Stability theory in finite variable logic

Marko Djordjević

Department of Mathematics
Uppsala University
UPPSALA 2000

Dissertation for the Degree of Doctor of Philosophy in Mathematical Logic
presented at Uppsala University in 2000

ABSTRACT

Djordjević, M., 2000: Stability theory in finite variable logic. *Uppsala Dissertations in Mathematics 16.* 60 pp. Uppsala. ISBN 91-506-1417-7

This thesis studies finite variable theories. To be more precise, complete L^n-theories, where L^n is the set of formulas in a first order language L in which at most n distinct variables occur. These need not be complete in the usual first order sense. We use ideas from infinite model theory, in particular stability theory, to define a class of complete L^n-theories which, as we show, has a tractable model theory, also with respect to finite models.

The three main properties of such theories that we consider are (1) a finite bound on the number of L^n-types, (2) an amalgamation property and (3) stability. We prove that any complete L^n-theory with an infinite model and with properties (1),(2) and (3) has an infinite model M which is ω-categorical and ω-stable from which it follows that it has arbitrarily large finite models. In fact, M almost admits elimination of quantifiers, in the sense that there exists an expansion of M by finitely many new n-ary relation symbols which admits elimination of quantifiers. This together with the stability of M allows us to obtain finer information about complete L^n-theories with properties (1)-(3).

We show that there exists a recursive function $f : \omega^2 \to \omega$ such that every theory T as above has a finite model of size at most $f(n, |S_n^n(T)|)$, where $S_n^n(T)$ is the set of L^n-types of T in n free variables.

Then we derive some results about forking in stable structures where there exists $n < \omega$ such that any type (with any number of free variables) over $\emptyset$ is determined by its subtypes with at most n free variables. We use this to give a different proof of a result due to Lachlan, saying that in a stable structure which almost admits elimination of quantifiers every strictly minimal set is indiscernible.

Finally, using the theory of stable structures which admit elimination of quantifiers, we show how to construct new (finite and infinite) models of L^n-theories T with an infinite model and properties (1)-(3). Moreover, every sufficiently saturated model of T which is L^n-elementarily embeddable in a stable structure which almost admits elimination of quantifiers can be constructed in this way and the amount of saturation that is needed can be effectively computed from $|S_n^n(T)|$.

Marko Djordjević, Department of Mathematics, Uppsala University,
Box 480, SE-751 06 Uppsala, Sweden; `marko@math.uu.se`

ISSN 1401-2049
ISBN 91-506-1417-7

Printed in Sweden by Reprocentralen, Ekonomikum, Uppsala 2000
Distributor: Department of Mathematics, Box 480, SE-751 06 Uppsala, Sweden

To all consumers!

Acknowledgements

I would like to thank my thesis advisor Viggo Stoltenberg-Hansen for his assistance and for supporting my endeavors into model theory. For this purpose I spent the academic year 1997/98 at University of Illinois at Chicago, where John T. Baldwin helped me to get started with my research. Ever since, I have benefitted much from the correspondence and meetings with him. Discussions with other people have also been rewarding, in particular with members and former members of the mathematical logic groups of University of Illinois at Chicago, University of Helsinki and Uppsala University.

Contents

Introduction

In model theory, when studying structures, also called models, some formal language often plays a role. Most commonly, this language is first-order predicate logic from which we get first-order theories, in the sequel usually just called theories. The attention in several branches of model theory, such as stability theory, has been on complete theories of infinite structures. If a complete theory has a finite model then all models of the theory are isomorphic. So, if we want to study theories that might have finite models then it is more interesting to study incomplete theories. Since this means that complete first-order theories say "too much" about finite structures, the following comment may be appropriate. In the branch of finite model theory called 'descriptive complexity theory' it is often said that first-order logic is too weak. This weakness refers to the inability of *one* first-order sentence to define classes of finite structures which are of interest with regard to questions in computational complexity theory. However, a complete first-order theory is usually not equivalent to one first-order sentence.

So, if we want to study theories that might have finite models then it is more interesting to study (possibly) incomplete theories. Our choice is to study theories in finite variable logic, that is, the fragment of first-order logic where we have a finite bound on the number of distinct variables that may occur in a formula. If L is a first-order language then the set of formulas in which at most n distinct variables occur is denoted by L^n, and formulas in L^n are called L^n-formulas. The choice to study L^n-theories, i.e. sets of sentences from L^n, where we fix some $n < \omega$, is not arbitrary. L^n has some attractive features. One is that two finite structures may satisfy the same L^n-formulas without being isomorphic (so L^n is not too strong). Another is that there is a nice game theoretic characterization of when any two L-structures satisfy exactly the same L^n-sentences. This characterization guarantees that some uniformly describable structural similarities must exist between two models which satisfy the same L^n-sentences. L^n also has certain closure properties: If only variables among $v_1, \ldots, v_n$ occur in the formulas φ and ψ and $\circ$ is a connective and Q a quantifier then $\neg\varphi$, $\varphi \circ \psi$ and $Qv_1\varphi$ are L^n-formulas.

Our approach in investigating L^n-theories is to see if methods and notions from infinite model theory, in particular stability theory, can be useful in understanding such theories. The general idea is that L^n-theories satisfying nice properties from infinite model theory, such as stability, adapted to L^n-theories, should have well behaved models, and in particular, well behaved finite models. Earlier work in this direction, which uses concepts from infinite model theory such as types and indiscernible sequences, but not stability, includes [9], [10], [14].

It is well known that some basic results from infinite model theory, such as the compactness theorem, fail if we replace all occurrences of 'model' by 'finite model'. We will not, however, restrict our attention to finite models, but instead move between the finite and infinite as we find suitable. In the typical situation we will assume that the theory under consideration has an infinite model.

We say that T is a complete L^n-theory if T is an L^n-theory such that for every L^n-sentence φ, $T \vdash \varphi$ or $T \vdash \neg\varphi$, (so every consistent L^n-theory can be extended to a consistent complete L^n-theory). The present work started by considering the following two questions:

> When does a complete L^n-theory with an infinite model have arbitrarily large finite models?
>
> Can we, by using methods from stability theory (in addition to already tested techniques), define a class of complete L^n-theories with infinite models, and prove that its members have well behaved models?

One idea was that theories in a class which positively answers the second question should have arbitrarily large finite models. A precise answer to the first question is not known, and I assume that this is a rather difficult problem since we might have to deal with many odd examples which are not amenable to a uniform treatment. Somewhat paradoxically, finding such odd examples is not trivial, but merely knowing that some exist (one appears in Section 5 and a family of others are constructed in [14]) makes it hard for me to hope for an easy solution which precisely answers the first question.

Having said this, the first chapter of this dissertation (which will appear in almost identical form in the Journal of Symbolic Logic) will address the second question, and at the end we get some information about the first. We will consider notions such as types, amalgamation properties and the order property and derive some results about them. From these results the "nice" properties, allowing classical model theoretic treatment, will emerge. Then the main results of Chapter I, about complete L^n-theories having these nice properties, are proved. We also give a couple of examples of what can happen if these properties fail.

The goal of the second chapter is to get finer information about the "well behaved" theories from the first chapter. This concerns finding recursive bounds on the smallest finite model of the theory, in terms of its number of L^n-types, and understanding the structure of its models better. Much of the analysis uses the fact that the theories that we study have a model which is infinite, stable and has an expansion by finitely many new relation symbols that admits elimination of quantifiers. Therefore the second chapter begins by generalizing some of the notions and results from the first chapter so that the class of "well behaved" complete L^n-theories that we study can also be characterized as the class of complete L^n-theories that have a model as above.

We will assume that the reader has a working knowledge of elementary model theory and stability theory. All model theoretic results and notions used in Chapter I, which are not explained, are to be found in [16] or [5] in one form or another. In Chapter II acquaintance with stability theoretic notions such as forking and ranks is assumed. The required knowledge about these can be found in any of the standard works on stability theory, such as [1] or [33].

CHAPTER I

Finite variable logic, stability and finite models

We will study complete L^n-theories and their models, where L^n is the set of first order formulas in which at most n distinct variables occur. Here, by a complete L^n-theory we mean a theory such that for every L^n-sentence, it or its negation is implied by the theory. Hence, a complete L^n-theory need not necessarily be complete in the usual sense. Our approach is to transfer concepts and methods from stability theory, such as the order property and counting types, to the context of L^n-theories. So, in one sense, we will develop some rudimentary stability theory for a particular class of (possibly) incomplete theories. To make the 'stability theoretic' arguments work, we need to assume that models, of the complete L^n-theory T which we consider, can be amalgamated in certain ways. If this condition is satisfied and T has infinite models then there will exist models of T which are sufficiently saturated with respect to L^n. This allows us to use some counting types arguments from stability theory. If, moreover, we impose some finiteness conditions on the number of L^n-types and the length of L^n-definable orders then a sufficiently saturated model of T will be ω-categorical and ω-stable. Using the theory of ω-categorical and ω-stable structures we derive that T has arbitrarily large finite models.

A different approach to combining stability theory with finite model theory is made by Hyttinen in [19] and [20].

In Section 2 we will study two amalgamation properties for complete L^n-theories with infinite models. We will see that if T has the weaker of these properties, the (L^n, ∞)-amalgamation property, then, assuming some restrictions on the vocabulary of L, there are models of T which are arbitrarily L^n-saturated (in a sense to be made precise). We will also see that in such a model, the type of any finite tuple in the model is determined by its L^n-fragment. In Section 3, we show, using well known results from stability theory, that stability and ω-stability in L^n are equivalent if T has the (L^n, ∞)-amalgamation property and the set of L^n-types in n free variables, $S_n^n(T)$, is finite. The main results are obtained in Section 4. There we use the results from the previous sections to derive that, under the given restrictions on the vocabulary, if T has the (L^n, ∞)-amalgamation property, $S_n^n(T)$ is finite and T is stable in L^n then T can be extended to a theory which is ω-categorical and ω-stable. By combining this with known results such as the finite axiomatizability of an L^n-theory with only finitely many L^n-types and the theorem of Cherlin, Harrington and Lachlan [6] saying that a sentence which is true in an ω-categorical and ω-stable structure M is true in a finite substructure of M, we will prove T has arbitrarily large finite models. In the last section we will give an example showing that for $n = 4$ this result fails if we only assume that $S^n_n(T)$ is finite and T is stable in L^n.

1 Definitions and preliminaries

We fix an infinite (countable) set of variables $V = \{v_1, v_2, v_3, \ldots\}$ (where $v_i \neq v_j$ if $i \neq j$). x, y, z, (sometimes with subscripts) will range over the variables in V and $\bar{x}$, $\bar{y}$, $\bar{z}$ (sometimes with subscripts) will denote finite sequences of variables. L is the set of first-order formulas over a given vocabulary (also called signature) where only variables from V are used. We will have no general restriction on the vocabulary except that it is countable and we will always assume that the identity symbol $=$ is included (and is interpreted as the identity relation in all structures that we consider). Let $\Gamma \subseteq V$ be a set of variables. $L^\Gamma \subseteq L$ is the set of formulas in which only variables from Γ occur. For $n < \omega$ we define

$$L^n = \bigcup_{\Gamma \subset \mathrm{V},\, |\Gamma| = n} L^\Gamma$$

so in other words L^n is the set of all formulas in L in which at most n different variables occur. An L^n*-theory* is a set of sentences from L^n. An L^n-theory T is called a *complete* L^n*-theory* if for every sentence $\varphi \in L^n$, $T \vdash \varphi$ or $T \vdash \neg\varphi$. If $\bar{x}$ is a sequence of variables (of any finite length) then $F^n_{\bar{x}}$ ($F_{\bar{x}}$) is the set of L^n-formulas (L-formulas) in which only variables from $\bar{x}$ (but not necessarily all) occur free. We define $F^n_m = F^n_{(v_1,\ldots,v_m)}$ and $F_m = F_{(v_1,\ldots,v_m)}$. If a formula is denoted by $\varphi(\bar{x})$ then we mean that only variables from $\bar{x}$ (but not necessarily all) occur free in that formula.

Let T be any consistent L-theory (i.e. any consistent set of L-sentences). We say that $p(\bar{x}) \subseteq F^n_{\bar{x}}$ is an $(L^n, \bar{x})$*-type* of T if $T \cup p(\bar{x})$ is consistent and for every $\varphi(\bar{x}) \in F^n_{\bar{x}}$, $\varphi(\bar{x}) \in p(\bar{x})$ or $\neg\varphi(\bar{x}) \in p(\bar{x})$. The set of all $(L^n, \bar{x})$-types of T is denoted by $S^n_{\bar{x}}(T)$, and we define $S^n_m(T) = S^n_{(v_1,\ldots,v_m)}(T)$ and $S^n(T) = \bigcup_{0<m<\omega} S^n_m(T)$. $(L, \bar{x})$*-types* are defined in the same way but with L^n and $F^n_{\bar{x}}$ replaced by L and $F_{\bar{x}}$ respectively. The set of all $(L, \bar{x})$-types of T is denoted by $S_{\bar{x}}(T)$, and we define $S_m(T) = S_{(v_1,\ldots,v_m)}(T)$ and $S(T) = \bigcup_{0<m<\omega} S_m(T)$. In what follows, whenever T is an L-theory (so in particular, if T is an L^n-theory) we will assume that T is consistent.

Elements of structures will be denoted a, b, c, etc. (sometimes with subscripts). Finite sequences of elements are denoted by $\bar{a}$, $\bar{b}$, $\bar{c}$, etc. (sometimes with subscripts). If we write $\bar{a} \in A$ then we usually mean that the elements of the sequence $\bar{a}$ belong to A (or, with different terminology, that the range of $\bar{a}$ is included in A) and not that the sequence itself belongs to A. If σ is a sequence then $|\sigma|$ is the length of σ and if A is a set then $|A|$ is the cardinality of A.

Remark 1.1 The definition of L^n given here is not standard. Usually L^n is defined to be $L^{\{v_1,\ldots,v_n\}}$, but this difference is not critical for the results about L^n-theories that will follow. To be more precise, define an $L^{\{v_1,\ldots,v_n\}}$-theory to be a consistent set of sentences from $L^{\{v_1,\ldots,v_n\}}$, and let us say that an $L^{\{v_1,\ldots,v_n\}}$-theory is complete if for every sentence $\varphi \in L^{\{v_1,\ldots,v_n\}}$, $T \vdash \varphi$ or $T \vdash \neg\varphi$. Then all results in this paper still remain true if we would replace 'L^n-theory' by '$L^{\{v_1,\ldots,v_n\}}$-theory' and replace 'complete L^n-theory' by 'complete $L^{\{v_1,\ldots,v_n\}}$-theory'. The reason is, of course, that if we are given an L^n-theory T, then for every sentence in T we can replace some variables in it (if necessary) and obtain an equivalent sentence which belongs to $L^{\{v_1,\ldots,v_n\}}$.

The following lemma will be used later.

Lemma 1.2 *If T is a complete L^n-theory and $T' \vdash T$ then $S^n_n(T) = S^n_n(T')$.*

Proof. It is clear that $S^n_n(T') \subseteq S^n_n(T)$ because if $p(v_1,\ldots,v_n) \subseteq F^n_n$ is consistent with T' then $p(v_1,\ldots,v_n)$ is consistent with T.

Now suppose that $p(v_1,\ldots v_n) \in S^n_n(T) - S^n_n(T')$. Then $T' \cup p$ is inconsistent so there are a sentence $\varphi \in L$ and $\psi(v_1,\ldots,v_n) \in p$ (because p is, up to equivalence, closed under conjunctions) such that $T' \vdash \varphi$ and $\varphi \wedge \psi(v_1,\ldots,v_n)$ is inconsistent. Hence

$$\vdash \varphi \rightarrow \neg\exists v_1,\ldots v_n \psi(v_1,\ldots,v_n) \tag{$*$}$$

Since T is a complete L^n-theory and $\exists v_1,\ldots v_n \psi(v_1,\ldots,v_n) \in L^n$ is consistent with T (because $\psi \in p$) we have

$$T \vdash \exists v_1,\ldots v_n \psi(v_1,\ldots,v_n)$$

which together with $(*)$ and $T' \vdash T \cup \{\varphi\}$ contradicts that T' is consistent. □

Among other things, we are interested in obtaining finite models of certain complete L^n-theories. The size of $S^n_n(T)$, where T is a complete L^n-theory, plays a role here. We state some known results concerning the size of $S^n_n(T)$.

Fact 1.3 *For any L-theory T, $S^n_n(T)$ is finite if and only if F^n_n is finite up to equivalence modulo T.*

Proof. This is a consequence of the Stone duality theorem for boolean algebras (see [16]). A direct proof is also simple. □

The next fact can (for example) be extracted from the the proof of a similar result by Dawar, Lindell and Weinstein in [10] (essentially, one constructs a Scott-sentence for a model of T and if $S^n_n(T)$ is finite this sentence is in L^n). It is also mentioned by Poizat in [29] (in exercise 4). The definition of quantifier rank is found in [16].

Fact 1.4 *Suppose that the vocabulary of L is finite and contains no function symbols. If T is a complete L^n-theory and $S^n_n(T)$ is finite then there is $\varphi \in L^n$ that axiomatizes T (i.e. $\varphi \vdash T$ and $T \vdash \varphi$). Moreover, we can choose φ so that its quantifier rank is at most $|S^n_n(T)| + n$.*

The following fact is one reason why we will (most of the time) restrict our attention to L^n-theories T for which $S^n_n(T)$ is finite.

Fact 1.5 *If T is a complete L^n-theory and $S_n^n(T)$ is infinite then T has no finite models.*

Proof. Assume that T has a finite model M. Let $k = |M|$. We will show that $|S_n^n(T)| \leq k^n$. Let $\bar{a}_i$, $1 \leq i \leq k^n$ be an enumeration of M^n, and let

$$p_i(\bar{v}) = \{\varphi(\bar{v}) \in F_n^n : \ M \models \varphi(\bar{a}_i)\}$$

where $\bar{v} = v_1, \ldots, v_n$. Clearly $p_i(\bar{v}) \in S_n^n(T)$. We show that

$$S_n^n(T) = \{p_1, \ldots, p_{k^n}\}.$$

If not, there is $p(\bar{v}) \in S_n^n(T)$ such that $p \neq p_i$ for all $1 \leq i \leq k^n$. Hence, for every $1 \leq i \leq k^n$ there is $\chi_i(\bar{v}) \in p$ such that $\neg\chi_i(\bar{v}) \in p_i$. Then we have $M \models \forall\bar{v} \bigvee_{i=1}^{k^n} \neg\chi_i(\bar{v})$ and since $\forall\bar{v} \bigvee_{i=1}^{k^n} \neg\chi_i(\bar{v})$ is equivalent to an L^n-sentence and T is complete we get $T \vdash \forall\bar{v} \bigvee_{i=1}^{k^n} \neg\chi_i(\bar{v})$. But then $p(\bar{v}) \cup T$ is not consistent which contradicts that $p(\bar{v}) \in S_n^n(T)$. $\square$

Remark 1.6 One may now ask if finiteness of $S_n^n(T)$ for an L^n-theory T is enough to guarantee the existence of finite models of T. The answer is no, and as a counter example we can let T be the L^3-theory of dense linear order without endpoints. Then $S_3^3(T)$ is finite (by quantifier elimination or ω-categoricity) but T has no finite models.

