Independence and the finite submodel property

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Abstract

We study a class $\mathcal{C}$ of $\aleph_0$-categorical simple structures such that every M in $\mathcal{C}$ has uncomplicated forking behavior and such that definable relations in M which do not cause forking are independent in a sense that is made precise; we call structures in $\mathcal{C}$ *independent*. The SU-rank of such M may be n for any natural number $n > 0$. The most well-known unstable member of $\mathcal{C}$ is the random graph, which has SU-rank one. The main result is that for every *strongly* independent structure M in $\mathcal{C}$, if a sentence φ is true in M then φ is true in a finite substructure of M. The same conclusion holds for every structure in $\mathcal{C}$ with SU-rank one; so in this case the word 'strongly' can be removed. A probability theoretic argument is involved and it requires sufficient independence between relations which do not cause forking. A stable structure M belongs to $\mathcal{C}$ if and only if it is $\aleph_0$-categorical, $\aleph_0$-stable and every definable strictly minimal subset of M^{eq} is indiscernible.

Introduction

As our starting point we can take the complete theory T_{rg} of the random graph (see [8], Section 7.4, for a definition of it). T_{rg} is countably categorical and unstable, but simple with uncomplicated forking behavior: for example, T_{rg} is 1-based, has SU-rank 1 and trivial forking. Every model M of T_{rg} also has the finite submodel property, by which we mean that if φ is a sentence which is true in M then φ is true in a finite substructure of M. This result owes to the fact that definable relations in a model of T_{rg} which do not cause forking are "sufficiently independent" and this allows one to prove the finite submodel property by a probability theoretic argument. In some cases, like the random graph, the independence of relations imply a stronger result, a 0-1 law for a set of finite structures. In the more general settings studied in [5] and in this paper, 0-1 laws are not necessarily a consequence of our arguments, but we get the finite submodel property.[1]

We will encounter three different ways of making precise the idea of sufficient independence: the n-embedding of types property, the n-independence hypothesis and the n-amalgamation property. The last two notions have been studied in [5] and [10], respectively. Without assuming sufficient independence we encounter some difficulties with respect to proving or refuting the finite submodel property, even if the theory under consideration has very uncomplicated forking behavior. For example, the random pyramid-free (3)-hyper graph (see [6] for instance) is $\aleph_0$-categorical, simple, 1-based, has

[1] When considering limit laws we have to decide (in a given context) which finite structures to take into account and what probability measure to use on them. For instance, there is a *strongly independent structure* M, in the sense of this paper, with SU-rank 1 (so algebraic closure is trivial) and the following property: Let $\mathcal{K}_n$ be the set of all structures with universe $\{1, \ldots, n\}$ which are isomorphic to some substructure of M. There is φ such that $M \models \varphi$, so φ is true in a finite substructure of M, but the proportion of structures in $\mathcal{K}_n$ in which φ is true approaches 0 as $n \to \infty$. However, there is another, in the context natural, probability measure (than the uniform one) on $\mathcal{K}_n$ such that for *every* $\psi \in Th(M)$ the probability that ψ is true in a member of $\mathcal{K}_n$ approaches 1 as $n \to \infty$. Results concerning 0-1 laws and finite substructures of $\aleph_0$-categorical structures will appear in a forthcoming paper of the author.

SU-rank 1 and trivial forking, but it is unknown (as far as the author knows) whether it has the finite submodel property. The random pyramid-free (3)-hyper graph does not, however, satisfy any of the three "sufficient independence" conditions considered in this article.

Here, we call a structure *strongly independent* if its complete theory T has the following properties: countable categoricity, simplicity, 1-basedness, trivial forking (which implies that T has finite SU-rank) and the n-embedding of types property (with respect to all generators) for every natural number n; in addition we will assume that the language of T has a finite upper bound on the arity of its function symbols. The main result is that every strongly independent structure has the finite submodel property. In the course of proving this we prove that every *independent* structure with SU-rank 1 has the finite submodel property. The difference between 'independent' and 'strongly independent' is that in the former case we only require the n-embedding of types property to hold with respect to simple generators; definitions are given in sections 3 and 4.

The class of independent structures includes as a subclass all $\aleph_0$-categorical $\aleph_0$-stable structures which satisfy that every definable strictly minimal set is indiscernible. The latter class was studied in [12] and contains all (infinite) countable finitely homogeneous stable structures (see [13] for a survey). Note that an $\aleph_0$-categorical $\aleph_0$-stable structure need not be independent since it need not have trivial forking, and an independent structure need not be smoothly approximable (a property which holds for every $\aleph_0$-categorical $\aleph_0$-stable structure [2], [9], [1]) since the bipartite random graph may be definable in it. But if an independent structure M is stable, then it is $\aleph_0$-stable and every definable strictly minimal subset of M^{eq} is indiscernible. In Section 7 an example is given of an unstable strongly independent structure with SU-rank $n+1$, for arbitrarily chosen $0 < n < \aleph_0$. More examples of unstable strongly independent structures are given in Section 7 and in Section 6 of [6].

The proof that a strongly independent structure has the finite submodel property is carried out in Sections 4 and 5 and it uses the main results from [5] and [6]. A rough outline of the proof goes as follows: Given a strongly independent M we find (by results in [6]) a *canonically embedded structure* N of M^{eq} which has the property that (N, acl_N) is a pregeometry (where acl_N is the algebraic closure operator on N) and $M \subseteq \text{acl}_{M^{\text{eq}}}(N)$. Then we are able to apply results from [5], an article which studies structures on which the algebraic closure forms a pregeometry, to prove that N has the finite submodel property. When this is done, we apply a result from [6] which roughly says that if M_0 is canonically embedded in M^{eq}, $M \subseteq \text{acl}_{M^{\text{eq}}}(M_0)$ and M_0 has the finite submodel property, then M also has it. In this way we conclude that every strongly independent structure has the finite submodel property.

In Section 3, we introduce different variants of the n-embedding of types property and prove that every stable theory has the strong n-embedding of types property with respect to simple generators, for every $2 \leq n < \aleph_0$; this is a consequence of the stationarity of types over algebraically closed sets in stable theories. In Section 6 we prove that if T is simple with SU-rank one, the algebraic closure coincides with the definable closure (when imaginaries are involved) and T has the k-amalgamation property for every $k \leq n+1$, then T has the k-embedding of types property for real types (where the free variables are of sort '=') with respect to simple generators, for every $2 \leq k \leq n$. If, in addition, forking is trivial, then the conclusion may be strengthened by removing the part "for real types". From this and [10] it follows that T_{rg} has the n-embedding of types property with respect to simple generators, for every $2 \leq n < \aleph_0$. In fact, T_{rg} has the (strong) n-embedding of types property with respect to *all* generators, for every $2 \leq n < \aleph_0$, which is explained in Section 7.

Section 2 reviews the main notions and results from [5] and Section 7 gives examples which illustrate the new concepts of this paper.

I would like to thank the anonymous referee for observing an error in an earlier version of the article.

1 Preliminaries

Notation and terminology. We use notation and terminology which is more or less standard. By $\bar{a}, \bar{b}$, $\bar{x}$, $\bar{y}$ etc., we denote sequences of elements or variables; unless said otherwise, sequences will be finite. For any sequences $\bar{a}$ and $\bar{b}$ the concatenation of them is denoted by $\bar{a}\bar{b}$. Occasionally we may consider a sequence $\bar{a}$ as a set (by disregarding the order of the elements in the sequence). With the notation $\bar{a} \in A$ we mean that each element in the sequence $\bar{a}$ belongs to A. For a sequence $\bar{a}$, $|\bar{a}|$ denotes its length; for a set A, $|A|$ denotes its cardinality. Sometimes we use the notation $\mathrm{rng}(\bar{a})$ to denote the set of all elements that occur in $\bar{a}$. Given sets A and B we sometimes write AB instead of $A \cup B$.

For a structure M, the complete theory of M is denoted by $Th(M)$. We write $\mathrm{dcl}_M(A)$, $\mathrm{acl}_M(A)$ and $tp_M(\bar{a}/A)$ for the definable closure of A in M, the algebraic closure of A in M and the type of $\bar{a}$ over A in M; if the subscript 'M' is clear from the context we may drop it. Two elements a and b are called interalgebraic if $\mathrm{acl}_M(a) = \mathrm{acl}_M(b)$ where M is the model under consideration. For a complete theory T, let $S_n(T)$ be the set of complete n-types of T. For a subset $A \subseteq M$, let $S_n^M(A)$ denote the set of n-types over A (which are realized in some elementary extension of M).

We say that M is $\aleph_0$-categorical/simple/supersimple if $Th(M)$ is it. We will frequently use the well-known characterization of $\aleph_0$-categorical theories (see [8] for example). An important consequence of this characterization is that if M is $\aleph_0$-categorical and $A \subseteq M$ is finite then $\mathrm{acl}_M(A)$ is finite.

If $\bar{a} \in M^{\mathrm{eq}}$ and $A \subseteq M^{\mathrm{eq}}$ then $\mathrm{SU}(\bar{a}/A)$ denotes the SU-rank of the type $tp_{M^{\mathrm{eq}}}(\bar{a}/A)$; and $\mathrm{SU}(\bar{a})$ means $\mathrm{SU}(\bar{a}/\emptyset)$. We define the SU-*rank* of a simple structure M to be $\sup\{\mathrm{SU}(p(x)) : p(x) \in S_1(Th(M))\}$, if the supremum exists. We say that M has *finite* SU-*rank* if this supremum is finite.

If T is supersimple, or $\aleph_0$-categorical and simple, then T has elimination of hyperimaginaries (see [16] for instance), so in this setting it is sufficient to consider the algebraic closure in situations where the general case (of simple theories) would require considerations of the bounded closure.

If we talk about sets or sequences of elements from some structure without specifying a structure, then we assume that the elements in these sets and sequences come from $\mathcal{M}^{hyp}$ where $\mathcal{M}$ is the monster model of the theory under consideration and $\mathcal{M}^{hyp}$ is the extension by hyperimaginaries. For a simple theory T and set A, $\mathrm{bdd}(A)$ denotes the bounded closure in $\mathcal{M}^{hyp}$. However, except for in a couple of definitions, the theories under consideration will have elimination of hyperimaginaries, so $\mathcal{M}^{hyp}$ may be replaced by $\mathcal{M}^{\mathrm{eq}}$ and bdd may be replaced by acl taken in $\mathcal{M}^{\mathrm{eq}}$.

Let T be simple. We say that T is *1-based* if for all sets A and B, A and B are independent over $\mathrm{bdd}(A) \cap \mathrm{bdd}(B)$. We say that T has *trivial dependence* (also called *trivial forking*) if whenever $A \underset{B}{\not\smile\!\!\!\!|} C_1C_2$, then $A \underset{B}{\not\smile\!\!\!\!|} C_i$ for $i = 1$ or for $i = 2$.

Suppose that L and L' are first order languages with vocabularies (or signatures) V and V', respectively. We say that L' is a *sublanguage* of L if $V' \subseteq V$. If L is a sublanguage of the language of M, then $M{\restriction}L$ denotes the reduct of M to L. Whenever M is $\aleph_0$-categorical we assume that its language is countable.

Definition 1.1 Let M be an L-structure.
(i) For every $0 < n < \aleph_0$ and every equivalence relation E on M^n which is $\emptyset$-definable (i.e. definable without parameters) L^{eq} contains a unary relation symbol P_E (not in L) which, in M^{eq}, is interpreted as the set of E-classes. By a *sort* (in M^{eq}) we mean a set of the form $S_E = \{a \in M^{\mathrm{eq}} : M^{\mathrm{eq}} \models P_E(a)\}$ for some E as above. If $A \subseteq M^{\mathrm{eq}}$ and there are only finitely many E such that $A \cap S_E \neq \emptyset$ then we say that *only finitely many sorts are represented in A*.
(ii) Any $\emptyset$-definable set $N \subseteq M^{\mathrm{eq}}$ may be considered as a structure in a language which, for every $0 < n < \aleph_0$ and every relation $R \subseteq N^n$ which is $\emptyset$-definable in M^{eq}, contains a relation symbol which is interpreted as R; and we assume that the language of N has no other relation (or function or constant) symbols. If a $\emptyset$-definable set $N \subseteq M^{\mathrm{eq}}$ is considered as a structure in this way, then we say that N is *canonically embedded* in M^{eq}.

Now we collect some facts that will be used in sections 4 and 5. More explanation concerning these facts is given in Section 1 of [6].

Fact 1.2 *Suppose that M is $\aleph_0$-categorical and that N is canonically embedded in M^{eq}. Then:*
(i) M^{eq} is $\aleph_0$-homogeneous.
(ii) For every $\bar{a} \in M^{\mathrm{eq}}$, $tp(\bar{a})$ is isolated.
(iii) For all $\bar{a}, \bar{b} \in N$, $tp_N(\bar{a}) = tp_N(\bar{b}) \iff tp_{M^{\mathrm{eq}}}(\bar{a}) = tp_{M^{\mathrm{eq}}}(\bar{b})$.
(iv) If $A \subseteq M^{\mathrm{eq}}$, only finitely many sorts are represented in A and $B \subset M$ is finite, then, for every $0 < n < \aleph_0$, only finitely many types from $S_n^{M^{\mathrm{eq}}}\big(\mathrm{acl}_{M^{\mathrm{eq}}}(B)\big)$ are realized by n-tuples from A^n.
(v) If only finitely many sorts are represented in N, then N is $\aleph_0$-categorical.
(vi) If M has finite SU-rank, then, for every $\bar{a} \in M^{\mathrm{eq}}$, $\mathrm{SU}(\bar{a}) < \aleph_0$.

Definition 1.3 An L-theory T has the *finite submodel property* if the following holds for any $M \models T$ and sentence $\varphi \in L$: If $M \models \varphi$ then there is a finite substructure $N \subseteq M$ such that $N \models \varphi$. A structure M has the *finite submodel property* if whenever φ is a sentence such that $M \models \varphi$, then there is a finite substructure $N \subseteq M$ such that $N \models \varphi$.

If $Th(M)$ has the finite submodel property then clearly M has it. The opposite direction holds if the language contains only finitely many relation, function and constant symbols; this is easy to see, but is also explained in Observation 1.6 in [5]. The next result is Corollary 2.5 in [6].

Theorem 1.4 *Suppose that M is $\aleph_0$-categorical and that $N \subseteq M^{\mathrm{eq}}$ is a canonically embedded structure such that only finitely many sorts are represented in N and $M \subseteq \mathrm{acl}_{M^{\mathrm{eq}}}(N)$. Also assume that for some $r < \aleph_0$, every function symbol in the language of M has arity at most r.*
(i) If N has the finite submodel property then so does M.
(ii) Suppose that for every formula $\varphi(\bar{x})$ (without parameters) in the language of M, there is a relation symbol R in the language of M such that $R^M = \{\bar{a} : M \models \varphi(\bar{a})\}$. Then M has the finite submodel property if and only if N has the finite submodel property.

2 Polynomial k-saturation and the k-independence hypothesis

In this section we review the main notions and results from [5], which will be essential for the proof of the main theorem (Theorem 5.1), which is carried out in Sections 4 and 5. These notions, polynomial k-saturation and the k-independence property, apply only to structures M such that (M, acl_M) is a pregeometry; the definition of a pregeometry can be found in [5] and in [8], for instance. If (M, acl_M) is a pregeometry, then we call it *trivial* (or *degenerate*) if, for every $A \subseteq M$, $\mathrm{acl}_M(A) = \bigcup_{a \in A} \mathrm{acl}_M(a)$.

Definition 2.1 (i) If M is a structure such that (M, acl_M) forms a pregeometry and $A \subseteq M$ then we define the *dimension of* A to be

$$\dim_M(A) = \inf \big\{ |B| : B \subseteq A \text{ and } A \subseteq \mathrm{acl}_M(B) \big\}.$$

(ii) For a structure M and a type $p(\bar{x})$ over $A \subseteq M$, we say that $p(\bar{x})$ is *algebraic* if it has only finitely many realizations (in any elementary extension of M); otherwise we say that $p(\bar{x})$ is *non-algebraic.*

Definition 2.2 Let $0 < k < \aleph_0$ and suppose that M is a structure such that (M, acl_M) forms a pregeometry. We say that M is *polynomially k-saturated* if there is a polynomial $P(x)$ such that for every $n_0 < \aleph_0$ there is a natural number $n \geq n_0$ and a finite substructure $N \subseteq M$ such that:

(1) $n \leq |N| \leq P(n)$.

(2) N is algebraically closed (in M).

(3) Whenever $\bar{a} \in N$, $\dim_M(\bar{a}) < k$ and $q(x) \in S_1^M(\bar{a})$ is non-algebraic, then there are distinct $b_1, \ldots, b_n \in N$ such that $M \models q(b_i)$ for each $1 \leq i \leq n$.

The random graph and infinite vector spaces over a finite field are examples of structures which are polynomially k-saturated for every $0 < k < \aleph_0$; see [5] for more information about examples. We have the following implication (see Lemma 1.8 in [5]):

Lemma 2.3 *If M is polynomially k-saturated for every $0 < k < \aleph_0$, then M has the finite submodel property.*

Lemma 2.4 *Suppose that M is an $\aleph_0$-categorical structure such that (M, acl_M) is a pregeometry and suppose that $\mathcal{L}$ is a sublanguage of the language of M. If $\mathrm{acl}_{M \restriction \mathcal{L}}$ coincides with acl_M and M is polynomially k-saturated then so is $M \restriction \mathcal{L}$.*

Proof. Suppose that M is polynomially k-saturated, so there is a polynomial $P(x)$ and for every $j < \aleph_0$ a number $j \leq n_j < \aleph_0$ and a substructure $N_j \subseteq M$ such that N_j satisfies (1) - (3) in the definition of polynomial k-saturation, with N_j in place of N and n_j in place of n. Then every N_j is algebraically closed in $M \restriction \mathcal{L}$ and hence in M; moreover, $\dim_M(\bar{a}) = \dim_{M \restriction \mathcal{L}}(\bar{a})$ for every $\bar{a} \in M$. If $q(x) \in S_1^{M \restriction \mathcal{L}}(\bar{a})$ is non-algebraic with respect to $Th(M \restriction \mathcal{L})$, then $q(x)$ is included in some $q'(x) \in S_1^M(\bar{a})$ which is non-algebraic with respect to $Th(M)$. It follows that, for every $j < \aleph_0$, (1) - (3) holds if M is replaced by $M \restriction \mathcal{L}$, N is replaced by $N_j \restriction \mathcal{L}$ and n is replaced by n_j. Hence $M \restriction \mathcal{L}$ is polynomially k-saturated. $\square$

Notation 2.5 (i) If $\bar{s} = (s_1, \ldots, s_n)$ is a sequence of objects and $I = \{i_1, \ldots, i_m\} \subseteq \{1, \ldots, n\}$, where we assume $i_1 < \ldots < i_m$, then $\bar{s}_I$ denotes the sequence $(s_{i_1}, \ldots, s_{i_m})$.
(ii) If $p(\bar{x})$ is a type and $\bar{x}'$ is a subsequence of $\bar{x}$, then $p{\restriction}\{\bar{x}'\}$ is the set of all formulas $\varphi(\bar{x}')$ such that $\varphi(\bar{x}') \in p(\bar{x})$; so $p{\restriction}\{\bar{x}'\}$ is a type.

Definition 2.6 Suppose that M is an $\aleph_0$-categorical L-structure such that (M, acl_M) is a pregeometry. Let $\mathcal{L}$ be a sublanguage of L. We say that M *satisfies the k-independence hypothesis over* $\mathcal{L}$ if the following holds for any $\bar{a} = (a_1, \ldots, a_n) \in M^n$ such that $\dim_M(\bar{a}) \leq k$:
If $I = \{i_1, \ldots, i_m\} \subseteq \{1, \ldots, n\}$ and $p(\bar{x}_I) \in S_m(Th(M))$ (where $\bar{x}_I = (x_{i_1}, \ldots x_{i_m})$) are such that

(a) $\mathrm{acl}_M(\bar{a}_I) = \mathrm{rng}(\bar{a}_I)$, $\dim_M(\bar{a}_I) < k$, $p(\bar{x}_I) \cap \mathcal{L} = tp_{M{\restriction}\mathcal{L}}(\bar{a}_I)$ and for every $J \subset I$ with $\dim_M(\bar{a}_J) < \dim_M(\bar{a}_I)$, $p{\restriction}\{\bar{x}_J\} = tp_M(\bar{a}_J)$,

then there is $\bar{b} = (b_1, \ldots, b_n) \in M^n$ such that

(b) $tp_{M{\restriction}\mathcal{L}}(\bar{b}) = tp_{M{\restriction}\mathcal{L}}(\bar{a})$, $tp_M(\bar{b}_I) = p(\bar{x}_I)$ and, for every $J \subset \{1, \ldots, n\}$ such that $\bar{a}_I \not\subseteq \mathrm{acl}_M(\bar{a}_J)$, $tp_M(\bar{a}_J) = tp_M(\bar{b}_J)$.

