = cx_1x_2$ attains its maximum in (b_1, b_2) under the constraints $x_1 + x_2 = \beta$, $x_1, x_2 \geq 0$, then $b_1 = b_2$. This is easily proved by (for example) using Lagrange multipliers [20].

Given that $a_1 = \ldots = a_k$, the constraints on $g_{k,m}$ imply that $a_i = \alpha/k$ for all i, and insertion of $(x_1, \ldots, x_k) = (a_1, \ldots, a_k)$ in the expression of $g_{k,m}$ shows that its maximum, subject to the constraints, is $\binom{k}{m}\big(\alpha/k\big)^m$. □

Remark 10.15. Lemma 10.14 is a special case of the result with number 52 in [21] (p. 52), which is attributed to Maclaurin [28]. In the notation of that result, but with the letter n replaced by k, we have "$p_1 > (p_2)^{1/2} > \ldots > (p_k)^{1/k}$ unless ...", so in particular "$p_1 > (p_m)^{1/m}$ unless ...", which, with the notation here and because $x_1 + \ldots + x_k = \alpha$, becomes $\alpha/k > \big(g_{k,m}(x_1, \ldots, x_k)/\binom{k}{m}\big)^{1/m}$ unless all x_i are equal. By raising both sides to the mth power we get the statement of the lemma.

Lemma 10.16. *Let $a > 0$. If $\gamma : [n] \to [l]$ is* not *$\frac{n}{a}$-rich, then*

$$\overline{\text{smult}}(n, \gamma, m) \ \leq \ \left[\frac{1}{(l-1)^m}\binom{l-1}{m} \ + \ \frac{1}{a(l-1)^{m-1}}\binom{l-1}{m-1}\right] n^m,$$

where the left term within the large parentheses vanishes if $m = l$.

Proof. Suppose that $a > 0$ and that $\gamma : [n] \to [l]$ is not $\frac{n}{a}$-rich. Then, for some $i \in [l]$, we have $p(n, \gamma, i) < \frac{n}{a}$. For simplicity of notation, and without loss of generality, assume that $i = l$. Then:

$$\begin{aligned}
\overline{\text{smult}}(n,\gamma,m) &= \sum_{1\le i_1<\ldots<i_m\le l} p(n,\gamma,i_1)\cdot\ldots\cdot p(n,\gamma,i_m) \quad \text{by (20)} \\
&= \sum_{1\le i_1<\ldots<i_m\le l-1} p(n,\gamma,i_1)\ldots p(n,\gamma,i_m) \\
&+ \sum_{1\le i_1<\ldots<i_{m-1}\le l-1} p(n,\gamma,i_1)\ldots p(n,\gamma,i_{m-1})p(n,\gamma,l) \\
&\quad \text{where the first sum vanishes if } m = l \\
&< \sum_{1\le i_1<\ldots<i_m\le l-1} p(n,\gamma,i_1)\ldots p(n,\gamma,i_m) \\
&+ \left[\sum_{1\le i_1<\ldots<i_{m-1}\le l-1} p(n,\gamma,i_1)\ldots p(n,\gamma,i_{m-1})\right]\frac{n}{a} \quad \text{by assumption} \\
&\le \binom{l-1}{m}\left(\frac{n-p(n,\gamma,l)}{l-1}\right)^m + \frac{n}{a}\binom{l-1}{m-1}\left(\frac{n-p(n,\gamma,l)}{l-1}\right)^{m-1} \\
&\quad \text{by Lemma 10.14, twice, with } k = l-1,\ \alpha = n-p(n,\gamma,l), \text{ and} \\
&\quad \text{with } m \text{ in the first application and } m-1 \text{ in the second application} \\
&\le \binom{l-1}{m}\frac{n^m}{(l-1)^m} + \binom{l-1}{m-1}\frac{n^m}{a(l-1)^{m-1}} \\
&= \left[\frac{1}{(l-1)^m}\binom{l-1}{m} + \frac{1}{a(l-1)^{m-1}}\binom{l-1}{m-1}\right]n^m. \qquad \square
\end{aligned}$$

10.4. **Proof of the second part of Theorem 10.5.** Let $m_{\max} = \max(r_1,\ldots,r_\rho)$, where $r_1,\ldots,r_\rho \ge 2$ are the arities of the relation symbols $R_1,\ldots,R_\rho$ of the vocabulary of L_{rel}. Suppose that at least one relation symbol has arity $\le l$. We use the notation from the previous section (Notation 10.13). Since for every $m > l$ and every $\gamma : [n] \to [l]$, *no* $(a_1,\ldots,a_m) \in [n]^m$ is strongly multichromatic with respect to γ, we may, without loss of generality, assume that $m_{\max} \le l$.