Let $\mathbf{L}$ be either L or L^n. Let A be a subset of a structure M. Then $\mathbf{L}(A)$ is the set of all formulas of the form $\varphi(\bar{x}, \bar{a})$ where $\varphi(\bar{x}, \bar{y}) \in \mathbf{L}$ and $\bar{a} \in A$. Observe that at most n distinct parameters from A can occur in a formula in $L^n(A)$ (and of course at most n distinct variables can occur free in an $L^n(A)$-formula). However, we may still denote a formula in $L^n(A)$ by $\varphi(\bar{x}, \bar{a})$, say, where $\bar{a}$ is a sequence of more then n distinct parameters and $\bar{x}$ is a sequence of more that n distinct variables, but then some of the parameters in $\bar{a}$ and some of the variables in $\bar{x}$ will not actually occur in the formula denoted by $\varphi(\bar{x}, \bar{a})$. For any finite sequence of variables $\bar{x}$, $F_{\bar{x}}^n(A)$ is the set of all $\varphi(\bar{x}, \bar{a}) \in L^n(A)$ such that only variables from $\bar{x}$ (but not necessarily all) occur free in $\varphi(\bar{x}, \bar{a})$. $F_m^n(A) = F_{(v_1,\ldots,v_m)}^n(A)$ (where $v_1, \ldots, v_m$ are the first m variables in $V = \{v_1, v_2, v_3, \ldots\}$). $F_{\bar{x}}(A)$ is defined as $F_{\bar{x}}^n(A)$ but with L^n replaced by L. If f is a function and $\bar{a} = (a_1, \ldots, a_k)$ where $a_1, \ldots, a_k$ belong to the domain of f, then (as usual) $(f(a_1), \ldots, f(a_n))$ is denoted by $f(\bar{a})$, and if $p \subseteq L(A)$ then $\{\varphi(\bar{x}, f(\bar{a})) : \ \varphi(\bar{x}, \bar{a}) \in p\}$ is denoted by $f(p)$.

Example 1.7 We give an example to illustrate some of the definitions. Suppose that the vocabulary of L contains a binary relation symbol R. Let $\varphi(v_1, v_5)$ be the formula

$$\exists v_8(\mathrm{R}(v_1, v_8) \wedge \exists v_4(\mathrm{R}(v_8, v_4) \wedge \exists v_8(\mathrm{R}(v_4, v_8) \wedge \mathrm{R}(v_8, v_5)\ )))$$

(So if R is interpreted as the edge relation in a graph then $\varphi(v_1, v_5)$ says that there is a path of length 4 between v_1 and v_5.)
Let $\varphi(v_5, v_2)$ be the formula which is obtained by simultaneously replacing in

$\varphi(v_1, v_5)$, v_1 by v_5 and v_5 by v_2, and let $\varphi(v_3, v_9)$ be obtained by simultaneously replacing, in $\varphi(v_1, v_5)$, v_1 by v_3 and v_5 by v_9. Then

$$\begin{aligned}
&\varphi(v_1, v_5),\ \varphi(v_5, v_2),\ \varphi(v_3, v_9) \in L^4,\\
&\varphi(v_1, v_5), \varphi(v_5, v_2) \in F^4_{(v_1,v_2,v_5,v_9)},\\
&\varphi(v_3, v_9) \notin F^4_{(v_1,v_2,v_5,v_9)}.
\end{aligned}$$

If $a, b \in A$, $c \notin A$, then

$$\begin{aligned}
&\varphi(v_1, a),\ \varphi(a, v_2),\ \varphi(v_3, b) \in L^4(A),\\
&\varphi(v_1, a), \varphi(a, v_2), \varphi(v_3, b) \in F^4_{(v_1,v_2,v_3)}(A),\\
&\varphi(v_3, b), \varphi(v_1, c) \notin F^4_{(v_1,v_2)}(A).
\end{aligned}$$

We define (where $\mathbf{L}$ is either L or L^n and M an L-structure)

$$Th_{\mathbf{L}}(M, A) = Th_{\mathbf{L}(A)}(M) = \{\varphi(\bar{a}) \in \mathbf{L}(A) :\ M \models \varphi(\bar{a})\}$$

and $Th_{\mathbf{L}}(M) = Th_{\mathbf{L}}((M, \emptyset))$. If $\bar{a} = (a_1, \ldots, a_m)$ and $a_1, \ldots, a_m \in M$ then $Th_{\mathbf{L}}(M, \bar{a}) = Th_{\mathbf{L}}(M, \{a_1, \ldots, a_m\})$. We sometimes, in particular in Chapter II, write $Th(M, \bar{a})$ (which is the usual notation) instead of $Th_L(M, \bar{a})$.

If M is a substructure of N and $Th_{\mathbf{L}(M)}(M) = Th_{\mathbf{L}(M)}(N)$, then we write $M \preccurlyeq_{\mathbf{L}} N$ and we say that M is an $\mathbf{L}$-*elementary substructure* (or $\mathbf{L}$-*elementary submodel*) of N and that N is an $\mathbf{L}$-*elementary extension* of M. If $a_i \in M$, $b_i \in N$ for $i < \lambda$ then we write

$$(M, (a_i : i < \lambda)) \equiv_{\mathbf{L}} (N, (b_i : i < \lambda))$$

if for every $m < \omega$ and $\varphi(x_1, \ldots x_m) \in \mathbf{L}$ and $\{i_1, \ldots, i_m\} \subseteq \lambda$,

$$M \models \varphi(a_{i_1}, \ldots, a_{i_m}) \text{ if and only if } N \models \varphi(b_{i_1}, \ldots, b_{i_m}).$$

If $A \subseteq M$ and $A \subseteq N$ and $Th_{\mathbf{L}(A)}(M) = Th_{\mathbf{L}(A)}(N)$ then we write

$$(M, A) \equiv_{\mathbf{L}} (N, A) \quad \text{or} \quad M \equiv_{\mathbf{L}(A)} N.$$

Note that $\preccurlyeq_L$ has the same meaning as $\preccurlyeq$ and $\equiv_L$ has the same meaning as $\equiv$ (where $\preccurlyeq$ and $\equiv$ have their usual meanings).

Let M be a structure and let $A \subseteq M$. We say that $p(\bar{x}) \subseteq F^n_{\bar{x}}(A)$ is an $(L^n, \bar{x})$-*type over* A with respect to M if $p(\bar{x}) \cup Th_{L^n}(M, A)$ is consistent and for every $\varphi(\bar{x}) \in F^n_{\bar{x}}(A)$ either $\varphi(\bar{x}) \in p$ or $\neg\varphi(\bar{x}) \in p$. $S^n_{\bar{x}}(A, M)$ is the set of all $(L^n, \bar{x})$-types over A with respect to M. $S^n_m(A, M) = S^n_{(v_1,\ldots,v_m)}(A, M)$ and $S^n(A, M) = \bigcup_{0<m<\omega} S^n_m(A, M)$.

We say that $p(\bar{x}) \subseteq F_{\bar{x}}(A)$ is an $(L, \bar{x})$-*type over* A with respect to M if $p(\bar{x}) \cup Th_L(M, A)$ is consistent and for every $\varphi(\bar{x}) \in F_{\bar{x}}(A)$ either $\varphi(\bar{x}) \in p$ or $\neg\varphi(\bar{x}) \in p$. $S_{\bar{x}}(A, M)$ is the set of all $(L, \bar{x})$-types over A with respect to M. $S_m(A, M) = S_{(v_1,\ldots,v_m)}(A, M)$ and $S(A, M) = \bigcup_{0<m<\omega} S_m(A, M)$.

Note that if $M \models T$, where T is a complete L^n-theory, then $S^n_{\bar{x}}(\emptyset, M) = S^n_{\bar{x}}(T)$. Also, if T is a complete L-theory (in the usual first-order sense) and $M \models T$ then $S_{\bar{x}}(\emptyset, M) = S_{\bar{x}}(T)$.

Lemma 1.8 *Let T be any L-theory. If $S_n^n(T)$ is finite then for every finite (nonempty) sequence of variables $\bar{x}$, $S_{\bar{x}}^n(T)$ is finite and $F_{\bar{x}}^n$ is finite up to equivalence modulo T.*

Proof. If $S_n^n(T)$ is finite then by Fact 1.3 F_n^n is finite up to equivalence modulo T, and then it is not hard to see that for any finite sequence of variables $\bar{x}$, $F_{\bar{x}}^n$ is finite up to equivalence modulo T, and hence $S_{\bar{x}}^n(T)$ is finite. $\square$

If $A \subseteq M$ where M is a structure and $\bar{b} \in M$ and $|\bar{b}| = m$, then we define

$$tp_L(\bar{b}/A) = tp(\bar{b}/A) = \{\varphi(v_1, \ldots, v_m) \in L(A) : \ M \models \varphi(\bar{b})\}$$

and

$$tp_{L^n}(\bar{b}/A) = \{\varphi(v_1, \ldots, v_m) \in L^n(A) : \ M \models \varphi(\bar{b})\}.$$

If $\bar{a} = (a_1, \ldots, a_m)$ and $\mathbf{L}$ is L or L^n then by $tp_{\mathbf{L}}(\bar{b}/\bar{a})$ we mean

$$tp_{\mathbf{L}}(\bar{b}/\{a_1, \ldots, a_m\})$$

and $tp_{\mathbf{L}}(\bar{b})$ means $tp_{\mathbf{L}}(\bar{b}/\emptyset)$. Observe that these definitions depend on the structure M but M does not appear in the notations $tp_L(\bar{b}/A)$ and $tp_{L^n}(\bar{b}/A)$. However, when these notations are used it should be clear from the context which M we have in mind.

We will often assume that the vocabulary of the language L in question does not contain function symbols. This is not such a big restriction as we now explain. Suppose that L is a language such that for some finite k the arity of every relation and function symbol is at most k. Let L_R be a language with the same vocabualary as L except that, for every $i \leq k$, every i-ary function symbol f (in the vocabulary of L) is replaced by an $(i+1)$-ary relation symbol R_f. Then, by the proof of Theorem 1.10 in [15], for every L^n-theory T, there exists an $(L_R)^{n+2k+1}$-theory T_R such that: For every L-structure M, if M_R is the L_R-structure with the same universe as M and in which all constant and relation symbols in $L - L_R$ are interpreted as in M and where, for every function symbol f in the vocabulary of L, R_f is interpreted as the graph of the interpretation of f in M (so for $\bar{a}, b \in M$, $M \models \mathrm{f}(\bar{a}) = b \Leftrightarrow M_R \models \mathrm{R}_\mathrm{f}(\bar{a}, b)$), then $M \models T$ if and only if $M_R \models T_R$. In special cases there might be $m < n+2k+1$ such that $M \models T$ if and only if $M_R \models T_R \cap L^m$.

2 Amalgamation and consequences

In this section T will be a complete L^n-theory which has infinite models.

A function $f : A \to N$, where M and N are L-structures and $A \subseteq M$, is called an *L^n-elementary embedding* if for every $\varphi(\bar{x}) \in L^n$ and $\bar{a} \in A$ with $|\bar{a}| = |\bar{x}|$, we have

$$M \models \varphi(\bar{a}) \quad \text{if and only if} \quad N \models \varphi(f(\bar{a})).$$

We say that T has the *L^n-amalgamation property* if the following holds:
If M_1 and M_2 are models of T and $a_\alpha \in M_1$, $b_\alpha \in M_2$, for $\alpha < \kappa$, and

$$(M_1, (a_\alpha : \alpha < \kappa)) \equiv_{L^n} (M_2, (b_\alpha : \alpha < \kappa))$$

then there are $N \succcurlyeq_{L^n} M_1$ and an L^n-elementary embedding $f : M_2 \to N$ such that $f(b_\alpha) = a_\alpha$ for all $\alpha < \kappa$. If the above condition holds for all infinite $M_1, M_2 \models T$ then we say that T has the *(L^n, ∞)-amalgamation property.*

Let κ be a cardinal. We say that an L-structure M is *(L^n, κ)-saturated* if M is infinite and for every $A \subseteq M$ with $|A| < \kappa$ and every $p \in S^n(A, M)$, p is realized in M. If M is infinite and for every $A \subseteq M$ with $|A| < \kappa$ and every $p \in S^n(A, M)$ which is realized in an infinite model, p is realized in M, then we say that M is *(L^n, κ, ∞)-saturated.*

We say that an L-structure M is *strongly (L^n, κ)-universal* if the following holds:
Whenever N is an L-structure such that $|N| < \kappa$ and $a_\alpha \in M,\ b_\alpha \in N$ for $\alpha < \lambda$ where $\lambda < \kappa$ and

$$(M, (a_\alpha : \alpha < \lambda)) \equiv_{L^n} (N, (b_\alpha : \alpha < \lambda))$$

then there is an L^n-elementary embedding $f : N \to M$ such that $f(b_\alpha) = a_\alpha$ for every $\alpha < \lambda$. If κ is infinite and the above condition holds for all infinite N with $|N| < \kappa$ then we say that M is *strongly (L^n, κ, ∞)-universal.*

Note that since we can take $\lambda = 0$ it follows that if M is strongly κ-universal, then $N \equiv_{L^n} M$ and $|N| < \kappa$ implies that there is an L^n-elementary embedding $f : N \to M$. Also observe that if $\kappa \leq \lambda$ and M is (L^n, λ)-saturated (strongly (L^n, λ)-universal) then M is (L^n, κ)-saturated (strongly (L^n, κ)-universal). Similar observations apply for (L^n, κ, ∞)-saturation and strong (L^n, κ, ∞)-universality.

If M is (L^n, κ)-saturated and $S_n^n(Th_{L^n}(M))$ is finite then M is κ-saturated (in the usual first-order sense; see [5],[16] or [33] for definitions). This is proved in Proposition 2.12. The author does not know if this holds without the assumption that $S_n^n(Th_{L^n}(M))$ is finite. The next example shows that an L-structure which is κ-saturated need not necessarily be (L^n, κ)-saturated.

Example 2.1 Let $n \geq 3$ and let κ be an arbitrary infinite cardinal. Let the vocabulary of L consist of a binary relation symbol R (and the equality symbol $=$). Let M_1 be an L-structure in which R is interpreted as an equivalence relation and assume that:

1. For every $0 < k < \omega$, M_1 has exactly one equivalence class with exactly k elements.
2. M_1 has κ equivalence classes with κ elements.
3. There are no other equivalence classes in M_1.

Then M_1 is κ-saturated (in the usual first-order sense) but not (L^n, κ)-saturated. To see the latter let A be the equivalence class with exactly n elements and consider the set of formulas

$$\Gamma = \{\mathrm{R}(x, a) :\ a \in A\} \cup \{\neg x = a :\ a \in A\}.$$

Γ is consistent with $Th_{L^n}(M_1, A)$ (because if we add a new element to A then the resulting structure is a model of $Th_{L^n}(M_1, A)$) and can therefore be extended to an L^n-type $p(x) \in S^n_x(A, M)$. But $p(x)$ is not realized in M_1 so M_1 is not (L^n, κ)-saturated, and neither is it (L^n, κ, ∞)-saturated, since p is realized in M_2 which we now define. Let M_2 be the L-structure (in which R is still interpreted as an equivalence relation) which is defined like M_1 except that we replace ω by n in clause 1. So M_2 has exactly one equivalence class with exactly k elements, for every $0 < k < n$, and M_2 has κ equivalence classes with κ elements and there are no other classes. We may suppose that A is a subset of one of the infinite classes. Then $(M_2, A) \equiv_{L^n} (M_1, A)$ and M_2 is (L^n, κ)-saturated as the reader may verify.

The next proposition is proved in a similar way as the theorem saying that for every (consistent) L-theory with infinite models and every cardinal κ there is a model of that theory which is κ-saturated and κ-universal (see [5] for example).

Proposition 2.2 *The following are equivalent* :
(i) *T has the L^n-amalgamation property.*
(ii) *For every cardinal κ and $M \models T$, there exists $N \succcurlyeq_{L^n} M$ such that N is (L^n, κ)-saturated.*
(iii) *For every cardinal κ and $M \models T$, there exists $N \succcurlyeq_{L^n} M$ such that N is strongly (L^n, κ)-universal.*

Before proving Proposition 2.2 we state and prove an elementary lemma which will be used later.