In [5] examples are given of structures which either satisfy or fail to satisfy the k-independence hypothesis over some sublanguage, for various k.

From [5] (Theorem 2.2) we have the following:

Theorem 2.7 *Let M be an $\aleph_0$-categorical L-structure such that (M, acl_M) forms a pregeometry. Suppose that there is a sublanguage $\mathcal{L} \subseteq L$ such that $\mathrm{acl}_{M{\restriction}\mathcal{L}}$ coincides with acl_M and, for every $0 < k < \aleph_0$, $M{\restriction}\mathcal{L}$ is polynomially k-saturated and M satisfies the k-independence hypothesis over $\mathcal{L}$. Then M is polynomially k-saturated, for every $0 < k < \aleph_0$, and M has the finite submodel property.*

Remark 2.8 The properties of polynomials which are used when applying the assumption of 'polynomial k-saturation' in the proof of Theorem 2.7 (given in [5]) are that polynomials are closed under composition and that if $P(x)$ is a polynomial and $0 < c < 1$, then $\lim_{n\to\infty} P(n)c^n = 0$. Any other class $\mathcal{F}$ of functions which is closed under composition with a polynomial (i.e. if $f(x) \in \mathcal{F}$ and $P(x)$ is a polynomial, then $f(P(x)), P(f(x)) \in \mathcal{F}$) and satisfies that $\lim_{n\to\infty} |f(n)|c^n = 0$, for every $f(x) \in \mathcal{F}$ and every $0 < c < 1$, would do; in fact it would be sufficient that the limit exists and is less than one.

3 The n-embedding of types property

In this section we introduce the n-embedding of types property (where $2 \leq n < \aleph_0$), or rather, a few variants of it. This notion is a way of making precise the idea that definable relations which do not cause forking are independent of each other. All stable theories have the (strong) n-embedding of types property with respect to simple generators, for every $2 \leq n < \aleph_0$, which is proved below. This is a consequence of the stationarity of strong types in stable theories. The random graph has the (strong) n-embedding of types property (with respect to all generators) for every $2 \leq n < \aleph_0$; this is explained in Example 7.1. Another simple unstable example, with SU-rank k for arbitrarily chosen $1 < k < \aleph_0$, which has the (strong) n-embedding of types property (with respect to all generators) for every $2 \leq n < \aleph_0$ is given in Section 7.

A related notion, studied in [10] and [4], is the n-amalgamation property, which generalizes a similar property from [11]. In Section 6 we will prove a relationship between the n-amalgamation property and the n-embedding of types propery in the case when the theory has SU-rank one. Throughout this section we assume that T is simple, with monster model $\mathcal{M}$, although this may be repeated; and we work in $\mathcal{M}^{\mathrm{hyp}}$, or in $\mathcal{M}^{\mathrm{eq}}$ if the theory under consideration has elimination of hyperimaginaries.

Definition 3.1 Let S be a partially ordered set with a least element 0, such that the greatest lower bound $s \wedge t$ of any two $s, t \in S$ exists, and if $s, t \in S$ have an upper bound, then there is a least upper bound $s \vee t$ of s and t.

(i) We call $(\{A_s : s \in S\},\ \{\pi^s_t : s \leq t \in S\})$ a *directed family of boundedly closed sets* if for each $s \in S$, $\pi^s_t : A_s \to A_t$ is an elementary map whenever $s \leq t \in S$ and π^s_s is the identity map for each $s \in S$.

(ii) A directed family of boundedly closed sets $(\{A_s : s \in S\},\ \{\pi^s_t : s \leq t \in S\})$ is called an *independent system of boundedly closed sets indexed by* S if the following hold for every $s \in S$:

(1) If $u, v \leq s$ and $t = u \wedge v$, then $\pi^u_s(A_u) \underset{\pi^t_s(A_t)}{\downarrow} \pi^v_s(A_v)$.

(2) If there is $t \in S$ such that $0 < t < s$ then $A_s = \mathrm{bdd}\Big(\bigcup_{t<s} \pi^t_s(A_t)\Big)$.

If all the maps π^s_t are inclusions then we write $\{A_s : s \in S\}$ instead of $(\{A_s : s \in S\},\ \{\pi^s_t : s \leq t \in S\})$.

(iii) For every $n < \aleph_0$, n also denotes the set $\{0, \ldots, n-1\}$ (or $\emptyset$ if $n = 0$). Let $\mathcal{P}(n)$ be the power set of n and let $\mathcal{P}^-(n) = \mathcal{P}(n) - \{n\}$. Note that $\mathcal{P}(n)$ and $\mathcal{P}^-(n)$ are partially ordered by inclusion and that $(\mathcal{P}(n), \subseteq)$ and $(\mathcal{P}^-(n), \subseteq)$ satisfy the requirements on S mentioned above.

(iv) Let $\mathcal{A} = \{A_w : w \in \mathcal{P}^-(n)\}$ be an independent system of boundedly closed sets with inclusion maps and suppose that A and A^0_i, $i \in n$, are such that $A \subseteq A_\emptyset$ and for every $i \in n$, $A_{\{i\}} = \mathrm{bdd}\Big(A \cup A^0_i\Big)$ and whenever $|w| = n-1$, then $A_w = \mathrm{bdd}\Big(\bigcup_{i \in w} A^0_i\Big)$. Then we call $\mathcal{G}_A = \{A^0_i : i \in n\}$ a *set of generators of* $\mathcal{A}$ *over* A, or say that $\mathcal{A}$ *is generated by* $\mathcal{G}_A = \{A^0_i : i \in n\}$ *over* A. We may also express this by saying that $(\mathcal{A}, \mathcal{G}_A)$ is an independent system of boundedly closed sets generated by $\mathcal{G}_A$ over A. If $\mathcal{G}_A$ is a generator of $\mathcal{A}$ over A and $A \subseteq A^0_i$ for every $i \in n$, then we call $\mathcal{G}_A$ a *simple generator* of $\mathcal{A}$ (over A).

(v) An *independent system of algebraically closed sets* $\mathcal{A}$ *(generated by* $\mathcal{G}_A$) is defined in the same way as an independent system of boundedly closed sets $\mathcal{A}$ (generated by $\mathcal{G}_A$), except that we replace 'boundedly closed' by 'algebraically closed' in (ii) and (iv).

(vi) Suppose that $(\mathcal{A}, \mathcal{G}_A)$ and $(\mathcal{B}, \mathcal{G}_B)$ are independent systems of algebraically closed sets, indexed by $\mathcal{P}^-(n)$, with inclusion maps and generated by $\mathcal{G}_A = \{A^0_i : i \in n\}$ over A and $\mathcal{G}_B = \{B^0_i : i \in n\}$ over B, respectively. If for every $w \in \mathcal{P}^-(n)$, f_w is an elementary map from A_w onto B_w, $f_w \restriction \bigcup_{i \in w} A^0_i$ extends $f_v \restriction \bigcup_{i \in v} A^0_i$ whenever $w \supseteq v$, and $f_{\{i\}}(A^0_i) = B^0_i$ for every $i \in n$, then we call $\{f_w : w \in \mathcal{P}^-(n)\}$ a *system of elementary maps from* $(\mathcal{A}, \mathcal{G}_A)$ *onto* $(\mathcal{B}, \mathcal{G}_B)$.

Definition 3.2 Let $n \geq 2$.

(i) Let $(\mathcal{A}, \mathcal{G}_A)$ and $(\mathcal{B}, \mathcal{G}_B)$ be independent systems of algebraically closed sets indexed by $\mathcal{P}^-(n)$, with inclusion maps and generated by $\mathcal{G}_A = \{A_i^0 : i \in n\}$ over A and $\mathcal{G}_B = \{B_i^0 : i \in n\}$ over B, respectively. Moreover suppose that $\mathcal{F} = \{f_w : w \in \mathcal{P}^-(n)\}$ is a system of elementary maps from $(\mathcal{A}, \mathcal{G}_A)$ onto $(\mathcal{B}, \mathcal{G}_B)$. We say that the triple $\big((\mathcal{A}, \mathcal{G}_A), (\mathcal{B}, \mathcal{G}_B), \mathcal{F}\big)$ has the *embedding of types property* if whenever

(1) $\text{rng}(\bar{a}) \cap \text{acl}\Big(\bigcup_{w \in \mathcal{P}^-(n)} A_w\Big) = \emptyset$ and

(2) $a \in \text{rng}(\bar{a})$ and $a \in \text{acl}\Big((\text{rng}(\bar{a}) - \{a\}) \cup \bigcup_{w \in \mathcal{P}^-(n)} A_w\Big)$ implies that $a \in \text{acl}\big(\text{rng}(\bar{a}) - \{a\}\big)$,

then

(3) there are $\bar{b}$ and for every $w \in \mathcal{P}^-(n)$ an elementary map g_w from $\text{rng}(\bar{a}) \cup A_w$ onto $\text{rng}(\bar{b}) \cup B_w$ such that $g_w(\bar{a}) = \bar{b}$, if $w \supseteq v$ then $g_w \restriction \bigcup_{i \in w} A_i^0$ extends $g_v \restriction \bigcup_{i \in v} A_i^0$, and $g_{\{i\}} \restriction A_i^0 = f_{\{i\}} \restriction A_i^0$ for every $i \in n$.

(ii) We say that $\big((\mathcal{A}, \mathcal{G}_A), (\mathcal{B}, \mathcal{G}_B), \mathcal{F}\big)$ has the *strong embedding of types property* if (1) implies (3).

(iii) We say that $\big((\mathcal{A}, \mathcal{G}_A), (\mathcal{B}, \mathcal{G}_B), \mathcal{F}\big)$ has the *embedding of types property for real types* (or *strong embedding of types property for real types*) if whenever $\bar{a}$ is a sequence of real elements (i.e. elements of sort '=') and (1) and (2) hold (or (1) holds), then (3) holds.

(iv) We say that T has the *(strong) n-embedding of types property (for real types) with respect to all generators* if whenever $(\mathcal{A}, \mathcal{G}_A)$ and $(\mathcal{B}, \mathcal{G}_B)$ are independent systems of algebraically closed sets indexed by $\mathcal{P}^-(n)$, with inclusion maps, with generators $\mathcal{G}_A$ over A and $\mathcal{G}_B$ over B, respectively, and $\mathcal{F} = \{f_w : w \in \mathcal{P}^-(n)\}$ is a system of elementary maps from $(\mathcal{A}, \mathcal{G}_A)$ onto $(\mathcal{B}, \mathcal{G}_B)$, then $\big((\mathcal{A}, \mathcal{G}_A), (\mathcal{B}, \mathcal{G}_B), \mathcal{F}\big)$ has the (strong) embedding of types property (for real types).

(v) We say that T has the *(strong) n-embedding of types property (for real types) with respect to simple generators* if whenever $\mathcal{A}$ and $\mathcal{B}$ are independent systems of algebraically closed sets indexed by $\mathcal{P}^-(n)$, with inclusion maps and with *simple generators $\mathcal{G}_{A_\emptyset}$ over $A_\emptyset$ and $\mathcal{G}_{B_\emptyset}$ over $B_\emptyset$*, respectively, and $\mathcal{F} = \{f_w : w \in \mathcal{P}^-(n)\}$ is a system of elementary maps from $(\mathcal{A}, \mathcal{G}_{A_\emptyset})$ onto $(\mathcal{B}, \mathcal{G}_{B_\emptyset})$ such that $f_w \restriction A_\emptyset = f_v \restriction A_\emptyset$ for all $w, v \in \mathcal{P}^-(n)$, then $\big((\mathcal{A}, \mathcal{G}_{A_\emptyset}), (\mathcal{B}, \mathcal{G}_{B_\emptyset}), \mathcal{F}\big)$ has the (strong) embedding of types property (for real types). *Note:* If $n \geq 3$ then the condition that all $f_w \in \mathcal{F}$ agree on $A_\emptyset$ follows from the other assumptions.

The next result will be used in Sections 6 and 7.

Lemma 3.3 *If T has the strong n-embedding of types property* for real types *with respect to all generators, then T has the strong n-embedding of types property with respect to all generators. The same implication holds if 'with respect to all generators' is replaced by 'with respect to simple generators'.*

Proof. Suppose that T has the strong n-embedding of types property for real types with respect to all generators. Let $(\mathcal{A}, \mathcal{G}_A)$ and $(\mathcal{B}, \mathcal{G}_B)$ be two independent systems of algebraically closed sets indexed by $\mathcal{P}^-(n)$, with inclusion maps and with generators $\mathcal{G}_A = \{A_i^0 : i \in n\}$ over A and $\mathcal{G}_B = \{B_i^0 : i \in n\}$ over B, respectively, and let $\mathcal{F} = \{f_w : w \in \mathcal{P}^-(n)\}$ be a system of elementary maps from $(\mathcal{A}, \mathcal{G}_A)$ onto $(\mathcal{B}, \mathcal{G}_B)$. Suppose that $\bar{a} = (a_1, \ldots, a_k) \in (\mathcal{M}^{\mathrm{eq}})^k$ is such that

$$\mathrm{rng}(\bar{a}) \cap \mathrm{acl}\Big(\bigcup_{w \in \mathcal{P}^-(m)} A_w \Big) = \emptyset. \qquad (*)$$

Then there are real tuples $\bar{a}_i' \in \mathcal{M}$ such that $a_i \in \mathrm{dcl}_{\mathcal{M}^{\mathrm{eq}}}(\bar{a}_i')$ for each $1 \le i \le k$. Let $\bar{a}^* = \bar{a}_1' \ldots \bar{a}_k'$. From $(*)$ we get

$$\mathrm{rng}(\bar{a}^*) \cap \mathrm{acl}\Big(\bigcup_{w \in \mathcal{P}^-(m)} A_w \Big) = \emptyset.$$

By assumption, T has the strong n-embedding of types property for real types with respect to all generators, so there are $\bar{b}^* \in \mathcal{M}$, and for every $w \in \mathcal{P}^-(n)$, an elementary map $g_w : \mathrm{rng}(\bar{a}^*) \cup A_w \to \mathrm{rng}(\bar{b}^*) \cup B_w$ such that $g_w(\bar{a}^*) = \bar{b}^*$, if $w \supseteq v$ then $g_w \restriction \bigcup_{i \in w} A_i^0$ extends $g_v \restriction \bigcup_{i \in v} A_i^0$, and $g_{\{i\}} \restriction A_i^0 = f_{\{i\}} \restriction A_i^0$ for every $i \in n$. For each $1 \le i \le k$ we have (by the choice of $\bar{a}_i'$) $a_i \in \mathrm{dcl}_{\mathcal{M}^{\mathrm{eq}}}(\bar{a}_i')$ and hence $\bar{a} \in \mathrm{dcl}_{\mathcal{M}^{\mathrm{eq}}}(\bar{a}^*)$, so there are $\bar{b} = (b_1, \ldots, b_k) \in \mathrm{dcl}_{\mathcal{M}^{\mathrm{eq}}}(\bar{b}^*)$ and, for every $w \in \mathcal{P}^-(n)$, an elementary map $h_w : \mathrm{rng}(\bar{a}^*\bar{a}) \cup A_w \to \mathrm{rng}(\bar{b}^*\bar{b}) \cup B_w$ which extends g_w. Then clearly, $h_w \restriction \bigcup_{i \in w} A_i^0$ extends $h_v \restriction \bigcup_{i \in v} A_i^0$ if $w \supseteq v$ and $h_{\{i\}} \restriction A_i^0 = g_{\{i\}} \restriction A_i^0 = f_{\{i\}} \restriction A_i^0$ for every $i \in n$. This proves that T has the strong n-embedding of types property with respect to all generators. The other statement of the theorem is proved in the same way; in this case we just assume that $\mathcal{G}_{A_\emptyset}$ and $\mathcal{G}_{B_\emptyset}$ are simple generators over $A_\emptyset$ and $B_\emptyset$, respectively. $\square$

Theorem 3.4 *If T is stable then T has the strong n-embedding of types property with respect to simple generators, for every $2 \le n < \aleph_0$.*

Proof. We use the following notation in this proof: If $f_1 : A_1 \to B_1$ and $f_2 : A_2 \to B_2$ are maps which agree on $A_1 \cap A_2$, then $f_1 \cup f_2$ denotes the map from $A_1 \cup A_2$ which extends both f_1 and f_2.

Suppose that T is stable and that $\mathcal{A} = \{A_w : w \in \mathcal{P}^-(n)\}$ and $\mathcal{B} = \{B_w : w \in \mathcal{P}^-(n)\}$ (where $2 \le n < \aleph_0$) are two independent systems of algebraically closed sets with inclusion maps. Moreover, assume that $\mathcal{G}_{A_\emptyset} = \{A_i^0 : i \in n\}$ and $\mathcal{G}_{B_\emptyset} = \{B_i^0 : i \in n\}$ are simple generators of $\mathcal{A}$ and $\mathcal{B}$ over $A_\emptyset$ and $B_\emptyset$, respectively. Also suppose that $\mathcal{F} = \{f_w : w \in \mathcal{P}^-(n)\}$ is a system of elementary maps from $(\mathcal{A}, \mathcal{G}_{A_\emptyset})$ onto $(\mathcal{B}, \mathcal{G}_{B_\emptyset})$ such that $f_w \restriction A_\emptyset = f_v \restriction A_\emptyset$ for all $w, v \in \mathcal{P}^-(n)$. By parts (ii) and (v) of Definition 3.2, we need to show that $\big((\mathcal{B}, \mathcal{G}_{A_\emptyset}), (\mathcal{B}, \mathcal{G}_{B_\emptyset}), \mathcal{F}\big)$ has the strong n-embedding of types property.

So suppose that

$$\mathrm{rng}(\bar{a}) \cap \mathrm{acl}\Big(\bigcup_{w \in \mathcal{P}^-(n)} A_w \Big) = \emptyset \quad \text{and let} \quad w_0 = n - \{0\} = \{1, \ldots, n-1\}.$$

Then we have $A_{\{0\}} \mathop{\smile\!\!\!\!\!|}_{A_\emptyset} A_{w_0}$ and $B_{\{0\}} \mathop{\smile\!\!\!\!\!|}_{B_\emptyset} B_{w_0}$. Moreover, from $A_{\{0\}} \mathop{\smile\!\!\!\!\!|}_{A_\emptyset} A_{w_0}$ and the assumption that $A_{\{0\}}$, $A_\emptyset$ and A_{w_0} are algebraically closed we get $A_{\{0\}} \cap A_{w_0} = A_\emptyset$; and in the same way we get $B_{\{0\}} \cap B_{w_0} = B_\emptyset$.

Since $\mathcal{G}_{A_\emptyset}$ and $\mathcal{G}_{B_\emptyset}$ are simple generators over $A_\emptyset$ and over $B_\emptyset$, respectively, we have $A_\emptyset \subseteq A_0^0 \subseteq A_{\{0\}}$ and $A_\emptyset \subseteq A_1^0 \subseteq A_{w_0}$, so both $f_{\{0\}}$ and f_{w_0} extend $f_\emptyset : A_\emptyset \to B_\emptyset$. As $A_\emptyset$ and $B_\emptyset$ are algebraically closed and T is stable, all types over algebraically closed sets are stationary. Hence $f_{\{0\}} \cup f_{w_0}$ is an elementary map from $A_{\{0\}} \cup A_{w_0}$ onto $B_{\{0\}} \cup B_{w_0}$ which extends $f_{\{0\}}$ and f_{w_0}; it follows that for every $i \in n$, $(f_{\{0\}} \cup f_{w_0}) \restriction A_i^0$ extends $f_{\{i\}} \restriction A_i^0$. We can extend $f_{\{0\}} \cup f_{w_0}$ to an elementary map f from $\mathrm{acl}(A_{\{0\}} \cup A_{w_0})$ onto $\mathrm{acl}(B_{\{0\}} \cup B_{w_0})$. By (2) in Definition 3.1 (ii) we have $A_w \subseteq \mathrm{acl}(A_{\{0\}} \cup A_{w_0})$ and $B_w \subseteq \mathrm{acl}(B_{\{0\}} \cup B_{w_0})$ for every $w \in \mathcal{P}^-(n)$. Then we can find $\bar{b}$ and extend f to an elementary map f' from $\mathrm{rng}(\bar{a}) \cup \mathrm{acl}(A_{\{0\}} \cup A_{w_0})$ onto $\mathrm{rng}(\bar{b}) \cup \mathrm{acl}(B_{\{0\}} \cup B_{w_0})$ such that $f'(\bar{a}) = \bar{b}$. Now let $g_w = f' \restriction \mathrm{rng}(\bar{a}) \cup A_w$ for every $w \in \mathcal{P}^-(n)$. Then $g_w(\bar{a}) = \bar{b}$ and if $w \supseteq v$ then g_w extends g_v. Moreover, for every $i \in n$, $g_{\{i\}} \restriction A_i^0 = f' \restriction A_i^0 = f \restriction A_i^0 = (f_{\{0\}} \cup f_{w_0}) \restriction A_i^0 = f_{\{i\}} \restriction A_i^0$. $\square$

Evidently, the '(strong) n-embedding of types property (for real types) *with respect to all generators*' implies the '(strong) n-embedding of types property (for real types) *with respect to simple generators*'. The next lemma says that under the assumption $n \geq 3$ and that the algebraic closure has very simple behaviour then the implication can be reversed. The implication cannot be reversed in general, as Example 7.4 shows; we say more about this issue in Examples 7.4 and 7.5.

