The finite submodel property and ω-categorical expansions of pregeometries

Marko Djordjević

Abstract

We prove, by a probabilistic argument, that a class of ω-categorical structures, on which algebraic closure defines a pregeometry, have the finite submodel property. This class includes any expansion of a pure set or of a vector space, projective space or affine space over a finite field such that the new relations are sufficiently independent of each other and over the original structure. In particular, the random graph belongs to this class, since it is a sufficiently independent expansion of an infinite set, with no structure. The class also contains structures for which the pregeometry given by algebraic closure is non-trivial.

Introduction

The random graph, random bipartite graph and random structure have the finite submodel property, which means that every first-order sentence which is true in the structure is also true in a finite substructure of it. This follows from the 0-1 law for each one of them, the proof of which uses a probabilistic argument (see [8], [9], for example). It is also known that all smoothly approximable structures have the finite submodel property, which follows rather easily from the definition [11]. The hard part is to show that certain structures are smoothly approximable. It has been shown that all ω-categorical ω-stable structures are smoothly approximable [4] and later that a structure is smoothly approximable if and only if it is Lie coordinatizable ([5]; partially proved in [11]).

All above mentioned structures are ω-categorical and simple. If M is simple with SU-rank 1, then (M, acl), where 'acl' denotes the algebraic closure operator, is a pregeometry. The random (bipartite) graph and random structure have SU-rank 1. Every smoothly approximable structure M has finite SU-rank and can be nicely described, via Lie coordinatizability, in terms of definable subsets of M^{eq} (so-called Lie geometries in [5]) of SU-rank 1.

A pregeometry (G, cl) can be viewed as a first-order structure $M = (G, P_n; n < \omega)$, where $M \models P_n(a_1, \ldots, a_{n+1})$ if and only if $a_{n+1} \in \mathrm{cl}(\{a_1, \ldots, a_n\})$. In such a structure we have a notion of dimension, defined in terms of the closure operator cl. In this article we will study ω-categorical structures M such that (M, acl) is a pregeometry. Since M may have relations which are not expressible in terms of the P_n's (now defined with cl $=$ acl), we will view such M as an expansion of $(M, P_n; n < \omega)$.

We will prove (Theorem 2.2) that if M is an L-structure and there is a sublanguage $\mathcal{L} \subseteq L$ such that, for every $k < \omega$, the following three points (which will be made precise later) are satisfied, then M has the finite submodel property:
(1) The algebraic closure operator in M is the same as the algebraic closure operator in $M{\restriction}\mathcal{L}$, where $M{\restriction}\mathcal{L}$ is the reduct of M to $\mathcal{L}$.
(2) The relations on tuples of dimension $\leq k$ which are definable in M but not in $M{\restriction}\mathcal{L}$ are sufficiently independent of each other.

(3) There is a polynomial $P(x)$ such that, for any $n_0 < \omega$ there is $n \geq n_0$ and a (finite) substructure A of $M{\restriction}\mathcal{L}$ such that $|A| \leq P(n)$ and

(a) A is algebraically closed in $M{\restriction}\mathcal{L}$ (and hence in M by (1)), and

(b) any non-algebraic 1-type (in $M{\restriction}\mathcal{L}$) over a subset of A of dimension $< k$ is realized by n distinct elements in A.

We will say that a structure which satisfies the precise version of condition (2) *satisfies the k-independence hypothesis over $\mathcal{L}$*. If $N = M{\restriction}\mathcal{L}$ satisfies the precise version of condition (3) then we say that N is *polynomially k-saturated*. We will see (Lemma 1.8) that being polynomially k-saturated, for every $k < \omega$, implies having the finite submodel property. So above, we are implicitly assuming that $M{\restriction}\mathcal{L}$ has the finite submodel property. The point is that under conditions (1)-(3) also M will have it; in fact M will satisfy the stronger condition of being polynomially k-saturated for every $k < \omega$, so this property is transferred from $M{\restriction}\mathcal{L}$ to M. If we only assume that (the precise versions of) conditions (1)-(3) hold for some particular k (and hence for all $l \leq k$) then we get a weaker conclusion (Theorem 2.1) which only says that every unnested sentence, in which at most k distinct variables occur, which is true in M has a finite model, but here we are not able to prove that the finite model can be emedded into M.

Structures which are Lie coordinatizable, or equivalently 'smoothly approximable', have the finite submodel property [5]. In the special cases of vector spaces, projective spaces or affine spaces over a finite field we can strengthen this and show (in Section 3.2) that these structures are polynomially k-saturated for every $k < \omega$. Hence any vector space over a finite field (or its projective or affine variants) is a good "base structure" which can potentially be expanded in a non-trivial way without loosing the finite submodel property and polynomial k-saturatedness; a particularly simple example of this is the "well-behaved" structure in Section 3.3. A vector space over a finite field is a linear geometry in the sense of [5]. A natural question, not answered in this article, is whether every linear geometry is polynomially k-saturated for every $k < \omega$.

In Section 3 we give two examples which do not satisfy the premises of Theorem 2.2 which where roughly stated as (1)-(3) above. One of the examples shows that if we remove these premises then the theorem fails, even if we assume that the structures under consideration are simple with SU-rank 1. For other results concerning expansions of non-trivial structures, including vector spaces over a finite field, see [1].

The main theorems and their proofs are given in Section 2; the prerequisites, which are stated in Section 1, include only basic model theory. In Section 3 examples are given of structures which have or don't have the main properties considered in this paper; here basic results about simple theories will be used as well as some more specialized results about structures obtained by amalgamation constructions, with or without a predimension.

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1 Preliminaries

Notation and terminology If L is a (first-order) language then its *vocabulary* (or signature) is the set of relation, function and constant symbols of L; we always assume that '$=$' belongs to the vocabulary and that L is countable. If $L' \subseteq L$ are languages and M is an L-structure then $M{\restriction}L'$ denotes the reduct of M to the language L'. If P is a symbol in the vocabulary of L then P^M is the interpretation of P in M. For simplicity, we will say things like "for P in L" when we actually mean "for P in the vocabulary of L". We will frequently speak about *unnested formulas*; a definition follows below. When considering complete theories we assume that they have infinite models and hence only infinite models.

Let M be an L-structure. $Th(M)$ denotes the complete theory of M. We say that M is ω-categorical (simple) if $Th(M)$ is ω-categorical (simple); see [12], [13] for the basics of simple structures. By $\mathrm{acl}_M(A)$ (or just $\mathrm{acl}(A)$) we mean the *algebraic closure* of A in M. By $\bar{a}$, $\bar{b}, \ldots$. we denote finite sequences of elements from some structure; $\mathrm{rng}(\bar{a})$ denotes the set of elements enumerated by $\bar{a}$ and $|\bar{a}|$ is the length of the sequence; we write $\mathrm{acl}(\bar{a})$ instead of $\mathrm{acl}(\mathrm{rng}(\bar{a}))$. By $\bar{x}$, $\bar{y}, \ldots$ we denote finite sequences of variables. By $\bar{a} \in A$ we mean that $\mathrm{rng}(\bar{a}) \subseteq A$. By $\bar{a} \in A^n$ we mean that $\mathrm{rng}(\bar{a}) \subseteq A$ and $|\bar{a}| = n$. For sequences $\bar{a}$, $\bar{b}$ we will sometimes write $\bar{a} \cap \bar{b}$ for $\mathrm{rng}(\bar{a}) \cap \mathrm{rng}(\bar{b})$, and occasionally we will view $\mathrm{rng}(\bar{a}) \cap \mathrm{rng}(\bar{b})$ as a sequence by assuming that it is listed somehow. For sequences $\bar{a} = (a_1, \ldots, a_n)$ and $\bar{b} = (b_1, \ldots, b_m)$, we frequently write $\bar{a}\bar{b}$ for the sequence $(a_1, \ldots, a_n, b_1, \ldots, b_m)$. If X is a set then $|X|$ denotes its cardinality. If $A \subseteq M$ then we say that $R \subseteq M^n$ is A-definable if there is $\varphi(\bar{x}, \bar{y}) \in L$ and $\bar{a} \in A$ such that $\bar{b} \in R$ if and only if $M \models \varphi(\bar{b}, \bar{a})$.

If T is a complete theory then, for $0 < n < \omega$, $S_n(T)$ denotes the set of complete n-types of T. If M is a structure and $A \subseteq M$ then M_A denotes the expansion of M obtained by adding to the language a new constant symbol for every $a \in A$ (and this constant symbol is also denoted by a) which is interpreted as a. For $A \subseteq M$, the set of complete n-types over A (with respect to M), denoted $S_n^M(A)$, is defined to be the set $S_n(Th(M_A))$; if it is clear in which structure we are working we may drop the superscript M. If $\bar{a} \in M^{\mathrm{eq}}$, $A \subseteq M^{\mathrm{eq}}$, then $tp_M(\bar{a}/A)$ denotes the complete type of $\bar{a}$ over A in M^{eq}, or in other words, the type of $\bar{a}$ in $(M^{\mathrm{eq}})_A$; if it is clear in which structure the type is taken then we just write $tp(\bar{a}/A)$. $tp(\bar{a})$ is an abbreviation of $tp(\bar{a}/\emptyset)$. A type is *algebraic* if it has only finitely many realizations; otherwise it is *non-algebraic*. If $p(\bar{x})$ is a type and $\bar{x}'$ is a subsequence of $\bar{x}$ then $p{\restriction}\{\bar{x}'\} = \{\varphi \in p :$ every free variable of φ occurs in $\bar{x}'\}$.

The SU-rank of a complete simple theory T is the supremum (if it exists) of the SU-ranks of types $tp(a)$ where a ranges over elements from models of T. The SU-rank of a simple structure M is defined to be the SU-rank of $Th(M)$.

Definition 1.1 An *unnested atomic formula* is a formula which has one of the following forms:

$$
\begin{aligned}
& x = y, \\
& c = y, \\
& f(\bar{x}) = y, \\
& P(\bar{x}),
\end{aligned}
$$

where x and y are variables, $\bar{x}$ a sequence of variables, c a constant symbol, f a function symbol and P a relation symbol. A formula is *unnested* if all of its atomic subformulas are unnested.

Every formula is logically equivalent to an unnested formula (by [9], Corollary 2.6.2, for instance).

Definition 1.2 Let G be a set and let $\mathrm{cl} : \mathcal{P}(G) \to \mathcal{P}(G)$ be a function, where $\mathcal{P}(G)$ is the powerset of G. We call cl a *closure operator* and say that (G, cl) is a *pregeometry* if

the following conditions are satisfied:
(1) If $A \subseteq G$ then $A \subseteq \mathrm{cl}(A)$ and $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$.
(2) If $A \subseteq B \subseteq G$ then $\mathrm{cl}(A) \subseteq \mathrm{cl}(B)$.
(3) If $A \subseteq G$, $a, b \in G$ and $a \in \mathrm{cl}(A \cup \{b\})$ then $a \in \mathrm{cl}(A)$ or $b \in \mathrm{cl}(A \cup \{a\})$.
(4) If $A \subseteq G$ and $a \in \mathrm{cl}(A)$ then there is a finite $B \subseteq A$ such that $a \in \mathrm{cl}(B)$.

The properties (1), (2) and (4) hold if we replace G by any structure M and cl by acl_M. In Section 3 we will consider simple structures which have SU-rank 1. For such a structure M, acl_M also satisfies (4) which is a consequence of the symmetry of forking , so (M, acl_M) is a pregeometry.

If (M, acl_M) is a pregeometry then we can speak about the dimension of any $A \subseteq M$, denoted $\dim_M(A)$ (or just $\dim(A)$), which is defined by

$$\dim_M(A) = \min\{|B| : B \subseteq A \text{ and } A \subseteq \mathrm{acl}_M(B)\}.$$

In particular, if $A \subseteq \mathrm{acl}_M(\emptyset)$ then $\dim_M(A) = 0$.
The following characterization (see [9] for example) of ω-categorical theories will often be used without reference:

Fact 1.3 *The following are equivalent for a complete theory T with infinite models:*
(1) T is ω-categorical.
(2) $S_n(T)$ is finite for every $0 < n < \omega$.
(3) For every $0 < n < \omega$ there are, up to equivalence in T, only finitely many formulas with all free variables among $x_1, \dots, x_n$.
(4) Every type in $S_n(T)$ is isolated, for every $0 < n < \omega$.

A consequence which is important in the present context is:

Fact 1.4 *If M is an ω-categorical structure and $A \subseteq M$ is finite then $\mathrm{acl}_M(A)$ is finite.*

Definition 1.5 An L-theory T has the *finite submodel property* if the following holds for any $M \models T$ and sentence $\varphi \in L$: If $M \models \varphi$ then there is a finite substructure $N \subseteq M$ such that $N \models \varphi$. A structure M has the *finite submodel property* if whenever φ is a sentence and $M \models \varphi$, then there exists a finite substructure $N \subseteq M$ such that $N \models \varphi$.

Observation 1.6 *(i) Suppose that the vocabulary of the language of M has only finitely many symbols. Then M has the finite submodel property if and only if $Th(M)$ has the finite submodel property.*
(ii) If a complete theory T with infinite models has the finite submodel property then T is not finitely axiomatizable.

Proof. (i) The direction from right to left is immediate from the definitions and does not need the given assumption about the language. Now suppose that the vocabulary of the language of M has only finitely many symbols and that M has the finite submodel property. Suppose that $M' \models Th(M)$ and $M' \models \varphi$. Then $M \models \varphi$, so φ is true in a finite substructure $A_\varphi \subseteq M$. By the assumption about the language, the isomorphism type of A_φ is described by a quantifier free formula $\psi(\bar{x})$ and we have $\exists \bar{x} \psi(\bar{x}) \in Th(M)$, so A_φ can be embedded in M'. Part (ii) is immediate since a finite structure cannot be elementarily equivalent with an infinite one. $\square$

Below we give the definitions of the main notions of this article.

Definition 1.7 Let $0 < k < \omega$ and suppose that M is a structure such that (M, acl) is a pregeometry. We say that M is *polynomially k-saturated* if there is a polynomial $P(x)$ such that for every $n_0 < \omega$ there is a natural number $n \geq n_0$ and a finite substructure $N \subseteq M$ such that:

(1) $n \leq |N| \leq P(n)$.

(2) N is algebraically closed.

(3) Whenever $\bar{a} \in N$, $\dim_M(\bar{a}) < k$ and $q(x) \in S_1^M(\bar{a})$ is non-algebraic there are distinct $b_1, \ldots, b_n \in N$ such that $M \models q(b_i)$ for each $1 \leq i \leq n$.

Examples of structures which are polynomially k-saturated, for every $0 < k < \omega$, include infinite vector spaces over finite fields and the random graph; more will be said about this in Section 3.

Lemma 1.8 *If M is polynomially k-saturated for every $0 < k < \omega$, then M has the finite submodel property.*

Proof. The proof uses Observation 1.10 below. Suppose that M is polynomially k-saturated for every $0 < k < \omega$. By Observation 1.10, it is sufficient to show that for any $k < \omega$ there is a finite substructure N such that condition (ii) in Observation 1.10 holds. So we fix an arbitrary k. Then there is $n \geq 1$ and a finite substructure $N \subseteq M$ for which (2) and (3) of Definition 1.7 hold. The notation 'L^k' is explained in Observation 1.10 below. Let $\varphi(\bar{x}, y) \in L^k$, where we may assume that $|\bar{x}| < k$, and suppose that $\bar{a} \in N$, $b \in M$ and $M \models \varphi(\bar{a}, b)$. If $b \in \mathrm{acl}_M(\bar{a})$ then (2) implies that $b \in N$ and we are done. Otherwise letting $p(\bar{x}, y) = tp_M(\bar{a}, b)$, $p(\bar{a}, y)$ is non-algebraic so by (3) there is $b' \in N$ such that $M \models p(\bar{a}, b')$ which implies $M \models \varphi(\bar{a}, b')$. $\square$

Remark 1.9 Note that, in the proof of Lemma 1.8, we only needed parts (2) and (3) from Definition 1.7.