Lemma 2.3 *Suppose that $\bar{a} \in M$, $\bar{b} \in N$ and $\psi(\bar{x}, \bar{y}) \in L$ where $|\bar{a}| = |\bar{b}| = |\bar{x}|$. If $tp_{L^n}(\bar{a}) = tp_{L^n}(\bar{b})$ and $\psi(\bar{a}, \bar{y})$ is consistent with $Th_{L^n}(M, \bar{a})$ then $\psi(\bar{b}, \bar{y})$ is consistent with $Th_{L^n}(N, \bar{b})$.*

Proof. Suppose that $tp_{L^n}(\bar{a}) = tp_{L^n}(\bar{b})$ and $\psi(\bar{a}, \bar{y})$ is consistent with $Th_{L^n}(M, \bar{a})$ but $\psi(\bar{b}, \bar{y})$ is inconsistent with $Th_{L^n}(N, \bar{b})$. Then by compactness there are

$$\varphi_1(\bar{x}), \ldots, \varphi_k(\bar{x}) \in L^n$$

such that

$$\varphi_1(\bar{b}), \ldots, \varphi_k(\bar{b}) \in Th_{L^n}(N, \bar{b}) \text{ and } \vdash \varphi_1(\bar{b}) \wedge \ldots \wedge \varphi_k(\bar{b}) \to \forall \bar{y} \neg \psi(\bar{b}, \bar{y}).$$

But then (since no $\bar{b}$'s occur in $\psi(\bar{x}, \bar{y})$ or in $\varphi_i(\bar{x})$)

$$\vdash \forall \bar{x} [\varphi_1(\bar{x}) \wedge \ldots \wedge \varphi_k(\bar{x}) \to \forall \bar{y} \neg \psi(\bar{x}, \bar{y})],$$

and since $tp_{L^n}(\bar{a}) = tp_{L^n}(\bar{b})$, for any $M' \models Th_{L^n}(M, \bar{a})$, we have

$$M' \models \varphi_1(\bar{a}) \wedge \ldots \wedge \varphi_k(\bar{a}) \quad \text{and hence} \quad M' \models \forall \bar{y} \neg \psi(\bar{a}, \bar{y}),$$

which contradicts our assumption that $\psi(\bar{a}, \bar{y})$ is consistent with $Th_{L^n}(M, \bar{a})$. $\square$

As a part of the proof of Proposition 2.2 we first prove the following:

Claim 2.4 *If T has the L^n-amalgamation property and M is any model of T then there is $N \succcurlyeq_{L^n} M$ such that for every $A \subseteq M$ and $p(\bar{x}) \in S^n(A, M)$, $p(\bar{x})$ is realized in N.*

Proof. For every $A \subseteq M$ and $p(\bar{x}) \in S^n(A, M)$ there is $N_p \models Th_{L^n(A)}(M)$ and $\bar{a}_p \in N_p$ such that $A \subseteq N_p \models p(\bar{a}_p)$. Let

$$T^* = T \cup \bigcup_{\substack{A \subseteq M \\ p \in S^n(A,M)}} p(\bar{\mathbf{c}}_p)$$

where $\bar{\mathbf{c}}_p$ is a tuple of *distinct new* constant symbols such that if $p \neq p'$ then every constant in $\bar{\mathbf{c}}_p$ is different from every constant in $\bar{\mathbf{c}}_{p'}$. Let $\Delta \subseteq T^*$ be finite. Then

$$\Delta \subseteq T \cup p_1(\bar{\mathbf{c}}_{p_1}) \cup \ldots \cup p_k(\bar{\mathbf{c}}_{p_k}).$$

Let A_i be the set of parameters from M that occur in p_i. Since T has the L^n-amalgamation property and $N_{p_1} \equiv_{L^n(A_1)} M$ there are $M_1 \succcurlyeq_{L^n} M$ and an L^n-elementary embedding $f_1 : N_{p_1} \to M_1$ such that $f_1 \restriction A_1$ is the identity function. We now have $A_2 \subseteq M \subseteq M_1$ and $N_{p_2} \equiv_{L^n(A_2)} M_1$ so, by the L^n-amalgamation property again, there are $M_2 \succcurlyeq_{L^n} M_1$ and an L^n-elementary embedding $f_2 : N_{p_2} \to M_2$ such that $f_2 \restriction A_2$ is the identity function. By continuing in this way we get a chain

$$M \preccurlyeq_{L^n} M_1 \preccurlyeq_{L^n} \ldots \preccurlyeq_{L^n} M_k$$

and L^n-elementary embeddings $f_i : N_{p_i} \to M_k$ such that $f_i \restriction A_i$ is the identity function. Let $\bar{b}_i = f_i(\bar{a}_{p_i})$ for $1 \leq i \leq k$. Then

$$M_k \models T \cup p_1(\bar{b}_1) \cup \ldots \cup p_k(\bar{b}_k)$$

so by interpreting $\bar{\mathbf{c}}_{p_i}$ as $\bar{b}_i$ we see that Δ is consistent and hence by compactness T^* is also consistent. If $N^* \models T^*$ then clearly the reduct of N^* to L satisfies the claim. $\square$

Proof of Proposition 2.2. (i)$\Rightarrow$(ii) Let κ be any infinite cardinal and let $M \models T$. By applying the L^n-amalgamation property to M and an infinite model of T (which we assume exists) it follows that there exists an infinite $M' \succcurlyeq_{L^n} M$. We define a chain of models M_α, $\alpha < \kappa^+$ as follows. Let $M_0 = M'$. If M_α is defined let $M_{\alpha+1}$ be a model that satisfies the conclusion of Claim 2.4 when we replace M by M_α in the hypothesis of Claim 2.4. If M_α is defined for all $\alpha < \delta$ where δ is a limit ordinal then let $M_\delta = \bigcup_{\alpha<\delta} M_\alpha$. Then $M_\alpha \preccurlyeq_{L^n} M_\beta$ for all $\alpha < \beta < \kappa^+$ (it is not difficult to verify that the union of an 'L^n-elementary chain' is an L^n-elementary extension of every structure in the chain). Let $N = \bigcup_{\alpha<\kappa^+} M_\alpha$. Then $M_\alpha \preccurlyeq_{L^n} N$ for all $\alpha < \kappa^+$ so in particular $N \models T$. Suppose that $A \subseteq N$ with $|A| \leq \kappa$. Since κ^+ is regular we have $A \subseteq M_\alpha$ for some $\alpha < \kappa^+$ and hence by the construction every $p \in S^n(A, M_\alpha)$ is realized in $M_{\alpha+1}$. Let $p \in S^n(A, N)$ be arbitrary. Since $p \cup Th_{L^n}(N, A)$ is consistent and $Th_{L^n}(N, A) = Th_{L^n}(M_\alpha, A)$ (because $A \subseteq M_\alpha \preccurlyeq_{L^n} N$) it follows that $p \cup Th_{L^n}(M_\alpha, A)$ is consistent, so $p \in S^n(A, M_\alpha)$, and therefore p is realized in $M_{\alpha+1}$. Then p is realized in N because $M_{\alpha+1} \preccurlyeq_{L^n} N$.

(ii)⇒(iii) Let κ be any infinite cardinal. It is sufficient to show that every $N \models T$ which is (L^n, κ)-saturated is also strongly (L^n, κ)-universal.
Let $N \models T$ be (L^n, κ)-saturated and suppose that $|M| < \kappa$ and

$$(N, (b_\alpha : \alpha < \lambda)) \equiv_{L^n} (M, (a_\alpha : \alpha < \lambda))$$

where $b_\alpha \in N$ and $a_\alpha \in M$. Let $(a_\alpha : \alpha < \mu)$, where $\mu < \kappa$, be an enumeration of all the elements of M that extends $(a_\alpha : \alpha < \lambda)$. We will inductively define L^n-elementary embeddings $f_\beta : (a_\alpha : \alpha < \beta) \to N$, for $\lambda \leq \beta < \mu$, such that if $\alpha < \lambda \leq \beta < \gamma < \mu$ then $f_\beta(a_\alpha) = b_\alpha$ and $f_\beta = f_\gamma \restriction (a_\alpha : \alpha < \beta)$. Define $f_\lambda(a_\alpha) = b_\alpha$ for every $\alpha < \lambda$. Now suppose that f_α is defined for every α such that $\lambda \leq \alpha < \beta$. If β is a limit ordinal then for $\alpha < \beta$ define $f_\beta(a_\alpha) = f_{\alpha+1}(a_\alpha)$. Suppose $\beta = \gamma + 1$. If $\alpha < \gamma$ define $f_\beta(a_\alpha) = f_\gamma(a_\alpha)$. Let $p(x) = tp_{L^n}(a_\gamma / \{a_\alpha : \alpha < \gamma\})$. Since $f_\gamma : (a_\alpha : \alpha < \gamma) \to N$ is an L^n-elementary embedding (by the induction hypothesis) it follows by compactness and Lemma 2.3 that $f_\gamma(p)$ is a consistent L^n-type over $\{f(a_\alpha) : \alpha < \gamma\}$ with respect to N. Since N is (L^n, κ)-saturated for L there is $c \in N$ that realizes $f_\gamma(p)$. Define $f_\beta(a_\gamma) = c$. Define $f : M \to N$ by $f(a_\alpha) = f_{\alpha+1}(a_\alpha)$ for $\alpha < \mu$. Then by the construction f is an L^n-elementary embedding.

(iii)⇒(i) Suppose that M_1 and M_2 are models of T and $a_\alpha \in M_1$, $b_\alpha \in M_2$, for $\alpha < \lambda$ and

$$(M_1, (a_\alpha : \alpha < \lambda)) \equiv_{L^n} (M_2, (b_\alpha : \alpha < \lambda)).$$

Let $\kappa > \sup(|M_1|, |M_2|)$ and by (iii) let $N \succcurlyeq M_1$ be strongly (L^n, κ)-universal. Let $\mu = |M_2|$ and let $(b_\alpha : \alpha < \mu)$ be an enumeration of M_2 that extends $(b_\alpha : \alpha < \lambda)$. We will inductively define L^n-elementary embeddings $f_\beta : (b_\alpha : \alpha < \beta) \to N$ for $\lambda \leq \beta < \mu$ such that if $\alpha < \lambda \leq \beta < \gamma < \mu$ then $f_\beta(b_\alpha) = a_\alpha$ and $f_\beta = f_\gamma \restriction (b_\alpha : \alpha < \beta)$. Define $f_\lambda(b_\alpha) = a_\alpha$ for every $\alpha < \lambda$. Now suppose that f_α is defined for every α such that $\lambda \leq \alpha < \beta$. If β is a limit ordinal then for $\alpha < \beta$ define $f_\beta(b_\alpha) = f_{\alpha+1}(b_\alpha)$. Suppose $\beta = \gamma + 1$. If $\alpha < \gamma$ define $f_\beta(b_\alpha) = f_\gamma(b_\alpha)$. Let $p(x) = tp_{L^n}(b_\gamma / \{b_\alpha : \alpha < \gamma\})$. By induction hypothesis $f_\gamma : (b_\alpha : \alpha < \gamma) \to N$ is an L^n-elementary embedding and by compactness and Lemma 2.3 it follows that $q = f_\gamma(p)$ is a consistent L^n-type over $\{f(b_\alpha) : \alpha < \gamma\}$ with respect to N. Then there is $M' \supseteq \{f(b_\alpha) : \alpha < \gamma\}$ and $c \in M'$ such that

$$(M', (f(b_\alpha) : \alpha < \gamma)) \equiv_{L^n} (N, (f(b_\alpha) : \alpha < \gamma))$$

and $M' \models q(c)$. By the Löwenheim-Skolem theorem we may assume that $|M'| < \kappa$ and since N is strongly (L^n, κ)-universal there is an L^n-elementary embedding $g : M' \to N$ which is the identity on $\{f(b_\alpha) : \alpha < \gamma\}$. Define $f_\beta(b_\gamma) = g(c)$. Define $f : M_2 \to N$ by $f(b_\alpha) = f_{\alpha+1}(b_\alpha)$ for $\alpha < \mu$. Then $f : M_2 \to N$ is the required L^n-elementary embedding. □

By minor changes in the proof of Proposition 2.2 we get:

Proposition 2.5 *The following are equivalent* :
(i) *T has the (L^n, ∞)-amalgamation property.*
(ii) *For every infinite cardinal κ and infinite $M \models T$, there exists $N \succcurlyeq_{L^n} M$ such*

that N is (L^n, κ, ∞)-saturated.
(iii) *For every infinite cardinal κ and infinite $M \models T$, there exists $N \succcurlyeq_{L^n} M$ such that N is strongly (L^n, κ, ∞)-universal.*

Structures that are (L^n, ω, ∞)-saturated will be of interest to us since they have the property that the L-type of a tuple of elements in the structure is determined by its restriction to L^n.

Proposition 2.6 *Suppose that the vocabulary of L contains no function symbols and that n is greater than or equal to the arity of every relation symbol in the vocabulary. If M and N are (L^n, ω, ∞)-saturated then for any $\bar{a} \in M$ and $\bar{b} \in N$ with $|\bar{a}| = |\bar{b}|$,*

$$\text{if} \quad (M, \bar{a}) \equiv_{L^n} (N, \bar{b}) \quad \text{then} \quad (M, \bar{a}) \equiv_L (N, \bar{b}).$$

Proof. Suppose that M and N are (L^n, ω, ∞)-saturated. Note that since we have assumed that the vocabulary of L contains no function symbols and that the arity of every symbol is $\leq n$, then for any tuples $\bar{c} \in M$ and $\bar{d} \in N$, if $(M, \bar{c}) \equiv_{L^n} (N, \bar{d})$ then $\bar{c}$ and $\bar{d}$ have the same atomic type (i.e. $\bar{c}$ and $\bar{d}$ satisfy the same quantifier free L-formulas). We want to show that if $(M, \bar{a}) \equiv_{L^n} (N, \bar{b})$ then $(M, \bar{a}) \equiv_L (N, \bar{b})$. It is enough to show that if $(M, \bar{a}) \equiv_{L^n} (N, \bar{b})$ then there exists a back and forth system I from $(M, \bar{a})$ to $(N, \bar{b})$ such that $(\bar{a}, \bar{b}) \in I$ (see [16] for definitions and results). Suppose that $(M, \bar{a}) \equiv_{L^n} (N, \bar{b})$. Then let

$$I = \{(\bar{c}, \bar{d}) : \ \bar{c} \in M, \ \bar{d} \in N, \ |\bar{c}| = |\bar{d}| \text{ and } (M, \bar{c}) \equiv_{L^n} (N, \bar{d})\}.$$

Clearly, $(\bar{a}, \bar{b}) \in I$ and for any $(\bar{c}, \bar{d}) \in I$, $\bar{c}$ and $\bar{d}$ have the same atomic type. So we only need to show that if $(\bar{c}, \bar{d}) \in I$ and $c \in M$, $d \in N$ then there are $c' \in M$, $d' \in N$ such that $(\bar{c}c, \bar{d}d') \in I$ and $(\bar{c}c', \bar{d}d) \in I$. (If $\bar{c} = (c_1, \ldots, c_m)$ then $\bar{c}c = (c_1, \ldots, c_m, c)$.)

Suppose that $(\bar{c}, \bar{d}) \in I$ and $c \in M$, $d \in N$. Then $(M, \bar{c}) \equiv_{L^n} (N, \bar{d})$ which means that $tp_{L^n}(\bar{c}) = tp_{L^n}(\bar{d})$. By Lemma 2.3, for every finite subset

$$\{\varphi_1(y, \bar{c}), \ldots, \varphi_k(y, \bar{c})\} \subseteq tp_{L^n}(c/\bar{c}),$$

$\varphi_1(y, \bar{d}) \wedge \ldots \wedge \varphi_k(y, \bar{d})$ is consistent with $Th_{L^n}(N, \bar{d})$. By compactness,

$$p(y) = \{\varphi(y, \bar{d}) : \ \varphi(y, \bar{c}) \in tp_{L^n}(c/\bar{c})\}$$

is consistent with $Th_{L^n}(N, \bar{d})$. Since N is (L^n, ω, ∞)-saturated and $|\bar{d}| < \omega$ there exists $d' \in N$ that realizes $p(y)$. We then have $tp_{L^n}(\bar{c}c) = tp_{L^n}(\bar{d}d')$, which implies $(M, \bar{c}c) \equiv_{L^n} (N, \bar{d}d')$ and $(\bar{c}c, \bar{d}d') \in I$. In the same way we can find $c' \in M$ such that $(\bar{c}c', \bar{d}d) \in I$. $\square$

A special case of Proposition 2.6 is:

Corollary 2.7 *Suppose that the vocabulary of L contains no function symbols and that n is greater than or equal to the arity of every relation symbol in the vocabulary. Then :*
(i) *Any two models of T which are (L^n, ω, ∞)-saturated are elementarily equivalent.*
(ii) *If $M \models T$ is (L^n, ω, ∞)-saturated then for any $\bar{a}, \bar{b} \in M$,*

$$\text{if} \quad tp_{L^n}(\bar{a}) = tp_{L^n}(\bar{b}) \quad \text{then} \quad tp_L(\bar{a}) = tp_L(\bar{b}).$$

In the rest of this section we will assume that T has the (L^n, ∞)-amalgamation property, the vocabulary contains no function symbols and the arity of any relation symbol is less than or equal to n. Observe that since we always assume that the identity symbol $=$ is in the vocabulary of L, it follows that $n \geq 2$.

Definition 2.8 The *canonical completion* of T is the theory $Th_L(M)$ where M is a model of T that is (L^n, ω, ∞)-saturated. By Corollary 2.7 (i), this definition does not depend on M. The canonical completion of T will be denoted by T^{c}. Observe that it follows from the definition that T^{c} is a complete L-theory with no finite models.

If $S^n_n(T)$ is finite then Proposition 2.6 yields the following:

Proposition 2.9 *Suppose that $S^n_n(T)$ is finite. If $p(\bar{x}) \in S_{\bar{x}}(T^{\mathrm{c}})$ and $q(\bar{x})$ is the restriction of p to L^n (i.e. $q(\bar{x}) = p(\bar{x}) \cap L^n$) then $T^{\mathrm{c}} \cup q(\bar{x}) \vdash p(\bar{x})$.*

Proof. Without loss of generality we may assume that $\bar{x} = (v_1, \ldots, v_m)$. Let $M \models T$ be (L^n, ω, ∞)-saturated. Since $S^n_n(T)$ is finite then by Lemma 1.8 $F^n_{\bar{x}}$ is finite up to equivalence modulo T. Let Δ be a finite subset of $F^n_{\bar{x}}$ such that every formula in $F^n_{\bar{x}}$ is equivalent modulo T to a formula in Δ. Let $\Gamma_{L^n} = \{tp_{L^n}(\bar{a}) : \bar{a} \in M,\ |\bar{a}| = |\bar{x}|\}$ and let $\Gamma_L = \{tp_L(\bar{a}) :\ \bar{a} \in M,\ |\bar{a}| = |\bar{x}|\}$. Then $\Gamma_{L^n} = S^n_{\bar{x}}(T)$ (because M is (L^n, ω, ∞)-saturated) and by Proposition 2.6, every $q(\bar{x}) \in \Gamma_{L^n}$ has exactly one extension in Γ_L. For every $p(\bar{x}) \in S_{\bar{x}}(T^{\mathrm{c}})$ let $\varphi_p(\bar{x})$ be the conjunction of the formulas in $\Delta \cap p(\bar{x})$. Then $T \cup \{\varphi_p(\bar{x})\} \vdash q(\bar{x})$, where $q(\bar{x}) = p(\bar{x}) \cap L^n$. Let $\psi(\bar{x}) \in L$ and $p(\bar{x}) \in \Gamma_L$ be arbitrary. If $\psi(\bar{x}) \in p(\bar{x})$ then for every $\bar{a} \in M$, $M \models \varphi_p(\bar{a}) \Rightarrow M \models q(\bar{a})$ (where $q(\bar{x}) = p(\bar{x}) \cap L^n$) $\Rightarrow M \models p(\bar{a})$ (by Proposition 2.6) $\Rightarrow M \models \psi(\bar{a})$. Hence, if $\psi(\bar{x}) \in p(\bar{x})$ then $M \models \forall \bar{x}[\varphi_p(\bar{x}) \rightarrow \psi(\bar{x})]$ so

$$\text{if } \psi(\bar{x}) \in p(\bar{x}) \text{ then } T^{\mathrm{c}} \vdash \forall \bar{x}[\varphi_p(\bar{x}) \rightarrow \psi(\bar{x})]. \tag{1}$$

In the same way we see that

$$\text{if } \psi(\bar{x}) \notin p(\bar{x}) \text{ then } T^{\mathrm{c}} \vdash \forall \bar{x}[\varphi_p(\bar{x}) \rightarrow \neg\psi(\bar{x})]. \tag{2}$$

Note that we have proved (1) and (2) for every $p \in \Gamma_L$ (but not necessarily for every $p \in S_{\bar{x}}(T^{\mathrm{c}})$). Let $p'(\bar{x}) \in S_{\bar{x}}(T^{\mathrm{c}})$ and let $q(\bar{x}) = p'(\bar{x}) \cap L^n$. Then $q(\bar{x}) \in \Gamma_{L^n}$ so $q(\bar{x})$ has a unique extension $p(\bar{x}) \in \Gamma_L$. From $q(\bar{x}) \vdash \varphi_p(\bar{x})$ (by definition of φ_p) and (1) and (2) (which hold for all $p \in \Gamma_L$) it follows that $T^{\mathrm{c}} \cup q(\bar{x}) \vdash p(\bar{x})$. Hence $p = p'$ so $T^{\mathrm{c}} \cup q(\bar{x}) \vdash p'(\bar{x})$ and the proof is complete (and it follows that $S_{\bar{x}}(T^{\mathrm{c}}) = \Gamma_L$). $\square$

Corollary 2.10 (i) *If $S^n_n(T)$ is finite then T^{c} is ω-categorical.*
(ii) *If $S^n_n(T)$ is finite and $M \models T$ is infinite, then there exists $N \succcurlyeq_{L^n} M$ such that N is ω-categorical.*

Proof. (i) Suppose that $S^n_n(T)$ is finite. Then by Lemma 1.8, for any $0 < m < \omega$, $S^n_m(T)$ is finite. By Proposition 2.9 every $p \in S_m(T^{\mathrm{c}})$ is determined by $p \cap L^n \in S^n_m(T)$, so $S_m(T^{\mathrm{c}})$ is finite and therefore T^{c} is ω-categorical.