Definition 3.5 (i) We say that the *algebraic closure and definable closure coincide* if $\mathrm{acl}(A) = \mathrm{dcl}(A)$ for all $A \subset \mathcal{M}^{\mathrm{eq}}$.
(ii) We say that the definable closure is *trivial* if the following holds: Whenever $a, \bar{b} \in \mathcal{M}^{\mathrm{eq}}$, $|\bar{b}| > 1$, $a \in \mathrm{dcl}(\bar{b})$ and $\bar{b} \notin \mathrm{dcl}(a)$, then there is a proper subsequence $\bar{b}'$ of $\bar{b}$ such that $a \in \mathrm{dcl}(\bar{b}')$.

Lemma 3.6 *Let T be simple. Suppose that the algebraic closure coincides with the definable closure and that the latter is trivial. If $n \geq 3$ and T has the (strong) n-embedding of types property (for real types)* with respect to simple generators, *then T has the (strong) n-embedding of types property (for real types)* with respect to all generators.

Proof. Suppose that T is simple, that acl coincides with dcl and that dcl is trivial. We only prove (explicitly) that if $n \geq 3$ and T has the n-embedding of types property *with respect to simple generators*, then T has the n-embedding of types property *with respect to all generators*, because the other variants of the statement are proved by making evident modifications in the proof below.

So, suppose that $n \geq 3$ and that T has the n-embedding of types property with respect to simple generators. Suppose that $(\mathcal{A}, \mathcal{G}_A)$ and $(\mathcal{B}, \mathcal{G}_B)$ are independent systems of algebraically closed sets indexed by $\mathcal{P}^-(n)$, with inclusion maps and with (not necessarily simple) generators $\mathcal{G}_A = \{A_i^0 : i \in n\}$ over A and $\mathcal{G}_B = \{B_i^0 : i \in n\}$ over B, respectively. Moreover, let $\mathcal{F} = \{f_w : w \in \mathcal{P}^-(n)\}$ be a system of elementary maps from $(\mathcal{A}, \mathcal{G}_A)$ onto $(\mathcal{B}, \mathcal{G}_B)$.

It is immediate from the definition that

$$\mathcal{G}'_{A_\emptyset} = \{A_\emptyset \cup A_i^0 : i \in n\} \text{ and } \mathcal{G}'_{B_\emptyset} = \{B_\emptyset \cup B_i^0 : i \in n\}$$

are simple generators of $\mathcal{A}$ over $A_\emptyset$ and of $\mathcal{B}$ over $B_\emptyset$, respectively. Since T has the n-embedding of types property with respect to simple generators it is sufficient to show that there is a system of elementary maps $\mathcal{F}' = \{f'_w : w \in \mathcal{P}^-(n)\}$ from $(\mathcal{A}, \mathcal{G}'_\emptyset)$ onto $(\mathcal{B}, \mathcal{G}'_\emptyset)$ such that, for every $w \in \mathcal{P}^-(n)$, $f'_w \restriction \bigcup_{i \in w} A_i^0 = f_w \restriction \bigcup_{i \in w} A_i^0$. We show that this follows from the following claim, and then prove the claim:

Claim. If $a \in \mathrm{dcl}(A_i^0) \cap \mathrm{dcl}(A_j^0)$ then $f_{\{i\}}(a) = f_{\{j\}}(a)$.

Since $\mathcal{G}_A$ generates $\mathcal{A}$ it follows that, for every w with $|w| = n-1$, $A_\emptyset \subseteq A_w \subseteq \mathrm{dcl}\Big(\bigcup_{i \in w} A_i^0\Big)$. Note that since $\mathcal{A}$ is an independent system of algebraically closed sets generated by $\mathcal{A}$ we must have $A_j^0 \cap \mathrm{acl}\Big(\bigcup_{i\in w} A_i^0\Big) = \emptyset$ whenever $j \notin w$, and it follows that if $a \in \mathrm{dcl}(A_i^0) \cap \mathrm{dcl}(A_j^0)$ and $i \neq j$, then a is not interalgebraic with any tuple of elements from A_i^0 (or from A_j^0). This together with the claim and the assumptions that acl coincides with dcl, where the latter is trivial, and that $\mathcal{F}$ is a system of elementary maps from $(\mathcal{A}, \mathcal{G}_A)$ onto $(\mathcal{B}, \mathcal{G}_B)$ implies that

$$\text{if } |v| = |w| = n-1, \text{ then } f_v \restriction A_\emptyset = f_w \restriction A_\emptyset. \tag{+}$$

Now we define a system of elementary maps from $(\mathcal{A}, \mathcal{G}'_{A_\emptyset})$ onto $(\mathcal{B}, \mathcal{G}'_{B_\emptyset})$. For every $u \in \mathcal{P}^-(n)$ chose (any) $\sigma_u \in \mathcal{P}^-(n)$ such that $u \subseteq \sigma_u$ and $|\sigma_u| = n-1$ and then let $f'_u = f_{\sigma_u} \restriction A_u$. By (+), for all $u, v \in \mathcal{P}^-(n)$, $f'_u \restriction A_\emptyset = f_{\sigma_u} \restriction A_\emptyset = f_{\sigma_v} \restriction A_\emptyset = f'_v \restriction A_\emptyset$. Since, for all $u \in \mathcal{P}^-(n)$, we have $f'_u \restriction \bigcup_{i\in u} A_i^0 = f_{\sigma_u} \restriction \bigcup_{i\in u} A_i^0 = f_u \restriction \bigcup_{i \in u} A_i^0$, it follws that whenever $u \supseteq v$, then $f'_u \restriction \bigcup_{i\in u} A_i^0$ extends $f'_v \restriction \bigcup_{i \in v} A_i^0$. Hence $\mathcal{F}' = \{f'_w : w \in \mathcal{P}^-(n)\}$ is a system of elementary maps from $(\mathcal{A}, \mathcal{G}'_{A_\emptyset})$ onto $(\mathcal{B}, \mathcal{G}'_{B_\emptyset})$ of the kind that we are looking for.

It remains to prove the claim. So suppose that $a \in \mathrm{dcl}(A_i^0) \cap \mathrm{dcl}(A_j^0)$. For a contradiction, suppose that $f_{\{i\}}(a) \neq f_{\{j\}}(a)$, so $i \neq j$. Since we assume that $n \geq 3$, $\{i,j\} \in \mathcal{P}^-(n)$ and hence $f_{\{i,j\}} \restriction \mathrm{dcl}(A_i^0 \cup A_j^0)$ extends both $f_{\{i\}} \restriction \mathrm{dcl}(A_i^0)$ and $f_{\{j\}} \restriction \mathrm{dcl}(A_j^0)$; hence $f_{\{i,j\}}(a) = f_{\{i\}}(a) \neq f_{\{j\}}(a) = f_{\{i,j\}}(a)$, a contradiction. □

4 Independent structures and canonically embedded structures with rank one

In this section and the next we study countably categorical structures M which are simple, with uncomplicated forking behaviour, and where $Th(M)$ has the n-embedding of types property for every $2 \leq n < \aleph_0$. Such strucures will be called *independent* or *strongly independent*, depending on whether we assume the 'n-embedding of types property *with respect to simple generators*' or the stronger version 'n-embedding of types property *with respect to all generators*' (definitions follow below). In this section we prove, in rough terms, that if N is infinite and canonically embedded in M^{eq}, where M is an independent structure, and for every $a \in N$, $\mathrm{SU}(a) = 1$, then N is polynomially n-saturated for every $n < \aleph_0$; consequently, every independent structure with SU-rank 1 has the finite submodel property. This result will be used in Section 5 in the proof of the main result that every strongly independent structure, regardless of its SU-rank, has the finite submodel property.

Definition 4.1 In this paper we call a countable complete theory T *independent* if it is $\aleph_0$-categorical, simple, 1-based, has trivial dependence, has the n-embedding of types property with respect to simple generators for every $2 \leq n < \aleph_0$, and there is $m < \aleph_0$ such that no function symbol in the language of T has arity greater than m. If, in addition, T has the n-embedding of types property with respect to all generators for every $2 \leq n < \aleph_0$, then we call T *strongly independent.* We say that a structure M is *(strongly) independent* if its complete theory is (strongly) independent.

Remark 4.2 By Corollary 4.7 in [7], every simple, 1-based and $\aleph_0$-categorical theory is supersimple with finite SU-rank. Hence, every independent theory is supersimple with finite SU-rank. The most well-known example of a strongly independent structure (with SU-rank 1) is the random graph; see Example 7.1. Section 7 gives another example, which is a variation of the random graph and which has SU-rank $n + 1$ for arbitrarily chosen $0 < n < \aleph_0$. Every supersimple $\aleph_0$-categorical theory with finite SU-rank and trivial dependence is 1-based [7], so in the above definition of independent structure one can replace '1-based' with 'finite SU-rank'.

Every $\aleph_0$-categorical $\aleph_0$-stable structure is 1-based (by Theorem 5.12 in [14] for example). From Corollary 3.23 in [3] and Theorem 3.5 in this paper, it follows that if M is $\aleph_0$-categorical and $\aleph_0$-stable (with a finite upper bound on the arity of its function symbols) and every definable strictly minimal subset of M^{eq} is indiscernible, then M is an independent structure. (See [2] for a definition of a strictly minimal set.)

Now suppose that M is independent and stable. The $\aleph_0$-categoricity and supersimplicity of M implies that M is superstable and hence $\aleph_0$-stable. Since M has trivial dependence (and is stable, so types over algebraically closed sets are stationary), every definable strictly minimal subset of M^{eq} is indiscernible. Hence the structures studied in [12] are precisely the independent structures which are stable.

Theorem 4.3 *Suppose that M is an independent structure, that N is canonically embedded in M^{eq} and that only finitely many sorts are represented in N.*
(i) If N is infinite and for every $a \in N$, $\text{SU}(a) \leq 1$ (where SU-rank is taken with respect to $Th(M^{\text{eq}})$) then N, as a structure in itself, has the finite submodel property.
(ii) Suppose that N is infinite and for every $a \in N$, $\text{SU}(a) = 1$ and $\text{acl}_N(a) = \{a\}$. Then N, as a structure in itself, is polynomially k-saturated for every $k < \aleph_0$.

Remark 4.4 From Theorem 4.3 (i) it follows that if M is independent with SU-rank 1, then M has the finite submodel property.

Proof of Theorem 4.3. Suppose that M is an independent structure, that N is canonically embedded in M^{eq} and that only finitely many sorts are represented in N. We first show that (ii) implies (i). So suppose (ii) holds and that N is infinite and for every $a \in N$, $\text{SU}(a) \leq 1$. In order to use the assumption that (ii) holds we will look at a quotient of N. Let N' be the set of all equivalence classes of the relation $\text{acl}_N(x) = \text{acl}_N(y)$ on $N - \text{acl}_N(\emptyset)$. Since M is $\aleph_0$-categorical and only finitely many sorts are represented in N, N is $\aleph_0$-categorical (see Fact 1.2) and this equivalence relation is $\emptyset$-definable in N and in M^{eq}; and each class of the relation is finite, so N' is infinite. Hence $N' \subseteq N^{\text{eq}}$, and since N' is $\emptyset$-definable in N^{eq} we can consider N' as a canonically embedded structure in N^{eq}, and hence in $(M^{\text{eq}})^{\text{eq}}$. As M^{eq} has elimination of imaginaries, each element of N' is interdefinable (in M^{eq}) with an element of M^{eq}. Thus N' may be identified with a $\emptyset$-definable subset of M^{eq}, and hence we can also see N' as a canonically embedded structure in M^{eq}; moreover, only finitely many sorts are represented in N'. The assumption that M is an independent structure (so dependence is trivial) and the definition of N' implies that for every $a \in N'$, $\text{acl}_{N'}(a) = \{a\}$ and $\text{SU}(a) = 1$. By the assumption that (ii) holds, N' is polynomially k-saturated for every $k < \aleph_0$. By Lemma 2.3, N' has the finite submodel property and, by Theorem 1.4 and the fact $N \subseteq \text{acl}_{N^{\text{eq}}}(N')$, it follows that N has the finite submodel property.

It remains to prove (ii). Recall that we assume that M is an independent structure, that N is canonically embedded in M^{eq} and that only finitely many sorts are represented in N. Moreover assume that N is infinite and for every $a \in N$, $\text{SU}(a) = 1$ and $\text{acl}_N(a) = \{a\}$. Also note that since N is canonically embedded in M^{eq} we have

$\mathrm{acl}_N(A) = \mathrm{acl}_{M^{\mathrm{eq}}}(A) \cap N$ for every $A \subseteq N$; by trivial dependence it follows that for every $A \subseteq N$, $\mathrm{acl}_N(A) = A$ and hence $\dim_N(A) = |A|$.

We will show that an expansion of N which we call N' is polynomially k-saturated for every $k < \aleph_0$. Since for this expansion N' we will have $\mathrm{acl}_N(A) = \mathrm{acl}_{N'}(A)$ for every $A \subseteq N$ it follows from Lemma 2.4 that N is polynomially k-saturated.

For each $n < \aleph_0$ and every $\mathrm{acl}_{M^{\mathrm{eq}}}(\emptyset)$-definable n-ary relation R on N, the language of N', which we call L, contains a relation symbol which is interpreted as R; there are no other relation (or function or constant) symbols in L. By Fact 1.2, N' is $\aleph_0$-categorical.

We will now prove that N' is polynomially k-saturated, for every $k < \aleph_0$, by proving that a reduct of N' is polynomially k-saturated (for every $k < \aleph_0$), that the algebraic closure in the reduct coincides with the algebraic closure in N, and that N' satisfies the k-independence hypothesis over the language of the reduct; then we apply Theorem 2.7 to conclude that N' is polynomially k-saturated.

Let $L_=$ be the language with vocabulary $\{=\}$, so the reduct $N' \restriction L_=$ is just an infinite set with the identity relation. $N' \restriction L_=$ has elimination of quantifiers and it is straightforward to verify that $N' \restriction L_=$ is polynomially k-saturated for every $k < \aleph_0$. Note that, for every $A \subseteq N$, $\mathrm{acl}_{N' \restriction L_=}(A) = A = \mathrm{acl}_N(A) = \mathrm{acl}_{N'}(A)$. We will prove that N' satisfies the k-independence hypothesis over $L_=$ for every $k < \aleph_0$; then Theorem 2.7 gives us the conclusion of (ii).

Fix some $k < \aleph_0$. Let $\bar{a} = (a_0, \ldots, a_{d-1}) \in N^d$ be such that $\dim_{N'}(\bar{a}) = d \leq k$, so no element occurs twice in $\bar{a}$, and assume that $I = \{i_1, \ldots, i_m\} \subseteq \{0, \ldots, d-1\}$ and $p(\bar{x}_I) \in S_m(Th(N'))$ (where $\bar{x}_I = (x_{i_1}, \ldots, x_{i_m})$) are such that

(a) $|\bar{a}_I| < k$, $p(\bar{x}_I) \cap L_= = tp_{N' \restriction L_=}(\bar{a}_I)$ and for every proper subset $J \subset I$, $p \restriction \{\bar{x}_J\} = tp_{N'}(\bar{a}_J)$.

We must show that there is $\bar{b} = (b_0, \ldots, b_{d-1}) \in N^d$ such that

(b) $tp_{N' \restriction L_=}(\bar{b}) = tp_{N' \restriction L_=}(\bar{a})$, $tp_{N'}(\bar{b}_I) = p(\bar{x}_I)$ and, for every $J \subset \{0, \ldots, d-1\}$ such that $\mathrm{rng}(\bar{a}_I) \not\subseteq \mathrm{rng}(\bar{a}_J)$, $tp_{N'}(\bar{a}_J) = tp_{N'}(\bar{b}_J)$.

Observe that, by (a), $m = |\bar{a}_I| = \dim_{N'}(\bar{a}_I)$ and $m < k$; and since we assume that $p(\bar{x}_I)$ has at least one free variable (because otherwise there is nothing to prove) we have $m > 0$. We get two cases to consider, the first being rather trivial. Recall that $d = |\bar{a}| = \dim_{N'}(\bar{a})$.

Case 1. $m = d$.
Then we have $\bar{a}_I = \bar{a}$ (and $\bar{x}_I = \bar{x} = (x_0, \ldots, x_{d-1})$), so $d < k$. Let $\bar{b} = (b_0, \ldots, b_{d-1}) \in N^d$ realize $p(\bar{x}_I)$. The conditions in (a) imply that $tp_{N' \restriction L_=}(\bar{b}) = tp_{N' \restriction L_=}(\bar{a})$ and, if $J \subset \{0, \ldots, d-1\}$ and $\mathrm{rng}(\bar{a}_I) \not\subseteq \mathrm{rng}(\bar{a}_J)$ (which in this case implies that $|\bar{a}_J| < m$) then $tp_{N'}(\bar{a}_J) = tp_{N'}(\bar{b}_J)$. Hence (b) is satisfied.

Case 2. $0 < m < d$.
By reordering if necessary, we may assume that $\bar{a}_I = (a_0, a_1, \ldots, a_{m-1})$ and $\bar{x}_I = (x_0, x_1, \ldots, x_{m-1})$. Let $(b_0, b_1, \ldots, b_{m-1}) \in N^m$ realize $p(\bar{x}_I)$.

If $m = 1$, then let b_0 be a realization of $p(x_0) = p(\bar{x}_I)$ which is different from all $a_1, \ldots, a_{d-1}$; then the tuple $\bar{b} = (b_0, a_1, \ldots, a_{d-1})$ satisfies (b).

In the rest of the proof we assume that $1 < m < d$. For every $w = \{j_0, \ldots, j_l\} \in \mathcal{P}^-(m)$ (where we assume $j_0 < \ldots < j_l$) let

$$\bar{a}_w = (a_{j_0}, \ldots, a_{j_l}), \quad \bar{b}_w = (b_{j_0}, \ldots, b_{j_l}),$$

$$A_w = \mathrm{acl}(\bar{a}_w) \quad \text{and} \quad B_w = \mathrm{acl}(\bar{b}_w),$$

where in the rest of the proof, acl denotes the algebraic closure in $\mathcal{M}^{\mathrm{eq}}$, where $\mathcal{M}$ is the monster model of $Th(M)$.

Claim 1. For all $v, w \in \mathcal{P}^-(m)$, $A_v \mathop{\downarrow}\limits_{A_{v\cap w}} A_w$ and $B_v \mathop{\downarrow}\limits_{B_{v\cap w}} B_w$.

Proof of Claim 1. Suppose that $A_v \mathop{\not\downarrow}\limits_{A_{v\cap w}} A_w$, which implies $\bar{a}_v \mathop{\not\downarrow}\limits_{A_{v\cap w}} \bar{a}_w$. By trivial dependence and symmetry, there are $i \in v - w$ and $j \in w - v$ such that $i \neq 0$ and $a_i \mathop{\not\downarrow}\limits_{A_{v\cap w}} a_j$ and hence $a_i \not\downarrow a_j A_{v\cap w}$ and $a_i \not\downarrow a_j \bar{a}_{v\cap w}$ By trivial dependence, there is $j' \in (v \cap w) \cup \{j\}$ such that $a_i \not\downarrow a_{j'}$. Since $\mathrm{SU}(a_i) = 1$ it follows that $a_i \in \mathrm{acl}(a_{j'})$ and hence $a_i \in \mathrm{acl}_N(a_{j'})$ so $a_i = a_{j'}$. This contradicts the assumption that no element occurs twice in the sequence $\bar{a} = (a_0, \ldots, a_{d-1})$. Hence we must have $A_v \mathop{\downarrow}\limits_{A_{v\cap w}} A_w$ for all $v, w \in \mathcal{P}^-(m)$.

Now suppose that $B_v \mathop{\not\downarrow}\limits_{B_{v\cap w}} B_w$. In the same way as above we find $i \in v - w$ and $j' \in w$ such that $i, j' \neq 0$ and $b_i \not\downarrow b_{j'}$, and hence $b_i \in \mathrm{acl}_N(b_{j'})$ which implies that $b_i = b_{j'}$. Since $\bar{b}_I = (b_0, \ldots, b_{m-1})$ realizes $p(\bar{x}_I)$ and $p(\bar{x}_I) \cap L_= = tp_{N' \restriction L_=}(\bar{a}_I)$, we have

$$tp_{N' \restriction L_=}(b_0, \ldots, b_{m-1}) = tp_{N' \restriction L_=}(a_0, \ldots, a_{m-1}).$$

Since $b_i = b_{j'}$ we get $a_i = a_{j'}$ which, since $i \neq j'$, contradicts that that no element occurs twice in the sequence $\bar{a} = (a_0, \ldots, a_{d-1})$. $\square$

By Claim 1, $\mathcal{A} = \{A_w : w \in \mathcal{P}^-(m)\}$ and $\mathcal{B} = \{B_w : w \in \mathcal{P}^-(m)\}$ are independent systems of algebraically closed sets with inclusion maps. Let $A_i^0 = \{a_i\} \cup \mathrm{acl}(\emptyset)$ and $B_i^0 = \{b_i\} \cup \mathrm{acl}(\emptyset)$ for $i \in m$. Then $\mathcal{G}_{A_\emptyset} = \{A_i^0 : i \in m\}$ is a simple generator of $\mathcal{A}$ over $A_\emptyset$, and $\mathcal{G}_{B_\emptyset} = \{B_i^0 : i \in m\}$ is a simple generator of $\mathcal{B}$ over $B_\emptyset$ (see Definition 3.1).