Observation 1.10 (Tarski-Vaught test for L^k) *Let M be an L-structure and let L^k denote the set of L-formulas in which at most k distinct variables occur, whether free or bound. If N is a substructure of M then the following are equivalent:*
(i) For every $\varphi(\bar{x}) \in L^k$ and $\bar{a} \in N^{|\bar{x}|}$, $M \models \varphi(\bar{a}) \iff N \models \varphi(\bar{a})$.
(ii) For every $\varphi(\bar{x}, y) \in L^k$ and $\bar{a} \in N^{|\bar{x}|}$, if $M \models \exists y \varphi(\bar{a}, y)$ then there is $b \in N$ such that $M \models \varphi(\bar{a}, b)$.

Proof. Observe that if $\psi(\bar{x}) \in L^k$ then any subformula of $\psi(\bar{x})$ also belongs to L^k. As for the proof of the original Tarski-Vaught test, one uses a straightforward induction on the complexity of formulas, which is left for the reader. $\square$

Notation 1.11 If $\bar{s} = (s_1, \ldots, s_n)$ is a sequence of objects and $I = \{i_1, \ldots, i_m\} \subseteq \{1, \ldots, n\}$ where we assume $i_1 < \ldots < i_m$ then $\bar{s}_I$ denotes the sequence $(s_{i_1}, \ldots, s_{i_m})$.

Definition 1.12 Suppose that M is an ω-categorical L-structure such that (M, acl_M) is a pregeometry. Let $\mathcal{L}$ be a sublanguage of L. We say that M *satisfies the k-independence hypothesis over $\mathcal{L}$* if the following holds for any $\bar{a} = (a_1, \ldots, a_n) \in M^n$ such that $\dim_M(\bar{a}) \leq k$:
If $I = \{i_1, \ldots, i_m\} \subseteq \{1, \ldots, n\}$ and $p(\bar{x}_I) \in S_m(Th(M))$ (where $\bar{x}_I = (x_{i_1}, \ldots x_{i_m})$) are such that

$\mathrm{acl}_M(\bar{a}_I) = \mathrm{rng}(\bar{a}_I)$, $\dim_M(\bar{a}_I) < k$, $p(\bar{x}_I) \cap \mathcal{L} = tp_{M\restriction\mathcal{L}}(\bar{a}_I)$ and for every $J \subset I$ with $\dim_M(\bar{a}_J) < \dim_M(\bar{a}_I)$, $p\restriction\{\bar{x}_J\} = tp_M(\bar{a}_J)$,

then there is $\bar{b} = (b_1, \ldots, b_n) \in M^n$ such that

$tp_{M\restriction\mathcal{L}}(\bar{b}) = tp_{M\restriction\mathcal{L}}(\bar{a})$, $tp_M(\bar{b}_I) = p(\bar{x}_I)$ and, for every $J \subset \{1, \ldots, n\}$ such that $\bar{a}_I \not\subseteq \mathrm{acl}_M(\bar{a}_J)$, $tp_M(\bar{a}_J) = tp_M(\bar{b}_J)$.

The above definition will be considered in the context when acl_M and $\mathrm{acl}_{M\restriction\mathcal{L}}$ coincide (i.e. $\mathrm{acl}_M(A) = \mathrm{acl}_{M\restriction\mathcal{L}}(A)$ for every $A \subseteq M$), so in this situation, for $p(\bar{x}_I)$ as in the definition, any realization of $p(\bar{x}_I) \cap \mathcal{L}$ is algebraically closed in M.

An introductory example will illustrate the main notions introduced above. In Section 3, more examples will be given of structures having, or not having, the properties defined above.

An introductory example

We say that a structure M has *trivial* (also called *degenerate*) algebraic closure if for any $A \subseteq M$, $\mathrm{acl}_M(A) = \bigcup_{a\in A} \mathrm{acl}_M(a)$.

Suppose that M is an L-structure which is ω-categorical and simple with SU-rank 1. Also assume that M has trivial algebraic closure. After adding some assumptions on L we will show that, for a particular sublanguage $\mathcal{L}$ (defined below) of L, acl_M and $\mathrm{acl}_{M\restriction\mathcal{L}}$ coincide, $M\restriction\mathcal{L}$ is polynomially k-saturated for every $0 < k < \omega$, and M satisfies the 3-independence hypothesis over $\mathcal{L}$. So by Theorem 2.1, for every sentence $\varphi \in L$, if at most 3 distinct variables occur in φ and $M \models \varphi$, then φ has a finite model.

In order to simplify one part of the argument, we assume that for any $\bar{a} \in M$ with $\dim_M(\bar{a}) \leq 3$, $tp_M(\bar{a}/\mathrm{acl}_{M^{\mathrm{eq}}}(\emptyset))$ is determined by $tp_M(\bar{a}/\mathrm{acl}_M(\emptyset))$ (where $\mathrm{acl}_{M^{\mathrm{eq}}}$ is the algebraic closure taken in M^{eq}). Since M is ω-categorical, if this assumption does not hold from the beginning then it can be satisfied by considering a finite number of elements from M^{eq} to be part of M; the new M thus obtained will be ω-categorical and simple with SU-rank 1 and have trivial algebraic closure.

By the ω-categoricity of M, there is $m < \omega$ such that $|\mathrm{acl}_M(a)| \leq m$ for every $a \in M$. We will suppose that L has relation symbols $P, Q, R_1, \ldots, R_m$ which are interpreted in the following way:

$$\begin{aligned} P^M &= \{a \in M : a \in \mathrm{acl}_M(\emptyset)\}, \\ Q^M &= \{(a,b) \in M^2 : a \in \mathrm{acl}_M(b)\}, \\ R_i^M &= \{a \in M - \mathrm{acl}_M(\emptyset) : |\mathrm{acl}_M(a) - \mathrm{acl}_M(\emptyset)| = i\} \quad \text{for } i = 1, \ldots, m. \end{aligned}$$

If such symbols are not originally in the vocabulary of L, then we can expand M so that the above holds without destroying the other assumptions on M. Let $\mathcal{L}$ be the language with vocabulary $\{=, P, Q, R_1, \ldots, R_m\}$.

Claim 1.13 *(i) $M\restriction\mathcal{L}$ has elimination of quantifiers.*
(ii) For any subset $A \subseteq M$, $\mathrm{acl}_{M\restriction\mathcal{L}}(A) = \mathrm{acl}_M(A)$.

Proof. (i) Straightforward back and forth argument, left for the reader.
(ii) If $b \in \mathrm{acl}_{M\restriction\mathcal{L}}(A)$ then, since $M\restriction\mathcal{L}$ is a reduct of M, we must have $b \in \mathrm{acl}_M(A)$. If $b \in \mathrm{acl}_M(A)$ then, since acl_M is trivial, we get $b \in \mathrm{acl}_M(\emptyset)$ or $b \in \mathrm{acl}_M(a)$, for some $a \in A$, and hence $M\restriction\mathcal{L} \models P(b)$ or $M\restriction\mathcal{L} \models Q(b,a)$, for some $a \in A$. By the ω-categoricity of M, the sets P^M and $\{b' : (b', a) \in Q^M\}$ are finite so $b \in \mathrm{acl}_{M\restriction\mathcal{L}}(A)$. $\square$

Claim 1.14 *$M{\restriction}\mathcal{L}$ is polynomially k-saturated for every $0 < k < \omega$.*

Proof. Let $0 < k < \omega$ be given. Define $a \sim b \iff \mathrm{acl}_M(a) = \mathrm{acl}_M(b)$. Then every $\sim$-class has at most m elements. Let $F(x) = m(k+x)+m$. For any $n_0 < \omega$ we put $n = n_0$ and choose a finite set of $\sim$-classes as follows: If $\mathrm{acl}_M(\emptyset)$ is non-empty then $\mathrm{acl}_M(\emptyset)$ is an $\sim$-class which we choose. If there exists a $\sim$-class different from $\mathrm{acl}_M(\emptyset)$ which contains exactly i elements, then there are infinitely many such, because the elements in such a class do not belong to $\mathrm{acl}_M(\emptyset)$. For every $i \in \{1, \dots, m\}$ such that there exists a $\sim$-class different from $\mathrm{acl}_M(\emptyset)$ which contains exactly i elements, we choose exactly $k+n$ distinct such $\sim$-classes. Now let A be the union of all the chosen classes. Then A with the $\mathcal{L}$-structure induced from $M{\restriction}\mathcal{L}$ is a substructure of $M{\restriction}\mathcal{L}$. The construction of A implies that $n \leq |A| \leq m(k+n)+m = F(n)$. Also by construction, if E is a $\sim$-class and $E \cap A \neq \emptyset$ then $E \subseteq A$, so A is algebraically closed by Claim 1.13. Now we have taken care of parts (1) and (2) of Definition 1.7. For part (3), assume that $\bar{a} \in A$, $\dim_{M{\restriction}\mathcal{L}}(\bar{a}) < k$ and that $q(x) \in S_1^{M{\restriction}\mathcal{L}}(\bar{a})$ is non-algebraic. Then $q(x)$ contains the formula $\neg P(x)$ and, for every $a \in \mathrm{rng}(\bar{a})$, $q(x)$ contains the formula $\neg Q(x, a)$. By Claim 1.13, $M{\restriction}\mathcal{L}$ has elimination of quantifiers, so the construction of A guarantees that we find distinct $b_1, \dots, b_n \in A$ (from distinct $\sim$-classes of appropriate size) such that $M{\restriction}\mathcal{L} \models q(b_i)$, for each i. □

The previous two claims do not need the assumption that M has SU-rank 1; it is sufficient that M is ω-categorical and that (M, acl_M) is a trivial pregeometry. The proof of the next claim will however use the hypothesis that M has SU-rank 1 together with the independence theorem for simple theories.

Claim 1.15 *M satisfies the 3-independence hypothesis over $\mathcal{L}$.*

Proof. Suppose that $\bar{a} = (a_1, \dots, a_n) \in M^n$ and $\dim_M(\bar{a}) = d \leq 3$. Suppose that $I = \{i_1, \dots, i_m\} \subseteq \{1, \dots, n\}$ and $p(\bar{x}_I) \in S_n(Th(M))$ are such that

(a) $\mathrm{acl}_M(\bar{a}_I) = \mathrm{rng}(\bar{a}_I)$, $\dim_M(\bar{a}_I) < 3$, $p(\bar{x}_I) \cap \mathcal{L} = tp_{M{\restriction}\mathcal{L}}(\bar{a}_I)$ and for every $J \subset I$ with $\dim_M(\bar{a}_J) < \dim_M(\bar{a}_I)$, $p{\restriction}\{\bar{x}_J\} = tp_M(\bar{a}_J)$,

We must show that there is $\bar{b} = (b_1, \dots, b_n) \in M^n$ such that

(b) $tp_{M{\restriction}\mathcal{L}}(\bar{b}) = tp_{M{\restriction}\mathcal{L}}(\bar{a})$, $tp_M(\bar{b}_I) = p(\bar{x}_I)$ and, for every $J \subset \{1, \dots, n\}$ such that $\bar{a}_I \not\subseteq \mathrm{acl}_M(\bar{a}_J)$, $tp_M(\bar{a}_J) = tp_M(\bar{b}_J)$.

Without loss of generality we may assume that $\bar{a}$ is algebraically closed (in M), and since algebraic closure is trivial we may assume that $\bar{a} = \bar{a}_0\bar{a}_1 \dots \bar{a}_d$, where $\bar{a}_0 = \mathrm{acl}_M(\emptyset)$ and for $i = 1, \dots, d$, $\dim_M(\bar{a}_i) = 1$ and $\bar{a}_i = \mathrm{acl}_M(\bar{a}_i) - \mathrm{acl}_M(\emptyset)$. Let $l = \dim_M(\bar{a}_I)$. We get different cases depending on l; the first three (in a sense "degenerate") cases only uses that M is ω-categorical and that (M, acl_M) forms a trivial pregeometry; the fourth and last case also uses the assumption that M is simple with SU-rank 1 and the independence theorem for simple theories.

Case 1: Suppose that $l = d$.

Since $\bar{a}_I$ is algebraically closed we have $\bar{a}_I = \bar{a}$ (and $\bar{x}_I = \bar{x} = (x_1, \dots, x_n)$), so $d < 3$ (because $l = \dim_M(\bar{a}_I) < 3$). Let $\bar{b} = (b_1, \dots, b_n) \in M^n$ realize $p(\bar{x}_I)$. The conditions in (a) imply that $tp_{M{\restriction}\mathcal{L}}(\bar{b}) = tp_{M{\restriction}\mathcal{L}}(\bar{a})$ and, if $J \subset \{1, \dots, n\}$ and $\bar{a}_I \not\subseteq \mathrm{acl}_M(\bar{a}_J)$ (which implies $\dim_M(\bar{a}_J) < d$) then $tp_M(\bar{a}_J) = tp_M(\bar{b}_J)$. Hence (b) is satisfied.

Case 2: Suppose that $l = 0 < d$.

Then $\bar{a}_I = \bar{a}_0 = \mathrm{acl}_M(\emptyset)$. Let $\bar{b}_0$ realize $p(\bar{x}_I)$, which means that $\bar{b}_0 = \bar{a}_0$ or that $\bar{b}_0$ is a reordering of $\bar{a}_0$. Let $\bar{b} = (b_1, \ldots, b_n) = \bar{b}_0\bar{a}_1 \ldots \bar{a}_d$ and observe that $\bar{b}_I = \bar{b}_0$. Then the first two conditions of (b) are trivially satisfied and, since $\bar{a}_I \subseteq \mathrm{acl}_M(\bar{a}_J)$ for every $J \subset \{1, \ldots, n\}$, the last condition of (b) is vacuously fulfilled.

Case 3: Suppose that $l = 1 < d$.

Without loss of generality, we may assume that $\bar{a}_I = \bar{a}_0\bar{a}_1$. Let $\bar{c}$ realize $p(\bar{x}_I)$. By the assumptions on $p(\bar{x}_I)$ in (a), it follows that $\bar{c}$ has the form $\bar{b}_0\bar{b}_1$ where $\bar{b}_0 = \bar{a}_0$ or $\bar{b}_0$ is a reordering of $\bar{a}_0$. Since $tp_M(\bar{b}_0) = tp_M(\bar{a}_0)$ (because of the last condition in (a)), we may assume that $\bar{c} = \bar{a}_0\bar{b}_1$, where $\bar{b}_1 = \mathrm{acl}_M(\bar{c}) - \mathrm{acl}_M(\emptyset)$. As $\mathrm{rng}(\bar{b}_1) \cap \mathrm{acl}_M(\emptyset) = \emptyset$ and $\dim_M(\bar{b}_1) = 1$, we may also assume that $\mathrm{rng}(\bar{b}_1) \cap \mathrm{rng}(\bar{a}_i) = \emptyset$ for $i = 2, \ldots, d$. Let $\bar{b} = (b_1, \ldots, b_n) = \bar{a}_0\bar{b}_1\bar{a}_2 \ldots \bar{a}_d$, so $\bar{b}_I = \bar{a}_0\bar{b}_1$ and hence $tp_M(\bar{b}_I) = p(\bar{x}_I)$. By the choice of $\bar{b}_1$ and the triviality of algebraic closure we get $tp_{M\restriction\mathcal{L}}(\bar{b}) = tp_{M\restriction\mathcal{L}}(\bar{a})$. If $J \subset \{1, \ldots, n\}$ and $\bar{a}_I \not\subseteq \mathrm{acl}_M(\bar{a}_J)$ then $\bar{a}_J$ contains no element from $\bar{a}_1$, so $\mathrm{rng}(\bar{a}_J) \subseteq \mathrm{rng}(\bar{a}_0\bar{a}_2 \ldots \bar{a}_d)$ and hence $\bar{b}_J = \bar{a}_J$ so $tp_M(\bar{b}_J) = tp_M(\bar{a}_J)$. Hence, $\bar{b}$ satisfies (b).