Let $\mathbf{X}_n \subseteq \mathbf{S}_n^I$ be the set of all $\mathcal{M} \in \mathbf{S}_n^I$ which have a strong l-colouring which is *not* $\frac{n}{a}$-rich, where $a > l$ is a number that will be specified after we have made some estimates. In order to prove part (ii) of Theorem 10.5 it is enough to prove that we can choose a so that

$$\lim_{n\to\infty}\frac{|\mathbf{X}_n|}{|\mathbf{S}_n^I|} = 0. \tag{21}$$

For every $n \ge l$, fix $\sigma_n : [n] \to [l]$ such that, for every $i \in [l]$,

$$\frac{n}{l} - 1 \le p(n,\sigma_n,i) \le \frac{n}{l} + 1. \tag{22}$$

Then every σ_n is $\frac{n}{a}$-rich (because $a > l$). For all sufficiently large n,

$$|\mathbf{S}_n^I| \ge 2^{\sum_{k\in[\rho]-I}\text{smult}(n,\sigma_n,r_k) + \sum_{k\in I}\overline{\text{smult}}(n,\sigma_n,r_k)}. \tag{23}$$

For every $n \in \mathbb{N}$, choose $\gamma_n : [n] \to [l]$ such that

$$\gamma_n \text{ is } \textit{not } \frac{n}{a}\text{-rich, and} \tag{24}$$

for every $\gamma : [n] \to [l]$ which is *not* $\frac{n}{a}$-rich,

$$\begin{aligned}(25)\qquad & \sum_{k\in[\rho]-I} \mathrm{smult}(n,\gamma,r_k) \ + \ \sum_{k\in I}\overline{\mathrm{smult}}(n,\gamma,r_k) \\ \leq\ & \sum_{k\in[\rho]-I} \mathrm{smult}(n,\gamma_n,r_k) \ + \ \sum_{k\in I}\overline{\mathrm{smult}}(n,\gamma_n,r_k).\end{aligned}$$

Since, for some $\beta > 0$, there are at most $l^n = 2^{\beta n}$ l-colourings $\gamma : [n] \to [l]$, and every $\mathcal{M} \in \mathbf{X}_n$ has an l-colouring which is not $\frac{n}{a}$-rich, it follows from (24) and (25) that

$$(26)\qquad \big|\mathbf{X}_n\big| \ \leq \ 2^{\beta n + \sum_{k\in[\rho]-I}\mathrm{smult}(n,\gamma_n,r_k) + \sum_{k\in[\rho]}\overline{\mathrm{smult}}(n,\gamma_n,r_k)}.$$

From (19), (23) and (26) it follows that in order to prove (21) it suffices to show that, for every $2 \leq m \leq m_{\max}$ there is a constant $\lambda_m > 0$ such that for all sufficiently large n,

$$(27)\qquad \overline{\mathrm{smult}}(n,\sigma_n,m) - \overline{\mathrm{smult}}(n,\gamma_n,m) \ \geq \ \lambda_m n^m.$$

We have

$$\begin{aligned}(28)\qquad \overline{\mathrm{smult}}(n,\sigma_n,m) \ &= \sum_{1\leq i_1<\ldots<i_m\leq l} p(n,\sigma_n,i_1)\ldots p(n,\sigma_n,i_m) \quad \text{by (20)} \\ &\geq \binom{l}{m}\left(\frac{n}{l}-1\right)^m \quad \text{by the definition of } \sigma_n \\ &= \binom{l}{m}\frac{n^m}{l^m} \ \pm \ O\big(n^{m-1}\big).\end{aligned}$$

By Lemma 10.16,

$$(29)\qquad \overline{\mathrm{smult}}(n,\gamma_n,m) \ \leq \ \left[\frac{1}{(l-1)^m}\binom{l-1}{m} \ + \ \frac{1}{a(l-1)^{m-1}}\binom{l-1}{m-1}\right]n^m,$$

where the left term within the parentheses vanishes if $m = l$. From (28) and (29) we get

$$\begin{aligned}&\overline{\mathrm{smult}}(n,\sigma_n,m) \ - \ \overline{\mathrm{smult}}(n,\gamma_n,m) \\ &\geq \ \left[\frac{1}{l^m}\binom{l}{m} \ - \ \frac{1}{(l-1)^m}\binom{l-1}{m} \ - \ \frac{1}{a(l-1)^{m-1}}\binom{l-1}{m-1}\right]n^m \ \pm \ O\big(n^{m-1}\big).\end{aligned}$$

Observe that the rightmost term within the large parentheses can be made arbitrarily small by choosing a large enough. Thus, to prove (27) it suffices to show that

$$(30)\qquad \frac{1}{l^m}\binom{l}{m} \ - \ \frac{1}{(l-1)^m}\binom{l-1}{m} \ > \ 0.$$

But (30) holds because whenever $2 \leq m \leq l$ we have

$$\begin{aligned}\frac{1}{l^m}\binom{l}{m} \ &= \ \frac{1}{l^m}\cdot\frac{\prod_{i=0}^{m-1}(l-i)}{m!} \ = \ \frac{1}{m!}\prod_{i=0}^{m-1}\frac{l-i}{l} \\ &= \ \frac{1}{m!}\prod_{i=0}^{m-1}\left(1-\frac{i}{l}\right) \ > \ \frac{1}{m!}\prod_{i=0}^{m-1}\left(1-\frac{i}{l-1}\right) \ = \ \frac{1}{(l-1)^m}\binom{l-1}{m}.\end{aligned}$$

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