By assumption (a) and the choice of $(b_0, \ldots, b_{m-1})$, whenever $w \in \mathcal{P}^-(m)$ we have $tp_{N'}(\bar{a}_w) = tp_{N'}(\bar{b}_w)$ and from the definition of N' it follows that

$$tp_{M^{\mathrm{eq}}}\big(\bar{a}_w/\mathrm{acl}(\emptyset)\big) = tp_{M^{\mathrm{eq}}}\big(\bar{b}_w/\mathrm{acl}(\emptyset)\big),$$

and the same holds with M^{eq} replaced by $\mathcal{M}^{\mathrm{eq}}$. Hence there are elementary maps f_w from A_w onto B_w, for all $w \in \mathcal{P}^-(m)$, such that $f_w(\bar{a}_w) = \bar{b}_w$, f_w is the identity on $\mathrm{acl}(\emptyset)$ and if $w \supseteq v$ then $f_w \restriction \bigcup_{i\in w} A_i^0$ extends $f_v \restriction \bigcup_{i\in v} A_i^0$. It follows that $\mathcal{F} = \{f_w : w \in \mathcal{P}^-(m)\}$ is a system of elementary maps from $(\mathcal{A}, \mathcal{G}_{A_\emptyset})$ onto $(\mathcal{B}, \mathcal{G}_{B_\emptyset})$.

The next claim tells us that conditions (1) and (2) from the definition of the m-embedding of types property (Definition 3.1) hold for the sequence $(a_m, \ldots, a_{d-1})$, in the role of the sequence called $\bar{a}$ in that definition. This puts us in a position to use the assumption that $Th(M)$ has the m-embedding of types property with respect to simple generators (as M is an independent structure).

Claim 2. If $a \in \{a_m, \ldots, a_{d-1}\}$ then

$$a \notin \mathrm{acl}\Big((\{a_m, \ldots, a_{d-1}\} - \{a\}) \cup \bigcup_{w \in \mathcal{P}^-(m)} A_w\Big).$$

Proof of Claim 2. Suppose, for a contradiction, that $a \in \{a_m, \ldots, a_{d-1}\}$ and

$$a \in \mathrm{acl}\Big((\{a_m, \ldots, a_{d-1}\} - \{a\}) \cup \bigcup_{w \in \mathcal{P}^-(m)} A_w\Big).$$

Then $a = a_i$ for some $i \in \{m, \ldots, d-1\}$ and from the definition of A_w we get

$$a_i \in \mathrm{acl}\Big((\{a_m, \ldots, a_{d-1}\} - \{a_i\}) \cup \bigcup_{w \in \mathcal{P}^-(m)} \bar{a}_w\Big).$$

The same holds when acl is replaced with $\mathrm{acl}_{N'}$ and hence $a_i \in \{a_0, \ldots, a_{d-1}\} - \{a_i\}$ which contradicts that no element occurs twice in $\bar{a} = (a_0, \ldots, a_{d-1})$. □

By Claim 2 and the assumption that $Th(M)$ has the m-embedding of types property with respect to simple generators (as M is an independent structure), there are $b_m, \ldots, b_{d-1} \in \mathcal{M}^{\mathrm{eq}}$ and for every $w \in \mathcal{P}^-(m)$ an elementary map g_w from $\{a_m, \ldots, a_{d-1}\} \cup A_w$ onto $\{b_m, \ldots, b_{d-1}\} \cup B_w$ such that $g_w(a_i) = b_i$ for $i = m, \ldots, d-1$, and $g_w \restriction \bigcup_{i \in w} A_i^0$ extends $g_v \restriction \bigcup_{i \in v} A_i^0$ if $w \supseteq v$, and $g_{\{i\}} \restriction A_i^0 = f_{\{i\}} \restriction A_i^0$ for $i = 0, \ldots, m-1$. It follows that for every $w \in \mathcal{P}^-(m)$, $g_w(\bar{a}_w) = \bar{b}_w$ and that g_w is the identity on $\mathrm{acl}(\emptyset)$. Hence, for every $w \in \mathcal{P}^-(m)$,

$$tp_{\mathcal{M}^{\mathrm{eq}}}\big((b_m, \ldots, b_{d-1})\bar{b}_w / \mathrm{acl}(\emptyset)\big) = tp_{\mathcal{M}^{\mathrm{eq}}}\big((a_m, \ldots, a_{d-1})\bar{a}_w / \mathrm{acl}(\emptyset)\big).$$

As M is $\aleph_0$-categorical it follows that M^{eq} is $\aleph_0$-homogeneous and therefore we may, without loss of generality, assume that $b_m, \ldots, b_{d-1} \in M^{\mathrm{eq}}$ which implies that $b_m, \ldots, b_{d-1} \in N$. Hence

$$tp_{N'}\big((b_m, \ldots, b_{d-1})\bar{b}_w\big) = tp_{N'}\big((a_m, \ldots, a_{d-1})\bar{a}_w\big) \text{ for every } w \in \mathcal{P}^-(m).$$

From the choice of $\bar{b}_I = (b_0, \ldots, b_{m-1})$, being a realization of $p(\bar{x}_I)$, and $(b_m, \ldots, b_{d-1})$ it follows that if $\bar{b} = (b_0, \ldots, b_{d-1})$, then (b) is satisfied. □

5 Independent structures of higher rank

We will prove the article's main result in this section:

Theorem 5.1 *If M is a strongly independent structure then M has the finite submodel property.*

The general plan of the proof is to show that, given a strongly independent structure M, there is a canonically embedded structure $N \subseteq M^{\mathrm{eq}}$ such that $M \subseteq \mathrm{acl}_{M^{\mathrm{eq}}}(N)$, N has the finite submodel property, and only finitely many sorts are represented in N. Then Theorem 1.4 can be applied to conclude that M has the finite submodel property.

In order to find such N we have to do some preparatory work, most of which is already carried out in [6]. The preparatory work will show that there are structures $N_1, \ldots, N_r$ which are canonically embedded in M^{eq} and satisfy the following:

(1) For every $1 \leq s \leq r$, $(N_s, \mathrm{acl}_{N_s})$ is a pregeometry.

(2) For every $1 \leq s \leq r$, only finitely many sorts are represented in N_s.

(3) $M \subseteq \mathrm{acl}_{M^{\mathrm{eq}}}(N_r)$.

In addition, we will see that elements from $\bigcup_{1 \leq s \leq r} N_s$ have some useful properties. Then, for $s = 1, \ldots, r$, we consider the "quotient" $N'_s = N_s/\sim$ under the equivalence relation $x \sim y \iff \mathrm{acl}(x) = \mathrm{acl}(y)$. Since M^{eq} has elimination of imaginaries N'_s may be identified with an $\emptyset$-definable subset of M^{eq} and can thus be viewed as a canonically embedded structure of M^{eq}. By induction on s, we will then show that, for each $1 \leq s \leq r$, N'_s is polynomially n-saturated for every $n < \aleph_0$, and hence N'_s has the finite submodel property (by Lemma 2.3). It is when doing this that we will use the assumption that $Th(M)$ has the n-embedding of types property for every $n < \aleph_0$ (as M is strongly independent). When it has been shown that N'_r has the finite submodel property we use (2), (3) and Theorem 1.4 to conclude that M has the finite submodel property.

For the rest of this section we assume that M ***is a strongly independent structure*** (according to Definition 4.1).

Notation for this section. If $\bar{a} \in M^{\text{eq}}$ and $A \subseteq M^{\text{eq}}$ then $tp(\bar{a}/A)$ and $\text{acl}(A)$ mean $tp_{M^{\text{eq}}}(\bar{a}/A)$ and $\text{acl}_{M^{\text{eq}}}(A)$. By $\text{SU}(\bar{a}/A)$ we denote the SU-rank of $tp(a/A)$ with respect to $Th(M^{\text{eq}})$. If $a, b \in M^{\text{eq}}$ then $a < b$ is an abbreviation for '$a \in \text{acl}(b)$ and $b \notin \text{acl}(a)$'.

5.1 Preparatory work: finding a canonically embedded pregeometry

We will state a sequence of definitions, constructions and lemmas from Sections 3 and 5 of [6]. In addition we prove some new lemmas which are needed in this paper.

Definition 5.2 We say that a set $A \subseteq M^{\text{eq}}$ is *self-coordinatized* if the following holds:

(1) If $a \in A$ and $\text{SU}(a) > 1$ then there is $b \in A \cap \text{acl}(a)$ such that $\text{SU}(a/b) = 1$ (and hence $\text{SU}(b) = \text{SU}(a) - 1$).

(2) If $a, b \in A$, $\text{SU}(a) > 1$, $b \in A \cap \text{acl}(a)$, $\text{SU}(a/b) = 1$ and there exists $c \in M^{\text{eq}}$ such that $c < a$ and $c \notin \text{acl}(b)$ then such c exists in A.

By the Lemma 3.4 and Construction 3.5 in [6] there exists a self-coordinatized set $C \subseteq M^{\text{eq}}$ such that C is $\emptyset$-definable, $M \subseteq C$, only finitely many sorts are represented in C, and if $c \in C$, $c' \in M^{\text{eq}}$ and $tp(c) = tp(c')$, then $c' \in C$. By the $\aleph_0$-categoricity of $Th(M)$ it follows that only finitely many 1-types over $\emptyset$ are realized in C (Recall Fact 1.2).

Now we can construct subsets $C_n \subseteq C$ and $N_n \subseteq C_n$ in the following way.

Construction 5.3 We define subsets $C_n \subseteq C$ inductively by: $C_0 = \emptyset$ and if C_n is defined and $C \not\subseteq \text{acl}(C_n)$ then

$$C_{n+1} = C_n \cup \big\{c \in C - \text{acl}(C_n) : \text{ there exists no } c' \in C - \text{acl}(C_n) \text{ such that } c' < c\big\}.$$

If $C \subseteq \text{acl}(C_n)$ then C_{n+1} is not defined. Since $C_0 = \emptyset$ (by definition) and M is infinite and $\aleph_0$-categorical it follows that C_1 is defined.

Remark 5.4 Since $Th(M)$ is $\aleph_0$-categorical with finite SU-rank and only finitely many 1-types are realized in C, there is $m < \aleph_0$ such that whenever $c_0, \ldots, c_n \in C$ and $c_0 < \ldots < c_n$, then $n \leq m$.

By Lemma 3.7 in [6], there is $r < \aleph_0$ such that $C \subseteq \text{acl}(C_r)$, and therefore C_{r+1} is undefined and $M \subseteq \text{acl}(C_r)$ (by the choice of C). ***We fix this*** r ***for the rest of Section 5.***

Construction 5.5 For $s = 1, \ldots, r$, let

$$N_s = \big\{c \in C_s : \text{ there exists no } c' \in C_s \text{ such that } c < c' \ \big\}.$$

Since $Th(M)$ is $\aleph_0$-categorical and only finitely many 1-types over $\emptyset$ are realized in N_s (because $N_s \subseteq C$) it follows N_s is $\emptyset$-definable.[2] ***Hence we can regard N_s as a canonically embedded structure of M^{eq}, and we will do this.***

Lemmas 5.6 - 5.10 below are results from Section 3 of [6].[3]

Lemma 5.6 *If $n < r$ and $c \in C_{n+1} - C_n$ then* $\text{SU}(c/C_n) = 1$.

Lemma 5.7 *If $a \in C_r$, $b \in C$, $A \subseteq M^{\text{eq}}$, $b < a$,* $\text{SU}(a/b) = 1$ *and $a \not\downarrow_b A$ then $a \in \text{acl}(A)$.*

Lemma 5.8 *If $a \in C_r$, $d_1, \ldots, d_k \in M^{\text{eq}}$ and $a \in \text{acl}(d_1, \ldots, d_k)$ then $a \in \text{acl}(d_i)$ for some $1 \leq i \leq k$.*

Lemma 5.9 *Let $1 \leq s \leq r$.*
(i) N_s is ω-categorical.
(ii) If $A \subseteq N_s$ then $\text{acl}_{N_s}(A) = \text{acl}_{M^{\text{eq}}}(A) \cap N_s$.
(iii) (N_s, acl_{N_s}) is a trivial (or degenerate) pregeometry.

Lemma 5.10 $M \subseteq C \subseteq \text{acl}(N_r)$.

We also need the following result, which is Lemma 5.1 in [6]:

Lemma 5.11 *If $1 \leq s \leq r$ and $a \in \text{acl}(C_s) \cap C$ then $a \in \text{acl}\big(\text{acl}(a) \cap C_s\big)$.*

In order to prove Theorem 5.1 we need to prove some new lemmas.

Lemma 5.12 *Suppose that $a, b \in C_n$ and $b < a$. Then there is $c \in \text{acl}(a) \cap C_{n-1}$ such that $b \in \text{acl}(c)$.*

Proof. Suppose that $a, b \in C_n$ and $b < a$. If $a \in C_{n-1}$ then the conclusion is trivial so assume that $a \in C_n - C_{n-1}$. Let $\bar{c}$ enumerate $\text{acl}(a) \cap C_{n-1}$. Since C is self-coordiatized there is $d \in \text{acl}(a) \cap C$ such that $\text{SU}(a/d) = 1$; and consequently $d < a$. From the definition of C_n and the assumption that $a \in C_n - C_{n-1}$ it follows that $d \in \text{acl}(C_{n-1})$. Since $d \in \text{acl}(a)$ it follows from Lemma 5.11 that $d \in \text{acl}(\bar{c})$, and hence $\text{SU}(a/\bar{c}) \leq 1$. But Lemma 5.6 says that $\text{SU}(a/C_{n-1}) = 1$ and since $\text{rng}(\bar{c}) \subseteq C_{n-1}$ we get $\text{SU}(a/\bar{c}) = 1$.

Suppose for a contradiction that $b \notin \text{acl}(\bar{c})$. Since $d \in \text{acl}(\bar{c})$ we get $b \notin \text{acl}(d)$ By assumption, $b \in \text{acl}(a)$ so $b \not\downarrow_d a$ and, as $\text{SU}(a/d) = 1$, we get $a \in \text{acl}(b)$ which contradicts the assumption that $b < a$. Hence we conclude that $b \in \text{acl}(\bar{c})$.

But from $b \in \text{acl}(\bar{c})$ and Lemma 5.8 it follows that for some $c \in \text{rng}(\bar{c})$, $b \in \text{acl}(c)$. □

Definition 5.13 For every $s \leq r$ and $\bar{a} \in M^{\text{eq}}$ we define $\text{crd}_s(\bar{a}) = \text{acl}(\bar{a}) \cap C_s$ and we abbreviate crd_r with crd.

Lemma 5.14 *For all $s \leq r$, $n < \aleph_0$ and $a_0, \ldots, a_n \in M^{\text{eq}}$,*

$$\text{crd}_s(a_0, \ldots, a_n) = \text{crd}_s(a_0) \cup \ldots \cup \text{crd}_s(a_n) \quad \textit{and} \quad \text{crd}(a_0, \ldots, a_n) = \text{crd}(a_0) \cup \ldots \cup \text{crd}(a_n).$$

Proof. Immediate consequence of Lemma 5.8. □

[2]See Remark 3.9 in [6] for more about why every C_s and N_s is $\emptyset$-definable.
[3]The results in question are Lemmas 3.12, 3.14, 3.15, 3.16 and 3.18 in [6]

Lemma 5.15 *Let $1 \leq s < r$. If $\bar{a} \in N_{s+1}$ then* $\mathrm{crd}_s(\bar{a}) = \mathrm{crd}_s\big(\mathrm{acl}(\bar{a}) \cap N_s\big)$.

Proof. An easy consequence of the definition of N_s and Lemmas 5.14 and 5.12. □

Lemma 5.16 *If $\bar{a}, \bar{b} \in C_r$ then $\bar{a}$ is independent from $\bar{b}$ over* $\mathrm{crd}(\bar{a}) \cap \mathrm{crd}(\bar{b})$.

Proof. Note that for any $a \in \mathrm{rng}(\bar{a})$, $\mathrm{crd}(a) \cap \mathrm{crd}(\bar{b}) \subseteq \mathrm{crd}(\bar{a}) \cap \mathrm{crd}(\bar{b}) \subseteq \mathrm{acl}(\bar{b})$. So if $a \in \mathrm{rng}(\bar{a})$ and $tp(a/\bar{b})$ forks over $\mathrm{crd}(\bar{a}) \cap \mathrm{crd}(\bar{b})$ then $tp(a/\bar{b})$ forks over $\mathrm{crd}(a) \cap \mathrm{crd}(\bar{b})$. By the assumption that independence is trivial it is therefore sufficient so prove that whenever $a, \bar{b} \in C_r$ then a is independent from $\bar{b}$ over $\mathrm{crd}(a) \cap \mathrm{crd}(\bar{b})$. We will do this by induction on $\mathrm{SU}(a/\bar{b})$.

Let $a, \bar{b} \in C_r$. First suppose that $\mathrm{SU}(a/\bar{b}) = 0$, that is, $a \in \mathrm{acl}(\bar{b})$. Then $a \in \mathrm{crd}(a) \subseteq \mathrm{crd}(\bar{b})$, so $a \in \mathrm{crd}(a) \cap \mathrm{crd}(\bar{b})$ and therefore a is independent from $\bar{b}$ over $\mathrm{crd}(a) \cap \mathrm{crd}(\bar{b})$.

Now suppose that $\mathrm{SU}(a/\bar{b}) > 0$. Since $C_0 = \emptyset$ there is n such that $a \in C_n - C_{n-1}$. Since C is self-coordinatized there is $d \in \mathrm{acl}(a) \cap C$ such that $\mathrm{SU}(a/d) = 1$. It follows that $d < a$ and, from the assumption that $a \in C_n - C_{n-1}$ it follows that $d \in \mathrm{acl}(C_{n-1})$. Let $\bar{c}$ enumerate $\mathrm{acl}(d) \cap C_{n-1}$. By Lemma 5.11, $\mathrm{acl}(d) = \mathrm{acl}(\bar{c})$, so $\mathrm{SU}(a/\bar{c}) = \mathrm{SU}(a/d) = 1$. If we would have $\mathrm{SU}(a/\bar{c}\bar{b}) = 0$ then $\mathrm{SU}(a/d\bar{b}) = 0$ and, since $\mathrm{SU}(a/d) = 1$, Lemma 5.7 would imply that $a \in \mathrm{acl}(\bar{b})$ so $\mathrm{SU}(a/\bar{b}) = 0$, which contradicts our assumption. Hence $\mathrm{SU}(a/\bar{c}\bar{b}) = 1$.

The Lascar equation now gives

$$\mathrm{SU}(a/\bar{b}) = \mathrm{SU}(a\bar{c}/\bar{b}) = \mathrm{SU}(a/\bar{c}\bar{b}) + \mathrm{SU}(\bar{c}/\bar{b}) = 1 + \mathrm{SU}(\bar{c}/\bar{b}),$$

and therefore $\mathrm{SU}(\bar{c}/\bar{b}) < \mathrm{SU}(a/\bar{b})$.

Claim 1. If $e \in \mathrm{acl}(a) \cap C_{n-1}$ then $e \in \mathrm{acl}(d)$.

Proof of Claim 1. Suppose that $e \in \mathrm{acl}(a) \cap C_{n-1}$ and $e \notin \mathrm{acl}(d)$. Then $a \not\perp_d e$, so by Lemma 5.7, $a \in \mathrm{acl}(e)$ and hence $\mathrm{SU}(a/C_{n-1}) = 0$, which contradicts the assumption that $a \in C_n - C_{n-1}$ and Lemma 5.6. □

Claim 2. $\mathrm{crd}(a) \cap \mathrm{crd}(\bar{b}) = \mathrm{crd}(\bar{c}) \cap \mathrm{crd}(\bar{b})$.

Proof of Claim 2. Since $\mathrm{rng}(\bar{c}) \subseteq \mathrm{acl}(a)$ we have $\mathrm{crd}(\bar{c}) \cap \mathrm{crd}(\bar{b}) \subseteq \mathrm{crd}(a) \cap \mathrm{crd}(\bar{b})$.