Case 4: Suppose that $l = 2 < d$.

Then $d = 3$ Without loss of generality, we may assume that $\bar{a}_I = \bar{a}_0\bar{a}_1\bar{a}_2$. Let $\bar{c}$ realize $p(\bar{x}_I)$. As in the previous case we may assume that $\bar{c} = \bar{a}_0\bar{b}_1\bar{b}_2$, where $\mathrm{acl}_M(\bar{b}_i) - \mathrm{acl}_M(\emptyset) = \bar{b}_i$ for $i = 1, 2$. By the assumptions on $p(\bar{x}_I)$ in (a), we have $tp_M(\bar{a}_0, \bar{b}_i) = tp_M(\bar{a}_0, \bar{a}_i)$ for $i = 1, 2$. Hence, by the ω-categorcity of M, there are $\bar{b}'_3, \bar{b}''_3 \in M$ such that

$$tp_M(\bar{a}_0, \bar{b}_1, \bar{b}'_3) = tp_M(\bar{a}_0, \bar{a}_1, \bar{a}_3) \quad \text{and} \quad tp_M(\bar{a}_0, \bar{b}_2, \bar{b}''_3) = tp_M(\bar{a}_0, \bar{a}_2, \bar{a}_3).$$

Recall that $\bar{a}_0 = \mathrm{acl}_M(\emptyset)$. By the assumption that, for any $\bar{d} \in M$, $tp_M(\bar{d}/\mathrm{acl}_{M^{\mathrm{eq}}}(\emptyset))$ is determined by $tp_M(\bar{d}/\mathrm{acl}_M(\emptyset))$, it follows that

$$tp_M(\bar{b}'_3/\mathrm{acl}_{M^{\mathrm{eq}}}(\emptyset)) = tp_M(\bar{b}''_3/\mathrm{acl}_{M^{\mathrm{eq}}}(\emptyset)),$$

and consequently, $\bar{b}'_3$ and $\bar{b}''_3$ realize the same strong type over $\bar{a}_0$ ($= \mathrm{acl}_M(\emptyset)$).

Since M has trivial algebraic closure and the SU-rank of M is 1, it follows from the assumptions on $p(\bar{x}_I)$ (in (a)) and the choices of the involved sequences that $\bar{b}_1$ is independent from $\bar{b}_2$ over $\bar{a}_0$, and the types $tp_M(\bar{b}'_3/\bar{a}_0\bar{b}_1)$ and $tp_M(\bar{b}''_3/\bar{a}_0\bar{b}_2)$ do not fork over $\bar{a}_0$. Since M is ω-categorical, Lascar strong types in $Th(M)$ are the same as strong types in $Th(M)$ ([13], Corollary 6.1.11) so the independence theorem for simple theories ([12], Theorem 5.8 or [13], Theorem 2.5.20) implies that there exists $\bar{b}_3 \in M$ such that

$$(*) \quad tp_M(\bar{a}_0, \bar{b}_1, \bar{b}_3) = tp_M(\bar{a}_0, \bar{a}_1, \bar{a}_3) \quad \text{and} \quad tp_M(\bar{a}_0, \bar{b}_2, \bar{b}_3) = tp_M(\bar{a}_0, \bar{a}_2, \bar{a}_3).$$

Let $\bar{b} = (b_1, \ldots, b_n) = \bar{a}_0\bar{b}_1\bar{b}_2\bar{b}_3$. The triviality of acl_M and the choices of $\bar{b}_1, \bar{b}_2, \bar{b}_3$ imply that $tp_{M\restriction\mathcal{L}}(\bar{b}) = tp_{M\restriction\mathcal{L}}(\bar{a})$. Since $\bar{b}_I = \bar{a}_0\bar{b}_1\bar{b}_2$ was chosen to realize $p(\bar{x}_I)$ we have $tp_M(\bar{b}_I) = p(\bar{x}_I)$. If $J \subset \{1, \ldots, n\}$ is such that $\bar{a}_I \not\subseteq \mathrm{acl}_M(\bar{a}_J)$ then $\bar{a}_J \subseteq \bar{a}_0\bar{a}_i\bar{a}_3$ where $i = 1$ or $i = 2$, so the last part of (b) follows from $(*)$.

Now we have proved that M satisfies the 3-independence hypothesis over $\mathcal{L}$. $\square$

Under the assumptions on M (and its language L) we get, by Theorem 2.1, the following:

Conclusion 1.16 *If φ is a sentence in the language of M such that at most 3 distinct variables occur in φ and $M \models \varphi$, then φ has arbitrarily large finite models.*

2 Results

Theorem 2.1 *Let $0 < k < \omega$ and let M be an ω-categorical L-structure such that (M, acl_M) forms a pregeometry. Suppose that there is a sublanguage $\mathcal{L} \subseteq L$ such that $\mathrm{acl}_{M\restriction\mathcal{L}}$ coincides with acl_M, $M\restriction\mathcal{L}$ is polynomially k-saturated and M satisfies the k-independence hypothesis over $\mathcal{L}$. If $\varphi \in L$ is an unnested sentence, in which at most k distinct variables occur, and $M \models \varphi$, then φ has arbitrarily large finite models.*

Proof. Combine Lemma 2.15 and Proposition 2.8. More precisely: Under the assumptions of the theorem we get part (3) of the conclusion of Lemma 2.15, for arbitrary $n_0 < \omega$. This serves as input for Proposition 2.8 which gives the desired conclusion. $\square$

Note that Theorem 2.1 only speaks about arbitrarily large finite models, but does not claim that these are embeddable in M.

Theorem 2.2 *Let M be an ω-categorical L-structure such that (M, acl_M) forms a pregeometry. Suppose that there is a sublanguage $\mathcal{L} \subseteq L$ such that $\mathrm{acl}_{M\restriction\mathcal{L}}$ coincides with acl_M and, for every $0 < k < \omega$, $M\restriction\mathcal{L}$ is polynomially k-saturated and M satisfies the k-independence hypothesis over $\mathcal{L}$. Then M is polynomially k-saturated, for every $0 < k < \omega$, and M has the finite submodel property.*

Proof. Combine Lemma 2.15 and Proposition 2.14. More precisely: The assumptions of the theorem allow us to use Lemma 2.15 for every $k < \omega$. The conclusions of this lemma, for every k, serve as input to Proposition 2.14 which gives the desired conclusions. $\square$

Remark 2.3 If, in Theorem 2.2, we remove the assumptions that there is $\mathcal{L} \subseteq L$ such that $\mathrm{acl}_{M\restriction\mathcal{L}}$ coincides with acl_M and, for every $0 < k < \omega$, $M\restriction\mathcal{L}$ is polynomially k-saturated and M satisfies the k-independence hypothesis over $\mathcal{L}$, then the conclusion fails, even if we assume that M is simple of SU-rank 1; an example showing this is given in section 3.3.

Definition 2.4 Let M and N be structures with the same language.
(i) For any $0 < n < \omega$, and $a_1, \ldots, a_n \in N$, $tp_N^{\mathrm{ua}}(a_1, \ldots, a_n)$ denotes the set of unnested atomic formulas $\varphi(x_1, \ldots, x_n)$, such that $N \models \varphi(a_1, \ldots, a_n)$; we don't insist that all x_i actually occur in φ, so $\varphi(x_1, \ldots, x_n)$ may for example have the form $P(x_1)$ for a unary relation symbol P even if $n > 1$. We call $p(\bar{x})$ an *unnested atomic type of N* if $p(\bar{x}) = tp_N^{\mathrm{ua}}(\bar{a})$ for some $\bar{a} \in N^{|\bar{x}|}$. If the structure N is clear from the context then we may write 'tp^{ua}' instead of 'tp_N^{ua}'.
(ii) N is *atomicly k-compatible with M* if every $\bar{a} \in N^k$ realizes an unnested atomic type of M, or in other words, there is $\bar{b} \in M^k$ such that $tp_N^{\mathrm{ua}}(\bar{a}) = tp_M^{\mathrm{ua}}(\bar{b})$.
(iii) N is *atomicly k-saturated with respect to M* if, whenever $m < k$ and $q(x_1, \ldots, x_m)$ and $p(x_1, \ldots, x_{m+1})$ are unnested atomic types of M, $q \subseteq p$ and $a_1, \ldots, a_m \in N$ realizes

q, then there is $a_{m+1} \in N$ such that $a_1, \ldots, a_m, a_{m+1}$ realizes p.
(iv) If $p(\bar{x}, y)$ is an unnested atomic type of M, $\bar{a} \in M^{|\bar{x}|}$ and there are only finitely many $b \in M$ such that $M \models p(\bar{a}, b)$ then we say that $p(\bar{b}, y)$ is *algebraic*; otherwise we say that $p(\bar{b}, y)$ is *non-algebraic.*

Observe the following:

Lemma 2.5 *Let M be an L-structure and let $\mathcal{L} \subseteq L$ be a sublanguage. Suppose that*

1. *M is ω-categorical,*
2. *(M, acl_M) is a pregeometry*
3. *$\mathrm{acl}_{M \restriction \mathcal{L}}$ coincides with acl_M,*
4. *$M \restriction \mathcal{L}$ is polynomially k-saturated, and*
5. *M satisfies the k-independence hypothesis over $\mathcal{L}$.*

Let M' be the expansion of M which is obtained by adding, for every $n < \omega$ and every $\emptyset$-definable (in M) relation $R \subseteq M^n$, a relation symbol which is interpreted as R. Let L' be the language of M' and let $\mathcal{L}' \subseteq L'$ be the language which we get from $\mathcal{L}$ by adding to it, for every $n < \omega$ and every relation $R \subseteq M^n$ which is $\emptyset$-definable in $M \restriction \mathcal{L}$, a relation symbol from L' which is interpreted as R in M'. Then 1-5 hold with M' and $\mathcal{L}'$ in place of M and $\mathcal{L}$.

Proof. Straightforward consequence of the ω-categoricity of M (using Fact 1.3) and the definitions of the notions involved. $\square$

From now on M is an ω-categorical L-structure such that (M, acl) forms a pregeometry. Moreover, we fix a sublanguage $\mathcal{L} \subseteq L$; we allow the possibilities that $\mathcal{L} = L$ or that the vocabulary of $\mathcal{L}$ contains only '$=$'.

By Lemma 2.5 the following assumption is harmless for our purposes of proving Theorems 2.1 and 2.2:

Assumption 2.6 (a) For every $0 < n < \omega$ and every relation $R \subseteq M^n$ which is $\emptyset$-definable in $M \restriction \mathcal{L}$, $\mathcal{L}$ has a relation symbol which is interpreted as R.
(b) For every $0 < n < \omega$ and every relation $R \subseteq M^n$ which is $\emptyset$-definable in M, L has a relation symbol which is interpreted as R.

Now we define the parts of L in which we will work most of the time.

Definition 2.7 For any $k < \omega$ let

$$\widehat{k} = \max\big\{|\mathrm{acl}_M(a_1, \ldots, a_k)| : a_1, \ldots, a_k \in M\big\}.$$

Observe that for any $\bar{a}, \bar{b} \in M$, if $tp_M(\bar{a}) = tp_M(\bar{b})$ then $\dim_M(\bar{a}) = \dim_M(\bar{b})$. Define inductively, for every $-1 \leq k < \omega$, a sublanguage $L_k \subseteq L$ by:

> $L_{-1} = \mathcal{L}$.
>
> When L_k is defined let $L_{k+1} \supseteq L_k$ be obtained from L_k by adding, for every $0 < n \leq \widehat{k+1}$ and every $p(\bar{x}) \in S_n(Th(M))$ such that $\dim_M(\bar{a}) = k + 1$ if $M \models p(\bar{a})$, one (and only one) relation symbol P from L such that $P^M = \{\bar{a} \in M : M \models p(\bar{a})\}$.

Note that $L_k - \mathcal{L}$ has only finitely many relation symbols and no function or constant symbols.

Proposition 2.8 *Let $0 < k < \omega$. Suppose that there are arbitrarily large finite L_k-structures which are atomicly k-compatible with $M \restriction L_k$ and atomicly k-saturated with respect to $M \restriction L_k$. If φ is an unnested sentence, in which at most k distinct variables occur, and $M \models \varphi$, then φ has arbitrarily large finite models.*

Proof. Suppose that there are arbitrarily large finite L_k-structures which are atomicly k-compatible with M and atomicly k-saturated with respect to M. Let A' be such an L_k-structure. Let $L_\varphi \supseteq L_k$ be obtained from L_k by adding to (the vocabulary of) L_k every symbol occuring in φ which is not already in L_k. For every unnested atomic formula $\psi(\bar{x}) \in L_\varphi$ there are atomic $P_1(\bar{x}), \ldots, P_n(\bar{x}) \in L_k$ such that $M \models \forall\bar{x}\big(\psi(\bar{x}) \leftrightarrow \big(P_1(\bar{x}) \vee \ldots \vee P_n(\bar{x})\big)\big)$. Expand A' to an L_φ-structure A by interpreting each symbol in $L_\varphi - L_k$ in such a way that for every $\psi(\bar{x})$ and P_i, $1 \leq i \leq n$, as above, $A \models \forall\bar{x}\big(\psi(\bar{x}) \leftrightarrow \big(P_1(\bar{x}) \vee \ldots \vee P_n(\bar{x})\big)\big)$. For relation symbols of arity $> k$, their interpretations on sequences containing more than k distinct elements can be made arbitrarily. For function symbols of arity $\geq k$, their interpretations on sequences containing more than $k-1$ distinct elements can be made arbitrarily. Since A' is atomicly k-compatible with $M \restriction L_k$ it follows that A is atomicly k-compatible with $M \restriction L_\varphi$. Also, A is atomicly k-saturated with respect to $M \restriction L_\varphi$ because, in both A and $M \restriction L_\varphi$, the unnested atomic L_φ-type of any l-tuple, $l \leq k$, is determined by its restriction to L_k (by Assumption 2.6).

We will prove that if ψ is an unnested L_φ-sentence in which at most k distinct variables occur, then $M \models \psi$ if and only if $A \models \psi$; clearly the proposition follows from this. It is sufficient to show that for any $\bar{a} \in A$, $\bar{b} \in M$, if $|\bar{a}| = |\bar{b}| \leq k$ and $tp^{\mathrm{ua}}_A(\bar{a}) = tp^{\mathrm{ua}}_{M \restriction L_\varphi}(\bar{b})$ then $\bar{a}$ and $\bar{b}$ satisfy the same unnested L_φ-formulas in which at most k distinct variables occur; then taking $\bar{a} = \bar{b} = ()$ gives the desired conclusion. This we show by induction on the complexity of formulas. We need only consider formulas in which $\forall$ does not occur since '$\forall x$' can be replaced by '$\neg\exists x\neg$'.