(ii) Suppose that $S^n_n(T)$ is finite. By Proposition 2.5 there is $N \succcurlyeq_{L^n} M$ such that N is ω-saturated for L^n. By part (i), $Th_L(N) = T^{\mathrm{c}}$ is ω-categorical. $\square$

Lemma 2.11 *Suppose that T' is a complete L-theory such that for every $p(\bar{x}) \in S_{\bar{x}}(T')$, if $q(\bar{x}) = p(\bar{x}) \cap L^n$ then $T' \cup q(\bar{x}) \vdash p(\bar{x})$. Then for every $\psi(\bar{x}) \in L$ there is a boolean combination of L^n-formulas $\theta(\bar{x})$ such that*

$$T' \vdash \forall \bar{x}[\psi(\bar{x}) \leftrightarrow \theta(\bar{x})].$$

Proof. Let $\{p_i(\bar{x}) : i \in I\}$ be the set of all types $p(\bar{x}) \in S_{\bar{x}}(T')$ such that $\psi(\bar{x}) \in p(\bar{x})$. For every $i \in I$ let $q_i(\bar{x}) = p_i(\bar{x}) \cap L^n$. By the assumption we have $T' \cup q_i(\bar{x}) \vdash p_i(\bar{x})$, for every $i \in I$. Then, for every $i \in I$, $T' \cup q_i(\bar{x}) \vdash \psi(\bar{x})$ and by compactness there is a formula $\varphi_i(\bar{x})$, which is a conjunction of finitely many formulas from $q_i(\bar{x})$, such that $T' \vdash \forall \bar{x}[\varphi_i(\bar{x}) \rightarrow \psi(\bar{x})]$. Now suppose that for every finite subset $J \subseteq I$, the set of formulas

$$T' \cup \{\psi(\bar{x})\} \cup \{\neg\varphi_i(\bar{x}) : i \in J\}$$

is consistent. Then it follows by compactness that

$$T' \cup \{\psi(\bar{x})\} \cup \{\neg\varphi_i(\bar{x}) : i \in I\}$$

is consistent, so there is a type $p(\bar{x}) \in S_{\bar{x}}(T')$ such that $\psi(\bar{x}) \in p(\bar{x})$ and $p_i \neq p$ for all $i \in I$. But this contradicts the assumption that $\{p_i(\bar{x}) : i \in I\}$ is the set of all types in $S_{\bar{x}}(T')$ that contain $\psi(\bar{x})$. Hence, we conclude that there exists a finite subset $J \subseteq I$ such that

$$T' \vdash \forall \bar{x}\big[\psi(\bar{x}) \rightarrow \bigvee_{i \in J} \varphi_i(\bar{x})\big].$$

Since also $T' \vdash \forall \bar{x}[\varphi_i(\bar{x}) \rightarrow \psi(\bar{x})]$ for all i we get

$$T' \vdash \forall \bar{x}\big[\psi(\bar{x}) \leftrightarrow \bigvee_{i \in J} \varphi_i(\bar{x})\big],$$

so let $\theta(\bar{x})$ be $\bigvee_{i \in J} \varphi_i(\bar{x})$ which is a boolean combination of L^n-formulas. □

Proposition 2.12 *Let κ be an infinite cardinal. If M is infinite, (L^n, κ, ∞)-saturated and $S^n_n(Th_{L^n}(M))$ is finite then M is κ-saturated (in the usual first-order sense).*

Proof. Suppose that M is (L^n, κ, ∞)-saturated, where $\kappa \geq \aleph_0$, and that $S^n_n(Th_{L^n}(M))$ is finite. Let $T = Th_{L^n}(M)$ and let $T' = Th_L(M)$. The argument in the proof of Proposition 2.9 still holds if we replace T^{c} by T'. It follows that

($*$) for any finite sequence of variables $\bar{y}$ and any $p(\bar{y}) \in S_{\bar{y}}(T')$, if $q(\bar{y}) = p(\bar{y}) \cap L^n$ then $T' \cup q(\bar{y}) \vdash p(\bar{y})$.

Let $A \subseteq M$ with $|A| < \kappa$ and let $p(\bar{x}) \in S_{\bar{x}}(A, M)$. We need to show that p is realized in M. Let $q(\bar{x}) = p(\bar{x}) \cap L^n(A)$. Then $q(\bar{x}) \cup Th_{L^n}(M, A)$ is consistent (because $p(\bar{x}) \cup Th_L(M, A)$ is consistent) so $q(\bar{x}) \in S^n_{\bar{x}}(A, M)$. Then, since M is (L^n, κ, ∞)-saturated, q is realized in M by $\bar{b}$, say. By ($*$) and Lemma 2.11 it follows that p is determined by q and hence $M \models p(\bar{b})$. □

3 Stability and ω-stability in L^n

In this section we assume that n is greater than or equal to the arity of all relation symbols in the vocabulary of L and that T is a complete L^n-theory with infinite models and the (L^n, ∞)-amalgamation property.

We say that a formula $\varphi(\bar{x}, \bar{y})$ has the *order property with respect to* T if there is a model M of T and $\bar{a}_i, \bar{b}_i \in M$ for $i < \omega$ such that $M \models \varphi(\bar{a}_i, \bar{b}_j) \Leftrightarrow i \leq j$. We say that T has the *order property in* L^n if there is a formula $\varphi(\bar{x}, \bar{y}) \in L^n$ that has the order property with respect to T.

We will show that if $S^n_n(T)$ is finite, then T does not have the order property for L^n if and only if there are only countably many L^n-types over any countable set. If we add the assumption that the vocabulary of L contains no function symbols then it follows that T^{c} is ω-stable.

Let M be an L-structure, $A \subseteq M$ and $\varphi(\bar{x}, \bar{y}) \in L$. We say that a set of formulas $p \subseteq F_{\bar{x}}(A)$ is a $(\varphi, \bar{x})$*-type over* A with respect to M if $p \cup Th_L(M, A)$ is consistent, every formula in p has the form $\varphi(\bar{x}, \bar{a})$ or $\neg\varphi(\bar{x}, \bar{a})$ for some $\bar{a} \in A$, and for all $\bar{a} \in A$ either $\varphi(\bar{x}, \bar{a}) \in p$ or $\neg\varphi(\bar{x}, \bar{a}) \in p$. The set of all $(\varphi, \bar{x})$-types over A with respect to M is denoted by $S_{(\varphi,\bar{x})}(A, M)$.

The proof, given here, of the next proposition, which much simplifies my original proof (in [11]), was suggested by the referee of the article version of this chapter.

Proposition 3.1 *If $S^n_n(T)$ is finite then the following are equivalent:*
(i) *T does not have the order property in L^n.*
(ii) *For every $M \models T$, if $A \subseteq M$ and $|A| \leq \aleph_0$ then $|S^n(A, M)| \leq \aleph_0$.*

Proof. (i)$\Rightarrow$(ii). Let $A \subseteq M \models T$ and suppose that $|A| \leq \aleph_0$. It is sufficient to show that for any finite (nonempty) sequence of variables $\bar{x}$, $|S^n_{\bar{x}}(A, M)| \leq \aleph_0$. If A is finite then this follows from Lemma 1.8, so let's assume that A is infinite. By Proposition 2.5 there is $N \succcurlyeq_{L^n} M$ such that N is $(L^n, \aleph_1, \infty)$-saturated. Let $\bar{y}$ be a sequence such that $|\bar{y}| = n - 1$ and no variables in $\bar{y}$ occur in $\bar{x}$. By Lemma 1.8, $F^n_{\bar{x},\bar{y}}$ is finite up to equivalence modulo T. Let

$$\varphi_1(\bar{x}, \bar{y}), \ldots, \varphi_r(\bar{x}, \bar{y})$$

be formulas in $F^n_{\bar{x},\bar{y}}$ such that any other formula in $F^n_{\bar{x},\bar{y}}$ is equivalent, modulo T, to $\varphi_i(\bar{x}, \bar{y})$ for some $1 \leq i \leq r$. Then we have

$$|S^n_{\bar{x}}(A, M)| \leq |S_{(\varphi_1,\bar{x})}(A, M) \times \ldots \times S_{(\varphi_r,\bar{x})}(A, M)|$$

so it suffices to show that $|S_{(\varphi_i,\bar{x})}(A, M)| \leq \aleph_0$ whenever $1 \leq i \leq r$. Since $\varphi_i(\bar{x}, \bar{y})$ does not have the order property with respect to T and N is $(L^n, \aleph_1, \infty)$-saturated it follows, by Theorem I.2.10 in [33], that $|S_{(\varphi_i,\bar{x})}(A, N)| \leq \aleph_0$. We also have $S_{(\varphi_i,\bar{x})}(A, M) = S_{(\varphi_i,\bar{x})}(A, N)$, because $A \subseteq M \preccurlyeq_{L^n} N$, and therefore $|S_{(\varphi_i,\bar{x})}(A, M)| \leq \aleph_0$.

(ii)$\Rightarrow$(i). If some $\varphi(\bar{x}, \bar{y}) \in L^n$ has the order property with respect to T then it follows by Theorem II.2.2 in [33] that there are $M \models T$ and $A \subseteq M$ such that $|A| \leq \aleph_0$ and $|S^n(A, M)| > \aleph_0$. $\square$

Definition 3.2 T is *stable in* L^n if T does not have the order property in L^n. T is ω-*stable in* L^n if for all $M \models T$ and all $A \subseteq M$, if $|A| \leq \aleph_0$ then $|S^n(A, M)| \leq \aleph_0$. We say that T is *unstable in* L^n if T is not stable for L^n.

With this terminology Proposition 3.1 can be expressed as follows.

Corollary 3.3 *If $S^n_n(T)$ is finite (so in particular, if T has finite models) then T is stable in L^n if and only if T is ω-stable in L^n.*

For a definition of stability and ω-stability for complete L-theories, see for example [33]. We can now relate stability of T (for L^n) with ω-stability of T^{c}.

Corollary 3.4 *If $S^n_n(T)$ is finite and the vocabulary of L contains no function symbols then T is stable in L^n if and only if T^{c} is ω-stable.*

Proof. First note that since the vocabulary of L contains no function symbols, T^{c} exists. Let $A \subseteq M \models T^{\mathrm{c}}$ and suppose that T is stable for L^n. By Proposition 2.9 and Lemma 2.11 it follows that if $p(\bar{x}) \in S_{\bar{x}}(A, M)$ then $p(\bar{x})$ is determined by $p(\bar{x}) \cap L^n(A) \in S^n_{\bar{x}}(A, M)$ so, by Corollary 3.3, if $|A| \leq \aleph_0$ then $|S_{\bar{x}}(A, M)| \leq \aleph_0$.

Now suppose that $\varphi(\bar{x}, \bar{y}) \in L^n$ is unstable for L^n, i.e. some $\varphi(\bar{x}, \bar{y}) \in L^n$ has the order property with respect to T. Then there are $M \models T$ and $\bar{a}_i, \bar{b}_i \in M$ for $i < \omega$ such that for all $i, j < \omega$

$$M \models \varphi(\bar{a}_i, \bar{b}_j) \text{ if and only if } i \leq j.$$

By Proposition 2.5 let $N \succcurlyeq_{L^n} M$ be (L^n, ω, ∞)-saturated. Then $N \models T^{\mathrm{c}}$ and

$$N \models \varphi(\bar{a}_i, \bar{b}_j) \text{ if and only if } i \leq j$$

so T^{c} is unstable and hence not ω-stable. □

The next example shows that the assumption that $S^n_n(T)$ is finite is necessary in Corollary 3.3. The author does not know if the (tacit) assumption in Corollary 3.3 that T has the L^n-amalgamation property is necessary.

Example 3.5 Let $T' = Th_L((\mathbb{Z}, +, 0))$ and let $T = T' \cap L^n$ for some fixed $n \geq 3$. So, for the moment, we assume that the vocabulary of L consists of a function symbol $+$, a constant symbol 0, and as always the identity symbol $=$. We will show that T is stable in L^n but not ω-stable for L^n. We use the fact that the complete theory of any abelian group is stable. (This is proved in [3] and as mentioned in [32] it also follows from results in [12] and [31].) First we show that T is not ω-stable in L^n and that $S^n_n(T)$ is infinite.

Let $P \subseteq \mathbb{Z}$ be the set of (positive) primes. For every $X \subseteq P$ we define a set of formulas Φ_X by

$$\Phi_X = \{\exists v_2(\underbrace{v_2 + \ldots + v_2}_{p \text{ times}} = v_1) : \ p \in X\}$$
$$\cup \{\neg\exists v_2(\underbrace{v_2 + \ldots + v_2}_{q \text{ times}} = v_1) : \ q \in P - X\}.$$

So Φ_X says that exactly the primes in X divide v_1. By compactness Φ_X is consistent with T for every $X \subseteq P$ and clearly $\Phi_X \cap \Phi_Y$ is inconsistent if $X \neq Y$. Since every Φ_X can be extended to a complete L^n-type it follows that $|S_1^n(T)| = 2^{\aleph_0}$, so T is not ω-stable in L^n.

It remains to show that T is stable in L^n. Suppose it is not. Then there are $\varphi(\bar{x}, \bar{y}) \in L^n$, $M \models T$ and $\bar{a}_i$, $\bar{b}_i \in M$, for $i < \omega$, such that

$$M \models \varphi(\bar{a}_i, \bar{b}_j) \quad \Leftrightarrow \quad i \leq j. \tag{$*$}$$

Since $M \models T$ and the axioms for abelian groups are axiomatizable by sentences in L^3 (with the given vocabulary) it follows that M is an abelian group, so $Th_L(M)$ is stable, which contradicts $(*)$. Hence we conclude that T is stable for L^n.

If $T = Th_{L^n}((\mathbb{Z}, G_+, 0))$ where $n \geq 5$ and G_+ is the graph of the addition function, (so now we assume that the vocabulary of L consists of a ternary relation symbol, a constant symbol, and the identity symbol) then T will be a complete L^n-theory, where the vocabulary of L contains no function symbols, such that T is stable in L^n but not ω-stable in L^n. The proof is similar to the one given above; we just have to express, in L^n, the axioms of abelian groups and the statements "p divides v_1" by using G_+ instead of $+$.

4 Finite models

We are now in position to collect the main results of this chapter.

Theorem 4.1 *Suppose that the vocabulary of L contains no function symbols and that n is greater than or equal to the arity of every relation symbol in the vocabulary. Let T be a complete L^n-theory with infinite models such that $S_n^n(T)$ is finite, T is stable in L^n and T has the (L^n, ∞)-amalgamation property. Then the canonical completion, T^c, exists and is a complete L-theory which extends T and is ω-categorical and ω-stable.*

Proof. By Corollary 2.10 (i) T^c is ω-categorical and by Corollary 3.4 T^c is ω-stable. □

Theorem 4.2 *Suppose that the vocabulary of L is finite and contains no function symbols. If M is an L-structure which is ω-categorical and ω-stable then for every $1 \leq n < \omega$ and every finite $A \subseteq M$ there is a finite $N \preccurlyeq_{L^n} M$ such that $A \subseteq N$.*

Proof. Let M be an ω-categorical and ω-stable L-structure and let $A \subseteq M$ be finite. Let $T = Th_{L^n}(M)$ and $T' = Th_L(M)$. $S_n(T')$ is finite because M is ω-categorical. Hence $S_n^n(T')$ is finite and by Lemma 1.2 $S_n^n(T)$ is finite. Then by Lemma 1.8 F_n^n is finite up to equivalence modulo T. Let

$$\varphi_1(v_1, \ldots, v_n), \ldots, \varphi_r(v_1, \ldots, v_n) \in F_n^n$$

be such that every formula in F_n^n is equivalent, modulo T, to one of the φ_i's. Let $R_1, \ldots, R_r$ be new n-ary relation symbols and let L_1 be the language obtained from L by adding R_i , for $1 \leq i \leq r$, to the vocabulary. Let M_1 be the expansion of M which is obtained by interpreting R_i as follows:

$$\text{for any } \bar{a} \in M^n,\ M_1 \models R_i(\bar{a}) \quad \Leftrightarrow \quad M \models \varphi_i(\bar{a}).$$

Then M_1 is ω-categorical and ω-stable. Let $T_1 = Th_{L_1^n}(M_1)$ and $T_1' = Th_{L_1}(M_1)$. Then

$$T_1 \vdash \forall v_1, \dots v_n[\varphi_i(v_1, \dots v_n) \leftrightarrow R_i(v_1, \dots v_n)],\ \text{for all } 1 \leq i \leq r. \qquad (*)$$

Observe that the formula above is in L_1^n. Since M_1 is ω-categorical, $S_n(T_1')$ is finite and then $S_n^n(T_1')$ is finite so by Lemma 1.2 $S_n^n(T_1)$ is finite. By Fact 1.4, T_1 is axiomatized by an L_1^n-sentence σ.

Let $M_2 = (M_1, A)$, so M_2 is the expansion of M_1 which is obtained by naming the finitely many elements in A. Then M_2 is ω-categorical and ω-stable. Also note that every substructure of M_2 includes A. We will now use a theorem, first proved by B. Zilber in the totally categorical case, and then generalized by Cherlin, Harrington and Lachlan in [6] to yield:

> If M' is an ω-categorical and ω-stable structure (in a language with at most finitely many function symbols) and ψ is a sentence such that $M' \models \psi$ then there exists a finite substructure $N' \subseteq M'$ such that $N' \models \psi$.

Since $M_1 \models \sigma$ we have $M_2 \models \sigma$ and hence there exists a finite substructure N_2 of M_2 such that $N_2 \models \sigma$. Let N_1 be the reduct of N_2 to L_1 and let N be the reduct of N_2 to L. Then $A \subseteq N$ and N is a substructure of M. It remains to show that N is an L^n-elementary substructure of M. Note that $N_1 \models \sigma$ and hence $N_1 \models T_1$. Let $\bar{a} \in N$. Then for $1 \leq i \leq r$,

$$\begin{array}{lll} & N \models \varphi_i(\bar{a}) & \\ \Leftrightarrow & N_1 \models \varphi_i(\bar{a}) & \text{since } \varphi_i \in L \\ \Leftrightarrow & N_1 \models R_i(\bar{a}) & \text{by } (*) \\ \Leftrightarrow & M_1 \models R_i(\bar{a}) & \text{since } N_1 \text{ is a substructure of } M_1 \\ \Leftrightarrow & M_1 \models \varphi_i(\bar{a}) & \text{by } (*) \\ \Leftrightarrow & M \models \varphi_i(\bar{a}) & \text{since } \varphi_i \in L. \end{array}$$

Hence $N \preccurlyeq_{L^n} M$. □

Remark 4.3 Since ω-categoricity and superstability together implies ω-stability it follows that Theorem 4.2 stays true if we replace ω-stable by superstable. Also note that Theorem 4.2 does not depend on any results from sections 2 or 3. The essential ingredients of the proof are Fact 1.4 and the above mentioned result from [6].