Now suppose that $e \in \mathrm{crd}(a) \cap \mathrm{crd}(\bar{b})$, so in particular $e \in \mathrm{acl}(a) \cap \mathrm{acl}(\bar{b})$. Recall that we assume that $a \in C_n - C_{n-1}$ and therefore $e \in C_n$. If $e \in C_n - C_{n-1}$ and $a \notin \mathrm{acl}(e)$ then we have a contradiction to the assumption that $a \in C_n - C_{n-1}$. If $e \in C_n - C_{n-1}$ and $a \in \mathrm{acl}(e)$ then $a \in \mathrm{acl}(\bar{b})$ which contradicts the assumption that $\mathrm{SU}(a/\bar{b}) > 0$. Hence $e \in C_{n-1}$ (and $e \in \mathrm{crd}(a) \subseteq \mathrm{acl}(a)$) so by Claim 1, $e \in \mathrm{acl}(d) = \mathrm{acl}(\bar{c})$. The assumption that $e \in \mathrm{crd}(a) \cap \mathrm{crd}(\bar{b})$ implies that $e \in C_r$ and since $e \in \mathrm{acl}(\bar{c})$ we get $e \in \mathrm{crd}(\bar{c}) \cap \mathrm{crd}(\bar{b})$. □

Above we showed that $\mathrm{SU}(\bar{c}/\bar{b}) < \mathrm{SU}(a/\bar{b})$, so for every $c \in \mathrm{rng}(\bar{c})$, $\mathrm{SU}(c/\bar{b}) < \mathrm{SU}(a/\bar{b})$. By the induction hypothesis, for every $c \in \mathrm{rng}(\bar{c})$, $tp(c/\bar{b})$ does not fork over $\mathrm{crd}(c) \cap \mathrm{crd}(\bar{b})$ so $tp(c/\mathrm{acl}(\bar{b}))$ does not fork over $\mathrm{crd}(c) \cap \mathrm{crd}(\bar{b})$. Since $\mathrm{crd}(c) \cap \mathrm{crd}(\bar{b}) \subseteq \mathrm{crd}(\bar{c}) \cap \mathrm{crd}(\bar{b}) \subseteq \mathrm{acl}(\bar{b})$ it follows by monotonicity that $tp(c/\mathrm{acl}(\bar{b}))$ does not fork over $\mathrm{crd}(\bar{c}) \cap \mathrm{crd}(\bar{b})$. Therefore c is independent from $\bar{b}$ over $\mathrm{crd}(\bar{c}) \cap \mathrm{crd}(\bar{b})$, for every $c \in \mathrm{rng}(\bar{c})$. By the triviality of dependence, $\bar{c}$ is independent from $\bar{b}$ over $\mathrm{crd}(\bar{c}) \cap \mathrm{crd}(\bar{b})$. Let $D = \mathrm{crd}(\bar{c}) \cap \mathrm{crd}(\bar{b}) = \mathrm{crd}(a) \cap \mathrm{crd}(\bar{b})$ (by Claim 2). With this notation we have proved that $\mathrm{SU}(\bar{c}/\bar{b}) = \mathrm{SU}(\bar{c}/D)$. Above we proved that $\mathrm{SU}(a/\bar{c}\bar{b}) = 1$, and since $\mathrm{SU}(a/\bar{c}) = \mathrm{SU}(a/d) = 1$ it follows that $\mathrm{SU}(a/\bar{c}\bar{b}) = \mathrm{SU}(a/\bar{c})$. By applying the Lascar equation twice we get:

$$\begin{aligned}
&\mathrm{SU}(a/\bar{b}) = \mathrm{SU}(a\bar{c}/\bar{b})\\
&= \mathrm{SU}(a/\bar{c}\bar{b}) + \mathrm{SU}(\bar{c}/\bar{b})\\
&= \mathrm{SU}(a/\bar{c}) + \mathrm{SU}(\bar{c}/D)\\
&= \mathrm{SU}(a/\bar{c}D) + \mathrm{SU}(\bar{c}/D) \quad \text{as } D \subseteq \mathrm{acl}(\bar{c})\\
&= \mathrm{SU}(a\bar{c}/D) = \mathrm{SU}(a/D).
\end{aligned}$$

This proves that a is independent from $\bar{b}$ over $D = \mathrm{crd}(a) \cap \mathrm{crd}(\bar{b})$. □

Construction 5.17 For each $s = 1, \dots, r$, let N'_s be the set of equivalence classes of the $\emptyset$-definable equivalence relation $\mathrm{acl}(x) = \mathrm{acl}(y)$ on N_s; by the $\aleph_0$-categoricity of N_s (Lemma 5.9) every equivalence class is finite. Since M^{eq} has elimination of imaginaries, N'_s can be identified with a $\emptyset$-definable subset of M^{eq}, so we will consider N'_s as structure, in its own right, which is canonically embedded in M^{eq}. Note that every element of N'_s is interalgebraic with an element of N_s and vice versa. Also observe that $C_s = \bigcup_{1 \le t \le s} N_t$. Let $C'_0 = \emptyset$ and for $s = 1, \dots, r$, let $C'_s = \bigcup_{1 \le t \le s} N'_t$.

Definition 5.18 For each $1 \le s \le r$ and $\bar{a} \in M^{\mathrm{eq}}$ define $\mathrm{crd}'_s(\bar{a}) = \mathrm{acl}(\bar{a}) \cap C'_s$ and we abbreviate crd'_r with crd'.

Since each element in N'_s is interalgebraic with an element in N_s, and vice versa, we have the following:

Lemma 5.19 *The lemmas 5.6 - 5.16 hold when, for* $s = 1, \dots, r$, C_s, N_s *and* crd_s *are replaced* C'_s, N'_s *and* crd'_s.

Remark 5.20 The only assumptions on M that are used for proving the results in Section 5.1 are that $Th(M)$ is $\aleph_0$-categorical, simple, 1-based and has trivial dependence.

5.2 Proof that M has the finite submodel property

In this subsection we prove that M has the finite submodel property. This will be done by first proving inductively that, for every $0 < s \le r$, N'_s is polynomially k-saturated for every $k < \aleph_0$ and hence (by Lemma 2.3) N'_s has the finite submodel property. When this is proved for N'_r, then, since N'_r is canonically embedded in M^{eq}, $M \subseteq \mathrm{acl}(N'_r)$ (by Lemma 5.10 and Lemma 5.19) and only finitely many sorts are represented in N'_r, we can apply Theorem 1.4 to conclude that M has the finite submodel property.

By lemmas 5.6 and 5.19, for every $a \in C'_1$, $\mathrm{SU}(a/C'_0) = 1$ and since $C'_0 = \emptyset$ (by definition) we have $\mathrm{SU}(a) = 1$ for every $a \in C'_1$. As $N'_1 \subseteq C'_1$ we get $\mathrm{SU}(a) = 1$ for every $a \in N'_1$. Since N'_1, as a structure, is canonically embedded in M^{eq} it follows from Theorem 4.3 that N'_1 is polynomially k-saturated for every $k < \aleph_0$.

For the induction step, suppose that N'_s (where $1 \le s < r$) is polynomially k-saturated for every $k < \aleph_0$. We will prove that N'_{s+1} is polynomially k-saturated for every $k < \aleph_0$. For this we will define a sublanguage $\mathcal{L}$ of the language of N'_{s+1} (as a canonically embedded structure) and show that $N'_{s+1} {\restriction} \mathcal{L}$ is polynomially k-saturated for every $k < \aleph_0$; here we use the induction hypothesis that N'_s is polynomially k-saturated for every $k < \aleph_0$. Then we show that N'_{s+1} satisfies the k-independence hypothesis over

$\mathcal{L}$ for every $k < \aleph_0$; here we use that $Th(M)$ has the k-embedding of types property for every $k < \aleph_0$ (as M is a strongly independent structure). And finally we apply Theorem 2.7 to conclude that N'_{s+1} is polynomially k-saturated for every $k < \aleph_0$.

Definition 5.21 The sublanguage $\mathcal{L}$ of the language of N'_{s+1} will be defined in a few steps.

(i) Let $0 < n < \aleph_0$. We define a $2n$-ary relation P_n on N'_{s+1} in the following way: Let $\bar{a} = (a_0, \ldots, a_{n-1}) \in (N'_{s+1})^n$ and $\bar{b} = (b_0, \ldots, b_{n-1}) \in (N'_{s+1})^n$. Then $P_n(\bar{a}\bar{b})$ if and only if, for every $i < n$, $\mathrm{acl}(a_i) \cap N'_s$ and $\mathrm{acl}(b_i) \cap N'_s$ can be ordered as $\bar{a}'_i$ and $\bar{b}'_i$, respectively, in such a way that

$$tp(\bar{a}'_0 \ldots \bar{a}'_{n-1}) = tp(\bar{b}'_0 \ldots \bar{b}'_{n-1}).$$

Note that P_n defines an equivalence relation on n-tuples from N'_{s+1} and that P_n has only finitely many equivalence classes (because N'_s is $\aleph_0$-categorical).

(ii) Let $0 < n < \aleph_0$ and let $A_{(n,0)}, \ldots, A_{(n,m_n)}$ be a list of all equivalence classes of P_n on $(N'_{s+1})^n$. Recall that N'_{s+1} is regarded as a canonically embedded structure in M^{eq}, so for every relation R on N'_{s+1} (of any arity) which is $\emptyset$-definable in M^{eq} there is a relation symbol in the language of N'_{s+1} which is interpreted as R. For each $i \leq m_n$, let $F_{(n,i)}$ be the relation symbol from the language of N'_{s+1} which is interpreted as the equivalence class $A_{(n,i)}$.

(iii) Let $\mathcal{L}$ be the language the vocabulary of which is

$$\{=\} \cup \{F_{(n,i)} : 0 < n < \aleph_0,\ i \leq m_n\}.$$

Then $\mathcal{L}$ is a sublanguage of the language of N'_{s+1}.

Remark 5.22 By the definition of N'_{s+1}, $\mathcal{L}$ and Lemma 5.19, for every $A \subseteq N'_{s+1}$, $\mathrm{acl}_{N'_{s+1}\restriction\mathcal{L}}(A) = \mathrm{acl}_{N'_{s+1}}(A) = \mathrm{acl}(A) \cap N'_{s+1} = A$ and consequently $\dim_{N'_{s+1}\restriction\mathcal{L}}(A) = \dim_{N'_{s+1}}(A) = |A|$.

Lemma 5.23 *Let $a_0, \ldots, a_n, b_0, \ldots, b_n \in N'_{s+1}$. Then the following are equivalent:*

(i) $tp_{N'_{s+1}\restriction\mathcal{L}}(a_0, \ldots, a_n) = tp_{N'_{s+1}\restriction\mathcal{L}}(b_0, \ldots, b_n)$.

(ii) For all $i, j \leq n$, $a_i = a_j \iff b_i = b_j$, and for each $i \leq n$, $\mathrm{crd}'_s(a_i)$ and $\mathrm{crd}'_s(b_i)$ can be ordered as $\bar{a}'_i$ and $\bar{b}'_i$, respectively, in such a way that $tp(\bar{a}'_0 \ldots \bar{a}'_n) = tp(\bar{b}'_0 \ldots \bar{b}'_n)$.

Proof. This is a straightforward consequence of the definition of $\mathcal{L}$ and Lemmas 5.14, 5.15 and 5.19. □

Lemma 5.24 *$N'_{s+1}\restriction\mathcal{L}$ has eliminination of quantifiers.*

Proof. By a back and forth argument. In this proof let $\bar{a} \equiv_{\mathcal{L}_{at}} \bar{b}$ mean that $\bar{a}$ and $\bar{b}$ satisfy the same atomic $\mathcal{L}$-formulas. Suppose that $\bar{a}, \bar{b}, c \in N'_{s+1}$ and $\bar{a} \equiv_{\mathcal{L}_{at}} \bar{b}$. We need to find $d \in N'_{s+1}$ such that $\bar{a}c \equiv_{\mathcal{L}_{at}} \bar{b}d$. The case when $c \in \mathrm{rng}(\bar{a})$ is trivial so we assume that $c \notin \mathrm{rng}(\bar{a})$ which implies that $c \notin \mathrm{acl}_{N'_{s+1}}(\bar{a})$.

Let $\bar{a} = (a_1, \ldots, a_n)$ and $\bar{b} = (b_1, \ldots, b_n)$. The assumption $\bar{a} \equiv_{\mathcal{L}_{at}} \bar{b}$ implies that $\mathrm{acl}(\bar{a}) \cap N'_s$ and $\mathrm{acl}(\bar{b}) \cap N'_s$ can be enumerated as $\bar{a}'$ and $\bar{b}'$, respectively, in such a way that $tp(\bar{a}') = tp(\bar{b}')$. Let $\bar{a}' = a'_1, \ldots, a'_m$, $\bar{b}' = b'_1, \ldots, b'_m$ and let $\bar{c}' = \mathrm{crd}(c)$. We get two cases.

First suppose that $\bar{c}' \subseteq \bar{a}'$, so $\bar{c}' = a'_{i_1}, \ldots, a'_{i_l}$ for some $i_1, \ldots, i_l \in \{1, \ldots, m\}$. Let $\bar{d}' = b'_{i_1}, \ldots, b'_{i_l}$. If it would be the case that $c \in N'_s$ then, since $\mathrm{acl}_{N'_s}$ defines a trivial pregeometry on N'_s, we would have $c \in \mathrm{acl}(c) \cap N'_s = \bar{c}' \subseteq \bar{a}' \subseteq \mathrm{acl}(\bar{a})$, contradicting that $c \notin \mathrm{acl}_{N'_{s+1}}(\bar{a})$. Hence $c \notin N'_s$ which, since $c \in N'_{s+1}$ and $\bar{c}' \in N'_s$, implies that $tp(c/\bar{c}')$ is non-algebraic. As $tp(\bar{c}') = tp(\bar{d}')$ there is $d \in N'_{s+1} - \mathrm{acl}_{N'_{s+1}}(\bar{b})$ such that $tp(d\bar{d}') = tp(c\bar{c}')$, and consequently $\mathrm{acl}(d) \cap N'_s = \bar{d}'$. Since, by lemmas 5.8 and 5.19, $\mathrm{acl}(\bar{b}d) \cap N'_s = \mathrm{acl}(\bar{b}) \cap N'_s$ and $\mathrm{acl}(\bar{a}c) \cap N'_s = \mathrm{acl}(\bar{a}) \cap N'_s$ it follows that $\bar{a}c \equiv_{\mathcal{L}_{at}} \bar{b}d$.

Now suppose that $\bar{c}' \not\subseteq \bar{a}'$. Let $\bar{c}_1$ be the subsequence of elements of $\bar{c}'$ which belong to $\bar{a}'$ and let $\bar{c}_2$ be the subsequence of elements of $\bar{c}'$ which do not belong to $\bar{a}'$. Let $\bar{d}_1$ be the subsequence of $\bar{b}'$ which corresponds to $\bar{c}_1$ in $\bar{a}'$; i.e. if $\bar{c}_1 = a'_{j_1}, \ldots, a'_{j_m}$ then $\bar{d}_1 = b'_{j_1}, \ldots, b'_{j_m}$. Since $tp(\bar{a}') = tp(\bar{b}')$ there is $\bar{d}_2 \in N'_s$ such that $tp(\bar{a}'\bar{c}_2) = tp(\bar{b}'\bar{d}_2)$. As $tp(\bar{c}_1\bar{c}_2) = tp(\bar{d}_1\bar{d}_2)$ there is $d \in N'_{s+1}$ such that $\mathrm{acl}(d) \cap N'_s = \bar{d}_1\bar{d}_2$. By lemmas 5.8 and 5.19, $\mathrm{acl}(\bar{a}c) \cap N'_s = \bar{a}'\bar{c}_2$ and $\mathrm{acl}(\bar{b}d) \cap N'_s = \bar{b}'\bar{d}_2$ where, by the choice of $\bar{d}_2$, $tp(\bar{a}'\bar{c}_2) = tp(\bar{b}'\bar{d}_2)$. Hence $\bar{a}c \equiv_{\mathcal{L}_{at}} \bar{b}d$. □

Lemma 5.25 *$N'_{s+1}\restriction\mathcal{L}$ is polynomially k-saturated for every $k < \aleph_0$.*

Proof. By assumption (the induction hypothesis), N'_s is polynomially k-saturated for every $k < \aleph_0$. Fix any $k < \aleph_0$. We will show that $N'_{s+1}\restriction\mathcal{L}$ is polynomially k-saturated. Let

$$k_0 = \max\big\{|\mathrm{acl}(a) \cap N_s| : a \in N_{s+1}\big\},$$
$$k_1 = 1 + \max\big\{|\mathrm{acl}(\bar{a}) \cap N_s| : \bar{a} \in N_{s+1},\ \dim_{N_{s+1}}(\bar{a}) < k\big\}.$$

Since N'_s is polynomially k_1-saturated it is sufficient to show that if $P(x)$ is a polynomial then there is a polynomial $Q(x)$ (depending only on $P(x)$, k, k_0) such that if $A \subseteq N'_s$ satisfies

(a) A is algebraically closed in N'_s,

(b) $n \leq |A| \leq P(n)$, and

(c) whenever $\bar{d} \in A$, $\dim_{N'_{s+1}}(\bar{d}) < k_1$ and $p(x) \in S_1^{N'_s}(\bar{d})$ is non-algebraic, then p has at least n distinct realizations in A,

then there is $B \subseteq N'_{s+1}$ satisfying

(a') B is algebraically closed in $N'_{s+1}\restriction\mathcal{L}$,

(b') $n \leq |B| \leq Q(n)$, and

(c') whenever $\bar{d} \in B$, $\dim_{N'_{s+1}}(\bar{d}) < k$ and $p(x) \in S_1^{N'_{s+1}\restriction\mathcal{L}}(\bar{d})$ is non-algebraic, then p has at least n distinct realizations in B.

So let a polynomial $P(x)$ be given. Then we take $Q(x) = P(x) + P(x)^{k_0}(k + x)$. This choice will be understood when we have constructed an appropriate B (satisfying (a') - (c')) for a given A (satisfying (a) - (b)).

Let $A \subseteq N'_s$ satisfy (a), (b) and (c). We construct $B \subseteq N'_{s+1}$ as follows:

(1) If $a \in A \cap N'_s \cap N'_{s+1}$, then put a into B.

(2) For every $D \subseteq A$ such that there is $a \in N'_{s+1} - N'_s$ such that $\mathrm{acl}(a) \cap N'_s = D$, we choose (any) distinct $a_1, \ldots, a_{k+n} \in N'_{s+1} - N'_s$ such that $\mathrm{acl}(a_i) \cap N'_s = D$, for each $1 \leq i \leq k+n$ and put $a_1, \ldots, a_{k+n}$ into B.

The set B contains only elements as specified by (1) and (2) above. Since $\mathrm{acl}_{N'_{s+1} \upharpoonright \mathcal{L}}(X) = \mathrm{acl}_{N'_{s+1}}(X) = X$ for every $X \subseteq N'_{s+1}$, it follows that B is algebraically closed in $N'_{s+1} \upharpoonright \mathcal{L}$, so (a') holds. The construction implies that

$$n \leq |B| \leq |A| + |A|^{k_0}(k+n) \leq P(n) + P(n)^{k_0}(k+n) = Q(n),$$

so (b') holds. It remains to prove (c').

Suppose that $\bar{d} \in B$, $\dim_{N'_{s+1}}(\bar{d}) < k$ and that $p(x) \in S_1^{N'_{s+1}}(\bar{d})$ is non-algebraic. Let $\bar{d}' = \mathrm{acl}(\bar{d}) \cap N'_s$, so $\bar{d}' \in A$. Let $a \in N'_{s+1}$ realize p. We consider two cases.

Case 1. $a \in N'_s$.

By the choice of k_1, $\dim_{N'_s}(\bar{d}') < k_1$. Since A satisfies (c) there are distinct $a_1, \ldots, a_n \in A$ such that for each i, $tp_{N'_s}(a_i\bar{d}') = tp_{N'_s}(a\bar{d}')$ and hence $tp(a_i\bar{d}') = tp(a\bar{d}')$. Since in particular, $tp(a_i) = tp(a)$, we must have $a_i \in A \cap N'_s \cap N'_{s+1}$, so $a_1, \ldots, a_n \in B$, by clause (1). Since $a, a_i, \bar{d}' \in N'_s$, our conclusion that $tp(a_i\bar{d}') = tp(a\bar{d}')$ implies that $a_i\bar{d}$ and $a\bar{d}$ satisfy the same atomic formulas in $\mathcal{L}$, for each i. As $N'_{s+1} \upharpoonright \mathcal{L}$ has elimination of quantifiers (Lemma 5.24) it follows that $tp_{N'_{s+1} \upharpoonright \mathcal{L}}(a_i\bar{d}) = tp_{N'_{s+1} \upharpoonright \mathcal{L}}(a\bar{d})$ for each i, so all $a_1, \ldots, a_n$ are realizations of p.

Case 2. $a \notin N'_s$.

Recall that a realizes $p(x)$ (so $a \in N'_{s+1}$) and that $\bar{d}' = \mathrm{acl}(\bar{d}) \cap N'_s$, where $\mathrm{rng}(\bar{d})$ is the domain of p. Let $\bar{a}' = \mathrm{acl}(a) \cap N'_s$. We have two subcases.