The base case concerning unnested atomic L_φ-formulas is trivial. The inductive step involving the connectives is also obvious so we only treat the case involving $\exists$. Let $\exists x\psi(x, \bar{y})$ be an L_φ-formula in which at most k distinct variables occur and x does not occur in $\bar{y}$, so $|\bar{y}| < k$. Suppose that $\bar{a} \in A^{|\bar{y}|}$, $\bar{b} \in M^{|\bar{y}|}$ and $tp^{\mathrm{ua}}_A(\bar{a}) = tp^{\mathrm{ua}}_{M \restriction L_\varphi}(\bar{b}) = q(\bar{y})$.

Suppose that $A \models \psi(a, \bar{a})$ for some $a \in A$. Let $p(x, \bar{y}) = tp^{\mathrm{ua}}_A(a, \bar{a})$. Since A is atomicly k-compatible with $M \restriction L_\varphi$, p is realized in M. By Assumption 2.6, q determines the (complete first-order) type of $\bar{b}$ in M, so there exists $b \in M$ such that $b\bar{b}$ realizes $p(x, \bar{y})$. By the induction hypothesis we get $M \models \psi(b, \bar{b})$.

Now suppose that $M \models \psi(b, \bar{b})$ for some $b \in M$. Let $p(x, \bar{y}) = tp^{\mathrm{ua}}_{M \restriction L_\varphi}(b, \bar{b})$. Since $\bar{a}$ realizes q and A is atomicly k-saturated with respect to $M \restriction L_\varphi$ there exists $a \in A$ such that $a\bar{a}$ realizes $p(x, \bar{y})$. By the induction hypothesis we get $A \models \psi(a, \bar{a})$. $\square$

From now on we assume that acl_M ***coincides with*** $\mathrm{acl}_{M \restriction \mathcal{L}}$***.***

Observation 2.9 Since $\mathcal{L} \subseteq L_r \subseteq L$, $\mathrm{acl}_{M \restriction L_r}$ is the same as acl_M and as $\mathrm{acl}_{M \restriction \mathcal{L}}$, for any $r < \omega$.

By Assumption 2.6, there is, for every $r < \omega$, an $(r+1)$-ary relation symbol P_r in $\mathcal{L}$ which is interpreted in $M \restriction \mathcal{L}$ so that for any $a_1, \ldots, a_r, a_{r+1} \in M$, $M \restriction \mathcal{L} \models P_r(a_1, \ldots, a_r, a_{r+1})$ if and only if $a_{r+1} \in \mathrm{acl}(a_1, \ldots, a_r)$.

Definition 2.10 Suppose that A is a structure (finite or infinite) such that the language of A includes $\mathcal{L}$. Let P_r, $r < \omega$, be the symbols from Observation 2.9. For $B \subseteq A$ define

$$\mathrm{cl}(B) = \big\{a \in A : A \models P_r(\bar{b}, a) \text{ for some } r < \omega \text{ and } \bar{b} \in B^r\big\}.$$

For a sequence $\bar{a} \in A$ define $\mathrm{cl}(\bar{a}) = \mathrm{cl}(\mathrm{rng}(\bar{a}))$. The meaning of '$\mathrm{cl}(\bar{a}) = \bar{a}$' is '$\mathrm{cl}(\bar{a}) = \mathrm{rng}(\bar{a})$'. We say that $B \subseteq A$ (or $\bar{a} \in A$) is *closed* if $\mathrm{cl}(B) = B$ (or $\mathrm{cl}(\bar{a}) = \bar{a}$). For $B \subseteq A$ define

$$\dim^{\mathrm{cl}}(B) = \min\big\{|C| : C \subseteq B \text{ and } B \subseteq \mathrm{cl}(C)\big\}.$$

Observe that if $B \subseteq M$ then $\mathrm{cl}(B) = \mathrm{acl}(B)$ and $\dim(B) = \dim^{\mathrm{cl}}(B)$. Hence 'closed' and 'algebraically closed' mean the same thing in M. The idea of introducing 'cl' is that, when we use it, it will imitate, in a *finite* structure, the behaviour of 'acl' on M.

Definition 2.11 Let $0 \leq r \leq k < \omega$. Suppose that A is an L_r-structure. We say that A is *strongly atomicly k-compatible with $M{\restriction}L_r$* if for any $\bar{a} \in A$ such that $\dim^{\mathrm{cl}}(\bar{a}) \leq k$ there is $\bar{b} \in M$ such that $tp_A^{\mathrm{ua}}(\bar{a}) = tp_{M\restriction L_r}^{\mathrm{ua}}(\bar{b})$.

Clearly, being strongly atomicly k-compatible with $M{\restriction}L_r$ implies being atomicly k-compatible with $M{\restriction}L_r$.

Remark 2.12 In the proof of the next lemma it will be convenient to use the following consequence of M satisfying the n-independence hypothesis over $\mathcal{L}$, under the standing assumptions, such that acl_M coincides with $\mathrm{acl}_{M\restriction\mathcal{L}}$. So assume that M satisfies the n-independence hypothesis over $\mathcal{L}$. Then, by the definition of $L_{-1}, L_0, \ldots, L_k$, the following holds:
If

- $E \subseteq B \subseteq M{\restriction}L_k$ where E and B are closed, $1 \leq \dim_M(E) \leq k < n$, $\dim_M(B) \leq n$, and
- E' is an L_k-structure which is strongly atomicly k-compatible with $M{\restriction}L_k$ and f is an isomorphism from $E'{\restriction}L_0$ to $E{\restriction}L_0$, such that whenever $E'' \subset E'$ is closed and $\dim^{\mathrm{cl}}(E'') < \dim^{\mathrm{cl}}(E')$ then the restriction of f to E'' is an isomorphism from E'' to E', as L_k-structures,

then there exists a substructure $C \subseteq M{\restriction}L_k$ and an isomorphism $g : B{\restriction}\mathcal{L} \to C{\restriction}\mathcal{L}$ such that

- gf is an isomorphism from E' to $gf(E')$, as L_k-structures, and
- whenever $G \subseteq B$ is closed and $E \not\subseteq G$, then the restriction of g to G is an isomorphism from G to $g(G)$, as L_k-structures.

From the assumption that acl_M coincides with $\mathrm{acl}_{M\restriction\mathcal{L}}$ it follows that C must be closed in M. From the last point above above it follows that g is, in fact, an isomorphism from $B{\restriction}L_0$ to $C{\restriction}L_0$.

Lemma 2.13 *Let $k < \omega$. Suppose that A is a finite L_k-structure which is strongly atomicly k-compatible with $M{\restriction}L_k$ and that $A{\restriction}L_0$ is isomorphic to a substructure of $M{\restriction}L_0$ which is algebraically closed. If M satisfies the $(|A|+1)$-independence hypothesis over $\mathcal{L}$ then A is isomorphic to a substructure of $M{\restriction}L_k$.*

Proof. Let $k < \omega$ and suppose that A is a finite L_k-structure which is strongly atomicly k-compatible with $M{\restriction}L_k$. Let $f : A{\restriction}L_0 \to B \subseteq M{\restriction}L_0$ be an isomorphism, where B is algebraically closed (and hence closed). Since $B \subseteq M$ we may also regard B as a substructure of $M{\restriction}L_k$ and hence as an L_k-structure. We will show, by induction on n, that for every $n = 0, 1, \ldots, k$, there are a closed substructure $B_n \subseteq M{\restriction}L_k$ and an isomorphism $f_n : A{\restriction}L_0 \to B_n{\restriction}L_0$ such that,

$(*)_n$ for every closed $E \subseteq A$ with $\dim^{\mathrm{cl}}(E) \leq n$, the restriction of f_n to E is an isomorphism from E to $f_n(E)$, as L_k-structures.

By the definition of $L_{-1}, L_0, \ldots, L_k$, it follows that when a closed substructure $B_k \subseteq M \restriction L_k$ and an isomorphism $f_k : A \restriction L_0 \to B_k \restriction L_0$ has been found such that $(*_n)$ holds with $n = k$, then A is isomorphic to B_k and the lemma is proved.

Step $n = 0$. Take $B_0 = B$, where B and B_0 are now regarded as L_k-structures. Then take $f_0 = f$. Suppose that $E \subseteq A$ is closed with $\dim^{\mathrm{cl}}(E) = 0$. Then $E = \mathrm{cl}(\emptyset)$ (where cl is taken in A). Let $\bar{e}$ enumerate E. Observe that, by the definition of $L_{-1}, L_0, \ldots, L_k$, for any $\bar{a} \in M$ with $\dim_M(\bar{a}) = 0$ we have $tp^{\mathrm{ua}}_{M \restriction L_k}(\bar{a}) = tp^{\mathrm{ua}}_{M \restriction L_0}(\bar{a})$. Since A is strongly atomicly k-compatible with $M \restriction L_k$ we have $tp^{\mathrm{ua}}_{A}(\bar{a}) = tp^{\mathrm{ua}}_{A \restriction L_0}(\bar{a})$ for any $\bar{a} \in A$ such that $\dim^{\mathrm{cl}}(\bar{a}) = 0$. By the assumption that f_0 $(= f)$ is an isomorphism from $A \restriction L_0$ to $B \restriction L_0 = B_0 \restriction L_0$ it follows that $tp^{\mathrm{ua}}_{A}(\bar{e}) = tp^{\mathrm{ua}}_{A \restriction L_0}(\bar{e}) = tp^{\mathrm{ua}}_{M \restriction L_0}(f_0(\bar{e})) = tp^{\mathrm{ua}}_{M \restriction L_k}(f_0(\bar{e}))$. Therefore the restriction of f_0 to E is an isomorphism from E to $f_0(E)$, as L_k-structures.

Step $n + 1$*, where* $0 \leq n < k$. Suppose that we have found a closed substructure $B_n \subseteq M \restriction L_k$ and an isomorphism $f_n : A \restriction L_0 \to B_n \restriction L_0$ such that, for every closed $E \subseteq A$ with $\dim^{\mathrm{cl}}(E) \leq n$, the restriction of f_n to E is an isomorphism from E to $f_n(E)$, as L_k-structures.

Let m be the number of (distinct) closed subsets E of A (recall that A is finite) with $\dim^{\mathrm{cl}}(E) = n + 1$, and let $E_0, \ldots, E_{m-1}$ enumerate all such subsets of A. Inductively we will find, for $i = 0, \ldots, m - 1$, a closed substructure $C_i \subseteq M \restriction L_k$ and an isomorphism $h_i : A \restriction L_0 \to C_i \restriction L_0$ such that

$(a)_i$ for $j \leq i$, the restriction of h_i to E_j is an isomorphism from E_j to $h_j(E_j)$, as L_k-structures, and

$(b)_i$ whenever $G \subseteq A$ is closed and $\dim^{\mathrm{cl}}(G) \leq n$, then the restriction of h_i to G is an isomorphism from G to $h_i(G)$, as L_k-structures.

Clearly, when we have found $C_{m-1} \subseteq M \restriction L_k$ and an isomorphism $h_{m-1} : A \restriction L_0 \to M \restriction L_0$ such that $(a)_i$ and $(b)_i$ hold with $i = m - 1$, then $B_{n+1} = C_{m-1}$ and $f_{n+1} = h_{m-1}$ satisfy $(*)_{n+1}$ (that is, $(*)_n$ above with n replaced by $n + 1$). We first show how to find C_{i+1} and h_{i+1} which satisfy $(a)_{i+1}$ and $(b)_{i+1}$ (that is, $(a)_i$ and $(b)_i$ above with i replaced by $i + 1$), provided that we are given C_i and h_i which satisfy $(a)_i$ and $(b)_i$. Then we explain how to slightly modify the argument to find C_0 and h_0 which satisfy $(a)_i$ and $(b)_i$ for $i = 0$.

> *Induction step.* Suppose that $0 \leq i < m - 1$ and that we have found a closed substructure $C_i \subseteq M \restriction L_k$ and an isomorphism $h_i : A \restriction L_0 \to C_i \restriction L_0$ such that $(a)_i$ and $(b)_i$ hold. Let $F_{i+1} = h_i(E_{i+1})$. Since C_i is closed (in M) and
> $h_i : A \restriction L_0 \to C_i \restriction L_0$ is an isomorphism and acl_M coincides with $\mathrm{acl}_{M \restriction \mathcal{L}}$, it follows that F_{i+1} is closed (in M) and $\dim^{\mathrm{cl}}(F_{i+1}) = n + 1$. Since E_{i+1} is strongly atomicly k-compatible with $M \restriction L_k$ (because A is), it follows from the assumption that M satisfies the $(|A| + 1)$-independence hypothesis over $\mathcal{L}$, applied in the form described in Remark 2.12, that there is a closed substructure $C_{i+1} \subseteq M \restriction L_k$ and an isomorphism $g_{i+1} : C_i \restriction L_0 \to C_{i+1} \restriction L_0$ such that
>
> - the restriction of g_{i+1} to F_{i+1} is an isomorphism from F_{i+1} to $g_{i+1}(F_{i+1})$, as L_k-structures, and

- whenever $G \subseteq C_i$ is closed and $F_{i+1} \not\subseteq G$ then the restriction of g_{i+1} to G is an isomorphism from G to $g_{i+1}(G)$, as L_k-structures.

If we let $h_{i+1} = g_{i+1}h_i$ then h_{i+1} is an isomorphism from $A\restriction L_0$ to $C_0\restriction L_0$ and, since $h_i(F_{i+1}) \not\subseteq h_i(F_j)$ if $j \leq i$, it follows that $(a)_{i+1}$ and $(b)_{i+1}$ (that is, $(a)_i$ and $(b)_i$ above with i replaced by $i+1$) are satisfied.

Base case: $i = 0$. We argue as in the induction step, except that we use B_n and f_n instead of C_i and h_i. In other words, we start by letting $F_0 = f_n(E_0)$. In the same way as in the induction step we find a closed substructure $C_0 \subseteq M\restriction L_k$ and an isomorphism $g_0 : B_n\restriction L_0 \to C_0\restriction L_0$ such that the two points in the induction step hold if we replace $i+1$ by 0, C_i by B_n and C_{i+1} by C_0. Then, letting $h_0 = g_0 f_n$, h_0 is an isomorphism from $A\restriction L_0$ to $C_0\restriction L_0$ and $(a)_i$ and $(b)_i$ are satisfied for $i = 0$.

□

Proposition 2.14 *Suppose that, for every $0 < k < \omega$, M satisfies the k-independence hypothesis over $\mathcal{L}$. Moreover, assume that, for every $0 < k < \omega$, there is a polynomial $Q_k(x)$ such that for any $n_0 < \omega$ there is $n \geq n_0$ and a finite L_k-structure A such that the following conditions are satisfied:*

(1) $n \leq |A| \leq Q_k(n)$.

(2) $A\restriction L_0$ is isomorphic to a substructure of $M\restriction L_0$ which is algebraically closed.

(3) A is strongly atomicly k-compatible with $M\restriction L_k$.

(4) Whenever $\bar{a} \in A$, $\bar{b}, b \in M$, $tp^{\mathrm{ua}}_A(\bar{a}) = tp^{\mathrm{ua}}_{M\restriction L_k}(\bar{b})$, $\dim^{\mathrm{cl}}(\bar{a}) < k$, $p(\bar{x}, y) = tp^{\mathrm{ua}}_{M\restriction L_k}(\bar{b}, b)$ and $p(\bar{b}, y)$ is non-algebraic, then there are distinct $c_1, \ldots, c_n \in A$ such that $A \models p(\bar{a}, c_i)$ for each $1 \leq i \leq n$.

Then M has the finite submodel property and is polynomially k-saturated, for every $0 < k < \omega$.