By combining Theorem 4.1 and Theorem 4.2 we get:

Corollary 4.4 *Suppose that the vocabulary of L is finite and contains no function symbols and that n is greater than or equal to the arity of every relation symbol in the vocabulary. Let T be a complete L^n-theory such that T has an infinite model, $S^n_n(T)$ is finite and T is stable in L^n.*
(i) *If T has the (L^n, ∞)-amalgamation property then T has arbitrarily large finite models (which can be taken as L^n-elementary substructures of any model of T^c).*
(ii) (Finite amalgamation for L^n.) *Suppose that T has the L^n-amalgamation property. Let M_1 and M_2 be finite models of T and suppose $\bar{a} \in M_1$, $\bar{b} \in M_2$, $|\bar{a}| = |\bar{b}|$ and $(M_1, \bar{a}) \equiv_{L^n} (M_2, \bar{b})$. Then there exist a finite $N \succcurlyeq_{L^n} M_1$ and an L^n-elementary embedding $f : M_2 \to N$ such that $f(\bar{b}) = \bar{a}$.*

Proof. (i) follows immediately from Theorem 4.1 and Theorem 4.2. Now we prove (ii). By the L^n-amalgamation property there exists $M \succcurlyeq_{L^n} M_1$ and an L^n-elementary embedding $f : M_2 \to M$ such that $f(\bar{b}) = \bar{a}$. If M is finite then we are done, so now suppose that M is infinite. By Proposition 2.2 there exists $N' \succcurlyeq_{L^n} M$ such that N' is (L^n, ω)-saturated. Then $Th_L(N') = T^c$ (by definition of T^c) so N' is ω-categorical and ω-stable (since T^c is, by Theorem 4.1) and hence, by Theorem 4.2, there exists a finite $N \preccurlyeq_{L^n} N'$ such that $M_1 \cup f(M_2) \subseteq N$. Since $M_1 \preccurlyeq_{L^n} M \preccurlyeq_{L^n} N'$ and $N' \succcurlyeq_{L^n} N \supseteq M_1$ it follows that $M_1 \preccurlyeq_{L^n} N$. Since $f : M_2 \to M$ is an L^n-elementary embedding and $M \preccurlyeq_{L^n} N'$ and $N' \succcurlyeq_{L^n} N \supseteq f(M_2)$ it follows that $f : M_2 \to N$ is an L^n-elementary embedding (and of course, we still have $f(\bar{b}) = \bar{a}$). □

Example 4.5 As an example of a theory which satisfies the conditions of the last corollary, including the L^n-amalgamation property, we can take the complete L^n-theory, where $n \geq 4$ and the vocabulary of L contains only one binary relation symbol, of a disjoint union of trees (where a tree is viewed as an undirected connected graph without cycles and the relation symbol of L is interpreted as the edge relation) such that,

for some $k < \omega$, there is no path (in any tree) of length more than k, and

(a) there are (at least) n trees that are isomorphic, *or*
(b) there is at least one vertex a (in some tree) which has (at least) n neighbours $b_1, \ldots, b_n$ such that if G_i is the subgraph which is induced by the set containing exactly b_i and every vertex that can be reached from b_i by a path which does not contain a, then, for all $1 \leq i, j \leq n$, there is an isomorphism from G_i onto G_j which maps b_i to b_j.

The (tedious) verification of this is left to the reader; note that "there exists a path of length m from x to y" is expressible in L^4 for any $m < \omega$ and that a tree which does not satisfy (b) is determined up to isomorphism by its L^n-theory.

After Remark 6.12 we will see a more general way of producing examples of complete L^n-theories which are stable and have the amalgamation property in a more general sense, but which still have arbitrarily large finite models.

5 When amalgamation fails

In this section we will see an example, due to Simon Thomas, of a complete L^4-theory T such that the vocabulary of L contains only a binary relation symbol and the identity symbol, T has infinite models, $S^4_4(T)$ is finite, T is stable in L^4 but T has only finitely many finite models. It follows from Corollary 4.4 (i) that T does not have the (L^4, ∞)-amalgamation property. Hence, in Corollary 4.4 (i) we can not omit the assumption that T has the (L^n, ∞)-amalgamation property.

Let the vocabulary of L consist of a binary relation symbol R and the identity symbol =. If M is an L-structure and $a, b \in M$ then we say that a and b are *adjacent* if $M \models \mathrm{R}(a, b) \vee \mathrm{R}(b, a)$. Otherwise we say that a and b are *nonadjacent*. Consider the following axioms:

A_0 : R is symmetric and irreflexive.

A_1 : There are at least 4 elements.

A_2 : There does *not* exist elements a, b, c such that $\mathrm{R}(a, b)$, $\mathrm{R}(b, c)$ and $\mathrm{R}(c, a)$.

A_3 : For all a and b, if $a \neq b$ and a and b are nonadjacent then there is a *unique* element which is adjacent to both a and b.

A_4 : For all a there are at least 3 elements which are adjacent to a.

$$\mathrm{A}_5 \ : \quad \forall v_1, v_2, v_3 \Big(\bigwedge_{1 \leq i < j \leq 3} v_i \neq v_j \rightarrow \exists v_4 \Big(\bigwedge_{1 \leq i \leq 3} v_4 \neq v_i \quad \wedge$$

$$\bigwedge_{1 \leq i \leq 3} \neg \mathrm{R}(v_4, v_i) \quad \wedge \quad \neg \exists v_1 \bigwedge_{2 \leq i \leq 4} \mathrm{R}(v_1, v_i) \Big) \Big).$$

Clearly all the above axioms can be expressed in L^4, and any model of A_0 is an undirected graph. By a graph we will mean an undirected graph. Let $T = \{\mathrm{A}_0, \ldots, \mathrm{A}_5\}$. Simon Thomas has shown (in unpublished notes) that T is a complete L^4-theory which has finitely many finite models and also infinite models. We will show this in essentially the same way as he has done. Then we will also show that T is stable in L^4.

It is not difficult to see that a finite graph which satisfies axioms A_1 - A_4 has diameter 2, is regular (i.e. all vertices have the same degree) and has no cycles of length less than 5. Hoffman and Singleton [17] has proved that a finite regular graph with diameter 2 which has no cycles of length less than 5 must have degree 3 or 7 or 57. Moreover they showed that there is a unique such graph with degree 3, the Petersen graph, and a unique such graph with degree 7, the Hoffman-Singleton graph. It is unknown whether there exists such a graph with degree 57.

By an easy counting argument it follows that a finite regular graph with degree d which satisfies A_1 - A_4 has exactly d^2+1 vertices. Hence there are at most finitely many finite models of T. The Hoffman-Singleton graph, with R interpreted as the edge relation, is one of them (and perhaps the only one). This is not too difficult to see by considering a construction of the Hoffman-Singleton graph, due to N.

Robertson, which can be found in [4]. We now show that T also has infinite models.

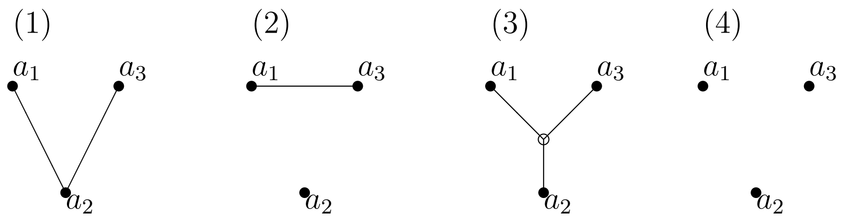
Let $G_0 = \emptyset$ and let G_1 be any nonempty (finite or infinite) graph such that there are no edges between any two elements of G_1. Inductively we define graphs G_k, for $1 < k < \omega$, such that G_i is an induced subgraph of G_{i+1} (in model theoretic terms this simply means that G_i is a substructure of G_{i+1}) for every $i < \omega$. Suppose that G_i is defined for $0 \leq i \leq k$ where $2 \leq k < \omega$. Let G_{k+1} contain the vertices and edges in G_k together with exactly the *new* vertices and edges specified by (i) and (ii) below:

(i) For every $\{a, b\} \subseteq G_k$ such that $a \neq b$, a and b are nonadjacent and there is no vertex in G_k which is adjacent to both a and b, add one new vertex c to G_{k+1} and join c with both a and b. Do not join c to any other vertex in G_{k+1}.

(ii) Add one new vertex d to G_{k+1} and do *not* join d to any vertex in G_{k+1}.

Let G be the graph with vertices $\bigcup_{k<\omega} V(G_k)$ and edges $\bigcup_{k<\omega} E(G_k)$, where $V(G_k)$ is the set of vertices of G_k and $E(G_k)$ is the set of edges of G_k. By step (ii), G is infinite. We may view G as an L-structure by interpreting R as the edge relation in G. Clearly G satisfies A_0 and A_1. Since G_1 satisfies A_2 it follows from the construction that every G_k, for $k > 1$, satisfies A_2 and hence also G satisfies A_2. By (i) in the construction, G satisfies A_3. It follows from (i) and (ii), iterated 3 times, that G satisfies A_4; in fact, every vertex in G has infinite degree. Next, we show that $G \models \mathrm{A}_5$. Let $a_1, a_2, a_3 \in G$ be distinct. Then $a_1, a_2, a_3 \in G_k$ for some k. By (ii) there is $a_4 \in G_{k+1} - G_k$ which is nonadjacent to a_1, a_2 and a_3. If a_2 and a_3 are adjacent then, by A_2, there is no $b \in G$ which is adjacent to a_2, a_3 and a_4. Now suppose that a_2 and a_3 are nonadjacent. Then by A_4 there is a unique b which is adjacent to both a_2 and a_3. Then $a_2, a_3, b \in G_l$ for some l. By (ii) there is $a'_4 \in G_{l+1} - G_l$ which is nonadjacent to a_2, a_3 and b. By the choice of b and a'_4 and the uniqueness part of axiom A_3 there can not exist $c \in G$ which is adjacent to a_2, a_3 and a'_4. Hence $G \models \mathrm{A}_0, \ldots, \mathrm{A}_5$ so G is an infinite model of T.

We now show that T is a complete L^4-theory. Let $M \models T$ and let $a_1, a_2, a_3 \in M$ be distinct. Then (a_1, a_2, a_3) has one of the following configurations, up to a permutation of $\{1, 2, 3\}$:



where (3) expresses that a_1, a_2 and a_3 are mutually nonadjacent and there exists a vertex which is adjacent to a_1, a_2 and a_3, and (4) expresses that a_1, a_2 and a_3 are mutually nonadjacent and there does not exist a vertex which is adjacent to a_1, a_2 and a_3. We will show that the configuration (as given above) of (a_1, a_2, a_3) determines the L^4-type of (a_1, a_2, a_3), that is,

(I) if $a_1, a_2, a_3 \in M \models T$ are distinct and $b_1, b_2, b_3 \in N \models T$ are distinct and (a_1, a_2, a_3) and (b_1, b_2, b_3) have the same configuration then

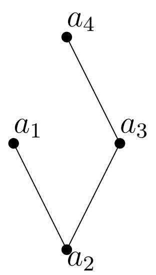
$$(M, a_1, a_2, a_3) \equiv_{L^4} (N, b_1, b_2, b_3).$$

Since every configuration is realized in every model of T it follows that any two models of T are L^4-elementarily equivalent. Hence T is a complete L^4-theory.

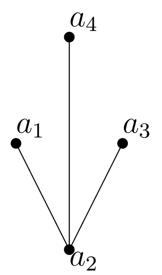
We prove (I) by showing that Duplicator (sometimes called player II, or $\exists$) has a winning strategy for the 4-pebble game i ω rounds, where 3 of the pebbles played on M are initially placed on a_1, a_2 and a_3, and 3 of the pebbles played on N are initially placed on b_1, b_2 and b_3. This game theoretic characterization of L^n-elementary equivalence comes from Immerman [22], and implicitly, Poizat [29]. It is enough to show that whenever $a_1, a_2, a_3 \in M \models T$ are distinct and $b_1, b_2, b_3 \in N \models T$ are distinct and (a_1, a_2, a_3) and (b_1, b_2, b_3) have the same configuration then for any $a_4 \in M - \{a_1, a_2, a_3\}$ there is $b_4 \in N - \{b_1, b_2, b_3\}$ such that (a_i, a_j, a_k) and (b_i, b_j, b_k) have the same configuration for all $1 \leq i < j < k \leq 4$.

So now suppose that $a_1, a_2, a_3 \in M \models T$ are distinct and $b_1, b_2, b_3 \in N \models T$ are distinct and (a_1, a_2, a_3) and (b_1, b_2, b_3) have the same configuration. Let $a_4 \in M - \{a_1, a_2, a_3\}$. If, up to a permutation of $\{1, 2, 3\}$, (a_1, a_2, a_3) has configuration (1) then, up to a permutation of $\{1, 2, 3\}$, (a_1, a_2, a_3, a_4) has one of the following configurations:

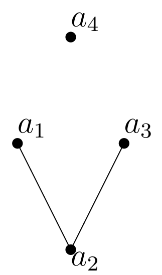
(1) (a)



(b)

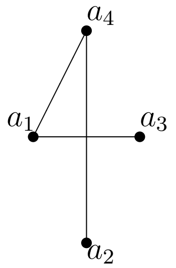


(c)

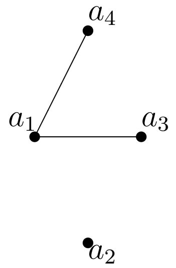


If, up to a permutation of $\{1, 2, 3\}$, (a_1, a_2, a_3) has configuration (2) then, up to a permutation of $\{1, 2, 3\}$, (a_1, a_2, a_3, a_4) has one of the following configurations:

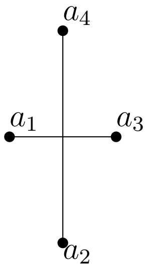
(2) (a)

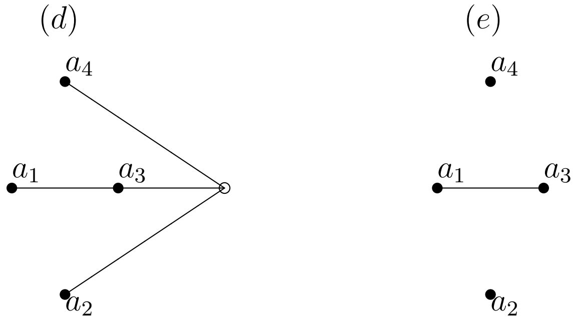


(b)



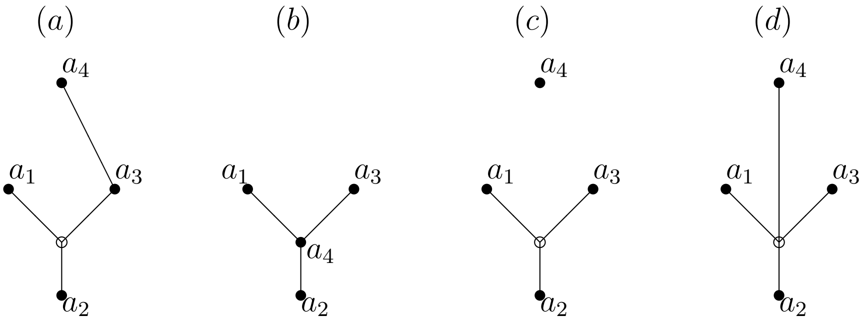
(c)





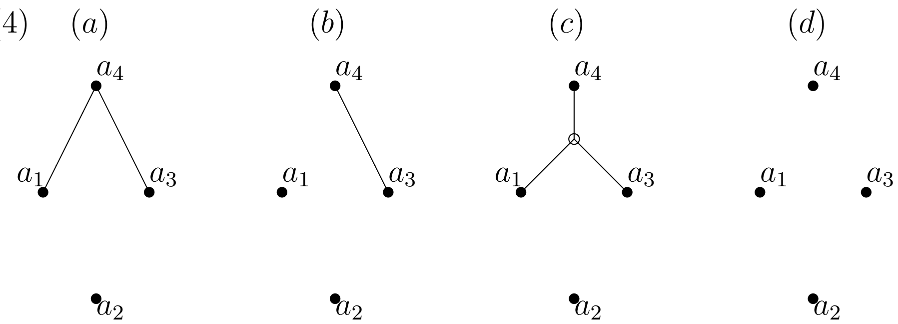
If, up to a permutation of $\{1, 2, 3\}$, (a_1, a_2, a_3) has configuration (3) then, up to a permutation of $\{1, 2, 3\}$, (a_1, a_2, a_3, a_4) has one of the following configurations:

(3)



If, up to a permutation of $\{1, 2, 3\}$, (a_1, a_2, a_3) has configuration (4) then, up to a permutation of $\{1, 2, 3\}$, (a_1, a_2, a_3, a_4) has one of the following configurations:

(4)



Since $N \models T$ we can, in all cases, find $b_4 \in N$ such that (a_1, a_2, a_3, a_4) and (b_1, b_2, b_3, b_4) have the same configuration. To verify this is left to the reader. In cases (3)(d) and (4)(c) it is useful to observe that every vertex in any model of T has degree at least 4; this follows from axioms A_3, A_4 and A_5. The proof of (I) is now finished.

Clearly all tuples which realize the same type $p \in S_4^4(T)$ have the same configuration. Conversely, the above proof of (I) also shows that any $p \in S_4^4(T)$ is determined by the configuration of any tuple (a_1, a_2, a_3, a_4) realizing it; in fact it follows that any $p \in S_4^4(T)$ is determined by the configurations of (a_i, a_j, a_k) for $1 \leq i < j < k \leq 4$ where (a_1, a_2, a_3, a_4) is any tuple which realizes p. In particular,

(II) if $p, q \in S_4^4(T)$ then $p = q$ if and only if $p \cap F^4_{(v_i,v_j,v_k)} = q \cap F^4_{(v_i,v_j,v_k)}$ for all $1 \leq i < j < k \leq 4$.

Hence $S_4^4(T)$ is finite. This also follows from the fact that T has a finite model. We will show that T is stable in L^4 by obtaining a contradiction from the assumption that some formula in L^4 has the order property with respect to T.