First, assume that $\bar{a}' \subseteq \bar{d}'$. Then, by clause (2) in the construction of B, there are distinct $a_1, \ldots, a_n \in B - \mathrm{acl}_{N'_{s+1}}(\bar{d})$ such that $\mathrm{acl}(a_i) \cap N'_s = \mathrm{acl}(a) \cap N'_s = \bar{a}'$ for each i. Hence, for each i, the sequences $a_i\bar{d}$ and $a\bar{d}$ satisfy the same atomic $\mathcal{L}$-formulas, so by elimination of quantifiers for $N'_{s+1} \upharpoonright \mathcal{L}$ each a_i realizes p.

Now assume that $\bar{a}' \not\subseteq \bar{d}'$. Let $\bar{a}_1$ contain all elements in $\bar{a}'$ which belong to $\bar{d}'$, and let $\bar{a}_2$ contain all elements in $\bar{a}'$ which do not belong to $\bar{d}'$. (Our assumption implies that $\bar{a}_2$ is non-empty.) By the choice of k_1 we have $|\bar{a}_2\bar{d}'| < k_1$. Since every subset of N'_s is algebraically closed in N'_s and $\mathrm{rng}(\bar{a}_2) \cap \mathrm{rng}(\bar{d}') = \emptyset$ it follows that $tp_{N'_s}(\bar{a}_2/\bar{d}')$ is non-algebraic. By (c), repeated times, there are distinct $\bar{a}_2^1, \ldots, \bar{a}_2^n \in A$ which realize $tp_{N'_s}(\bar{a}_2/\bar{d}')$. Since N'_s is canonically embedded in M^{eq} we have

(∗) $tp(\bar{a}_2^i\bar{d}') = tp(\bar{a}_2\bar{d}')$ for each i.

In particular, $tp(\bar{a}_2^i\bar{a}_1) = tp(\bar{a}_2\bar{a}_1)$ for each i, so there are $e_1, \ldots, e_n \in N'_{s+1} - N'_s$ with $\mathrm{acl}(e_i) \cap N'_s = \bar{a}_2^i\bar{a}_1$ for each i. By Clause (2) in the construction of B, there are distinct $a_1, \ldots, a_n \in B - \mathrm{acl}_{N'_{s+1}}(\bar{d})$ such that $\mathrm{acl}(a_i) \cap N'_s = \bar{a}_2^i\bar{a}_1$ for each i. From (∗) it follows that the sequences $a_i\bar{d}$ and $a\bar{d}$ satisfy the same atomic $\mathcal{L}$-formulas, for every i. By elimination of quantifiers for $N'_{s+1} \upharpoonright \mathcal{L}$, each a_i realizes p. Now we have verified (c'). □

Lemma 5.26 *N'_{s+1} satisfies the k-independence hypothesis over $\mathcal{L}$ for every $k < \aleph_0$.*

Proof. Recall that by the definitions of N'_{s+1} and $\mathcal{L}$ we have, for every $A \subseteq N'_{s+1}$, $\mathrm{acl}_{N'_{s+1} \upharpoonright \mathcal{L}}(A) = \mathrm{acl}_{N'_{s+1}}(A) = \mathrm{acl}(A) \cap N'_{s+1} = A$ and hence $\dim_{N'_{s+1} \upharpoonright \mathcal{L}}(A) = \dim_{N'_{s+1}}(A) =$

$|A|$. We verify that for arbitrary $k < \aleph_0$, N'_{s+1} satisfies the k-independence hypothesis over $\mathcal{L}$. Recall Notation 2.5. Let $\bar{a} = (a_0, \ldots, a_{d-1}) \in (N'_{s+1})^d$ be such that $\dim_{N'_{s+1}}(\bar{a}) = d \leq k$, so no element occurs twice in $\bar{a}$. Suppose that $I = \{i_1, \ldots, i_m\} \subseteq \{0, \ldots, d-1\}$ and that $p(\bar{x}_I) \in S_m(Th(N'_{s+1}))$, where $\bar{x}_I = (x_{i_1}, \ldots x_{i_m})$, is such that

(a) $|\bar{a}_I| < k$, $p(\bar{x}_I) \cap \mathcal{L} = tp_{N'_{s+1}\restriction\mathcal{L}}(\bar{a}_I)$ and for every proper subset $J \subset I$, $p\restriction\{\bar{x}_J\} = tp_{N'_{s+1}}(\bar{a}_J)$.

We must show that there is $\bar{b} = (b_0, \ldots, b_{d-1}) \in (N'_{s+1})^d$ such that

(b) $tp_{N'_{s+1}\restriction\mathcal{L}}(\bar{b}) = tp_{N'_{s+1}\restriction\mathcal{L}}(\bar{a})$, $tp_{N'_{s+1}}(\bar{b}_I) = p(\bar{x}_I)$ and, for every $J \subset \{0, \ldots, d-1\}$ such that $\mathrm{rng}(\bar{a}_I) \not\subseteq \mathrm{rng}(\bar{a}_J)$, $tp_{N'_{s+1}}(\bar{a}_J) = tp_{N'_{s+1}}(\bar{b}_J)$.

By reordering if necessary, we may assume that $\bar{a}_I = (a_0, \ldots, a_{m-1})$ and $\bar{x}_I = (x_0, \ldots, x_{m-1})$.

We assume that $p(\bar{x}_I)$ has at least one free variable and, as noted above, any nonempty subset of N'_{s+1} has dimension at least one. Hence, we must have $m > 0$. If $m = d$ then $I = \{0, \ldots, d-1\}$ and letting $\bar{b} = (b_0, \ldots, b_{d-1})$ realize $p(\bar{x}_I)$ then, trivially, all conditions in (b) are satisfied.

Now assume that $0 < m < d$. If $m = 1$, then by the assumption that $p(\bar{x}_I) \cap \mathcal{L} = tp_{N'_{s+1}\restriction\mathcal{L}}(\bar{a}_I)$ in (a) and the definition of $\mathcal{L}$, we find b_0 that realizes $p(x_0) = p(\bar{x}_I)$ and satisfies $\mathrm{crd}'(b_0) = \mathrm{crd}'(a_0)$. If we take $\bar{b} = (b_0, a_1, \ldots, a_d)$ then (b) is satisfied.

In the rest of the proof we assume that $1 < m < d$. Observe that, by the definition of $\mathcal{L}$, if $\bar{c}, \bar{d} \in N'_{s+1} \cap N'_s$ and $tp_{N'_{s+1}\restriction\mathcal{L}}(\bar{c}) = tp_{N'_{s+1}\restriction\mathcal{L}}(\bar{d})$, then $tp(\bar{c}) = tp(\bar{d})$. Suppose that for every $i < m$, $a_i \in N'_s$ (so $a_i \in N'_{s+1} \cap N'_s$). Then the assumption that $p(\bar{x}_I) \cap \mathcal{L} = tp_{N'_{s+1}\restriction\mathcal{L}}(\bar{a}_I)$ implies that $\bar{a}_I = (a_0, \ldots, a_{m-1})$ realizes $p(\bar{x}_I)$. Hence, if $\bar{b} = \bar{a} = (a_0, \ldots, a_{d-1})$ then (b) is trivially satisfied.

Now suppose that for some $i < m$, $a_i \notin N'_s$. By reordering if necessary, we may assume that there is $0 < m_0 \leq m$ such that for every $i < m$, $a_i \in N'_s$ if and only if $i \geq m_0$. Let $a^*_0, \ldots, a^*_{m_0-1} \in N'_{s+1} - \{a_0, \ldots, a_{d-1}\}$ be distinct (and hence non-interalgebraic) elements such that, for each $0 \leq i < m_0$, $\mathrm{acl}(a^*_i) \cap N'_s = \mathrm{acl}(a_i) \cap N'_s$. Then let $\bar{b}_I = (b_0, \ldots, b_{m-1})$ realize $p(\bar{x}_I)$. Whenever $w = \{i_1, \ldots, i_\mu\} \in \mathcal{P}(m)$ and $i_1 < \ldots < i_\mu$, let $\bar{a}_w = (a_{i_1}, \ldots, a_{i_\mu})$ and $\bar{b}_w = (b_{i_1}, \ldots, b_{i_\mu})$.

From now on the proof consists of two steps. First we will find $b^*_0, \ldots, b^*_{m_0-1}$ such that, for $i < m_0$, $\mathrm{acl}(b^*_i) \cap N'_s = \mathrm{acl}(b_i) \cap N'_s$ and

$$\text{for every } w \in \mathcal{P}^-(m),\ \ tp((b^*_0, \ldots, b^*_{m_0-1})\bar{b}_w) = tp((a^*_0, \ldots, a^*_{m_0-1})\bar{a}_w). \tag{1}$$

(This step is made for the purpose that the final tuple $\bar{b}$ that we are looking for will have the same type as $\bar{a}$ in $N'_{s+1}\restriction\mathcal{L}$.) Then we will be able to find $b_m, \ldots b_{d-1} \in N'_{s+1}$ such that

$$\begin{aligned}&\text{for every } w \in \mathcal{P}^-(m),\\ &tp\big((b_m, \ldots, b_{d-1})\bar{b}_w(b^*_0, \ldots, b^*_{m_0-1})\big) = tp\big((a_m, \ldots, a_{d-1})\bar{a}_w(a^*_0, \ldots, a^*_{m_0-1})\big).\end{aligned} \tag{2}$$

From (2), with $u = \{m_0, \ldots, m-1\}$ (so $u = \emptyset$ if $m_0 = m$), we get in particular that

$$tp\big((b_m, \ldots, b_{d-1})\bar{b}_u(b^*_0, \ldots, b^*_{m_0-1})\big) = tp\big((a_m, \ldots, a_{d-1})\bar{a}_u(a^*_0, \ldots, a^*_{m_0-1})\big),$$

and from this, and the definition of $\mathcal{L}$, it follows that

$$tp_{N'_{s+1}\restriction\mathcal{L}}(b_0, \ldots, b_{d-1}) = tp_{N'_{s+1}\restriction\mathcal{L}}(a_0, \ldots, a_{d-1}). \tag{3}$$

If we now take $\bar{b} = (b_0, \dots, b_{d-1})$ then it follows from (2) and (3) that $\bar{b}$ satisfies the requirements in (b) above. We start by finding $b_0^*, \dots, b_{m_0-1}^* \in N'_{s+1}$ which satisfy (1). Define

$$A = \bigcap \big\{ \mathrm{crd}'(\bar{a}_w) : w \in \mathcal{P}^-(m),\ |w| = m-1 \big\},$$
$$B = \bigcap \big\{ \mathrm{crd}'(\bar{b}_w) : w \in \mathcal{P}^-(m),\ |w| = m-1 \big\}.$$

Since $A \subseteq \mathrm{acl}(\bar{a}_w) \cap C'_r$ and $B \subseteq \mathrm{acl}(\bar{b}_w) \cap C'_r$ if $|w| = m-1$ and only finitely many sorts are represented in C'_r it follows that A and B are finite. Note that by the definition of crd' and Lemma 5.8, we have

$$A = \{a \in C'_r : \exists i, j \in m,\ i \neq j,\ a \in \mathrm{crd}'(a_i) \cap \mathrm{crd}'(a_j)\},$$
$$B = \{b \in C'_r : \exists i, j \in m,\ i \neq j,\ a \in \mathrm{crd}'(b_i) \cap \mathrm{crd}'(b_j)\}.$$

Also observe that $\mathrm{crd}'(A) = A$ and $\mathrm{crd}'(B) = B$. For every $w \in \mathcal{P}^-(m)$ let

$$A_w = \mathrm{acl}(\bar{a}_w A), \quad B_w = \mathrm{acl}(\bar{b}_w B).$$

Now we will show that $\mathcal{A} = \{A_w : w \in \mathcal{P}^-(m)\}$ and $\mathcal{B} = \{B_w : w \in \mathcal{P}^-(m)\}$ are independent systems of algebraically closed sets with inclusion maps.

Claim 1. For all $v, w \in \mathcal{P}^-(m)$ the following holds:
(i) $\bar{a}_v$ is independent from $\bar{a}_w$ over $\bar{a}_{v\cap w}A$, and hence $A_v \underset{A_{v\cap w}}{\downarrow} A_w$.
(ii) $\bar{b}_v$ is independent from $\bar{b}_w$ over $\bar{b}_{v\cap w}B$, and hence $B_v \underset{B_{v\cap w}}{\downarrow} B_w$.

Proof of Claim 1. Let $v, w \in \mathcal{P}^-(m)$. Parts (i) and (ii) are proved in the same way so we only prove (i). Suppose for a contradiction that $\bar{a}_v$ is not independent from $\bar{a}_w$ over $\bar{a}_{v\cap w}A$. By the triviality of dependence (and symmetry) there are $i \in v - w$ and $j \in w - v$ such that

$$a_i \text{ is not independent from } a_j \text{ over } \bar{a}_{v\cap w}A. \tag{$*$}$$

As noted above, we have $\mathrm{crd}'(A) = A$ and if $i', j' < m$ and $i' \neq j'$ then $\mathrm{crd}'(a_{i'}) \cap \mathrm{crd}'(a_{j'}) \subseteq A$. It follows (with the use of Lemmas 5.14 and 5.19) that

$$\begin{aligned}
&\mathrm{crd}'(a_i) \cap \mathrm{crd}'(a_j \bar{a}_{v\cap w} A) \\
&= \big(\mathrm{crd}'(a_i) \cap \mathrm{crd}'(a_j)\big) \cup \Big(\bigcup_{i' \in v\cap w} \big(\mathrm{crd}'(a_i) \cap \mathrm{crd}'(a_{i'})\big)\Big) \cup \big(\mathrm{crd}'(a_i) \cap \mathrm{crd}'(A)\big) \\
&\subseteq A.
\end{aligned}$$

Since $\{a_i, a_j\} \cup \mathrm{rng}(\bar{a}_{v\cap w}) \cup A \subseteq C'_r$, it follows from Lemmas 5.16 and 5.19 that a_i is independent from $a_j \bar{a}_{v\cap w} A$ over $\mathrm{crd}'(a_i) \cap \big(\mathrm{crd}'(a_j \bar{a}_{v\cap w} A)$. But as shown above,

$$\mathrm{crd}'(a_i) \cap \mathrm{crd}'(a_j \bar{a}_{v\cap w} A) \subseteq A \subseteq \bar{a}_{v\cap w} A,$$

so by monotonicity, a_i is independent from $a_j \bar{a}_{v\cap w} A$ over $\bar{a}_{v\cap w} A$ and hence a_i is independent from a_j over $\bar{a}_{v\cap w} A$, which contradicts $(*)$. □

By Claim 1, $\mathcal{A} = \{A_w : w \in \mathcal{P}^-(m)\}$ and $\mathcal{B} = \{B_w : w \in \mathcal{P}^-(m)\}$ are independent systems of sets with inclusion maps. Let $\mathcal{G}_A = \big\{\{a_i\} : i \in m\big\}$ and $\mathcal{G}_B = \big\{\{b_i\} : i \in m\big\}$. Then $\mathcal{A}$ is generated by $\mathcal{G}_A$ over A, and $\mathcal{B}$ is generated by $\mathcal{G}_B$ over B.

By assumption we have $tp(\bar{a}_w) = tp(\bar{b}_w)$ for every $w \in \mathcal{P}^-(m)$ and if $|w| = m-1$ then $A \subseteq \operatorname{acl}(\bar{a}_w)$ and $B \subseteq \operatorname{acl}(\bar{b}_w)$. Hence, for every $w \in \mathcal{P}^-(m)$ with $|w| = m-1$ there is an elementary map f_w from $A_w = \operatorname{acl}(\bar{a}_w A) = \operatorname{acl}(\bar{a}_w)$ onto $B_w = \operatorname{acl}(\bar{b}_w B) = \operatorname{acl}(\bar{b}_w)$ such that $f_w(\bar{a}_w) = \bar{b}_w$. For every $v \in \mathcal{P}^-(m)$ with $|v| < m-1$ choose (arbitrary) $w \in \mathcal{P}^-(m)$ such that $|w| = m-1$ and $v \subseteq w$ and let $f_v = f_w \restriction A_v$; then f_v is an elementary map from A_v onto B_v and $f_v(\bar{a}_v) = \bar{b}_v$. From the definition it follows that if $v, w \in \mathcal{P}^-(m)$ and $v \subseteq w$, then $f_w \restriction \bigcup_{i \in w}\{a_i\}$ extends $f_v \restriction \bigcup_{i \in v}\{a_i\}$. It follows that $\{f_w : w \in \mathcal{P}^-(m)\}$ is a system of elementary maps from $(\mathcal{A}, \mathcal{G}_A)$ onto $(\mathcal{B}, \mathcal{G}_B)$. The next claim shows that conditions (1) and (2) in the definition of the m-embedding of types property (Definition 3.2) are satisfied for the sequence $(a_0^*, \ldots, a_{m_0-1}^*)$.

Claim 2. For every $i < m_0$,

$$a_i^* \notin \operatorname{acl}\Big((\{a_0^*, \ldots, a_{m_0-1}^*\} - \{a_i^*\}) \cup \bigcup_{w \in \mathcal{P}^-(m)} A_w\Big).$$

Proof of Claim 2. Suppose that $i < m_0$ and

$$a_i^* \in \operatorname{acl}\Big((\{a_0^*, \ldots, a_{m_0-1}^*\} - \{a_i^*\}) \cup \bigcup_{w \in \mathcal{P}^-(m)} A_w\Big).$$

By definition, $A_w = \operatorname{acl}(\bar{a}_w A)$ and if $|w| = m-1$ then $A \subseteq \operatorname{acl}(\bar{a}_w)$, so we get

$$a_i^* \in \operatorname{acl}\Big((\{a_0^*, \ldots, a_{m_0-1}^*\} - \{a_i^*\}) \cup \bigcup_{w \in \mathcal{P}^-(m)} \bar{a}_w\Big).$$

From the fact that every subset of N'_{s+1} is algebraically closed in N'_{s+1} and that $a_0^*, \ldots, a_{m_0-1}^*$ are distinct, it follows that there is $j \in \{0, \ldots, m-1\}$ such that $a_i^* = a_j$. But this contradicts that each a_i^* was chosen so that it does not belong to $\{a_0, \ldots, a_{d-1}\}$. □

By Claim 2 and the assumption that $Th(M)$ has the m-embedding of types property with respect to all generators (as M is a strongly independent structure), it follows that there are $b_0^*, \ldots, b_{m_0-1}^* \in \mathcal{M}^{\mathrm{eq}}$, where $\mathcal{M}$ is the monster model of $Th(M)$, and, for every $w \in \mathcal{P}^-(m)$, an elementary map g_w from $\{a_0^*, \ldots a_{m_0-1}^*\} \cup A_w$ onto $\{b_0^*, \ldots b_{m_0-1}^*\} \cup B_w$ such that, for $i = 0, \ldots, m_0-1$, $g_w(a_i^*) = b_i^*$, if $w \supseteq v$ then $g_w \restriction \bigcup_{i \in w}\{a_i\}$ extends $g_v \restriction \bigcup_{i \in v}\{a_i\}$, and for every $i = 0, \ldots, m-1$, $g_{\{i\}} \restriction \{a_i\} = f_{\{i\}} \restriction \{a_i\}$. Hence

$$\text{for every } w \in \mathcal{P}^-(m),\ tp_{\mathcal{M}^{\mathrm{eq}}}\big((b_0^*, \ldots, b_{m_0-1}^*)\bar{b}_w\big) = tp_{\mathcal{M}^{\mathrm{eq}}}\big((a_0^*, \ldots, a_{m_0-1}^*)\bar{a}_w\big). \quad (4)$$

Since M^{eq} is $\aleph_0$-homogeneous we may assume that $b_0^*, \ldots, b_{m_0-1}^* \in M^{\mathrm{eq}}$ and hence they belong to N'_{s+1}. Note that, by the choice of $a_0^*, \ldots, a_{m_0-1}^*$ and lemmas 5.8 and 5.19,

$$b_i^* \notin \operatorname{acl}\Big((\{b_0^*, \ldots, b_{m_0-1}^*\} - \{b_i^*\}) \cup \{b_0, \ldots, b_{m-1}\}\Big), \quad \text{for every } i < m_0.$$

Now we will find $b_m, \ldots, b_{d-1} \in N'_{s+1}$ such that (2) and (3) hold. For every $w \in \mathcal{P}^-(m)$ define

$$A'_w = \operatorname{acl}\big(\bar{a}_w(a_0^*, \ldots, a_{m_0-1}^*)A\big),$$
$$B'_w = \operatorname{acl}\big(\bar{b}_w(b_0^*, \ldots, b_{m_0-1}^*)B\big).$$

The next claim shows that $\mathcal{A}' = \{A'_w : w \in \mathcal{P}^-(m)\}$ and $\mathcal{B}' = \{B'_w : w \in \mathcal{P}^-(m)\}$ are independent systems of algebraically closed sets with inclusion maps.