Proof. By Lemma 1.8 it is sufficient to show that M is polynomially k-saturated for every $0 < k < \omega$. Fix arbitrary $0 < k < \omega$ and let $Q_k(x)$ be as in the proposition and assume that for every n_0 there is $n \geq n_0$ and a finite L_k-structure A for which (1)-(4) hold. By Assumption 2.6 and the definition of L_k it is sufficient to show that every A satisfying (1)-(4) can be embedded into $M\restriction L_k$, but this follows from (2), (3) and Lemma 2.13 because M satisfies the k-independence hypothesis over $\mathcal{L}$ for every $0 < k < \omega$. □

Theorems 2.1 and 2.2 follow from Proposition 2.8, Proposition 2.14 and:

Lemma 2.15 *Let $0 < k < \omega$. Suppose that $\mathrm{acl}_{M\restriction\mathcal{L}}$ coincides with acl_M, $M\restriction\mathcal{L}$ is polynomially k-saturated and that M satisfies the k-independence hypothesis over $\mathcal{L}$. Then there is a polynomial $Q(x)$ and for any $n_0 < \omega$ there is $n \geq n_0$ and a finite L_k-structure A such that:*

(1) $n \leq |A| \leq Q(n)$.

(2) $A\restriction L_0$ is isomorphic to a substructure of $M\restriction L_0$ which is algebraically closed.

(3) A is strongly atomicly k-compatible with $M\restriction L_k$ and atomicly k-saturated with respect to $M\restriction L_k$.

(4) Whenever $\bar{a} \in A$, $\bar{b}, b \in M$, $tp^{\mathrm{ua}}_A(\bar{a}) = tp^{\mathrm{ua}}_{M\restriction L_k}(\bar{b})$, $\dim^{\mathrm{cl}}(\bar{a}) < k$, $p(\bar{x}, y) = tp^{\mathrm{ua}}_{M\restriction L_k}(\bar{b}, b)$ and $p(\bar{b}, y)$ is non-algebraic, then there are distinct $c_1, \ldots, c_n \in A$ such that $A \models p(\bar{a}, c_i)$ for each $1 \le i \le n$.

Proof of Lemma 2.15

Fix** $0 < k < \omega$. **Assume that** $\mathrm{acl}_{M\restriction\mathcal{L}}$ **coincides with** acl_M, $M\restriction\mathcal{L}$ **is polynomially** k**-saturated and that** M **satisfies the** k**-independence hypothesis over** $\mathcal{L}$**. Recall that Assumption 2.6 is in action.

An outline of the proof goes as follows. First, we find a strictly increasing sequence $(n_m : m < \omega)$ of natural numbers, a polynomial $Q(x)$ and substructures A_m of $M\restriction L_0$ so that (1), (2) and (4) of the lemma are satisfied. For this we use the assumption that $M\restriction\mathcal{L}$ is polynomially k-saturated. Then we show by induction on r, where $r \le k$, and a probabilistic argument, that there exists a strictly increasing sequence $(n'_m : m < \omega)$ of natural numbers, a polynomial $Q_r(x)$ and L_r-structures B_m such that B_m is an expansion of A_m, B_m is strongly atomicly k-compatible with $M\restriction L_r$ and (1), (2) and a condition resembling (4) hold with Q_r, B_m and n'_m in place of Q, A and n_m, respectively. When we have this for $r = k$ we put things together to get Lemma 2.15.

The next two lemmas will be used in the proof of Lemma 2.22.

Lemma 2.16 *Let $0 \le r < k$. Suppose that A is an L_{r+1}-structure which is strongly atomicly k-compatible with $M\restriction L_{r+1}$. Suppose that $\bar{a} \in A$ where $\mathrm{cl}(\bar{a}) = \bar{a}$ and $r < \dim^{\mathrm{cl}}(\bar{a}) \le k$. Let $p(\bar{x}) = tp^{\mathrm{ua}}_A(\bar{a})$ and $p'(\bar{x}) = p \cap L_r$. Suppose that $q(\bar{x})$ is an unnested atomic type of $M\restriction L_{r+1}$ such that $p' \subset q$ and let A' be the result of changing (if necessary) the interpretations of symbols in L_{r+1} on $\bar{a}$ so that $A' \models q(\bar{a})$, but not changing the interpretations on any other sequences of elements from A. Then A' is strongly atomicly k-compatible with $M\restriction L_{r+1}$.*

Proof. First we show that:

($*$) For any $\bar{b} \in A'$ such that $\mathrm{cl}(\bar{b}) = \bar{b}$ and $\dim^{\mathrm{cl}}(\bar{b}) \le r+1$ there is $\bar{c} \in M$ such that $tp^{\mathrm{ua}}_{A'}(\bar{b}) = tp^{\mathrm{ua}}_{M\restriction L_{r+1}}(\bar{c})$.

We may assume that $\dim^{\mathrm{cl}}(\bar{b}) = r+1$ because $A\restriction L_r = A'\restriction L_r$ and if $\bar{b} \in A'$ and $\dim^{\mathrm{cl}}(\bar{b}) \le r$ then $tp^{\mathrm{ua}}_{A'}(\bar{b}) = tp^{\mathrm{ua}}_{A'\restriction L_r}(\bar{b})$. If $\bar{b}$ is a subsequence of $\bar{a}$ the conclusion is clear because $A' \models q(\bar{a})$. If $\bar{b} \cap \bar{a} = \emptyset$ then the conclusion also follows directly, because we did not change the structure on any tuple which does not contain elements from $\bar{a}$. So suppose that $\bar{b}$ is not a subsequence of $\bar{a}$ and that $\bar{b} \cap \bar{a} \neq \emptyset$. Then $\dim^{\mathrm{cl}}(\bar{b} \cap \bar{a}) \le r$ so

$$tp^{\mathrm{ua}}_{A'}(\bar{b} \cap \bar{a}) = tp^{\mathrm{ua}}_{A'\restriction L_r}(\bar{b} \cap \bar{a}) = tp^{\mathrm{ua}}_{A\restriction L_r}(\bar{b} \cap \bar{a}) = tp^{\mathrm{ua}}_{A}(\bar{b} \cap \bar{a}).$$

If $\bar{b} \cap \bar{a} \subset \bar{c} \subseteq \bar{b}$ then for any unnested atomic formula $\varphi(\bar{x}) \in L_{r+1}$ $A \models \varphi(\bar{c})$ if and only if $A' \models \varphi(\bar{c})$, by the definition of A'. Therefore $tp^{\mathrm{ua}}_{A'}(\bar{b}) = tp^{\mathrm{ua}}_{A}(\bar{b})$ and since A is strongly atomicly k-compatible with $M\restriction L_{r+1}$ we have proved ($*$).

Let $\bar{c} = (c_1, \ldots, c_n) \in (A')^n$ and suppose that $\dim^{\mathrm{cl}}(\bar{c}) \le k$. We need to show that $tp^{\mathrm{ua}}_{A'}(\bar{c})$ is realized in $M\restriction L_{r+1}$. We may assume that $\bar{c}$ is closed. If $r+1 = k$ then by ($*$), $tp^{\mathrm{ua}}_{A'}(\bar{c})$ is realized in $M\restriction L_{r+1}$. So suppose that $r+1 < k$. Since $A'\restriction L_r = A\restriction L_r$ which is strongly atomicly k-compatible with $M\restriction L_r$ there is $\bar{d} \in M$ such that $tp^{\mathrm{ua}}_{A'\restriction L_r}(\bar{c}) = tp^{\mathrm{ua}}_{M\restriction L_r}(\bar{d})$. We use the notation from Notation 1.11. By ($*$), for every $I \subseteq \{1, \ldots, n\}$ such

that $\bar{c}_I$ is closed and $\dim^{\mathrm{cl}}(\bar{c}_I) = r+1$, there is $\bar{e} \in M$ with $tp^{\mathrm{ua}}_{A'}(\bar{c}_I) = tp^{\mathrm{ua}}_{M\restriction L_{r+1}}(\bar{e})$. Let $I_1, \ldots, I_m$ be subsets of $\{1, \ldots, n\}$ such that $\mathrm{rng}(\bar{c}_{I_1}), \ldots, \mathrm{rng}(\bar{c}_{I_m})$ enumerates all closed $E \subseteq \{c_1, \ldots, c_n\}$ with $\dim^{\mathrm{cl}}(E) = r+1$, without repetitions. We are assuming that $r+1 < k$ so, by repeated uses of the k-independence hypothesis over $\mathcal{L}$ (similarly as in the proof of Lemma 2.13), we find $\bar{e} = (e_1, \ldots, e_n) \in M^n$ such that $tp^{\mathrm{ua}}_{A'\restriction L_r}(\bar{c}) = tp^{\mathrm{ua}}_{M\restriction L_r}(\bar{e})$ and, for every $1 \leq i \leq m$, $tp^{\mathrm{ua}}_{A'}(\bar{c}_{I_i}) = tp^{\mathrm{ua}}_{M\restriction L_{r+1}}(\bar{e}_{I_i})$. This means that $tp^{\mathrm{ua}}_{A'}(\bar{c}) = tp^{\mathrm{ua}}_{M\restriction L_{r+1}}(\bar{e})$.
□

Lemma 2.17 *If $0 \leq r < k$ then any L_r-structure A, which is strongly atomicly k-compatible with $M\restriction L_r$, can be expanded to an L_{r+1}-structure A' which is strongly atomicly k-compatible with $M\restriction L_{r+1}$.*

Proof. Let $0 \leq r < k$ and suppose that A is an L_r-structure which is strongly atomicly k-compatible with $M\restriction L_r$. We get A' from A by performing the following operation to every closed $B \subseteq A$ such that $\dim^{\mathrm{cl}}(B) = r+1$: Order B as $\bar{b}$. By assumption, there exists $\bar{c} \in M$ such that $tp^{\mathrm{ua}}_{M\restriction L_r}(\bar{c}) = tp^{\mathrm{ua}}_{A}(\bar{b})$. In A' we interpret the symbols in $L_{r+1} - L_r$ on B in such a way that $tp^{\mathrm{ua}}_{M\restriction L_{r+1}}(\bar{c}) = tp^{\mathrm{ua}}_{A'}(\bar{b})$.

A' is well-defined because, if $B, C \subseteq A$ are closed and $\dim^{\mathrm{cl}}(B) = \dim^{\mathrm{cl}}(C) = r+1$ and $B \neq C$, then $\dim^{\mathrm{cl}}(B \cap C) \leq r$ and for every $\bar{a} \in M$ and unnested atomic $P(\bar{x}) \in L_{r+1} - L_r$, $M \models P(\bar{a})$ implies $\dim^{\mathrm{cl}}(\bar{a}) = r+1$.

If $r+1 = k$ then it immediately follows that A' is strongly atomicly k-compatible with $M\restriction L_{r+1}$. Suppose that $r+1 < k$. Let $\bar{b} \in A'$ with $\dim^{\mathrm{cl}}(\bar{b}) \leq k$. We need to show that $tp^{\mathrm{ua}}_{A'}(\bar{b})$ is realized in $M\restriction L_{r+1}$. We may assume that $\mathrm{cl}(\bar{b}) = \bar{b}$. Since A is strongly atomicly k-compatible with $M\restriction L_r$ there is $\bar{c} \in M$ such that $tp^{\mathrm{ua}}_{M\restriction L_r}(\bar{c}) = tp^{\mathrm{ua}}_{A}(\bar{b})$. As in the proof of Lemma 2.16, we find, by repeated uses of the k-independence hypothesis over $\mathcal{L}$, $\bar{d} \in M$ such that $tp^{\mathrm{ua}}_{A'}(\bar{b}) = tp^{\mathrm{ua}}_{M\restriction L_{r+1}}(\bar{d})$. □

Definition 2.18 Suppose that A is a structure in a language which includes $\mathcal{L}$ and that $A\restriction\mathcal{L}$ is isomorphic to a substructure of $M\restriction\mathcal{L}$ which is algebraically closed. Let $\bar{a}, \bar{b}, \bar{c} \in A$. We say that $\bar{a}$ *is* cl-*independent from* $\bar{b}$ *over* $\bar{c}$ if for any $a \in \mathrm{rng}(\bar{a})$, $a \in \mathrm{cl}(\bar{b}\bar{c}) \implies a \in \mathrm{cl}(\bar{c})$. By the given assumptions on A and the assumption that (M, acl) (which is the same as (M, cl)) is a pregeometry, $\bar{a}$ is cl-independent from $\bar{b}$ over $\bar{c}$ if and only if $\bar{b}$ is cl-independent from $\bar{a}$ over $\bar{c}$.

We introduced cl-independence because we want to be able to talk about independence ("induced" by acl_M, which is the same as $\mathrm{acl}_{M\restriction\mathcal{L}}$) in a *finite* structure A such that $A\restriction\mathcal{L}$ is embeddable in $M\restriction\mathcal{L}$.

Definition 2.19 (i) If a is a real number then $\lfloor a \rfloor$ denotes the greatest integer $n \leq a$.
(ii) Let $0 \leq r \leq k$. We say that an L_r-structure A is (n,k)-*saturated* if the following holds:
If $k' \leq k$, $p(\bar{x}\bar{y})$ and $q(\bar{x}) = p\restriction\{\bar{x}\}$ are unnested atomic L_r-types of $M\restriction L_r$ such that,

1. whenever $\bar{a}\bar{b} \in M$ realizes $p \cap \mathcal{L}$ then $\mathrm{cl}(\bar{a}) = \bar{a}$, $\mathrm{cl}(\bar{a}\bar{b}) = \bar{a}\bar{b}$ and $k' = \dim^{\mathrm{cl}}(\bar{a}\bar{b}) = \dim^{\mathrm{cl}}(\bar{a}) + 1$, and
2. $A \models q(\bar{c})$,

then there are $\bar{d}_1, \ldots, \bar{d}_n \in A$ such that $A \models p(\bar{c}\bar{d}_i)$ and $\bar{d}_i$ is cl-independent from $\bar{d}_j$ over $\bar{c}$ whenever $i \neq j$.

By induction on r, we will first prove "approximations" to Lemma 2.15.

Lemma 2.20 (Base case) *There exist a polynomial $P(x)$, a sequence $(n_m : m < \omega)$ of natural numbers with $\lim_{m\to\infty} n_m = \infty$ and L_0-structures A_m such that, for every $m < \omega$:*

(a) $n_m \leq |A_m| \leq P(n_m)$.

(b) $A_m{\restriction}L_0$ is isomorphic to a substructure of $M{\restriction}L_0$ which is algebraically closed.

(c) A_m is strongly atomicly k-compatible with $M{\restriction}L_0$.

(d) A_m is (n_m, k)-saturated.

Proof. We are assuming that $M{\restriction}\mathcal{L}$ is polynomially k-saturated, so there exist a polynomial $Q(x)$, a sequence $(l_m : m < \omega)$ with $\lim_{m\to\infty} l_m = \infty$ and finite substructures N_m of $M{\restriction}\mathcal{L}$ such that N_m is algebraically closed and, with $Q(x)$, l_m and N_m in place of $P(x)$, n_m and A_m, (a) holds, and

($*$) whenever $\bar{a} \in N_m$, $\dim_M(\bar{a}) < k$ and $p(x) \in S_1^{M{\restriction}\mathcal{L}}(\bar{a})$ is non-algebraic, then there are distinct $b_1, \ldots, b_{l_m} \in N_m$ such that $M{\restriction}\mathcal{L} \models p(b_i)$ for each i.