Suppose that $\varphi(\bar{x}, \bar{y}) \in L^4$ has the order property with respect to T. Without loss of generality we may assume that $\bar{x} = v_1, \ldots, v_s$, $\bar{y} = v_{s+1}, \ldots, y_{s+t}$, $1 \leq s, t \leq 3$, $s+t = 4$ and $\varphi(\bar{x}, \bar{y}) \vdash v_k \neq v_l$ if $1 \leq k, l \leq 4$ and $k \neq l$. Then there are $M \models T$ and $\bar{a}_i, \bar{b}_i \in M$ for $i < \omega$ such that $|\bar{a}_i| = |\bar{x}|$, $|\bar{b}_i| = |\bar{y}|$ for all $i < \omega$ and

$$M \models \varphi(\bar{a}_i, \bar{b}_j) \quad \Leftrightarrow \quad i \leq j \quad \text{for all } i, j < \omega.$$

By Ramsey's theorem and the fact that $S_4^4(T)$ is finite there are infinite subsequences $(\bar{a}'_i)_{i<\omega}$ and $(\bar{b}'_i)_{i<\omega}$ of $(\bar{a}_i)_{i<\omega}$ and $(\bar{b}_i)_{i<\omega}$, respectively, and $p(\bar{x}, \bar{y})$, $q(\bar{x}, \bar{y}) \in S_4^4(T)$ such that $p \neq q$, $M \models p(\bar{a}'_i, \bar{b}'_j)$ if $i \leq j$ and $M \models q(\bar{a}'_i, \bar{b}'_j)$ if $i > j$. By (II) there are $1 \leq k < l < m \leq 4$ and $p'(v_k, v_l, v_m), q'(v_k, v_l, v_m) \in S^4_{(v_k, v_l, v_m)}(T)$ such that $p' \subset p$, $q' \subset q$ and $p' \neq q'$. Then there are $\bar{c}_i, \bar{d}_i \in M$, for $i < \omega$, such that $|\bar{c}_i| = |\bar{c}_j|$, $|\bar{d}_i| = |\bar{d}_j|$, $|\bar{c}_i| + |\bar{d}_i| = 3$, for all $i, j < \omega$, and

(III) $M \models p'(\bar{c}_i, \bar{d}_j)$ if $i \leq j$ and $M \models q'(\bar{c}_i, \bar{d}_j)$ if $i > j$.

Suppose that $|\bar{c}_i| = 2$ and $|\bar{d}_i| = 1$. The other case is treated in a similar way.

If we assume that $\mathrm{R}(v_k, v_m) \in p'$ or $\mathrm{R}(v_l, v_m) \in p'$ then from (III) we will get a contradiction to the uniqueness part of axiom A_3. In the same way we get a contradiction if we assume that $\mathrm{R}(v_k, v_m) \in q'$ or $\mathrm{R}(v_l, v_m) \in q'$. Hence, $\neg\mathrm{R}(v_k, v_m), \neg\mathrm{R}(v_l, v_m) \in p'$ and $\neg\mathrm{R}(v_k, v_m), \neg\mathrm{R}(v_l, v_m) \in q'$

If we assume that $\mathrm{R}(v_k, v_l) \in p'$ then it follows from (III) that $\mathrm{R}(v_k, v_l) \in q'$. Since $p' \neq q'$ it follows (by (I)) that there is $i \in \{k, l\}$ such that $\mathrm{R}(v_i, v_m) \in p'$ or $\mathrm{R}(v_i, v_m) \in q'$ which we just showed is impossible. Hence we must have $\neg\mathrm{R}(v_k, v_l) \in p'$. By a symmetric argument we also have $\neg\mathrm{R}(v_k, v_l) \in q'$.

Now we have shown that $\neg\mathrm{R}(v_k, v_m)$, $\neg\mathrm{R}(v_l, v_m)$, $\neg\mathrm{R}(v_k, v_l) \in p' \cap q'$, and since $p' \neq q'$, the only alternatives are that every realization of p' has configuration (3) and every realization of q' has configuration (4), or vice versa. It is left to the reader to verify that in both cases the uniqueness part of axiom A_3 is contradicted.

CHAPTER II

Finite variable logic and elimination of quantifiers

This chapter continues the studies from the the first. Let's start by looking at a complete L^n-theory T such that T has an infinite model, $S^n_n(T)$ is finite, T has the (L^n, ∞)-amalgamation property and is stable in L^n. By Theorem 4.1 T has a complete extension T^{c} which is ω-categorical and ω-stable. Since $S^n_n(T)$ is finite, Proposition 2.9 and Lemma 2.11 implies that if $M \models T^{\mathrm{c}}$ then we can expand M by finitely many new relation symbols so that the expansion admits elimination of quantifiers. We say that a structure with this property *almost* admits elimination of quantifiers. The main results in this chapter, applied to T, are consequences of the fact that T has an infinite model which is stable and almost admits elimination of quantifiers. This observation suggests that we can generalize the notions of amalgamation and stability from the first chapter so that the main steps in the development there work out for all complete L^n-theories with such a model. In particular, Corollary 4.4 (i) can be stated in the more general setting. We do these things in Section 6.

Then we turn to the main subjects of this chapter, namely, for a "nice" L^n-theory T, investigating the structure of models of T and finding recursive upper bounds of the smallest model, computed from $|S^n_n(T)|$. In Section 7 the recursive bounds are derived from a decidability result due to Cherlin and Hrushovski.

The section thereafter is concerned with forking and strictly minimal sets in stable structures for which types of arbitrarily long tuples are determined by the types of all subtuples of length at most n (for some $n < \omega$). The main result, that in such structures which are also ω-categorical, strictly minimal sets are indiscernible, is a reformulation of a result due to Lachlan, but we will give a proof different from his.

In the final section we will use the theory of stable homogeneous structures, in the sense of [27],[25] and [28], to show how new models (finite and infinite) of a complete L^n-theory can be constructed if it has an infinite stable model which almost admits elimination of quantifiers. The method by which models are constructed is a generalization of the "shrinking" technique of Lachlan. We will also show that any sufficiently saturated substructure of an infinite stable structure M, which admits elimination of quantifiers in an expansion by finitely many new relation symbols of arity at most n, can be constructed in this way. The amount of saturation that is necessary can be effectively computed from the number of types in n free variables that are realized in M. Finally, we draw some conclusions from these results which more directly apply to the context of Section 6.

6 Amalgamation revisited

The amalgamation property studied in this section, i.e. the existence of a Φ-amalgamation class (a variation, among many, of a notion due to Fraïssé [13]), is different from the amalgamation properties studied in the previous chapter and [2], in, among other things, that we only require amalgamation for a possibly proper subclass of the class of all models of the theory in question, and that we don't restrict ourselves to L^n. The same approach as here is made in [18] but there some extra conditions are used which are not needed here because we impose some more restrictions (stronger than in [18]) on the set of formulas which we pay attention to when we amalgamate. Even though most of the work is carried out with some, less specified, set of formulas Φ instead of L^n it turns out that, in the situations that are of primary interest, Φ can be replaced by L^n, for some n. We finally arrive at the result that an L^n-theory, with a finite bound on its number of L^n-types, has an L^m-amalgamation class which is stable in L^m, for some m with $n \leq m < \omega$, if and only if the theory has a stable model which admits elimination of quantifiers in some expansion by finitely many new relation symbols. This generalizes Corollary 4.4 (i) and gives a way to generate examples of complete L^n-theories for which the main results of this chapter apply.

Let L be any countable first-order language and let $\Phi \subseteq L$. If M and N are L-structures and $a_i \in M$, $b_i \in N$, for $i < \lambda$, then we write

$$(M, (a_i : i < \lambda)) \equiv_\Phi (N, (b_i : i < \lambda))$$

if for every $m < \omega$ and $\varphi(x_1, \ldots x_m) \in \Phi$ and $\{i_1, \ldots, i_m\} \subseteq \lambda$,

$$M \models \varphi(a_{i_1}, \ldots, a_{i_m}) \text{ if and only if } N \models \varphi(b_{i_1}, \ldots, b_{i_m}).$$

A function $f : A \to N$, where $A \subseteq M$, is called a Φ-*elementary embedding* if for every $\varphi(\bar{x}) \in \Phi$ and $\bar{a} \in A$ with $|\bar{a}| = |\bar{x}|$, we have

$$M \models \varphi(\bar{a}) \quad \text{if and only if} \quad N \models \varphi(f(\bar{a})).$$

If M and N are L-structures and M is a substructure of N such that for every $\varphi(\bar{x}) \in \Phi$ and every $\bar{a} \in M$ with $|\bar{a}| = |\bar{x}|$, $M \models \varphi(\bar{a})$ if and only if $N \models \varphi(\bar{a})$, then we say that M is a Φ-elementary substructure of N and that N is a Φ-elementary extension of M, which we write as $M \preccurlyeq_\Phi N$. We say that a nonempty class, $\mathcal{A}$, of *countable* L-structures is a Φ-*amalgamation class* if:

1. $\mathcal{A}$ is closed under isomorphisms.

2. $\mathcal{A}$ is closed under Φ-elementary substructures, i.e. if $N \in \mathcal{A}$ and $M \preccurlyeq_\Phi N$ then $M \in \mathcal{A}$.

3. Whenever $M_1, M_2 \in \mathcal{A}$, $a_i \in M_1$, $b_i \in M_2$, for $i < k < \omega$, and

$$(M_1, (a_i : i < k)) \equiv_\Phi (M_2, (b_i : i < k)),$$

then there are $N \in \mathcal{A}$ and a Φ-elementary embedding $f : M_2 \to N$ such that $M_1 \preccurlyeq_\Phi N$ and $f(b_i) = a_i$ for all $i < k$.

We say that a theory T *has a Φ-amalgamation class* if there exists a Φ-amalgamation class $\mathcal{A}$ such that every structure in $\mathcal{A}$ is a model of T.

Remark 6.1 Observe that a Φ-amalgamation class need not have the joint embedding property (see [16] for a definition). Also note that if $\mathcal{A}'$ is a class of (not necessarily countable) structures which satisfies 1,2 and 3 and $\mathcal{A}$ is the class of all countable structures in $\mathcal{A}'$ then, by the downward Löwenheim-Skolem theorem, $\mathcal{A}$ is a Φ-amalgamation class. It follows that if a complete L^n-theory with infinite models has the L^n-amalgamation property ((L^n, ∞)-amalgamation property) then the class of all countable models of T (infinite countable models of T) is an L^n-amalgamation class.

We say that $p \subseteq \Phi$ is a $(\Phi, \mathcal{A})$-*type* if there are $M \in \mathcal{A}$ and a finite sequence $\bar{a} \in M$ such that

$$p = \{\varphi(\bar{x}) \in \Phi :\ M \models \varphi(\bar{a})\}.$$

If $\mathcal{A}$ is a Φ-amalgamation class and κ a cardinal then we say that M is $(\Phi, \kappa, \mathcal{A})$-*saturated* if whenever $\lambda < \kappa$, $b \in N \in \mathcal{A}$, $a_i \in M \cap N$, for $i < \lambda$, and

$$(M, (a_i : i < \lambda)) \equiv_\Phi (N, (a_i : i < \lambda)),$$

then there exists $c \in M$ such that $(M, c, (a_i : i < \lambda)) \equiv_\Phi (N, b, (a_i : i < \lambda))$.

If T is a theory and $\varphi(\bar{x})$ and $\psi(\bar{y})$ are two formulas, then let's say that φ and ψ are *essentially equivalent, modulo T*, written $\varphi \sim_T \psi$, if $\bar{x}$ and $\bar{y}$ have the same length and for all $M \models T$ and all $\bar{a} \in M$ of the same length as $\bar{x}$ and $\bar{y}$ we have $M \models \varphi(\bar{a})$ if and only if $M \models \psi(\bar{a})$. Note that $\sim_T$ is an equivalence relation (on any set of formulas). If $T = \emptyset$ then we just say that φ and ψ are *essentially equivalent* and write $\varphi \sim \psi$. We say that a set of formulas Φ is *essentially closed under subformulas* if for any $\varphi \in \Phi$ and subformula ψ of φ, there is a formula in Φ which is essentially equivalent to ψ.

Lemma 6.2 *Suppose that Φ is essentially closed under subformulas and that $M_i \preccurlyeq_\Phi M_{i+1}$, for $i < \kappa$. Let $M = \bigcup_{i<\kappa} M_i$. Then $M_i \preccurlyeq_\Phi M$ for all $i < \kappa$.*

Proof. Induction on complexity of formulas in Φ. □

Lemma 6.3 *Suppose Φ is essentially closed under subformulas. If $\mathcal{A}$ is a Φ-amalgamation class such that the set of all $(\Phi, \mathcal{A})$-types is countable, then there exists a structure M, such that*

(i) *M is $(\Phi, \omega, \mathcal{A})$-saturated,*

(ii) *for every finite $\bar{a} \in M$ there exists $N \in \mathcal{A}$ such that $\bar{a} \in N$ and $(M, \bar{a}) \equiv_\Phi (N, \bar{a})$, and*

(iii) *if T is an L-theory such that $T \subseteq \Phi$ and all structures in $\mathcal{A}$ are models of T then $M \models T$.*

Proof. We use the idea in the proof of Fraïssé's theorem in [16] to construct $M_i \in \mathcal{A}$, for $i < \omega$, such that

$M_i \preccurlyeq_\Phi M_{i+1}$, for all $i < \omega$, and

for any $i < \omega$, finite $\bar{a} \in M_i$ and $N \in \mathcal{A}$, if $\bar{a}, b \in N$ and $(M_i, \bar{a}) \equiv_\Phi (N, \bar{a})$, then there exists $j \geq i$ and $c \in M_j$ such that $(M_j, \bar{a}c) \equiv_\Phi (N, \bar{a}b)$.

Let $\pi : \omega^3 \to \omega$ be a bijection such that $\pi(i,j,k) \geq i,j,k$ for all i,j,k and let p_k, $k < \omega$, be an enumeration of all $(\Phi, \mathcal{A})$-types. Let $M_0 \in \mathcal{A}$ be arbitrary. Now suppose that M_i is defined for all $i < \ell + 1$, $M_i \preccurlyeq_\Phi M_{i+1}$ for all $i < \ell$ and that $\bar{a}_i^j$, $j < \omega$, is an enumeration of all finite sequences of elements from M_i, for $i < \ell + 1$. Suppose that $\ell = \pi(i,j,k)$. If there exists $N \in \mathcal{A}$ and $b \in N$ such that $\bar{a}_i^j \in N$, $p_k = \{\varphi(\bar{x}, y) \in \Phi : \ N \models \varphi(\bar{a}_i^j, b)\}$ and $(M_i, \bar{a}_i^j) \equiv_\Phi (N, \bar{a}_i^j)$, then $(M_\ell, \bar{a}_i^j) \equiv_\Phi (N, \bar{a}_i^j)$ so, by condition 3 in the definition of a Φ-amalgamation class, there are $M_{\ell+1} \in \mathcal{A}$ and $c \in M_{\ell+1}$ such that $M_\ell \preccurlyeq_\Phi M_{\ell+1}$ and $(M_{\ell+1}, \bar{a}c) \equiv_\Phi (N, \bar{a}b)$. Otherwise let $M_{\ell+1} = M_\ell$.

Let $M = \bigcup_{i<\omega} M_i$. Since Φ is closed under subformulas it follows from Lemma 6.2 that $M_i \preccurlyeq_\Phi M$, for all $i < \omega$, and from this we get (i). (ii) and (iii) follows from the construction of M. $\square$

If $\mathcal{A}$ is a Φ-amalgamation class as in Lemma 6.3 and M satisfies (i) and (ii) in the same lemma, then we say that M is a *limit of* $\mathcal{A}$.

Lemma 6.4 *Suppose that Φ is essentially closed under subformulas and that every atomic formula of L is essentially equivalent to a formula in Φ. Let $\mathcal{A}$ be a Φ-amalgamation class and suppose that M and N are limits of $\mathcal{A}$. Then for any finite sequences $\bar{a} \in M$ and $\bar{b} \in N$ with $|\bar{a}| = |\bar{b}|$,*

$$\textit{if} \quad (M, \bar{a}) \equiv_\Phi (N, \bar{b}) \quad \textit{then} \quad (M, \bar{a}) \equiv_L (N, \bar{b}). \qquad (*)$$

Proof. Back and forth argument as in the proof of Proposition 2.6, where we replace L^n by Φ and use the assumption that M and N are limits of $\mathcal{A}$ and an appropriate variation of Lemma 2.3. $\square$

If $(*)$ in Lemma 6.4 holds with $M = N$ and for all $\bar{a}, \bar{b} \in M$, with $|\bar{a}| = |\bar{b}|$, then we say that M is Φ-*determined*.

Remark 6.5 It is possible that $\mathcal{A}$ as in Lemma 6.3 has two limits M_1, M_2 such that $M_1 \not\equiv M_2$, or even $M_1 \not\equiv_\Phi M_2$. But if $\mathcal{A}$ is an L^n-amalgamation class such that all structures in $\mathcal{A}$ are L^n-elementarily equivalent, the vocabulary of L does not contain any function symbols and the arity of every relation symbol $\leq n$, then by Lemma 6.3 (ii) (applied to $\bar{a}$ = 'the empty sequence') and Lemma 6.4 it follows that any two limits of $\mathcal{A}$ are elementarily equivalent.

Lemma 6.6 *If $T \subseteq \Phi$ is a set of sentences and M a model of T which is Φ-determined and either finite, or infinite and ω-saturated, then T has a Φ-amalgamation class.*

Proof. Let $\mathcal{A}$ be the class of all countable $N \preccurlyeq_\Phi M$ and all structures isomorphic to these. It is not difficult to verify that $\mathcal{A}$ is a Φ-amalgamation class $\mathcal{A}$ such that every structure in $\mathcal{A}$ is a model of T. $\square$

We say that a structure M *almost admits elimination of quantifiers* (or *quantifier elimination*) if some expansion of M by finitely many new relation symbols admits elimination of quantifiers. It is easy to see (using the Ryll-Nardzewski theorem) that if M is an L-structure which almost admits elimination of quantifiers and the vocabulary of L is finite and contains no function symbols, then M is ω-categorical or finite. Also observe that (by a back and forth argument) a countable structure M, in a language with no function symbols, admits elimination of quantifiers if and only if every partial isomorphism between finite substructures of M extends to an automorphism on M. Countable structures that satisfy these two conditions are called *homogeneous* in [26], [27], [25] and [28], which we will refer to later.

If Φ is a set of formulas and $k < \omega$ then let

$$\Phi_k = \{\varphi \in \Phi : \text{ at most } k \text{ distinct variables occur free in } \varphi\}.$$

Proposition 6.7 *Let T be an L-theory and let Φ be a set of L-formulas such that*
(1) *$T \subseteq \Phi$.*
(2) *Φ is essentially closed under subformulas and every atomic formula in L is essentially equivalent to a formula in Φ.*
(3) *For every $k < \omega$, $\Phi_k/\sim_T$ is finite.*
The following are equivalent :

(i) *T has a Φ-amalgamation class.*

(ii) *There exists $M \models T$ which is Φ-determined, and hence, if M is infinite, then M is ω-categorical.*

Remark 6.8 If, in addition to (1), (2) and (3), Φ satisfies:

(4) there is $n < \omega$ such that every formula in Φ is essentially equivalent to a boolean combination of formulas in Φ_n (so in particular, if $\Phi = \Phi_n$),

then M from (ii) in Proposition 6.7 will be Φ_n-determined and therefore almost admit quantifier elimination.

Proof of Proposition 6.7. (i) $\Rightarrow$ (ii) follows from Lemma 6.3, Lemma 6.4 and the third condition on Φ. Now suppose that (ii) holds. If M is infinite then M is ω-saturated, because M is ω-categorical, so (i) follows by applying Lemma 6.6. $\square$
The next lemma and the following proposition show that the generality which is a priori gained by considering arbitrary $\Phi \subseteq L$ instead of L^n is actually rather vacuous in the cases that we are most interested in.