Claim 3. For all $v, w \in \mathcal{P}^-(m)$ the following hold:
(i) $\bar{a}_v$ is independent from $\bar{a}_w$ over $\bar{a}_{v\cap w}(a^*_0, \ldots, a^*_{m_0-1})A$, and hence $A'_v \underset{A'_{v\cap w}}{\downarrow} A'_w$.
(ii) $\bar{b}_v$ is independent from $\bar{b}_w$ over $\bar{b}_{v\cap w}(b^*_0, \ldots, b^*_{m_0-1})B$, and hence $B'_v \underset{B'_{v\cap w}}{\downarrow} B'_w$.

Proof of Claim 3. The proof is similar to the proof of Claim 1. Parts (i) and (ii) are proved in the same way so we only prove (i). Let $\bar{a}^* = (a^*_0, \ldots, a^*_{m_0-1})$. By the triviality and symmetry of dependence it is sufficient to prove that if $i \in v - w$ and $j \in w - v$ then a_i is independent from a_j over $\bar{a}_{v\cap w}\bar{a}^*A$.

Let $i \in v - w$ and $j \in w - v$. By Lemma 5.16 and Lemma 5.19, a_i is independent from $a_j\bar{a}_{v\cap w}\bar{a}^*A$ over $\mathrm{crd}'(a_i) \cap \mathrm{crd}'\big(a_j\bar{a}_{v\cap w}\bar{a}^*A\big)$. Since for any $D \subseteq M^{\mathrm{eq}}$, $\mathrm{crd}'(D) \subseteq \mathrm{acl}(D)$, it follows that a_i is independent from $a_j\bar{a}_{v\cap w}\mathrm{crd}'(\bar{a}^*)A$ over $\mathrm{crd}'(a_i) \cap \mathrm{crd}'\big(a_j\bar{a}_{v\cap w}\mathrm{crd}'(\bar{a}^*)A\big)$.

As shown in the proof of Claim 1, we have

$$\mathrm{crd}'(a_i) \cap \big(\mathrm{crd}'(a_j\bar{a}_{v\cap w}A)\big) \subseteq A.$$

Note that $\mathrm{crd}'(\mathrm{crd}'(D)) = \mathrm{crd}'(D)$ for every $D \subseteq M^{\mathrm{eq}}$. It follows (using Lemma 5.14) that

$$\begin{aligned}
&\mathrm{crd}'(a_i) \cap \mathrm{crd}'\big(a_j\bar{a}_{v\cap w}\mathrm{crd}'(\bar{a}^*)A\big)\\
&= \big(\mathrm{crd}'(a_i) \cap \mathrm{crd}'(a_j\bar{a}_{v\cap w}A)\big) \cup \big(\mathrm{crd}'(a_i) \cap \mathrm{crd}'(\bar{a}^*)\big)\\
&\subseteq A \cup \mathrm{crd}'(\bar{a}^*)\\
&\subseteq \bar{a}_{v\cap w} \cup \mathrm{crd}'(\bar{a}^*) \cup A.
\end{aligned}$$

By monotonicity it now follows that a_i is independent from $a_j\bar{a}_{v\cap w}\mathrm{crd}'(\bar{a}^*)A$ over $\bar{a}_{v\cap w}\mathrm{crd}'(\bar{a}^*)A$ and therefore a_i is independent from a_j over $\bar{a}_{v\cap w}\mathrm{crd}'(\bar{a}^*)A$ and hence also over $\bar{a}_{v\cap w}\bar{a}^*A$.
□

By Claim 3, $\mathcal{A}' = \{A'_w : w \in \mathcal{P}^-(m)\}$ and $\mathcal{B}' = \{B'_w : w \in \mathcal{P}^-(m)\}$ are independent systems of algebraically closed sets with inclusion maps. For every $i \in m$ let

$$\hat{A}^0_i = \{a_i\} \cup \{a^*_0, \ldots, a^*_{m_0-1}\} \quad \text{and} \quad \hat{B}^0_i = \{b_i\} \cup \{b^*_0, \ldots, b^*_{m_0-1}\}.$$

Then $\mathcal{A}$ is generated by $\mathcal{G}'_A = \{\hat{A}^0_i : i \in m\}$ over A, and $\mathcal{B}$ is generated by $\mathcal{G}'_B = \{\hat{B}^0_i : i \in m\}$ over B.

From (4) it follows that, for every $w \in \mathcal{P}^-(m)$ with $|w| = m-1$, there is an elementary map f'_w from A'_w onto B'_w such that $f'_w(\bar{a}_w) = \bar{b}_w$ and $f'_w(a^*_i) = b^*_i$ for $i = 0, \ldots, m_0 - 1$. For every $v \in \mathcal{P}^-(m)$ with $|v| < m - 1$ we choose (arbitrary) $w \in \mathcal{P}^-(m)$ such that $|w| = m - 1$ and $v \subseteq w$ and let $f'_v = f'_w \restriction A_v$; then f'_v is an elementary map from A_v onto B_v. It follows that whenever $v, w \in \mathcal{P}^-(m)$ and $v \subseteq w$, then $f'_w \restriction \bigcup_{i\in w} \hat{A}^0_i$ extends $f'_v \restriction \bigcup_{i\in v} \hat{A}^0_i$. Hence $\{f_w : w \in \mathcal{P}^-(m)\}$ is a system of elementary maps from $(\mathcal{A}', \mathcal{G}'_A)$ onto $(\mathcal{B}', \mathcal{G}'_B)$.

The next claim show that conditions (1) and (2) from the defintion of m-embedding of types property (Definition 3.2) hold for the sequence $(a_m, \ldots, a_{d-1})$.

Claim 4. If $a \in \{a_m, \dots, a_{d-1}\}$ then

$$a \notin \operatorname{acl}\Big((\{a_m, \dots, a_{d-1}\} - \{a\}) \cup \bigcup_{w \in \mathcal{P}^-(m)} A'_w\Big).$$

Proof of Claim 4. Suppose for a contradiction that $a \in \{a_m, \dots, a_{d-1}\}$ and

$$a \in \operatorname{acl}\Big((\{a_m, \dots, a_{d-1}\} - \{a\}) \cup \bigcup_{w \in \mathcal{P}^-(m)} A'_w\Big).$$

Since $A'_w = \operatorname{acl}(\bar{a}_w(a^*_0, \dots, a^*_{m_0-1})A)$ and $|w| = m - 1$ implies $A \subseteq \operatorname{acl}(\bar{a}_w)$, we get

$$a \in \operatorname{acl}_{N'_{s+1}}\Big((\{a_m, \dots, a_{d-1}\} - \{a\}) \cup \{a^*_0, \dots, a^*_{m_0-1}\} \cup \bigcup_{w \in \mathcal{P}^-(m)} \bar{a}_w\Big),$$

and as every subset of N'_{s+1} is algebraically closed in N'_{s+1} we get either $a = a_i$ for some $i \in \{0, \dots, d-1\}$ or $a = a^*_i$ for some $i \in \{0, \dots, m_0 - 1\}$. But in either case we have a contradiction because all elements in $\bar{a}$ are distinct and $a^*_0, \dots, a^*_{m_0-1}$ where chosen to be different from all elements in $\bar{a}$. □

Since $Th(M)$ has the m-embedding of types property with respect to all generators, Claim 4 implies that there are $b_m, \dots, b_{d-1} \in \mathcal{M}^{\mathrm{eq}}$ and, for every $w \in \mathcal{P}^-(m)$, an elementary map g'_w from $\{a_m, \dots, a_{d-1}\} \cup A'_w$ onto $\{b_m, \dots, b_{d-1}\} \cup B'_w$ such that $g'_w(a_i) = b_i$ for $i = m, \dots, d-1$, $g'_w \restriction \bigcup_{i \in w} \hat{A}^0_i$ extends $g'_v \restriction \bigcup_{i \in v} \hat{A}^0_i$ if $w \supseteq v$, and $g'_{\{i\}} \restriction \hat{A}^0_i = f'_{\{i\}} \restriction \hat{A}^0_i$ for $i = 0, \dots, m-1$. It follows that, for every $w \in \mathcal{P}^-(m)$, $g'_w(\bar{a}_w) = \bar{b}_w$ and $g_w(a^*_i) = b^*_i$ for $i = 0, \dots, m_0 - 1$. Therefore,

$$\begin{aligned}
&\text{for every } w \in \mathcal{P}^-(m),\\
&tp_{\mathcal{M}^{\mathrm{eq}}}\big((b_m, \dots, b_{d-1})\bar{b}_w(b^*_0, \dots, b^*_{m_0-1})\big) =\\
&tp_{\mathcal{M}^{\mathrm{eq}}}\big((a_m, \dots, a_{d-1})\bar{a}_w(a^*_0, \dots, a^*_{m_0-1})\big).
\end{aligned} \tag{5}$$

By the $\aleph_0$-homogeneity of M^{eq} we may assume that $b_m, \dots, b_{d-1} \in M^{\mathrm{eq}}$, so $b_m, \dots, b_{d-1} \in N'_{s+1}$. As N'_{s+1} is canonically embedded in M^{eq} it follows from (5) that

$$\text{for every } w \in \mathcal{P}^-(m),\ tp_{N'_{s+1}}\big((b_m, \dots, b_{d-1})\bar{b}_w\big) = tp_{N'_{s+1}}\big((a_m, \dots, a_{d-1})\bar{a}_w\big). \tag{6}$$

Let $u = \{m_0, \dots, m-1\}$ (so $u = \emptyset$ if $m_0 = m$). Since, by assumption, $m_0 > 0$, we have $u \in \mathcal{P}^-(m)$, so from (5) we get

$$tp\big((b_m, \dots, b_{d-1})\bar{b}_u(b^*_0, \dots, b^*_{m_0-1})\big) = tp\big((a_m, \dots, a_{d-1})\bar{a}_u(a^*_0, \dots, a^*_{m_0-1})\big). \tag{7}$$

From the choice of $a^*_0, \dots, a^*_{m_0-1}, b^*_0, \dots, b^*_{m_0-1}$ and u (and lemmas 5.8 and 5.19) it follows that

$$\begin{aligned}
\operatorname{acl}(a_0, \dots, a_{d-1}) \cap N'_s &= \operatorname{acl}((a^*_0, \dots, a^*_{m_0-1})\bar{a}_u) \cap N'_s, \quad \text{and}\\
\operatorname{acl}(b_0, \dots, b_{d-1}) \cap N'_s &= \operatorname{acl}((b^*_0, \dots, b^*_{m_0-1})\bar{b}_u) \cap N'_s.
\end{aligned}$$

The definition of $\mathcal{L}$ and (7) now gives

$$tp_{N'_{s+1} \restriction \mathcal{L}}(b_0, \dots, b_{d-1}) = tp_{N'_{s+1} \restriction \mathcal{L}}(a_0, \dots, a_{d-1}). \tag{8}$$

Let $\bar{b} = (b_0, \dots, b_{d-1})$. Since $\bar{b}_I = (b_0, \dots, b_{m-1})$ realizes $p(\bar{x}_I)$ it follows from (6) and (8) that $\bar{b}$ satisfies the conditions in (b), so the lemma is proved. □

Lemma 5.27 *N'_{s+1} is polynomially k-saturated for every $k < \aleph_0$.*

Proof. By the definition of $\mathcal{L}$, $\mathrm{acl}_{N'_{s+1}\restriction\mathcal{L}}$ and $\mathrm{acl}_{N'_{s+1}}$ coincide. By Lemma 5.25, $N'_{s+1}\restriction\mathcal{L}$ is polynomially k-saturated for every $k < \aleph_0$. By Lemma 5.26, N'_{s+1} satisfies the k-independence hypothesis over $\mathcal{L}$ for every $k < \aleph_0$. Hence, by Theorem 2.7, N'_{s+1} is polynomially k-saturated for every $k < \aleph_0$. □

Corollary 5.28 *N'_r is polynomially k-saturated for every $k < \aleph_0$.*

Proof. This follows by induction, since N'_1 is polynomially k-saturated for every $k < \aleph_0$, as pointed out in the beginning of Section 5.2, and we have proved that N'_{s+1} is polynomially k-saturated for every $k < \aleph_0$ under the assumption that N'_s is polynomially k-saturated for every $k < \aleph_0$. □

Now we can complete the proof of the main theorem:

Theorem 5.1 *If M is a strongly independent structure then M has the finite submodel property.*

Proof. Under the assumption that M is a strongly independent structure we have derived that M^{eq} has a canonically embedded structure N'_r which, by Corollary 5.28, is polynomially k-saturated for every $k < \aleph_0$. It follows (by Lemma 2.3) that N'_r has the finite submodel property. Since M is a strongly independent structure, there is a finite bound on the arity of function symbols in the language of M, so Theorem 1.4 and the fact that (by construction) $M \subseteq \mathrm{acl}_{M^{\mathrm{eq}}}(N'_r)$ and only finitely many sorts are represented in N'_r implies that M has the finite submodel property. □

Remark 5.29 Recall that the difference between 'independent structure' and 'strongly independent structure' is that in the latter case we assume the n-embedding of types property with respect to *all* generators, while in the former case we only assume the n-embedding of types property with respect to *simple* generators over the "base sets" $A_\emptyset$ and $B_\emptyset$. By Theorem 3.4, all stable theories have the n-embedding of types property, and from Theorem 4.3 it follows that every independent (not necesserarily strongly independent) structure with SU-rank 1 has the finite submodel property. It would be pleasing if one could show that every independent structure has the finite submodel property, or show that the assumption on *strong* independence is necessary; an issue not settled in this paper.

6 The n-amalgamation property

The n-amalgamation property was introduced and studied in [10] and generalizes an earlier variant of it studied in [11]. Here we will prove a result which relates the n-embedding of types property and the n-amalgamation of types property in the case when the theory under consideration has SU-rank one.

We start by giving the definition of the n-amalgamation property as well as the definition of a coherent system of types, a notion also comming from [10].

Definition 6.1 We say that T has the *n-amalgamation property* if whenever

$$(\{A_s : s \in \mathcal{P}^-(n)\},\ \{\pi^s_t : s \subseteq t \in \mathcal{P}^-(n)\})$$

is an independent system of boundedly closed sets indexed by $\mathcal{P}^-(n)$, there exist a boundedly closed A_n and elementary maps $\pi_n^u : A_u \to A_n$ for every $u \in \mathcal{P}^-(n)$ such that $(\{A_s : s \in \mathcal{P}(n)\},\ \{\pi_t^s : s \subseteq t \in \mathcal{P}(n)\})$ is an independent system of boundedly closed sets indexed by $\mathcal{P}(n)$.

Definition 6.2 Let $\{A_w : w \in \mathcal{P}^-(n)\}$ be an independent system of boundedly closed sets with inclusion maps. We say that $\{p_w(\bar{x}_w) : w \in \mathcal{P}^-(n)\}$, where $p_w(\bar{x}_w) \in S(A_w)$ for each $w \in \mathcal{P}^-(n)$, is a *coherent system of types over* $\{A_w : w \in \mathcal{P}^-(n)\}$ if the following hold:

(1) If C_w realizes p_w then $C_w \supset A_w$ (so $\bar{x}_w$ is an infinite sequence of variables).

(2) If $w \subseteq v$ then $\bar{x}_w \subseteq \bar{x}_v$ and $p_w \subseteq p_v$.

(3) For every $w \in \mathcal{P}^-(n)$ there is a bijection $f_w : C_w \to \bar{x}_w$ such that if $C_w^{\emptyset} = f_w^{-1} \circ f_{\emptyset}(C_{\emptyset})$, then

(4) $C_w = \mathrm{bdd}(A_w \cup C_w^{\emptyset})$ and $C_w^{\emptyset} \underset{A_{\emptyset}}{\downarrow} A_w$ (for every $w \in \mathcal{P}^-(n)$).

From [10] we have:

Theorem 6.3 *Let T be simple and let $n \geq 3$. Then the following are equivalent:*

(i) T has the k-amalgamation property for every $k \leq n+1$.

(ii) For every $k \leq n$ and coherent system of types $\{p_w(\bar{x}_w) : w \in \mathcal{P}^-(k)\}$ over an independent system of boundedly closed sets $\{A_w : w \in \mathcal{P}^-(k)\}$, there is C_k which realizes p_w for every $w \in \mathcal{P}^-(k)$ and $C_k^{\emptyset} \underset{A_{\emptyset}}{\downarrow} \bigcup_{i \in k} A_{\{i\}}$.

Now we can use Theorem 6.3 to prove the following lemma which has the technical content of the next theorem:

Lemma 6.4 *Let T be simple with SU-rank 1 and with the k-amalgamation property for every $k \leq n+1$, where $n \geq 3$. Suppose that $\mathcal{A} = \{A_w : w \in \mathcal{P}^-(k)\}$ and $\mathcal{B} = \{B_w : w \in \mathcal{P}^-(k)\}$ are independent systems of algebraically closed sets with inclusion maps and that, for every $w \in \mathcal{P}^-(k)$, f_w is an elementary map from A_w onto B_w, and f_w extends f_v whenever $w \supseteq v$. Let $\bar{a}$ be such that*

(1) $\mathrm{rng}(\bar{a}) \cap \mathrm{acl}\Big(\bigcup_{w \in \mathcal{P}^-(k)} A_w\Big) = \emptyset.$

(i) If T has trivial dependence then there are $\bar{b}$ and, for every $w \in \mathcal{P}^-(k)$, an elementary map from $A_w \cup \mathrm{rng}(\bar{a})$ onto $B_w \cup \mathrm{rng}(\bar{b})$ such that f_w extends g_w.
(ii) If $\bar{a}$ is a real *tuple which i addition to (1) also satisfies that*

(2) if $a \in \mathrm{rng}(\bar{a})$ and $a \in \mathrm{acl}\Big((\mathrm{rng}(\bar{a}) - \{a\}) \cup \bigcup_{w \in \mathcal{P}^-(k)} A_w\Big)$ then $a \in \mathrm{acl}\big(\mathrm{rng}(\bar{a}) - \{a\}\big)$,

then there are $\bar{b}$ and, for every $w \in \mathcal{P}^-(k)$, an elementary map from $A_w \cup \mathrm{rng}(\bar{a})$ onto $B_w \cup \mathrm{rng}(\bar{b})$ such that f_w extends g_w. (Here we do not assume trivial dependence.)

Proof. Let $n \geq 3$ and suppose that T is simple with SU-rank 1 and with the k-amalgamation property for every $k \leq n+1$. Let $k \leq n$. We will prove (ii) and then tell how to modify the proof so that it becomes a proof of (i). Let $\mathcal{A}$, $\mathcal{B}$ and f_w, for $w \in \mathcal{P}^-(k)$, be as in the assumptions of the lemma. Then let $\bar{a}$ be a tuple of real elements satisfying (1) and (2).

Since T has SU-rank 1, we have $\bar{a} \downarrow A_w$ for every $w \in \mathcal{P}^-(k)$. For every $w \in \mathcal{P}^-(k)$, let $C_w = \mathrm{acl}(\bar{a}A_w)$. Since each f_w is an elementary map, there are, for every $w \in \mathcal{P}^-(k)$, $\bar{b}_w$ and an elementary map f'_w from C_w onto $D_w = \mathrm{acl}(\bar{b}_w B_w)$ such that f'_w extends f_w and $f'_w(\bar{a}) = \bar{b}_w$.

We now transform the above data into a coherent system of types over $\{B_w : w \in \mathcal{P}^-(k)\}$. For each $w \in \mathcal{P}^-(k)$, let

$$p_w(\bar{x}_w) = tp(D_w/B_w),$$

so $\bar{x}_w$ is an infinite sequence of (distinct) variables of length $|D_w|$. The assumption that f_w extends f_v if $w \supseteq v$ implies that we may assume that if $v \subseteq w$ then $\bar{x}_v \subseteq \bar{x}_w$ and $p_v \subseteq p_w$. For every $w \in \mathcal{P}^-(k)$, let $h_w : D_w \to \bar{x}_w$ be the bijection which is implicit in the definition of p_w, and let $D^{\emptyset}_w = h_w^{-1} \circ h_{\emptyset}(D_{\emptyset})$; it follows that $D^{\emptyset}_w = \mathrm{acl}(\bar{b}_w B_{\emptyset})$ and hence $\bar{b}_w \subseteq D^{\emptyset}_w$. We need to verify that $D_w = \mathrm{acl}\big(B_w \cup D^{\emptyset}_w\big)$ and $D^{\emptyset}_w \underset{B_{\emptyset}}{\downarrow} B_w$, for every $w \in \mathcal{P}^-(k)$.

Let $w \in \mathcal{P}^-(k)$. Since $D_w = \mathrm{acl}(\bar{b}_w B_w)$ and $D^{\emptyset}_w = \mathrm{acl}(\bar{b}_w B_{\emptyset})$ we get $D_w = \mathrm{acl}\big(B_w \cup D^{\emptyset}_w\big)$. We already noted that $\bar{a} \downarrow A_w$ and hence $\bar{a} \underset{A_{\emptyset}}{\downarrow} A_w$, and since f'_w is an elementary map it follows that $\bar{b}_w \underset{B_{\emptyset}}{\downarrow} B_w$; as $D^{\emptyset}_w = \mathrm{acl}(\bar{b}_w B_{\emptyset})$ we get $D^{\emptyset}_w \underset{B_{\emptyset}}{\downarrow} B_w$.