Let A_m be the substructure of $M{\restriction}L_0$ with the same universe as N_m (so $A_m{\restriction}\mathcal{L} = N_m$). Then (b) and (c) hold. Without loss of generality, assume that $Q(a) < Q(b)$ if $0 < a < b$. Let $P(x) = Q\big(\widehat{k}(x+1)\big)$ and $n_m = \lfloor l_m/\widehat{k} \rfloor$ (see Definition 2.7 for the meaning of $\widehat{k}$). Then

$$n_m \leq l_m \leq |A_m| \leq Q(l_m) \leq Q\big(\widehat{k}(n_m+1)\big) = P(n_m),$$

so (a) holds.

Now we prove (d). Suppose that $k' \leq k$, $p(\bar{x}\bar{y})$ and $q(\bar{x}) = p{\restriction}\{\bar{x}\}$ are unnested atomic L_0-types of $M{\restriction}L_0$ such that,

(1) whenever $\bar{a}\bar{b} \in M$ realizes $p \cap \mathcal{L}$ then $\mathrm{cl}(\bar{a}) = \bar{a}$, $\mathrm{cl}(\bar{a}\bar{b}) = \bar{a}\bar{b}$ and $k' = \dim^{\mathrm{cl}}(\bar{a}\bar{b}) = \dim^{\mathrm{cl}}(\bar{a}) + 1$,

(2) and $A \models q(\bar{c})$.

From (1) and the definition of L_0 it follows that, whenever $\bar{d}$ is such that $M \models p(\bar{c}\bar{d})$ and $\bar{e}$ is a subsequence of $\bar{c}\bar{d}$ which contains at least one element from $\bar{d}$, then $M \models \neg R(\bar{e})$ for every symbol R which is in L_0 but not in $\mathcal{L}$. Let $p' = p \cap \mathcal{L}$. It follows that if $M \models p'(\bar{c}\bar{d})$ then $M \models p(\bar{c}\bar{d})$, so it suffices to find $\bar{d}_1, \ldots, \bar{d}_{n_m} \in A_m$ such that $M \models p'(\bar{c}\bar{d}_i)$, for $i = 1, \ldots, n_m$, and $\bar{d}_i$ is cl-independent from $\bar{d}_j$ over $\bar{c}$ if $i \neq j$. Let $\bar{y} = (y_1, \ldots, y_t)$. By (1), there is i such that if $p''(\bar{x}y_i) = p'{\restriction}\{\bar{x}y_i\}$ then $p''(\bar{c}, y_i)$ is non-algebraic. Without loss of generality we may assume that if $p''(\bar{x}y_1) = p'{\restriction}\{\bar{x}y_1\}$ then $p''(\bar{c}, y_1)$ is non-algebraic. From now on we assume this. Since $M{\restriction}\mathcal{L}$ has elimination of quantifiers (by Assumption 2.6), the (unique) complete extension of p'' to a type in $S_1^{M{\restriction}\mathcal{L}}(\bar{c})$ is non-algebraic. So, by ($*$), there are distinct $d_1, \ldots, d_{l_m} \in A_m$ (because $A_m{\restriction}\mathcal{L} = N_m$) such that $M{\restriction}\mathcal{L} \models p''(\bar{c}d_i)$ for each i. Since $p''(\bar{c}, y_1)$ is non-algebraic we have $d'_i \notin \mathrm{cl}(\bar{c})$ for each i and since p'' is an unnested atomic type we get $A_m \models p''(\bar{c}d_i)$ for each i. By the definition of $\widehat{k}$ and the choice of n_m, there is a subsequence of (distinct) elements $d'_1, \ldots, d'_{n_m}$ of the sequence $d_1, \ldots, d_{l_m}$ such that d'_i is cl-independent from d'_j over $\bar{c}$ whenever $i \neq j$.

From the assumption that M is ω-categorical it follows (using characterization (2) in Fact 1.3) that $M{\restriction}\mathcal{L}$ is ω-categorical and hence (by characterization (4) in Fact 1.3 and

Assumption 2.6), for every $0 < n < \omega$, every type $r(\bar{z}) \in S_n(Th(M{\restriction}\mathcal{L}))$ is isolated by an unnested atomic formula in $r(\bar{z})$. Recall that $\bar{y} = (y_1, \ldots, y_t)$ and $p''(\bar{x}y_1) = p'{\restriction}\{\bar{x}y_1\}$. Since the complete type of $Th(M{\restriction}\mathcal{L})$ which extends $p''(\bar{x}y_1)$ is isolated by an unnested atomic $\mathcal{L}$-formula in p'' there are $\bar{d}_1, \ldots, \bar{d}_{n_m} \in M$ such that $d'_i \in \mathrm{rng}(\bar{d}_i)$ and $M{\restriction}\mathcal{L} \models p'(\bar{c}\bar{d}_i)$ for $i = 1, \ldots, n_m$. By its definition, A_m is an algebraically closed substructure of $M{\restriction}L_0$, so in particular A_m is algebraically closed in $M{\restriction}\mathcal{L}$ (by the assumption that $\mathrm{acl}_{M\restriction\mathcal{L}}$ coincides with acl_M). As $d'_i \in A_m$ it follows from (1) that $\bar{d}_i \in A_m$ for each i.

Suppose for a contradiction that $\bar{d}_i$ is not cl-independent from $\bar{d}_j$ over $\bar{c}$ for some $i \neq j$. Then there is $d \in \mathrm{rng}(\bar{d}_i)$ such that $d \in \mathrm{cl}(\bar{c}\bar{d}_j) - \mathrm{cl}(\bar{c})$. By (1), $\dim^{\mathrm{cl}}(\bar{c}\bar{d}_i) = \dim^{\mathrm{cl}}(\bar{c}) + 1$ and since $d \notin \mathrm{cl}(\bar{c})$ we must have $\mathrm{rng}(\bar{d}_i) \subseteq \mathrm{cl}(\bar{c}d)$ and hence $\mathrm{rng}(\bar{d}_i) \subseteq \mathrm{cl}(\bar{c}\bar{d}_j)$. We have already noted that $d'_i, d'_j \notin \mathrm{cl}(\bar{c})$ (because $p''(\bar{c}, y_1)$ is non-algebraic) so it follows from (1) that $\mathrm{cl}(\bar{c}d'_j) = \mathrm{cl}(\bar{c}\bar{d}_j)$ and hence $d'_i \in \mathrm{rng}(\bar{d}_i) \subseteq \mathrm{cl}(\bar{c}d'_j)$. But this contradicts that d'_i is cl-independent from d'_j over $\bar{c}$. $\square$

Definition 2.21 For any $0 \leq r < k$ and finite L_r-structure A which is strongly atomicly k-compatible with $M{\restriction}L_r$, let $\mathcal{S}_{r+1}(A)$ be the set of L_{r+1}-structures B such that $B{\restriction}L_r = A$ and B is strongly atomicly k-compatible with $M{\restriction}L_{r+1}$. We consider each $\mathcal{S}_{r+1}(A)$ as a probability space by giving it the uniform probability measure. In other words, for any $X \subseteq \mathcal{S}_{r+1}(A)$ and $B \in \mathcal{S}_{r+1}(A)$, the probability that $B \in X$ is $|X|/|\mathcal{S}_{r+1}(A)|$.

Lemma 2.22 (Induction step) *Let $r < k$. Suppose that there is a polynomial $P(x)$, a sequence $(n_m : m < \omega)$ of natural numbers with $\lim_{m\to\infty} n_m = \infty$ and L_r-structures A_m such that, for every $m < \omega$:*

(a) $n_m \leq |A_m| \leq P(n_m)$.

(b) $A_m{\restriction}L_0$ is isomorphic to a substructrure of $M{\restriction}L_0$ which is algebraically closed.

(c) A_m is strongly atomicly k-compatible with $M{\restriction}L_r$.

(d) A_m is (n_m, k)-saturated.

Then there is a polymial $Q(x)$, a sequence $(n'_m : m < \omega)$ of natural numbers with $\lim_{m\to\infty} n'_m = \infty$ and L_{r+1}-structures B_m such that, for every $m < \omega$, (a), (b), (c) and (d) hold if we replace P, n_m, A_m and r with Q, n'_m, B_m and $r+1$, respectively. Moreover, the probability that $B \in \mathcal{S}_{r+1}(A_m)$ is (n'_m, k)-saturated approaches 1 as m approaches ∞.

Proof. Suppose that $r < k$ and that $P(x)$, n_m and A_m satisfy the assumptions of the lemma. We may, without loss of generality, assume that if a, b are real numbers and $0 < a < b$ then $P(a) < P(b)$. Define $n'_m = \lfloor \sqrt{n_m} \rfloor$ and $Q(x) = P\big((x+1)^2\big)$.

By Lemma 2.17, $\mathcal{S}_{r+1}(A_m) \neq \emptyset$ for every m. Observe that for every m and every $B \in \mathcal{S}_{r+1}(A_m)$ we have

$$n'_m \leq n_m \leq |B| \leq P(n_m) \leq P\big((\lfloor \sqrt{n_m} \rfloor + 1)^2\big) = Q(n'_m).$$

By the definition of $\mathcal{S}_{r+1}(A_m)$, every $B \in \mathcal{S}_{r+1}(A_m)$ is strongly atomicly k-compatible with $M{\restriction}L_{r+1}$ and $B{\restriction}L_0 = A_m{\restriction}L_0$. Hence it is sufficient to prove that the probability that $B \in \mathcal{S}_{r+1}(A_m)$ is (n'_m, k)-saturated approaches 1 as m approaches ∞.

Fix arbitrary m and let $B \in \mathcal{S}_{r+1}(A_m)$. We will calculate the probability that B is *not* (n'_m, k)-saturated with respect to $M{\restriction}L_{r+1}$. Let $k' \leq k$ and suppose that $p(\bar{x}\bar{y})$ and $q(\bar{x}) = p{\restriction}\{\bar{x}\}$ are unnested atomic L_{r+1}-types of $M{\restriction}L_{r+1}$ such that whenever $\bar{a}\bar{b} \in M$

realizes $p \cap \mathcal{L}$ then $\mathrm{cl}(\bar{a}) = \bar{a}$, $\mathrm{cl}(\bar{a}\bar{b}) = \bar{a}\bar{b}$ and $k' = \dim^{\mathrm{cl}}(\bar{a}\bar{b}) = \dim^{\mathrm{cl}}(\bar{a}) + 1$. We may, without loss of generality, assume that $|\bar{x}\bar{y}| \leq \widehat{k}$ (see Definition 2.7 for meaning of $\widehat{k}$).

Suppose that $\bar{c} \in B$ realizes $q(\bar{x})$. Let $q_0 = q \cap L_r$ and $p_0 = p \cap L_r$. Then $B{\restriction}L_r \models q_0(\bar{c})$. Since $B{\restriction}L_r = A_m$ is (n_m, k)-saturated with respect to $M{\restriction}L_r$ there are distinct $\bar{d}_1, \ldots, \bar{d}_{n_m} \in B$ such that $B{\restriction}L_r \models p_0(\bar{c}, \bar{d}_i)$ and $\bar{d}_i$ and $\bar{d}_j$ are cl-independent over $\bar{c}$ if $i \neq j$.

Let Φ be the set of all unnested atomic L_{r+1}-types of $M{\restriction}L_{r+1}$ in some fixed set of $\widehat{k}$ distinct variables. By Lemma 2.16, for any $1 \leq i \leq n_m$ the probability that $B \models p(\bar{c}\bar{d}_i)$ (that is, the probability that $B \in \{C \in \mathcal{S}_{r+1}(A_m) : C \models p(\bar{c}, \bar{d}_i)\}$) is at least $1/|\Phi|$. Fix an arbitrary natural number s such that $0 \leq s < n'_m$ and recall that $n'_m = \lfloor \sqrt{n_m} \rfloor$. Since $\bar{d}_i$ is cl-independent from $\bar{d}_j$ over $\bar{c}$ if $i \neq j$, it follows (by Lemma 2.16 again) that the probability that there is *no* $\bar{d}_i$ such that $sn'_m < i \leq (s+1)n'_m$ and $B \models p(\bar{c}\bar{d}_i)$ is less than or equal to

$$\Big(\frac{|\Phi| - 1}{|\Phi|}\Big)^{n'_m}.$$

Since $|B| \leq Q(n'_m)$ there are at most $k \cdot |\Phi|^2 \cdot Q(n'_m)^{\widehat{k}}$ ways in which we can choose k', q, p and $\bar{c}$ as above. Therefore the probability that there are k', q, p and $\bar{c}$ as above, but no $\bar{d}_i$ such that $sn'_m < i \leq (s+1)n'_m$ and $B \models p(\bar{c}\bar{d}_i)$, is less than or equal to

$$f(m) = k \cdot |\Phi|^2 \cdot Q(n'_m)^{\widehat{k}} \cdot \Big(\frac{|\Phi| - 1}{|\Phi|}\Big)^{n'_m}.$$

Observe that if B is not (n'_m, k)-saturated with respect to $M{\restriction}L_{r+1}$ then there will exist k', q, p, $\bar{c}$, as above, and $\bar{d}_1, \ldots, \bar{d}_{n_m} \in B$, mutually cl-independent over $\bar{c}$, such that

- if $p_0 = p \cap L_r$ then $B{\restriction}L_r \models p_0(\bar{c}\bar{d}_i)$ for each i, but
- for some $0 \leq s < n'_m$, there is *no* i such that $sn'_m < i \leq (s+1)n'_m$ and $B \models p(\bar{c}\bar{d}_i)$.

Hence the probability that B is not (n'_m, k)-saturated with respect to $M{\restriction}L_{r+1}$ is at most $f(m)$. Since $k \cdot |\Phi|^2 \cdot Q(n'_m)^{\widehat{k}}$ is a polynomial in n'_m and $\lim_{m\to\infty} n'_m = \infty$, it follows that $\lim_{m\to\infty} f(m) = 0$. Therefore the probability that $B \in \mathcal{S}_{r+1}(A_m)$ is (n'_m, k)-saturated approaches 1 as $m \to \infty$. $\square$

Now we can prove:

Lemma 2.15 *Let $0 < k < \omega$. Suppose that $\mathrm{acl}_{M{\restriction}\mathcal{L}}$ coincides with acl_M, $M{\restriction}\mathcal{L}$ is polynomially k-saturated and that M satisfies the k-independence hypothesis over $\mathcal{L}$. Then there is a polynomial $Q(x)$ and for any $n_0 < \omega$ there is $n \geq n_0$ and a finite L_k-structure A such that:*

(1) $n \leq |A| \leq Q(n)$.

(2) $A{\restriction}L_0$ is isomorphic to a substructure of $M{\restriction}L_0$ which is algebraically closed.

(3) A is strongly atomicly k-compatible with $M{\restriction}L_k$ and atomicly k-saturated with respect to $M{\restriction}L_k$.

(4) Whenever $\bar{a} \in A$, $\bar{b}, c \in M$, $tp^{\mathrm{ua}}_A(\bar{a}) = tp^{\mathrm{ua}}_{M{\restriction}L_k}(\bar{b})$, $\dim^{\mathrm{cl}}(\bar{a}) < k$, $p(\bar{x}, y) = tp^{\mathrm{ua}}_{M{\restriction}L_k}(\bar{b}, c)$ and $p(\bar{b}, y)$ is non-algebraic, then there are distinct $c_1, \ldots, c_n \in A$ such that $A \models p(\bar{a}, c_i)$ for each $1 \leq i \leq n$.

Proof. By Lemma 2.20, Lemma 2.22 and induction, there are a polynomial $Q(x)$, a sequence $(n_m : m < \omega)$ of natural numbers with $\lim_{m\to\infty} n_m = \infty$ and L_k-structures A_m such that, for every $m < \omega$:

(a) $n_m \le |A_m| \le Q(n_m)$.

(b) $A_m{\restriction}L_0$ is isomorphic to a substructrure of $M{\restriction}L_0$ which is algebraically closed.