Lemma 6.9 *If $\Phi \subseteq L$ is essentially closed under subformulas and every formula in Φ has at most n free variables then every formula in Φ is essentially equivalent to a formula in L^n. In particular, every sentence in Φ is equivalent to a sentence in L^n.*

Proof. By induction on complexity of formulas in Φ. $\square$

Proposition 6.10 *Let $\Phi \subseteq L$ and T be an L-theory. Suppose that Φ and T satisfy the conditions* (1) - (4)*, above. If T has a Φ-amalgamation class then T has an L^n-amalgamation class.*

Proof. Suppose that Φ and T satisfy (1) - (4). If there exists a Φ-amalgamation class in which all structures are models of T then by Proposition 6.7 and Remark 6.8 there exists $M \models T$ which almost admits elimination of quantifiers. We may assume that M is countable, so if M is infinite then it is ω-saturated. By Lemma 6.9 every formula in Φ_n is essentially equivalent to an L^n-formula. Therefore M is L^n-determined so by Lemma 6.6 there exists an L^n-amalgamation class in which every structure is a model of T. $\square$

If $\mathcal{A}$ is a Φ-amalgamation class, then we say that $\mathcal{A}$ is *stable in* Φ if for every $\varphi(\bar{x}, \bar{y}) \in \Phi$ there exists $k < \omega$ such that there does *not* exist $M \in \mathcal{A}$ and $\bar{a}_i, \bar{b}_i \in M$, for $i < k$, satisfying $M \models \varphi(\bar{a}_i, \bar{b}_j) \;\Leftrightarrow\; i \leq j$. Let's adopt the convention that every finite structure is *stable*. By one of the usual definitions we say that an infinite L-structure M is *stable* if there does *not* exist $\varphi(\bar{x}, \bar{y}) \in L$, $N \equiv M$ and $\bar{a}_i, \bar{b}_i \in N$, for $i < \omega$, such that $N \models \varphi(\bar{a}_i, \bar{b}_j) \;\Leftrightarrow\; i \leq j$.

Proposition 6.11 *Suppose that L has a finite vocabulary with no function symbols and in which all relation symbols have arity at most n. Let T be an L^n-theory. Then T has a stable model which almost admits quantifier elimination if and only if for some $m \geq n$, there are a complete L^m-theory $T' \vdash T$, such that $S^m_m(T')$ is finite and T' has an L^m-amalgamation class which is stable in L^m.*

Proof. Suppose that T has a stable model M which almost admits quantifier elimination. Then for some $m \geq n$, M is L^m-determined and as claimed in the proof of Lemma 6.6 the class $\mathcal{A}$ of all countable $N \preccurlyeq_{L^m} M$ and all structures isomorphic to these is an L^m-amalgamation class such that every structure in $\mathcal{A}$ is a model of $T' = Th_{L^m}(M)$. By the fact that M almost admits elimination of quantifiers and the choice of m, $S^m_m(T')$ is finite. From the stability of M and the definition of $\mathcal{A}$ it also follows that $\mathcal{A}$ is stable in L^m.

Now suppose that $m \geq n$, $T' \vdash T$ is a complete L^m-theory such that $S^m_m(T')$ is finite and that $\mathcal{A}$ is an L^m-amalgamation class which is stable in L^m and every structure in $\mathcal{A}$ is a model of T'. By Lemma 6.3, $\mathcal{A}$ has a limit, say M. Lemma 6.4 implies that M is L^m-determined. Since $S^m_m(T)$ is finite M almost admits elimination of quantifiers. If M is not stable then M is infinite and by Ramsey's theorem and the facts that M is L^m-determined and $S^m_m(T)$ finite it follows that there exists $\varphi(\bar{x}, \bar{y}) \in L^m$ such that for all $k < \omega$ there are $\bar{a}^k_i, \bar{b}^k_i \in M$, for $i \leq k$, such that $M \models \varphi(\bar{a}^k_i, \bar{b}^k_j) \;\Leftrightarrow\; i \leq j$. Since, by definition, a limit of $\mathcal{A}$ satisfies (ii) in Lemma 6.3, it follows that, for every $k < \omega$, there is $N_k \in \mathcal{A}$ such that $\bar{a}^k_i, \bar{b}^k_i \in N_k$, for $i \leq k$, and

$$(N_k, \bar{a}^k_0\bar{b}^k_0 \ldots \bar{a}^k_k\bar{b}^k_k) \equiv_{L^m} (M, \bar{a}^k_0\bar{b}^k_0 \ldots \bar{a}^k_k\bar{b}^k_k),$$

contradicting that $\mathcal{A}$ is stable in L^m. $\square$

Remark 6.12 Suppose that L and n are as in Proposition 6.11. If T in an L^n-theory that satisfies the two equivalent conditions in Proposition 6.11 then, by

Theorem 4.2 and the fact that every infinite stable structure which almost admits quantifier elimination is ω-categorical and ω-stable, it follows that T has arbitrarily large finite models. By Remark 6.1 this is a generalization of Corollary 4.4 (i).

Proposition 6.11 gives us a way of generating examples of complete L^n-theories T such that $S_n^n(T)$ is finite and there exists an L^n-amalgamation class which is stable in L^n and in which every structure is a model of T: Let M be an infinite and stable L-structure that almost admits elimination of quantifiers. For some $k < \omega$, there will be an expansion of M by finitely many new relation symbols which admits elimination of quantifiers and such that every symbol in the expanded language has arity at most k. There exists a number $\ell < \omega$ such that every type in $S_k(Th_L(M))$ is isolated by a formula in L^ℓ. For every natural number $n \geq \ell$, if $T = Th_{L^n}(M)$ then (by the proof of Proposition 6.11) T satisfies the above conditions.

A few examples of stable structures that admit elimination of quantifiers are given in [25] and [28] (where a countable structure that admits elimination of quantifiers is called 'homogeneous'). Some of those examples are not infinite but of arbitrarily large finite cardinality. From the (easily verified) fact that the class of all L-structures which admit elimination of quantifiers, where L is a language with finite relational vocabulary, is axiomatizable and the main result in [8] (also described in [25] and [28]) it follows that if $\mathcal{C}$ is a class of finite L-structures which admit elimination of quantifiers then any model of the theory of $\mathcal{C}$ (i.e. the set of L-sentences that are true in all members of $\mathcal{C}$) is stable. Consequently, we can use the examples in [28] which are uniformly described for arbitrarily large cardinalities to get, by compactness, infinite stable structures which admit elimination of quantifiers. In the same way; from the fact that there are arbitrarily large finite directed graphs admitting quantifier elimination which are not trivial, by which I mean that they have two vertices that are adjacent and two vertices that are not adjacent, it follows that there are nontrivial directed graphs which are infinite, stable and admit elimination of quantifiers. Finite directed graphs that admit elimination of quantifiers are studied in detail in [26].

It follows from Proposition 6.7 that the framework for studying L^n-theories which was developed in Chapter I and continued here is too narrow to include L^n-theories for which no infinite model is L^n-determined. Consider the following example: Let V be an infinite vector space over a finite field F, and consider V as an L-structure by letting the vocabulary of L consist of a binary function symbol, interpreted in V as vector addition, and for every $f \in F$ a unary function symbol f, interpreted in V as scalar multiplication by f. Let $T_V^n = Th_{L^n}(V)$, where $n \geq 3$. Then every model of T_V^n is a vector space over F, so T_V^n is stable in L^n and $S_n^n(T_V^n)$ is finite. Suppose that there exists an L^n-amalgamation class $\mathcal{A}$ in which every structure is a model of T_V^n and $\mathcal{A}$ has a limit M such that $|M| \geq |F|^{n+1}$. Since $M \models T_V^n$ (by Lemma 6.3 (iii)) M is a vector space over F and since M is big enough we can find $n+1$ linearly independent elements $a_1, \ldots, a_{n+1} \in M$. Then the tuples $(a_1, \ldots a_n, a_{n+1})$ and $(a_1, \ldots, a_n, b)$, where $b = a_1 + \ldots + a_n$, have the same L^n-type but different L-types. This contradicts Lemma 6.4, so an L^n-amalgamation class like $\mathcal{A}$ can not exist. If we want a similar example in a language with no function symbols then we can replace the function symbols in the above example by relation symbols in the obvious way and assume that $n \geq 7$.

In Section 8 we will see that the type of amalgamation considered here rules out all examples (of theories) for which all infinite models include (in an extension by imaginary elements) a nontrivial strictly minimal set. Since T_V^n, as above, is an axiomatization of vector spaces over F, several nice properties follow, such as stability, finiteness of $S_m(T_V^n)$, for any $m < \omega$, and the existence of arbitrarily large finite models. This motivates looking for a more general notion of amalgamation which is satisfied by T_V^n (and similar theories). Baldwin and Lessmann [2] have done this and found a notion of amalgamation which generalizes the L^n-amalgamation property from Chapter I so that, for example T_V^n, satisfies this property. Also Hyttinen [21] has recently generalized the framework that appears here and given a couple of interesting examples which are then included. Most results in the remaining sections do not automatically carry over to theories within the frameworks of [2] or [21] because such theories need not have models that almost admit quantifier elimination. Corollary 7.2 can, however, sometimes be applied to theories which do not have any of the amalgamation properties considered here. For example, if $n \geq 4$ (or $n \geq 8$ in a relational language) it follows from Corollary 7.2 that T_V^n has a model of size at most $\mathfrak{f}(n, |S_n^n(T_V^n)|)$, where $\mathfrak{f}$ is defined in the next section.

The results for L^n-theories in this chapter rely on the existence of stable models that almost admit quantifier elimination and we have a correspondence between "stability-amalgamation" and the existence of stable models that almost admit quantifier elimination (Proposition 6.11). Since stable structures that almost admit quantifier elimination are smoothly approximable (to be defined below) and the theory of smoothly approximable structures is a generalization of the theory of stable structures that almost admit elimination of quantifiers, it is conceivable that one can find a correspondence between "simplicity-amalgamation" and the existence of smoothly approximable models (of an L^n-theory), which would give similar results about L^n-theories as the ones we will see in the next few sections. I wrote "simplicity-amalgamation" because smoothly approximable structures need not be stable, but they are always simple. Definitions and results concerning simple theories can be found in [24]. The property 'smoothly approximable' is defined below, and results about structures having it are found in [7] and [23].

7 Recursive bounds

In this section we use a decidability result from the theory of smoothly approxiamable structures to derive a couple of results about recursive bounds on the size of the least model of particular kinds of sentences or L^n-theories. The relevant theorem is the following one, due to Cherlin and Hrushovski [7]:

Theorem 7.1 *We can effectively decide whether for a given sentence and $k < \omega$, that sentence has a finite model M such that $|S_4(Th(M))| = k$.*

We define a recursive function $\mathfrak{f} : \omega^2 \to \omega$ as follows. Let $\mathfrak{f}(n, k) = k$ if $n < 2$ or $k = 0$. Now suppose that $n \geq 2$ and $k \geq 1$. Let $\varphi_1, \ldots, \varphi_m$ be an enumeration of all sentences (up to equivalence) of quantifier rank at most $k + n$ in a language with k constant symbols and k i-ary relation symbols for every $1 \leq i \leq n$, and we assume that $=$ is one of the binary relation symbols. For $1 \leq i \leq m$ use Theorem 7.1 to decide if φ_i has a finite model M_i such that $|S_4(Th(M_i))| \leq k$; if such a model of φ_i exists then search until we find such M_i and let $\ell_i = |M_i|$; otherwise let $\ell_i = 0$. Then let $\mathfrak{f}(n, k) = \max\{\ell_1, \ldots, \ell_m\}$.

We say that an L-structure M is *smoothly approximable* if M is ω-categorical and if any L-sentence which is true in M is true in a finite substructure $N \subseteq M$ such that

> for any $\theta(\bar{x}) \in L$ there is $\chi(\bar{x}) \in L$ such that $\{\bar{a} \in N : M \models \theta(\bar{a})\} = \{\bar{a} \in N : N \models \chi(\bar{a})\}$, and
>
> any $\bar{a}, \bar{b} \in N$, with $|\bar{a}| = |\bar{b}|$, have the same type in N if and only if they have the same type in M.

Corollary 7.2 *Let $n \geq 4$ and let φ be a sentence such that no function symbols occur in φ, the arity of every relation symbol in φ is at most n and at most n distinct variables occur in φ. If $M \models \varphi$, where M is smoothly approximable and $|S_4^n(Th(M))| = |S_4(Th(M)|$, then φ has a finite model of cardinality at most $\mathfrak{f}(n, |S_n^n(Th(M))|)$.*

Proof. Suppose that φ, n and M satisfies the premises of the corollary. Without loss of generality we may assume that M is an L-structure and that φ is an L-sentence, where L is the language that occurs in the definition of $\mathfrak{f}$. Since M is smoothly approximable it follows that M has a finite substructure N such that $N \models \varphi$ and $|S_4(Th(N))| \leq |S_4(Th(M))|$. By Fact 1.4, $Th_{L^n}(M)$ is axiomatized by an L^n-sentence with quantifier rank at most $|S_n^n(Th(M))| + n$. Since $|S_4(Th(M))| = |S_4^n(Th(M))| \leq |S_n^n(Th(M))|$ it follows from the definition of $\mathfrak{f}$ that $Th_{L^n}(M)$ and hence φ has a finite model of cardinality at most $\mathfrak{f}(n, |S_n^n(Th(M))|)$. □

Corollary 7.3 *Let $n \geq 4$ and let L be a language with finite vocabulary which contains no function symbols and in which all relation symbols have arity at most n. If T is a complete L^n-theory such that $S_n^n(T)$ is finite and T has an L^n-amalgamation class which is stable in L^n, then T has a model of cardinality at most $\mathfrak{f}(n, |S_n^n(T)|)$.*

Proof. Suppose that T satisfies the above conditions. First note that (by Fact 1.4) T is axiomatized by an L^n-sentence with quantifier rank $\leq |S_n^n(T)| + n$. By the same argument as in the proof of Proposition 6.11 it follows that T has a stable model M which is L^n-determined, so $|S_4^n(Th(M))| = |S_4(Th(M)|$. Since (by results in [27]) every stable L-structure which almost admits elimination of quantifiers is smoothly approximable it follows from Corollary 7.2 that T has a model of cardinality at most $\mathfrak{f}(n, |S_n^n(T)|)$. □

Corollary 7.3 can also be proved by using weaker decidability results than Theorem 7.1 from the theory of countable stable structures that almost admit quantifier elimination (see [28]), which the theory of smoothly approximable structures generalizes.

Remark 7.4 If $1 < n < 4$ then Corollary 7.2 and Corollary 7.3 still hold if we replace $S_n^n(Th(M))$ by $S_4^n(Th(M))$ and $S_n^n(T)$ by $S_4^n(T)$, respectively.

Grohe [14] has shown that for $n \geq 3$ there does *not* exist a recursive function $f_n : \omega \to \omega$ such that for every complete L^n-theory T with finite models, $\min\{|M| : M \models T\} \leq f_n(|S_n^n(T)|)$. Hence we have a nonexistence result and a couple of existence results. Except for Corollary 7.2 above (from which Corollary 7.3 follows) an existence result is obtained by Dawar in [9]. We may ask: How general can a class, $\mathcal{T}$, of complete L^n-theories with finite models be if we require that there exists a recursive function f such that $\min\{|M| : M \models T\} \leq f(|S_n^n(T)|)$ for all $T \in \mathcal{T}$? Another problem is to determine such a function f more precisely, perhaps starting with some smaller class of theories on which we have more control. I recently learned that progress in this direction has been made by Julián Mariño.

8 Forking and strictly minimal sets

In this section we will study forking and strictly minimal sets in structures which are n-determined (as defined below). The main result, Proposition 8.7, is a corollary to a result of Lachlan (Lemma 8.2 in [27]). We will give a different proof which uses the properties of forking in the particular context and a trichotomy theorem for strictly minimal sets. The results of this section apply to any limit of an amalgamation class $\mathcal{A}$ as in Proposition 6.11. In particular, by the argument at the very beginning of this chapter, they apply to any model of T^{c} from Theorem 4.1.

Throughout this section we assume that n is an integer such that $n \geq 2$. We say that a structure M is *n-determined* if for any m such that $n \leq m < \omega$ and any $a_1, \ldots, a_m, b_1, \ldots, b_m \in M$, if

$$(M, (a_{i_1}, \ldots, a_{i_n})) \equiv (M, (b_{i_1}, \ldots, b_{i_n}))$$

whenever $1 \leq i_1 < \ldots < i_n \leq m$, then

$$(M, (a_1, \ldots, a_m)) \equiv (M, (b_1, \ldots, b_m)).$$

For example, every structure that almost admits quantifier elimination, in a language where there is a bound on the arity of all symbols, is n-determined for some $n < \omega$. In particular, this holds for every model of T^{c} from Section 2.

In what follows, when M is an L-structure we will often, as is customary in stability theory, work in the L^{eq}-structure M^{eq} where the many sorted language $L^{\mathrm{eq}} \supseteq L$ and M^{eq} are obtained from M and L as described in [1] (or one of the other standard books on stability theory). As before, L will always denote a one-sorted first order language. The notion of type extends straightforwardly to L^{eq}. If $A \subseteq M^{\mathrm{eq}}$ then $S(A)$ denotes the set of L^{eq}-types over A with respect to M. $S(A)$ depends on $Th_L(M, A)$ but the M we have in mind will be clear from the context. The variables which occur in formulas in L, that is, the variables in V in the beginning of Section 1, are called variables of sort $=$. If $A \subseteq M$, $p(\bar{x}) \in S(A)$ and $\bar{x}$ is a finite sequence of variables of sort $=$ then $p(\bar{x})$ is an $(L, \bar{x})$-type in the sense of Section 1.

When we write $\bar{a} \in M$ then we mean (as usual) that $\bar{a}$ is a finite sequence of elements, all of which belong to M. If we write $\bar{a} \in M^{\mathrm{eq}}$ we mean that $\bar{a}$ is a finite sequence of elements, all of which belong to M^{eq} (but not necessarily to M). $\mathrm{rng}(\bar{a})$ is the set of elements occurring in the sequence $\bar{a}$, i.e. the range of $\bar{a}$. If $p(\bar{x})$ is a type over the set B and $A \subseteq B$ then $p \restriction A = \{\varphi(\bar{x}, \bar{a}) : \varphi(\bar{x}, \bar{a}) \in p \text{ and } \bar{a} \in A\}$.

Let acl and dcl denote the *algebraic closure* and *definable closure* respectively, in M^{eq}, where M is the structure that we have in mind. We often write $\mathrm{acl}(\bar{a})$ $(\mathrm{dcl}(\bar{a}))$ for $\mathrm{acl}(\mathrm{rng}(\bar{a}))$ $(\mathrm{dcl}(\mathrm{rng}(\bar{a})))$.