Now we have proved that $\{p_w(\bar{x}_w) : w \in \mathcal{P}^-(k)\}$ is a coherent system of types over $\{B_w : w \in \mathcal{P}^-(k)\}$. By assumption, T has the k-amalgamation property for every $k \leq n+1$, so Theorem 6.3 implies that there is D which realizes p_w for every $w \in \mathcal{P}^-(k)$. But then there is a sequence of elements $\bar{b} \in D$ and, for every $w \in \mathcal{P}^-(k)$, an elementary map $g_w : \mathrm{rng}(\bar{a}) \cup A_w \to \mathrm{rng}(\bar{b}) \cup B_w$ such that g_w extends f_w. This completes the proof of (ii).

Part (i) is proved essentially in the same way as (ii). Since we now assume that T has trivial dependence, if $\bar{a}$ is a real tuple which satisfies (1) then $\bar{a} \downarrow A_w$ for every $w \in \mathcal{P}^-(k)$. Then we can argue precisely as in the proof of (ii) to find $\bar{b}$ and elementary maps g_w from $A_w \cup \mathrm{rng}(\bar{a})$ onto $B_w \cup \mathrm{rng}(\bar{b})$ such that for each w, g_w extends f_w. But by an analogous argument as in the proof of Lemma 3.3 it follows that the assumption that $\bar{a}$ is a *real* tuple is not necessary. □

Theorem 6.5 *Suppose that T is simple with SU-rank 1. Moreover, suppose that* $\mathrm{acl}(A) = \mathrm{dcl}(A)$ *for every $A \subset \mathcal{M}^{\mathrm{eq}}$, where $\mathcal{M}$ is the monster model of T. Let $n \geq 3$.*
(i) If T has the k-amalgamation property for every $k \leq n+1$, then T has the k-embedding of types property for real types with respect to simple generators, for every $2 \leq k \leq n$.
(ii) If T has trivial dependence and the k-amalgamation property for every $k \leq n+1$, then T has the strong k-embedding of types property with respect to simple generators, for every $2 \leq k \leq n$.

Proof. Let $n \geq 3$ and suppose that T is simple with SU-rank 1. Then T is supersimple and therefore it has elimination of hyperimaginaries, so we can replace the bounded closure by the algebraic closure in the definition of the n-amalgamation property. Moreover, assume that the algebraic closure coincides with the definable closure.

By Lemma 6.4, it is sufficient to show the following for $k \leq n$: Whenever

(a) $(\mathcal{A}, \mathcal{G}_A)$ and $(\mathcal{B}, \mathcal{G}_B)$ are independent systems of algebraically closed sets, indexed by $\mathcal{P}^-(k)$ with inclusion maps and simple generators $\mathcal{G}_{A_\emptyset} = \{A_i^0 : i \in k\}$ over $A_\emptyset$ and $\mathcal{G}_{B_\emptyset} = \{B_i^0 : i \in k\}$ over $B_\emptyset$, respectively, and

(b) $\{f_w : w \in \mathcal{P}^-(k)\}$ is a system of elementary maps from $(\mathcal{A}, \mathcal{G}_{A_\emptyset})$ onto $(\mathcal{B}, \mathcal{G}_{B_\emptyset})$,

then, for every $w \in \mathcal{P}^-(k)$, f_w extends f_v if $w \supseteq v$.

By the definitions of independent system of algebraically closed sets and of simple generators (so $A_\emptyset \subseteq A_i^0$, $B_\emptyset \subseteq B_i^0$) and the assumption that acl coincides with dcl we have the following for every $w \in \mathcal{P}^-(k)$:

$$A_w = \mathrm{acl}\Big(\bigcup_{i\in w} A_{\{i\}}\Big) = \mathrm{acl}\Big(\bigcup_{i\in w} (A_\emptyset \cup A_i^0)\Big) = \mathrm{acl}\Big(\bigcup_{i\in w} A_i^0\Big) = \mathrm{dcl}\Big(\bigcup_{i\in w} A_i^0\Big).$$

In the same way we get $B_w = \mathrm{dcl}\Big(\bigcup_{i\in w} B_i^0\Big)$ for every $w \in \mathcal{P}^-(k)$. So for every $w \in \mathcal{P}^-(k)$, every elementary map from $\bigcup_{i\in w} A_i^0$ onto $\bigcup_{i\in w} B_i^0$ can be extended to an elementary map from A_w onto B_w in one unique way. Since we assume that $\{f_w : w \in \mathcal{P}^-(k)\}$ is a system of elementary maps from $(\mathcal{A}, \mathcal{G}_{A_\emptyset})$ onto $(\mathcal{B}, \mathcal{G}_{B_\emptyset})$ it follows that f_w extends f_v whenever $w \supseteq v$. □

7 Examples

In all examples, when passing from statements concerning real elements (those of sort '=') to imaginary elements, we tacitly use the fact (see [14] or [15] for example) that T^{eq} is determined by T in the sense that for every $\varphi(x_1, \ldots, x_m) \in L^{\mathrm{eq}}$ and $\emptyset$-definable equivalence relations S_i, $i = 1, \ldots, m$, on $\mathcal{M}^{s_i}$ (where $\mathcal{M}$ is the monster model of T) with corresponding functions f_i sending $\bar{a} \in \mathcal{M}^{s_i}$ to its S_i-class, there is a formula $\psi(\bar{y}_1, \ldots, \bar{y}_m) \in L$ such that

$$T^{\mathrm{eq}} \models \forall \text{ real } \bar{y}_1 \ldots \bar{y}_m \big(\psi(\bar{y}_1, \ldots, \bar{y}_m) \leftrightarrow \varphi(f_1(\bar{y}_1), \ldots, f_m(\bar{y}_m))\big).$$

When this needs to be used in the examples, ψ can be chosen to be quantifier free.

Example 7.1 *The random graph:* In [10] it is shown that the complete theory of the random graph, denoted T_{rg}, has the n-amalgamation property for every $n < \aleph_0$. By Theorem 6.5 (ii) and Lemma 3.6, T_{rg} has the strong n-embedding of types property with respect to all generators, for every $3 \leq n < \aleph_0$; and one can verify "by hand" that the same holds for $n = 2$. Hence T_{rg} is strongly independent.

Example 7.2 *The strong 4-embedding of types property with respect to simple generators does not imply the 4-amalgamation property:* According to Theorem 3.4 every stable theory has the strong n-embedding of types property with respect to simple generators for every $2 \leq n < \aleph_0$. In [4] an example is given of a stable theory without the 4-amalgamation property.

Example 7.3 *A strongly independent structure with SU-rank $k+1$, $k > 0$ arbitrary:* We have seen that random graph is an example of a strongly independent structure of SU-rank 1. Another example, of SU-rank $k+1$, for arbitrary $k > 0$, can be constructed as follows. We use the basic theory of Fraïssé-limits; see [8], Chapter 7 (in particular, Theorems 7.1.2 and 7.4.1). Let the vocabulary of the language L be $\{=,\ E_0, \ldots, E_k,\ R\}$

and let $\mathcal{K}$ be the class of all finite L-structures A such that $E_0, \ldots, E_k$ are interpreted as equivalence relations, where E_{i+1} refines E_i for each $i < k$, and R is interpreted as a symmetric and irreflexive binary relation. It is easy to verify that $\mathcal{K}$ has the hereditary property and amalgamation property, which implies that $\mathcal{K}$ has the joint embedding property, so $\mathcal{K}$ has a so-called Fraïssé-limit M. The Fraïssé-limit M has the properties that $Th(M)$ eliminates quantifiers (so it is $\aleph_0$-categorical) and

(1) every finite substructure of M belongs to $\mathcal{K}$, and

(2) for every finite substructure $A \subset M$ (where we may have $A = \emptyset$) and $B \in \mathcal{K}$ such that $A \subseteq B$, there is an embedding $f : B \to M$ such that $f \restriction A$ is the identity map.

Let $\mathcal{M}$ be the monster model of $Th(M)$ and for $a \in \mathcal{M}$, let $[a]_i$ denote the E_i-class to which a belongs, so $[a]_i \in \mathcal{M}^{\text{eq}}$, and let f_i be the function which sends $a \in \mathcal{M}$ to its E_i-class. It follows that if $\bar{a} \in \mathcal{M}$ and $A \subseteq B \subseteq \mathcal{M}^{\text{eq}}$, then $tp(\bar{a}/B)$ forks over A if and only if, for some $a \in \text{rng}(\bar{a})$,

$$\exists b \in B - A\Big(b = a \vee \exists i \in \{0, \ldots, k\} \\ \Big[\Big(E_i(a,b) \wedge \forall a' \in A(\neg E_i(a,a'))\Big) \vee \Big(f_i(a) = b \wedge \forall a' \in A(f_i(a) \neq a')\Big)\Big]\Big).$$

From this, one can show that $Th(M)$ is simple with SU-rank $k+1$, that $Th(M)$ is 1-based and has trivial dependence. From the definition of $\mathcal{K}$ and (2) it follows that algebraic closure and definable closure always coincide (also when imaginary elements are involved) and that the latter is trivial. In order to verify that $Th(M)$ has the strong n-embedding of types property with respect to all generators, it is, by Lemma 3.3 sufficient to consider real types, and for $n > 2$ it is, by Lemma 3.6, sufficient to consider simple generators.

Example 7.4 *The random bipartite graph has the 2-embedding of types property with respect to simple generators, but not with respect to all generators; this happens for trivial reasons and the discrepancy disappears in a natural expansion of the random bipartite graph:* Let the language L have two binary relation symbols E and R. Let $\mathcal{K}_{rb}$ be the class of all of all finite L-structures A in which E is interpreted as an equivalence relation with exactly two classes and such that $A \models \forall xy\big(R(x,y) \to \neg E(x,y)\big)$. We call the Fraïssé limit M_{rb} of $\mathcal{K}_{rb}$ the *random bipartite graph* and let $T_{rb} = Th(M_{rb})$. Then T_{rb} has the (strong) 2-embedding of types property with respect to simple generators, *but not* with respect to all generators. To see the latter, first observe that any two distinct elements are independent of each other over $\emptyset$ and then consider distinct elements a_0, a_1 in the same E-class, and distinct elements b_0, b_1 not in the same E-class. Then the unique maps $f_{\{i\}} : \{a_i\} \to \{b_i\}$, for $i = 0, 1$, are elementary and there exists a which is adjacent (with respect to R) to both a_0 and a_1, but there is no b which is adjacent to both b_0 and b_1, since they are in different E-classes. This problem vanishes when we consider the (strong) 2-embedding of types property with respect to *simple generators* since in this case we must assume that a_i has the same type as b_i *over* $\text{acl}(\emptyset)$ and this puts a_i and b_i in the same E-class for $i = 0, 1$.

Now suppose that $n \geq 3$ and that $\mathcal{A} = \{A_w : w \in \mathcal{P}^-(n)\}$ and $\mathcal{B} = \{B_w : w \in \mathcal{P}^-(n)\}$ are systems of algebraically closed sets with (not necessarily simple) generators $\mathcal{G}_A = \{A_i^0 : i \in n\}$ and $\mathcal{G}_B = \{B_i^0 : i \in n\}$ over A and B, respectively. Assume that $\mathcal{F} = \{f_w : w \in \mathcal{P}^-(n)\}$ is a system of elementary maps from $(\mathcal{A}, \mathcal{G}_A)$ onto $(\mathcal{B}, \mathcal{G}_B)$. If $a_i \in A_i^0$ and $a_j \in A_j^0$ are in the same E-class then, since $f_{\{i,j\}}$ is elementary, $f_{\{i,j\}}(a_i)$

and $f_{\{i,j\}}(a_i)$ are in the same E-class, and vice versa. So for $n \geq 3$ the problem that appeared when $n = 2$ (with respect to *all* generators) does not appear. From this, one can deduce that T_{rb} has the strong n-embedding of types property with respect to all generators for every $n \geq 3$.

Now consider the expansion M'_{rb} of M_{rb} obtained by adding a unary predicate which is interpreted as (exactly) one of the E-classes and let $T'_{rb} = Th(M'_{rb})$. So if a and b have the same type with respect to T'_{rb}, then they are in the same E-class, and in fact they have the same type over $\text{acl}(\emptyset)$ where acl is taken in $\mathcal{M}^{\text{eq}}$ for a monster model $\mathcal{M}$ of T'_{rb}. From this it follows that T'_{rb} has the strong n-embedding of types property with respect to all generators for every $2 \leq n < \aleph_0$.

The author lacks an example of a complete theory T such that, for some n, T has the n-embedding of types property for simple generators, but not for all generators, and there does *not* exist a theory $T' \supseteq T$, in an expanded language, such that T' has the n-embedding of types property for all generators. In other words, I don't know of an example where a discrepancy between n-embedding of types property with respect to *simple* generators, and with respect to *all* generators, appears and cannot be fixed by just expanding the language in a way that preserves all other relevant properties of T (simplicity, $\aleph_0$-categoricity, 1-basedness etc).

Example 7.5 *Failure to extend generators and systems of elementary maps in general:* Here we construct a theory T and systems of algebraically closed sets $\mathcal{A} = \{A_w : w \in \mathcal{P}^-(n)\}$ and $\mathcal{B} = \{B_w : w \in \mathcal{P}^-(n)\}$ with (nonsimple) generators $\mathcal{G}_A = \{A_i^0 : i \in n\}$ and $\mathcal{G}_B = \{B_i^0 : i \in n\}$ over A and B, respectively. It will easily follow that there is a system of elementary maps from $(\mathcal{A}, \mathcal{G}_A)$ onto $(\mathcal{B}, \mathcal{G}_B)$. However, we will show that whenever $\mathcal{G}'_{A_\emptyset}$ and $\mathcal{G}'_{B_\emptyset}$ are simple generators for $\mathcal{A}$ and $\mathcal{B}$, respectively, then there is *no* elementary system of maps from from $(\mathcal{A}, \mathcal{G}'_{A_\emptyset})$ onto $(\mathcal{B}, \mathcal{G}'_{B_\emptyset})$. The example T will have very uncomplicated behaviour of forking and algebraic closure, but algebraic and definable closures will *not* coincide. Since T has the strong n-embedding of types property for every $2 \leq n < \aleph_0$ with respect to all generators, it does not show that the assumptions on algebraic and definable closures in Lemma 3.6 are necessary, but only that the method of "extending" systems of elementary maps to simple generators, applied in the proof of that lemma, may fail if algebraic and definable closures are different.

Let the language L_0 have unary relation symbols P, Q, a ternary relation symbol R, a binary relation symbol S and unary function symbols f_1 and f_2. Let $\mathcal{K}_0$ consist of all finite L_0-structures A which satisfy the following axioms:

$$\begin{aligned}
&A: \ \forall x\big(P(x) \vee Q(x)\big),\\
&B: \ \forall x\big(P(x) \rightarrow \neg Q(x)\big),\\
&C: \ \forall x\big(P(x) \rightarrow \big(Q(f_1(x)) \wedge Q(f_2(x)) \wedge f_1(x) \neq f_2(x)\big)\big),\\
&D: \ \forall x\big(Q(x) \rightarrow \big(f_1(x) = x \wedge f_2(x) = x\big)\big),\\
&E: \ \forall xy\big(S(x,y) \leftrightarrow \big(P(x) \wedge Q(y) \wedge \big(f_1(x) = y \vee f_2(x) = y\big)\big)\big),\\
&F: \ \forall xyz\big(R(x,y,z) \rightarrow \big(P(x) \wedge P(y) \wedge Q(z)\big).
\end{aligned}$$

Then $\mathcal{K}_0$ is closed under substructures and has the amalgamation property, so the Fraïssé limit M_0 of K_0 exists and has elimination of quantifiers. Since M_0 is uniformly locally finite it is $\aleph_0$-categorical. Note that, as M_0 is the Fraïssé limit of K_0, we have the following: Whenever A is a substructure of $B \in K_0$ and $f : A \rightarrow M_0$ is an embedding, then there is an embedding $g : B \rightarrow M_0$ which extends f; it follows that for any two

$a_1, a_2 \in Q^{M_0}$ there are infinitely many $b \in P^{M_0}$ such that $f_1(b) = a_1$ and $f_2(b) = a_2$. The theory $T_0 = Th(M_0)$ is simple with SU-rank 3, 1-based and with trivial dependence and with the strong n-amalgamation of types property with respect to all generators, for every $2 \leq n < \aleph_0$ (left for the reader to verify).

Now let $L \subseteq L_0$ be the language with the symbols P, Q, R and S (but not f_1 and f_2). Then let $M = M_0 \restriction L$ and $T = Th(M)$. Note that, by axioms C and E, whenever $a \in P^M$, then there are exactly two elements b_1, b_2 such that $(a, b_i) \in S^M$ and both these elements belong to Q^M. Let's call a subset $A \subseteq M$ *closed* if it is the universe of a substructure of M_0, or equivalently, if it satisfies that whenever $a \in A \cap P^M$ and $(a, b) \in S^M$ then $b \in A$. It follows (using properties of N) that every isomorphism $\sigma : A \to B$ where A and B are closed substructures of M can be extended to an automorphism of M; hence the quantifier-free type of a tuple $\bar{a} \in M$ such that $\text{rng}(\bar{a})$ is closed determines its type over $\emptyset$. Also, if $\mathcal{K}$ consists of all L-reducts of structures in $\mathcal{K}_0$, then it follows that whenever $A \in \mathcal{K}$ and A is a substructure of $B \in K$ and $f : A \to M$ is an embedding, then there is an embedding $g : B \to M$ which extends f.

Thus there are distinct $a_0, a_1, a_2 \in P^M$ and distinct $a'_0, a'_1 \in Q^M$ such that the substructure of M with universe $\{a_0, a_1, a_2, a'_0, a'_1\}$ satisfies the following atomic relations, and no others:

$$S(a_i, a'_j) \text{ for every } i \in \{0,1,2\} \text{ and every } j \in \{0,1\}, \text{ and}$$
$$R(a_0, a_1, a'_0) \text{ and } R(a_1, a_2, a'_0).$$

Then we can also find $b_0, b_1, b_2 \in P^M$ and distinct $b'_0, b'_1 \in Q^M$ such that the substructure of M with universe $\{b_0, b_1, b_2, b'_0, b'_1\}$ satisfies the following atomic relations, and no others:

$$S(b_i, b'_j) \text{ for every } i \in \{0,1,2\} \text{ and every } j \in \{0,1\}, \text{ and}$$
$$R(b_0, b_1, b'_0) \text{ and } R(b_1, b_2, b'_1).$$

Note that the with respect to the mapping $a_i \mapsto b_i$, $a'_i \mapsto b'_i$, the only difference is that we have $R(a_1, a_2, a'_0)$ for the first set of elements and $R(b_1, b_2, b'_1)$ for the other set. Also observe that for all $i, j \in 3$, (a_i, a_j, a'_0, a'_1) and (b_i, b_j, b'_0, b'_1) are closed, and hence their quantifier-free type determines their type over $\emptyset$. It follows that (a_i, a_j) has the same type as (b_i, b_j) over $\emptyset$. But there does *not* exist elementary maps

$$g_{\{0,1\}} : \{a_0, a_1, a'_0, a'_1\} \to \{b_0, b_1, b'_0, b'_1\}, \text{ and}$$
$$g_{\{1,2\}} : \{a_1, a_2, a'_0, a'_1\} \to \{b_1, b_2, b'_0, b'_1\}$$

such that $g_{\{0,1\}} \restriction \{a'_0, a'_1\} = g_{\{1,2\}} \restriction \{a'_0, a'_1\}$, $g_{\{0,1\}}$ maps (a_0, a_1) to (b_0, b_1) and $g_{\{1,2\}}$ maps (a_1, a_2) to (b_1, b_2).

Now let $\mathcal{M}$ be the monster model in which M is elementarily embedded and let acl the algebraic closure in $\mathcal{M}^{\text{eq}}$. From the above it follows that the claims made in the beginning of this example about $\mathcal{A}$, $\mathcal{G}_A$, $\mathcal{B}$ and $\mathcal{G}_B$ hold if we let $n = 3$, $A = \{a'_0, a'_1\}$, $A_i^0 = \{a_i\}$ for $i \in 3$, and $A_w = \text{acl}\Big(\bigcup_{i \in w} A_i^0\Big)$ for $w \in \mathcal{P}^-(3)$; and the same with 'b' and 'B' in place of 'a' and 'A'.

One can also show that T has the strong n-embedding of types property with respect to all generators for every $2 \leq n < \aleph_0$. The construction of M could have been carried out in the same fashion with a k-ary relation R, for any $k \geq 3$, and the requirement that if $R(x_1, \ldots, x_k)$ holds then $P(x_i)$ holds for $i = 1, \ldots, k-1$ and $Q(x_k)$ holds. Then the assertions in the beginning of the example would follow for $n = k$.

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