(c) A_m is strongly atomicly k-compatible with $M{\restriction}L_k$.

(d) A_m is (n_m, k)-saturated.

It is sufficient to show that (1)-(4) are true for each A_m; and in the case of (4), n is replaced by n_m. We see that (a), (b), (c) correspond to (1), (2) and the first part of (3). The proof of the second part of (3) will use (4), so we prove (4) first.

Suppose that $\bar{a} \in A_m$, $\bar{b}, c \in M$, $tp^{\mathrm{ua}}_A(\bar{a}) = tp^{\mathrm{ua}}_{M\restriction L_k}(\bar{b})$, $\dim^{\mathrm{cl}}(\bar{a}) < k$, $p(\bar{x}, y) = tp^{\mathrm{ua}}_{M\restriction L_k}(\bar{b}, c)$ and $p(\bar{b}, y)$ is non-algebraic. We may assume that $\bar{a}$ and $\bar{b}$ are closed. Let $\bar{b}\bar{c}$ enumerate $\mathrm{cl}(\bar{b}c)$, so we have $c \in \mathrm{rng}(\bar{c})$, and let $p'(\bar{x}, \bar{y}) = tp^{\mathrm{ua}}_{M\restriction L_k}(\bar{b}, \bar{c})$. Since A_m is (n_m, k)-saturated there are $\bar{d}_1, \ldots, \bar{d}_{n_m} \in A_m$ such that $A_m \models p'(\bar{a}\bar{d}_i)$ for every i, and $\bar{d}_i$ is cl-independent from $\bar{d}_j$ over $\bar{a}$ if $i \neq j$. Then, for every i, there is $e_i \in \mathrm{rng}(\bar{d}_i)$ such that $A_m \models p(\bar{a}, e_i)$, and $e_i \neq e_j$ if $i \neq j$. So (4) is proved.

It remains to prove the second part of (3) for A_m; i.e. that A_m is atomicly k-saturated with respect to $M{\restriction}L_k$. Let $l < k$ and let $q(x_1, \ldots, x_l)$ and $p(x_1, \ldots, x_{l+1})$ be unnested atomic types of $M{\restriction}L_k$ such that $q \subseteq p$ and suppose that $\bar{a} = (a_1, \ldots, a_l) \in (A_m)^l$ is such that $A_m \models q(\bar{a})$. By Assumption 2.6, the assumption that M is ω-categorical (using characterization (4) of Fact 1.3) and the definition of L_k, it follows that the unique complete l-type of $Th(M{\restriction}L_k)$ which extends q is isolated by a formula in q, and similarly for p. So whenever $M{\restriction}L_k \models q(\bar{b})$ there is $b \in M$ such that $M{\restriction}L_k \models p(\bar{b}b)$. And if $p(\bar{b}, x_{l+1})$ is algebraic for some $\bar{b} \in M^l$ such that $M{\restriction}L_k \models q(\bar{b})$, then $p(\bar{b}, x_{l+1})$ is algebraic for every $\bar{b} \in M^l$ such that $M{\restriction}L_k \models q(\bar{b})$.

Suppose that $p(\bar{b}, x_{l+1})$ is algebraic for some (and hence every) $\bar{b} \in M^l$ such that $M{\restriction}L_k \models q(\bar{b})$. Then,

(∗) whenever $\bar{b} \in M^l$ and $M{\restriction}L_k \models q(\bar{b})$, there exists $b \in \mathrm{cl}(\bar{b})$ such that $M{\restriction}L_k \models p(\bar{b}b)$.

If *no* $a \in A_m$ exists such that $A_m \models p(\bar{a}a)$ then, letting $\bar{a}' = \mathrm{cl}(\bar{a})$, it follows from (∗) that there exists *no* $\bar{b}' \in M$ such that $tp^{\mathrm{ua}}_{A_m}(\bar{a}') = tp^{\mathrm{ua}}_{M\restriction L_k}(\bar{b}')$, and, as $A_m \models q(\bar{a})$ and $\dim^{\mathrm{cl}}(\bar{a}') < k$, we have a contradiction to (c). Hence, if $p(\bar{b}, x_{l+1})$ is algebraic for some $\bar{b} \in M^l$ such that $M{\restriction}L_k \models q(\bar{b})$, then there exists $a \in A_m$ such that $A_m \models p(\bar{a}a)$.

Now suppose that $p(\bar{b}, x_{l+1})$ is non-algebraic for every $\bar{b} \in M^l$ such that $M{\restriction}L_k \models q(\bar{b})$. Let $\bar{b}$ and b be such that $M{\restriction}L_k \models p(\bar{b}b)$. Then $tp^{\mathrm{ua}}_{A_m}(\bar{a}) = tp^{\mathrm{ua}}_{M\restriction L_k}(\bar{b})$, $\dim^{\mathrm{cl}}(\bar{a}) < k$, $p(x_1, \ldots, x_{l+1}) = tp^{\mathrm{ua}}_{M\restriction L_k}(\bar{b}b)$ and $p(\bar{b}, x_{l+1})$ is non-algebraic, so by (4), there is $a \in A_m$ such that $A \models p(\bar{a}a)$. Now we have proved that A_m is atomicly k-saturated with respect to $M{\restriction}L_k$, so (3) is proved. □

3 Examples

We give examples of structures M which have, or do not have, the properties of being polynomially k-saturated or of having a sublanguage $\mathcal{L}$ such that acl_M and $\mathrm{acl}_{M\restriction\mathcal{L}}$ coincide and M satisfies the k-independence hypothesis over $\mathcal{L}$. The second example in Section 3.3 shows the necessity of the assumptions in Theorem 2.2. All examples will be simple with SU-rank 1, so 'acl' is a closure operator on these structures. We start by looking at examples with trivial algebraic closure. Then we show that all infinite vector spaces, projective spaces and affine spaces over a finite field are polynomially k-saturated for every $k < \omega$. Finally we study two non-Lie coordinatizable structures which have non-trivial algebraic closure operator.

In this section we will frequently use the Fraissé construction of a structure as a so-called Fraissé limit of a class of finite structures. The reader is refered to [9] for definitions and results (in particular Theorems 7.1.2 and 7.4.1 in [9]).

3.1 Simple structures with trivial algebraic closure

We say that a structure M has *trivial* (also called *degenerate*) algebraic closure if for any $A \subseteq M$, $\mathrm{acl}(A) = \bigcup_{a \in A} \mathrm{acl}(a)$. The general case of an ω-categorical simple structure with SU-rank 1 and trivial algebraic closure was treated as the 'introductory example' at the end of Section 1. We now look at a couple of particular cases.

Random structure and random (bipartite) graph: It is well-known that the random structure (in a finite relational language), the random graph (see [9]) and the random bipartite graph (see [11], end of section 4) are ω-categorical and simple with SU-rank 1 and have elimination of quantifiers. This follows from the construction as a Fraissé limit of the particular class of finite structures used in each case. It also follows that each of these examples satisfies the k-independence hypothesis over the language $\mathcal{L}$ with vocabulary $\{=\}$, for every $k < \omega$. Also, if M is any one of these structures then, for any $A \subseteq M$, $\mathrm{acl}_M(A) = A$. Hence acl_M and $\mathrm{acl}_{M \upharpoonright \mathcal{L}}$ coincide, and $M \upharpoonright \mathcal{L}$, which is just an infinite set, is polynomially k-saturated for every $0 < k < \omega$. Theorem 2.2 implies that M is polynomially k-saturated for every $0 < k < \omega$

Random pyramid-free (3)-hypergraph: In contrast to the random graph, this example will not satisfy the 4-independence hypothesis over the language with vocabulary $\{=\}$. Let the vocabulary of L be $\{=, R\}$ where R is a ternary relation symbol. We call an L-structure M a (3)-hypergraph, or just hypergraph, if $(a, b, c) \in R^M$ implies that a, b, c are distinct and that every permutation of (a, b, c) belongs to R^M. A hypergraph M is *pyramid-free* if there are no distinct $a_1, \ldots, a_4 \in M$ such that for any distinct $i, j, k \in \{1, 2, 3, 4\}$, $M \models R(a_i, a_j, a_k)$. Let K be the class of all finite pyramid-free hypergraphs. It is easy to see that K has the hereditary property, the the joint embedding property and the amalgamation property, so the Fraissé limit of K exists and is ω-categorical with elimination of quantifiers. Let M be the Fraissé limit of K. First we show that M is simple of SU-rank 1 and has trivial algebraic closure. That algebraic closure is trivial is a consequence of quantifier elimination and that every member of K is embeddable in M. To show that M is simple of SU-rank 1, it is (as in the case of the random graph) sufficient to show that $tp(\bar{a}/B)$ divides over $A \subseteq B$ if and only if $a \in B - A$ for some $a \in \mathrm{rng}(\bar{a})$. This follows if we can show that for any quantifier free $\varphi(\bar{x}, \bar{y})$ which does not express $x_i = y_j$ for any $x_i \in \mathrm{rng}(\bar{x})$ and $y_j \in \mathrm{rng}(\bar{y})$, any $n < \omega$ and $\bar{b}_i \in M$, $i \leq n$, if $M \models \bigwedge_{i \leq n} \exists \bar{x} \varphi(\bar{x}, \bar{b}_i)$ then $M \models \exists \bar{x}\big(\bigwedge_{i \leq n} \varphi(\bar{x}, \bar{b}_i)\big)$. Since M is the Fraissé limit of K, this is a consequence of the following:

Observation : Suppose that A_0, A_1, A_2 are pyramid-free hypergraphs such that:
(a) For $i = 1, 2$, the substructure of A_i with universe $A_i \cap A_0$ is identical to the substructure of A_0 with universe $A_i \cap A_0$, and
(b) the substructure of A_1 with universe $A_1 - A_0$ is identical to the substructure of A_2 with universe $A_2 - A_0$.
Then the hypergraph B with universe $A_0 \cup A_1$ ($= A_0 \cup A_2$), where $(a, b, c) \in R^B$ if and

only if $(a, b, c) \in R^{A_0}$ or $(a, b, c) \in R^{A_1}$ or $(a, b, c) \in R^{A_2}$, is pyramid-free.

If one assumes that the observation is false then one easily gets a contradiction to the assumption that A_i is pyramid-free for $i = 1, 2, 3$, or to the definition of B.

If we let $\mathcal{L}$ be the language with vocabulary $\{=\}$ then, in contrast to the case of the random graph, one easily checks that since M is pyramid-free it does not satisfy the 4-independence hypothesis over $\mathcal{L}$. It is not known to the author whether any sentence which is true in M must be true in a finite hypergraph or whether M is polynomially k-saturated for $k \geq 4$. This question has a similar taste as the the better known problem [2] whether any sentence which is true in the random (also called 'generic') triangle-free graph, which is the Fraissé limit of the class of all finite triangle-free graphs, must be true in a finite triangle-free graph.

3.2 Vector spaces, projective spaces and affine spaces

A vector space V over a finite field K may be regarded as a first-order structure $M = (V, +, h \in K, 0)$, where V is the universe of M, $+$ is a binary function symbol which is interpreted as vector addition, h is a unary function symbol interpreted as scalar multiplication by h (i.e. $h^M(a) = ha$) for every $h \in K$ and $a \in V$, and the constant symbol 0 is interpreted as the zero vector. It is well-known that an infinite vector space over a finite field is ω-categorical and ω-stable, so in particular it is Lie coordinatizable (see [5] for a definition). Any Lie coordinatizable structure has the finite submodel property [5]. In an infinite vector space over a finite field, linear span coincides with algebraic closure. Moreover, any countable infinite vector space over a finite field is isomorphic to a Fraissé limit of a class of finite structures and therefore any infinite (not necessarily countable) vector space over a finite field has elimination of quantifiers. In this section we show that any infinite vector space, projective space or affine space (defined below) over a finite field is polynomially k-saturated for any $0 < k < \omega$.

Definition 3.1 Suppose that M is an ω-categorical structure such that (M, acl) is a pregeometry. Then $\mathrm{acl}(x) = \mathrm{acl}(y)$ is a $\emptyset$-definable equivalence relation on M. For any $a \in M$ we define $[a]$ to be the equivalence class of a with respect to this equivalence relation. Also define $[M] = \{[a] : a \in M\}$. Observe that $[M]$ is a sort of M^{eq}. We will regard $[M]$ as a structure which, for every n and every $\emptyset$-definable (in M^{eq}) relation $R \subseteq [M]^n$, has a relation symbol which is interpreted as R; the vocabulary of the language of $[M]$ contains no other symbols.

Lemma 3.2 *Suppose that M is an ω-categorical structure such that (M, acl) is a pregeometry. For any $k < \omega$, M is polynomially k-saturated if and only if $[M]$ is polynomially k-saturated.*

Proof. This is a straightforward consequence of the definition of being polynomially k-saturated, because (by ω-categoricity) there is $m < \omega$ such that for any $a \in M$, $|[a]| \leq m$, and for any n and $a_1, \ldots, a_n \in M$, $\dim(a_1, \ldots, a_n) = \dim([a_1], \ldots, [a_n])$. $\square$

Definition 3.3 (i) By a *projective space* over a finite field K we mean a structure of the form $[M]$ where M is a vector space over K.
(ii) An *affine space* M^A over a finite field K is a structure obtained from an infinite vector space M over K and a set A (disjoint from the universe of M) which satisfies:

(a) The universe of M^A is $V \cup A$ where V is the vector space which is the universe of M and the structure on V is that of M (i.e. a vector space over K).
(b) The vector space V, as a group, acts regularly on A; i.e. for any $a, b \in A$, there is a unique $v \in V$ with $va = b$. This action $V \times A \to A$ is represented in M^A by a relation symbol which is interpreted as its graph.
(c) There is no other structure on M^A.
Note that A is a $\emptyset$-definable subset of M^A.

Proposition 3.4 *Any infinite vector space, projective space or affine space over a finite field is polynomially k-saturated for every $0 < k < \omega$.*

The above proposition is a consequence of Lemmas 3.2, 3.5 and 3.7.

Lemma 3.5 *Any infinite vector space over a finite field is polynomially k-saturated for every $0 < k < \omega$.*

Proof. Suppose that $M = (V, +, h \in K, 0)$ where V is infinite and K is a finite field. Fix arbitrary $0 < k < \omega$. We will show that M is polynomially k-saturated. Let $P(x) = |K|^k \cdot (x + 1)$. Let $n_0 < \omega$ be arbitrary.

Choose m such that $|K|^{m-k} > n_0$ and let $n = |K|^{m-k} - 1$. Let $V' \subseteq V$ be a subspace of dimension m and let A be the substructure of M with universe V'. Now we have

$$n = |K|^{m-k} - 1 \leq |K|^m = |A| = |K|^k \cdot |K|^{m-k} = |K|^k \cdot (n + 1) = P(n),$$

so we have verified part (1) of the definition of being polynomially k-saturated. We mentioned in the beginning of this section that algebraic closure in M coincides with linear span, so A is algebraically closed in M. Therefore part (2) of Definition 1.7 holds.

In order to complete the proof we need to show that if $\bar{a} \in A$, $\dim_M(\bar{a}) < k$ (where the model theoretic '$\dim_M$' in this case happens to coincide with the 'dimension' in the usual sense for vector spaces) and $p(\bar{x}, y)$ is a quantifier-free type of M (recall that M has elimination of quantifiers) such that $p(\bar{a}, y)$ is non-algebraic, then there are distinct $b_1, \ldots, b_n \in A$ such that $M \models p(\bar{a}, b_i)$ for $1 \leq i \leq n$. We may assume that $\bar{a}$ is algebraically closed, so $W = \text{rng}(\bar{a})$ is a subspace of V'. Since $p(\bar{a}, y)$ is assumed to be non-algebraic, any realization of $p(\bar{a}, y)$ must be outside of W. As M has elimination of quantifiers, any $b \in V - W$ will realize $p(\bar{b}, y)$, so it is sufficient to find distinct $b_1, \ldots, b_n \in V' - W$. We have $|W| \leq |K|^k$ and $|V'| = |K|^m = |K|^k \cdot |K|^{m-k} = |K|^k \cdot (n+1)$ by the choice of n, so $|V' - W| \geq n$ and we are done. $\square$

Lemma 3.6 *Let M^A be an affine space over a finite field and let V be the vector space of M which acts on A. Let $\bar{v} = (v_1, \ldots, v_n)$ be an enumeration of a subspace of V and let $a \in A$ and $\bar{a} = (v_1 a, \ldots, v_n a)$. Then $tp_{M^A}(\bar{v}\bar{a})$ is determined by $tp_M(\bar{v})$.*

Proof. We may assume that M^A is countable because otherwise we could just consider a countable elementary substructure of M instead.

It is sufficient to show that:

If for $i = 1, 2$,
(1) $\bar{v}_i = (v_i^1, \ldots, v_i^n)$ enumerates a subspace of V,
(2) $a_i \in A$, $\bar{a}_i = (v_i^1 a_i, \ldots, v_i^n a_i)$, and
(3) $tp_M(\bar{v}_1) = tp_M(\bar{v}_2)$,
then there is an automorphism of M^A which maps $\bar{v}_1\bar{a}_1$ onto $\bar{v}_2\bar{a}_2$.

First we show that for any $a_1, a_2 \in A$ there is an automorphism f of M^A which fixes V pointwise and sends a_1 to a_2. Let $a_1, a_2 \in A$. We define f in the following way. Since V acts regularly on A there is a unique $v \in V$ such that $va_1 = a_2$. Let f restricted to V be the identity and, for every $a \in A$, let $f(a) = va$. It follows that $f(a_1) = a_2$. Also, for any $a, a' \in A$ and $w \in V$ we have $wa = a' \iff v(wa) = va' \iff (v+w)a = va' \iff (w+v)a = va' \iff w(va) = va' \iff f(w)f(a) = f(a')$, so f is an automorphism of M^A.

For $i = 1, 2$ let $\bar{v}_i, a_i$, and $\bar{a}_i$ satisfy (1)-(3) above. An affine space over a finite field is an ω-categorical structure (by [5], Lemma 2.3.19, for instance) so M^A is ω-categorical and hence ω-homogeneous (see [9], for example). Since M^A is assumed to be countable, a standard back and forth argument gives an automorphism g of M^A which maps $\bar{v}_1$ to $\bar{v}_2$. As shown above, there is an automorphism f of M^A which fixes V pointwise and maps $g(a_1)$ to a_2. Then fg maps $\bar{v}_1$ to $\bar{v}_2$ and a_1 to a_2; it follows that fg maps $\bar{v}_1\bar{a}_1$ to $\bar{v}_2\bar{a}_2$. □

Corollary 3.7 *If M^A is an affine space over a finite field then M^A is polynomially k-saturated for every $0 < k < \omega$.*

Proof. Let M^A be an affine space over a finite field. Then M is an infinite vector space (over the same field) which, by Lemma 3.5, is polynomially k-saturated, where $0 < k < \omega$ is arbitrary. Suppose that $B \subset M$ is an algebraically closed finite substructure such that

(∗) for any $\bar{b} \in B$ with $\dim_M(\bar{b}) < k$ and any non-algebraic $p(x) \in S_1^M(\bar{b})$ there are distinct $c_1, \ldots, c_n$ such that $M \models p(c_i)$, for $i = 1, \ldots, n$.

Let a be an element of A and let $B^A = B \cup \{va : v \text{ is a vector in } B\}$. By Lemma 3.6, (∗) holds with B^A and M^A in place of B and M. Since $|B^A| \leq 2|B|$, it follows that M^A is polynomially k-saturated. □

Problem 3.8 Infinite vector spaces over a finite field are special cases of the structures called 'linear geometries' in [5]. Is it the case that if M is any linear geometry in the sense of [5], then M is polynomially k-saturated for every $0 < k < \omega$? If the answer is 'yes' then, by the definition of being Lie coordinatizable, modifications of results here and Corollary 2.5 in [3] (which corrects Lemma 2.4.8 in [5]), it follows that any Lie coordinatizable structure with SU-rank 1 is polynomially k-saturated for every $0 < k < \omega$. The problem that the author could not overcome was dealing (successfully) with the quadratic forms that are present in other linear geometries than pure vector spaces; such quadratic forms posed a problem since they may be non-trivial but trivial on some (perhaps large) subspaces.

3.3 Non-Lie coordinatizable structures with non-trivial algebraic closure

The random bipartite graph is not smoothly approximable (which is explained in [11], end of section 4) and hence not Lie coordinatizable [5] but has trivial algebraic closure as mentioned in Section 3.1. We now give two examples of ω-categorical simple structures with SU-rank 1 which are not Lie coordinatizable and have non-trivial algebraic closure. In the case of the first, "well-behaved" example M, there is a sublanguage $\mathcal{L}$ of the language of M, such that $\mathrm{acl}_{M\restriction\mathcal{L}}$ and acl_M coincide, $M\restriction\mathcal{L}$ is polynomially k-saturated and M satisfies the k-independence hypothesis over $\mathcal{L}$, for every $k < \omega$; by Theorem 2.2, M has the finite submodel property and is polynomially k-saturated for every $k < \omega$. In the case of the second, "badly behaved" example, which does not have the finite submodel property, there is no such sublanguage $\mathcal{L}$.

A "well-behaved" example: Let K be the class of all finite structures
$N = (V, P, E, +, f_0, f_1, 0)$ such that:
1. V, the universe of N, is a vector space over the field $F = \{0, 1\}$.
2. P is a unary relation.
3. E is a binary relation symbol interpreted as an irreflexive and symmetric relation.
4. $+$ is a binary function symbol interpreted as vector addition and the constant symbol 0 is interpreted as the zero vector.
5. $f_i(v) = i \cdot v$, for $i = 0, 1$ and any $v \in V$ (so f_i represents scalar multiplication).
6. $N \models \forall xy\big(\ E(x,y) \to \big[\ E(y,x) \wedge\ \big[\ \big(P(x) \wedge \neg P(y)\big)\ \vee\ \big(\neg P(x) \wedge P(y)\big)\ \big]\ \big]\ \big)$.
7. $N \models P(0)$.

It is easy to verify that K is nonempty and has the hereditary property, the joint embedding property and the amalgamation property and is uniformly locally finite. Hence the Fraissé limit of K, which we call M, exists and is ω-categorical and has elimination of quantifiers. Since the reduct of M to the language with vocabulary $\{=, P, E\}$ is the random bipartite graph, M is not Lie coordinatizable (by [5], Theorem 7, or see the example at the end of Section 4 in [11]).

Being a Fraissé limit, M has the property that for any $\bar{a} \in M$, $tp(\bar{a})$ is determined by the isomorphism type of the finite substructure of M that $\bar{a}$ generates, that is, by the subspace spanned by $\bar{a}$. It follows that for $A \subseteq M$ and $a \in M$, $a \in \mathrm{acl}_M(A)$ if and only if a belongs to the subspace spanned by A. Also, for any $\bar{a}$ and $A \subseteq B$ taken from any model of $Th(M)$, $tp(\bar{a}/B)$ divides over A if and only if there is $a \in \mathrm{rng}(\bar{a})$ such that a belongs to the subspace spanned by B but not to the subspace spanned by A. It follows that M is simple and has SU-rank 1.

Let $\mathcal{L} \subseteq L$ be the sublanguage which contains all symbols of L except P and E. Then $M{\restriction}\mathcal{L}$ is a vector space over a finite field so it is a linear geometry ([5], Definition 2.1.4) and by Lemma 3.5, $M{\restriction}\mathcal{L}$ is polynomially k-saturated, for every $k < \omega$. Since acl_M is linear span, acl_M and $\mathrm{acl}_{M{\restriction}\mathcal{L}}$ coincide. From the facts that M has elimination of quantifiers an every member of K is embeddable in M (since M is the Fraissé limit of K) it follows that M satisfies the k-independence hypothesis over $\mathcal{L}$, for every $k < \omega$. By Theorem 2.2, M has the finite submodel property and is polynomially k-saturated for every $k < \omega$.

A "badly behaved" example: This example was first given in [10] which is not published, but also occurs as Example 6.2.27 in [13]. It is obtained by an amalgamation construction with a predimension. We will not repeat all the details of the construction or the proofs, but only collect the facts which will be of use here. Let the language L contain only a ternary relation symbol R (and $=$). For any L-structure M let $R(M) = \big\{(a,b,c) : a,b,c \in M,\ M \models R(a,b,c)\big\}$. Let K be the class of all finite L-structures A such that $A \models \forall xyz\big(R(x,y,z) \to (x \neq y \wedge x \neq z \wedge y \neq z)\big)$. We consider $\emptyset$ as a structure so $\emptyset \in K$. For any $A \in K$ let $\delta(A) = |A| - |R(A)|$ and for any substructure $A \subseteq B \in K$ define $A \leq B$ if and only if $\delta(C) > \delta(A)$ whenever $A \subset C \subseteq B$. Let $f : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ be defined by

$$f(x) = \begin{cases} x & \text{if } 0 \leq x < 1, \\ \log_3(x) + 1 & \text{if } 1 \leq x. \end{cases}$$

Let

$$K_f = \big\{A \in K : \delta(B) \geq f(|B|) \text{ for any } B \subseteq A\big\}.$$

The argument in Example 6.2.27 in [13] now tells us that there exists a so-called generic model for K_f, which we denote by M_f, which is ω-categorical and simple; with the given set-up we can also get the same result by applying Theorems 3.5 and 3.6 in [7]. From parts of the construction and proof which we don't give here, it follows that:
1. Every finite substructure of M_f belongs to K_f and every $A \in K_f$ is isomorphic to a substructure of M_f.
2. For any finite $A \subseteq M_f$, A is algebraically closed if and only if whenever $A \subset B \subseteq M_f$ and B is finite then $\delta(B) > \delta(A)$.
3. For every $\bar{a} \in M_f$, $tp(\bar{a})$ is determined by the isomorphism type of $\mathrm{acl}(\bar{a})$.
Hence, all elements of M_f have the same type and $\mathrm{acl}(\emptyset) = \emptyset$ and $\mathrm{acl}(a) = \{a\}$ for any $a \in M_f$. It also follows that given any two distinct $a, b \in M_f$, exactly one of the following two cases holds:
(i) $\mathrm{acl}(a, b) = \{a, b\}$, that is, there is no third element c such that some permutation of (a, b, c) belongs to $R(M_f)$.
(ii) For some $c \in M_f$, $\mathrm{acl}(a, b) = \{a, b, c\}$, in which case some permutation of (a, b, c) belongs to $R(M_f)$.

The fact that for any $a \in M_f$ and algebraically closed $A \subseteq M_f$, $\delta(\mathrm{acl}(\{a\} \cup A)) - \delta(A)$ is either 0 or 1 (because either $a \in A$ in which case we get 0, or $a \notin A$ in which case the assumption that A is closed implies that there can be no $b \in A$ and $c \in M_f$ such that some permutation of (a, b, c) belongs to $R(M)$, so we get 1) implies that M_f has SU-rank 1.

Now we show that M_f does not have the finite submodel property, which implies that M_f is not Lie coordinatizable. Let φ be the sentence

$$\exists x(x = x) \ \wedge \ \forall x \exists yz\big(y \neq z \ \wedge \ \exists u R(x, y, u) \ \wedge \ \exists u R(x, z, u)\big).$$

Then $M_f \models \varphi$. If A is finite and $A \models \varphi$ then $|R(A)| \geq 2|A|$ so $\delta(A) = |A| - |R(A)| \leq |A| - 2|A| < 0$ and therefore A can not be a substructure of M_f (or of any model of $Th(M_f)$). For a more general statement concerning the finite submodel property and structures obtained by amalgamation constructions with predimension see the last section of [6].

Hence, for any $\mathcal{L} \subseteq L$, either acl_{M_f} does not coincide with $\mathrm{acl}_{M_f \restriction \mathcal{L}}$ or there must exist a k such that one of the other premises of Theorem 2.2 fails for this k. We give a direct argument which shows this, or more precisely, we claim that:
For no $\mathcal{L} \subseteq L$ is it the case that the following three conditions are satisfied:

- acl_{M_f} and $\mathrm{acl}_{M_f \restriction \mathcal{L}}$ coincides,
- $M_f \restriction \mathcal{L}$ is polynomially 2-saturated,
- M_f satisfies the 4-independence hypothesis over $\mathcal{L}$.

There are two cases to consider; the first when the vocabulary of $\mathcal{L}$ contains only $=$, the second when $\mathcal{L} = L$. Suppose that the vocabulary of $\mathcal{L}$ contains only $=$, so the structure $M_f \restriction \mathcal{L}$ is just an infinite set, which has trivial algebraic closure. M_f does not have trivial algebraic closure so the first point fails. Also, the third point fails and it might be instructive to see why.

By calculation, the structure A with universe $\{a_1, \ldots, a_6\}$ where

$$R(A) = \{(a_1, a_2, a_4), (a_2, a_3, a_5), (a_3, a_1, a_6)\}$$

belongs to K_f. By 1, we may assume $A \subset M_f$. By 2, A is algebraically closed and, by (i) and (ii), $\dim(A) = 3$. By 3, the formula $R(x_4, x_5, x_6)$ isolates a complete type

$p(x_4, x_5, x_6)$. Our conclusions so far together with (i), (ii) imply that $\{a_4, a_5, a_6\}$ is algebraically closed and has dimension 3. Trivially, we also have $p \cap \mathcal{L} = tp_{M_f \restriction \mathcal{L}}(a_4, a_5, a_6)$. If M_f satisfies the 4-independence hypothesis over $\mathcal{L}$, then there are $b_1, \ldots, b_6 \in M_f$ such that

$$\begin{aligned} tp(a_1, a_2, a_4) &= tp(b_1, b_2, b_4) \\ tp(a_2, a_3, a_5) &= tp(b_2, b_3, b_5) \\ tp(a_3, a_1, a_6) &= tp(b_3, b_1, b_6) \\ p(x_4, x_5, x_6) &= tp(b_4, b_5, b_6). \end{aligned}$$

Letting B be the substructure with universe $\{b_1, \ldots, b_6\}$ we get $\delta(B) = 6 - 4 = 2 < \log_3(6) + 1 = f(|B|)$ so $B \notin K_f$ which contradicts 1.

Now suppose that $\mathcal{L} = L$, so we have $M_f \restriction \mathcal{L} = M_f$. Let $p(x, y) = tp(a, b)$ where $a \neq b$ and $M_f \models \exists x R(a, b, x)$. Note that for every $a' \in M_f$, $p(a', y)$ is a non-algebraic type. Suppose for a contradiction that M_f is polynomially 2-saturated. Then there exists a finite substructure $A \subseteq M_f$ which is algebraically closed and for any $a' \in A$ there are distinct $b_1, b_2 \in A$ such that $M_f \models p(a', b_i)$ for $i = 1, 2$. Since A is algebraically closed, (i) and (ii) imply that $A \models \varphi$, where φ is the sentence previously defined, which is a contradiction.

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Dept. of Mathematics, Uppsala University, Box 480, 75106 Uppsala, Sweden.
E-mail address: marko@math.uu.se