R denotes the Morley rank (in the structure that we have in mind) and if $\bar{a} \in M^{\mathrm{eq}}$ and $A \subseteq M^{\mathrm{eq}}$ then $\mathrm{R}(\bar{a}/A) = \mathrm{R}(tp(\bar{a}/A))$. U denotes the U-rank and if $\bar{a} \in M^{\mathrm{eq}}$ and $A \subseteq M^{\mathrm{eq}}$ then $\mathrm{U}(\bar{a}/A) = \mathrm{U}(tp(\bar{a}/A))$. Recall that if M is ω-stable then $\mathrm{R}(p) = \mathrm{U}(p)$ for any type p (over any subset of M). Basic knowledge about forking and these ranks (see [1]) is assumed in this section and the next. For the rest of this section we fix some (arbitrary) countable language L and all structures considered are L-structures.

Lemma 8.1 *Suppose that M is n-determined. Let $A \subseteq M$, $p(\bar{x}), q(\bar{x}) \in S(A)$, where $\bar{x}$ is a finite sequence of variables of sort $=$, and assume that p and q are realized in M. If for every $A' \subseteq A$ such that $|A'| \leq n-1$, $p \restriction A' = q \restriction A'$ then $p = q$.*

Proof. Suppose that for every $A' \subseteq A$ such that $|A'| \leq n-1$, $p \restriction A' = q \restriction A'$. Since M is n-determined and p and q are realized in M it follows that for every *finite* $A'' \subseteq A$, $p \restriction A'' = q \restriction A''$ and this implies that $p = q$. □

Proposition 8.2 *Suppose that M is stable, κ-saturated and n-determined. Let $A, B \subseteq M$, where $|A|, |B| < \kappa$, and $p(\bar{x}) \in S(A \cup B)$, where $\bar{x}$ is a finite sequence of variables of sort $=$. If p forks over B then for some $A' \subseteq A$ with $|A'| \leq n-1$, $p \restriction A' \cup B$ forks over B.*

Proof. Suppose that p forks over B. We may assume that A is finite. Let $C = \mathrm{acl}(B)$ and let $p' \in S(A \cup C)$ be an extension of p. Then p' forks over C. It is sufficient to show that for some $A' \subseteq A$ with $|A'| \leq n-1$, $p' \restriction A' \cup C$ forks over C. Towards a contradiction suppose that $|A| > n-1$ and that $p' \restriction A' \cup C$ does not fork over C, for every $A' \subseteq A$ such that $|A'| \leq n-1$. Now let $A' \subseteq A$ be arbitrary such that $|A'| \leq n-1$. Let $q(\bar{x}) \in S(A \cup C)$ be the nonforking extension of $p' \restriction C$. Then $q \restriction A' \cup C$ does not fork over C. But $p' \restriction C = q \restriction C$ is stationary, since C is algebraically closed, so

$$p' \restriction A' \cup C = q \restriction A' \cup C.$$

which gives

$$p' \restriction A' \cup B = q \restriction A' \cup B.$$

Since the above equality holds for any $A' \subseteq A$ such that $|A'| \leq n-1$ it follows from Lemma 8.1 that $p = p' \restriction A \cup B = q \restriction A \cup B$ which contradicts that p forks over B but $q \restriction A \cup B$ does not. □

Corollary 8.3 *Suppose that M is stable, κ-saturated and n-determined. Let $A, B \subseteq M$, where $|A|, |B| < \kappa$, $p(\bar{x}) \in S(B)$ and $q(\bar{x}) \in S(A \cup B)$, where $\bar{x}$ is a finite sequence of variables of sort $=$. If q extends p, $\mathrm{U}(p) < \omega$ and $\mathrm{U}(p) - \mathrm{U}(q) = k$, then there exists $A' \subseteq A$ such that $|A'| \leq k(n-1)$ and $\mathrm{U}(q \restriction A' \cup B) = \mathrm{U}(q)$.*

Proof. By induction on k. For $k = 0$ the corollary is trivial. Suppose that the corollary holds for all $l \leq k$ and that $\mathrm{U}(p) - \mathrm{U}(q) = k + 1$. Then q is a forking extension of p so by Proposition 8.2 there is $A_1 \subseteq A$ such that $|A_1| \leq n - 1$ and $q{\restriction}A_1 \cup B$ forks over B. If $\mathrm{U}(q{\restriction}A_1 \cup B) = \mathrm{U}(q)$ then $A' = A_1$ satisfies the corollary so now suppose that $\mathrm{U}(q{\restriction}A_1 \cup B) > \mathrm{U}(q)$. Let $l = \mathrm{U}(q{\restriction}A_1 \cup B) - \mathrm{U}(q)$. Then $l \leq k$ so by the induction hypothesis there is $A_2 \subseteq A - A_1$ such that $|A_2| \leq l(n-1)$ and $\mathrm{U}(q{\restriction}A_2 \cup A_1 \cup B) = \mathrm{U}(q)$. Now $A' = A_1 \cup A_2$ satisfies the corollary. $\square$

We immediately get:

Corollary 8.4 *Suppose that M is stable, κ-saturated and n-determined. Let $A, B \subseteq M$, $|A|, |B| < \kappa$. If $a \in \mathrm{acl}(A \cup B) \cap M$ and $\mathrm{U}(a/B) = k$, where $k < \omega$, then there is $A' \subseteq A$ such that $|A'| \leq k(n-1)$ and $a \in \mathrm{acl}(A' \cup B)$.*

We will prove the final result of this section by using the properties of forking under the given conditions together with the next theorem and the following fact about projective and affine geometries. The theorem is a trichotomy theorem for strictly minimal sets which is due to Cherlin, Mills and Zilber, independently. For its proof, the reader may consult [6] or [34].

Theorem 8.5 *Let M be an ω-categorical and ω-stable structure. If H is a $\emptyset$-definable strictly minimal set in M^{eq} then the geometry of H is isomorphic to one of the following:*
1. The trivial (also called degenerate) geometry.
2. A projective geometry of infinite dimension over a finite field.
3. An affine geometry of infinite dimension over a finite field.

A proof of the following fact is given since I could not find one in the litterature.

Fact 8.6 *If (G, cl) is either a projective geometry of infinite dimension over a finite field or an affine geometry of infinite dimension over a finite field, then for every $m < \omega$ there are $k > m$ and distinct $a_1, \ldots, a_k, a \in G$ such that $\{a_1, \ldots, a_k\}$ is independent, $a \in \mathrm{cl}(\{a_1, \ldots, a_k\})$ and for every proper subset A of $\{a_1, \ldots, a_k\}$, $a \notin \mathrm{cl}(A)$.*

Proof. First suppose that (G, cl) is a projective geometry of infinite dimension over a finite field, where cl denotes the closure operation on subsets of G. Then we may identify G with the set of all one dimensional subspaces of an infinite dimensional vector space V over a finite field F, and for $A \subseteq G$, $\mathrm{cl}(A)$ is the set of one dimensional subspaces of the subspace of V spanned by $\bigcup A$. Let $2 \leq k < \omega$ and take a linearly independent subset $\{v_1, \ldots, v_k\}$ of V, where we assume $v_i \neq v_j$ if $i \neq j$, and let $v = v_1 + \ldots + v_k$. Then v is not in the linear span of any proper subset of $\{v_1, \ldots, v_k\}$. If we now let $a_i \in G$ be the subspace spanned by v_i, for $1 \leq i \leq k$, and let $a \in G$ be the subspace spanned by v then $a_1, \ldots, a_k, a$ satisfy the conclusion of the fact.

Now suppose that (G, cl) is an affine geometry of infinite dimension over a finite field. Then we may identify G with an infinite dimensional vector space V over a finite field F, and for $A \subseteq G$, $\mathrm{cl}(A)$ is the least affine subspace of V which includes

A, where an affine subspace of V is a subset of the form $\{w + v : v \in W\}$ where $w \in V$ and W is a subspace of V. If $A \subseteq V$ then let $\langle A \rangle$ denote the subspace spanned by A. Two elementary results about affine spaces (see [30], chapter 15, and note that, although only finite dimensional vector spaces are considered there, the relevant results hold in our setting) are that, for any $v_1, \dots v_m \in V$ (where $m < \omega$), if $1 \leq i \leq m$ then

$$\mathrm{cl}(\{v_1, \dots, v_m\}) = v_i + \langle \{v_1 - v_i, \dots, v_{i-1} - v_i, v_{i+1} - v_i, \dots, v_m - v_i\} \rangle,$$

and if $v_1, \dots v_m$ are linearly independent then $v_1, \dots v_m$ are independent with respect to cl. Let p be the characteristic of F and let $k = pr + 1$ where r is an arbitrary positive integer. Take a subset $\{v_1, \dots, v_k\} \subset V$ which is linearly independent. Then $\{v_1, \dots, v_k\}$ is also independent with respect to cl. Let $w = v_k + (v_1 - v_k) + \dots + (v_{k-1} - v_k)$. Then $w \in \mathrm{cl}(\{v_1, \dots, v_k\})$ and since $k-1 = pr$ where p is the characteristic of F we have

$$w = v_1 + \dots + v_{k-1} + v_k - (\underbrace{v_k + \dots + v_k}_{k-1 \text{ times}}) = v_1 + \dots + v_{k-1} + v_k. \qquad (*)$$

Towards a contradiction suppose that for some proper subset $A \subset \{v_1, \dots, v_k\}$, $w \in \mathrm{cl}(A)$. Without loss of generality we may suppose that

$$A = \{v_1, \dots, v_k\} \ - \{v_i\}$$

for some $1 \leq i \leq k$. Then, for any $j \in \{1, \dots, k\} - \{i\}$, if

$$A_j = \big\{ v - v_j : v \in A - \{v_j\} \big\}$$

then $\mathrm{cl}(A) = v_j + \langle A_j \rangle$, so, since $w \in \mathrm{cl}(A)$, w is a linear combination of $v_1, \dots, v_{i-1}$, $v_{i+1}, \dots, v_k$. But $(*)$ now implies that v_i is a linear combination of $v_1, \dots, v_{i-1}$, $v_{i+1}, \dots, v_k$, contradicting the choice of $v_1, \dots, v_k$. Hence, with $a_i = v_i$, for $1 \leq i \leq k$, and $a = w$ the conclusion of the fact is satisfied. $\square$

Proposition 8.7 (Lachlan) *Suppose that M is an L-structure which is ω-categorical, stable and n-determined. If H is a definable strictly minimal set in M^{eq} then the geometry of H is trivial, i.e. H is an indiscernible set in M^{eq}.*

Proof. Suppose that M is an L-structure which is ω-categorical, stable and n-determined. First note that if A is a finite set of elements from M over which H is definable, then (M, A) is ω-categorical, stable and n-determined so it is sufficient to prove that every $\emptyset$-definable strictly minimal set $H \subseteq M^{\mathrm{eq}}$ is indiscernible.

We first explain how this follows from a result due to Lachlan [27]. We may assume that the vocabulary of L is relational by replacing constant symbols by unary predicates and function symbols by relation symbols representing their graphs. Since M is ω-categorical it follows that $S_n(Th(M))$ is finite and since M is n-determined we get, by adding a relation symbol for every type in $S_n(Th(M))$, an expansion M^* of M such that every formula in the expanded language is equivalent, modulo $Th(M^*)$, to a quantifier free formula. By the fact that $S_k(Th(M))$ is finite for every $k < \omega$ and M is n-determined we may assume that the vocabulary of the language of M^* is finite. Since M^* is also stable we can apply Lemma 8.2 in [27], which says that every $\emptyset$-definable strictly minimal set in $(M^*)^{\mathrm{eq}}$ is indiscernible; clearly the same must then hold for M^{eq}.

Now we give an alternative proof that every $\emptyset$-definable strictly minimal set $H \subseteq M^{\mathrm{eq}}$ is indiscernible. (Here we need not transform M into a structure in a language with finite relational vocabulary.) Let H be a $\emptyset$-definable strictly minimal set in M^{eq}. By the argument above M almost admits elimination of quantifiers so stability implies ω-stability (see [27] or [28]). Hence M is ω-categorical and ω-stable so it follows from Theorem 8.5 that the geometry of H, as above, is either trivial or isomorphic to a projective geometry of infinite dimension over a finite field or isomorphic to an affine geometry of infinite dimension over a finite field. Suppose that one of the latter two cases holds. From this we will derive a contradiction.

We assume that elements of H are of sort E where $E(\bar{x}, \bar{y})$ is a $\emptyset$-definable equivalence relation on $M^{|\bar{x}|}$. Hence the elements of H are equivalence classes of E. Let $a \in H$ and let $\bar{b} \in M$ be such that $\bar{b}/E = a$. Then $a \in \mathrm{dcl}(\bar{b})$. Let $r = \mathrm{R}(\bar{b}/\{a\})$ (so r is finite). The Lascar equation implies

$$\mathrm{R}(\bar{b}) = \mathrm{R}(\bar{b}a) = \mathrm{R}(\bar{b}/\{a\}) + \mathrm{R}(a)$$

which gives $\mathrm{R}(\bar{b}) = r + 1$.

By Fact 8.6 there are $m > (n-1)$, an independent set $\{a_1, \ldots, a_m\} \subseteq H$ (with respect to the geometry on H given by acl restricted to H) and $a_{m+1} \in \mathrm{acl}(a_1, \ldots, a_m) \cap H$ such that $\{a_1, \ldots, a_m, a_{m+1}\} - \{a_i\}$ is independent for every $1 \leq i \leq m+1$. Note that all a_i realize $tp(a)$ because H is strictly minimal.

Inductively define $\bar{b}_i \in M$, for $1 \leq i \leq m+1$, such that $\bar{b}_i/E = a_i$, $tp(a_i, \bar{b}_i) = tp(a, \bar{b})$ and $\bar{b}_i$ is independent from

$$\{a_j \, : \, 1 \leq j \leq m+1, \, j \neq i\} \, \cup \, \{\bar{b}_j \, : \, 1 \leq j < i\}$$

over $\{a_i\}$. We can find such $\bar{b}_i$'s in M because M is ω-saturated, which follows from the assumption that M is ω-categorical. (This choice of $\bar{b}_i$'s and the following argument was suggested to me by T. Hyttinen and it simplifies my original alternative proof.) Then we have $\mathrm{R}(\bar{b}_{m+1}) = r + 1$ and

$$\begin{aligned} & \mathrm{R}(\bar{b}_{m+1}/\{a_1, \ldots, a_m\}) \\ = \; & \mathrm{R}(\bar{b}_{m+1}/\{a_1, \ldots, a_m, a_{m+1}\}) \qquad \text{(because } a_{m+1} \in \mathrm{acl}(a_1, \ldots, a_m)\text{)} \\ \leq \; & \mathrm{R}(\bar{b}_{m+1}/\{a_{m+1}\}) = r \end{aligned}$$

so $tp(\bar{b}_{m+1}/\{a_1, \ldots, a_m\})$ forks over $\emptyset$. Since $\{a_1, \ldots, a_m\} \subseteq \mathrm{dcl}(\bar{b}_1 \ldots \bar{b}_m)$ it follows that $tp(\bar{b}_{m+1}/\mathrm{rng}(\bar{b}_1) \cup \ldots \cup \mathrm{rng}(\bar{b}_m))$ forks over $\emptyset$. Then, by Proposition 8.2, there exists $1 \leq i_1 < \ldots < i_{n-1} \leq m$ such that if $B = \mathrm{rng}(\bar{b}_{i_1}) \cup \ldots \cup \mathrm{rng}(\bar{b}_{i_{n-1}})$ then

$$tp(\bar{b}_{m+1}/B) \text{ forks over } \emptyset. \tag{1}$$

Let $A = \{a_{i_1}, \ldots, a_{i_{n-1}}\}$. By the choice of the $\bar{b}_i$'s,

$$B \underset{A}{\downarrow} a_{m+1}. \tag{2}$$

By the choice of the a_i's,

$$A \mathop{\smile\!\!\!\!\!\!\!\;\;|}\, a_{m+1}$$

and together with (2) and transitivity and monotonicity this implies that

$$B \mathop{\smile\!\!\!\!\!\!\!\;\;|}\, a_{m+1}. \tag{3}$$

By the choice of the $\bar{b}_i$'s,

$$B \mathop{\smile\!\!\!\!\!\!\!\;\;|}_{a_{m+1}} \bar{b}_{m+1}$$

and together with (3), transitivity and monotonicity this implies that

$$B \mathop{\smile\!\!\!\!\!\!\!\;\;|}\, \bar{b}_{m+1}$$

which contradicts (1). □

9 Coordinatization

The main results of this section will, under some restrictions on L, give a characterization of all sufficiently saturated L^n-elementary substructures of a structure M which is stable and admits elimination of quantifiers in an expansion by finitely many new relation symbols of arity at most n. One part of the work will give a construction of (finite and infinite) L^n-elementary substructures of such a structure M. This is done by a less restricted way of "shrinking" M than in Lachlan's work (see [27], [25], [28]), because we only want to obtain an L^n-elementary substructure, and not necessarily one which admits elimination of quantifiers. Then we show that all sufficiently saturated substructures of M can be obtained by this kind of shrinking. Finally, these results are connected with the notions from Section 6.

Until the end of the proof of Theorem 9.6 L will be a fixed language with finite relational vocabulary and n is fixed and larger than or equal to the arity of every relation symbol. (So $n \geq 2$, because we always assume that the binary relation $=$ belongs to the vocabulary.)

For any $k < \omega$ let $s_k(L)$ be the largest number such that there exists a set V_k of exactly k variables and $s_k(L)$ logically inequivalent quantifier free L-formulas in which only variables from V_k occur. By the theory of countable stable structures which admit elimination of quantifiers (as can be found in [27], [25], [28] in different levels of detail) there are natural numbers $m_0(L,n)$ and $m_1(L,n)$, *effectively computable* from L and n such that, even in the uncountable case, with the definitions given below, the essential Proposition 9.3 holds.

Until the end of the proof of Theorem 9.6 we assume that M is an infinite L-structure which is stable and admits elimination of quantifiers.

We say that a family $\mathcal{I}$ of subsets of M^{eq} is *mutually indiscernible over* $A \subseteq M^{\text{eq}}$ if for every $I \in \mathcal{I}$, I is indiscernible over $A \cup \bigcup(\mathcal{I} - \{I\})$. If $A = \emptyset$ then we say that $\mathcal{I}$ is a *mutually indiscernible* family. A pair of quantifier free L-formulas $(\varphi_0(\bar{v}), \varphi_1(\bar{v}))$, where $\bar{v} = (v_1, v_2, v_3, v_4)$, is called a *nice pair for M* if the following two conditions hold:

1. φ_0 and φ_1 define equivalence relations E_0 and E_1, respectively, on a $\emptyset$-definable set $D \subseteq M^2$, $E_1 \subseteq E_0$ (so E_1 is a refinement of E_0) and D is the solution set of a 2-type over $\emptyset$.

2. $\{C/E_1 : C \in D/E_0\}$ is a mutually indiscernible family and $|C/E_1| \geq m_0(L, n)$, for every $C \in D/E_0$.

For any nice pair $\boldsymbol{\varphi} = (\varphi_0(\bar{v}), \varphi_1(\bar{v}))$ for M, define $\mathcal{F}_M(\boldsymbol{\varphi}) = \{C/E_1 : C \in D/E_0\}$, where D, E_0 and E_1 are as in the definition of a nice pair. We will often treat the members of $\mathcal{F}_M(\boldsymbol{\varphi})$ as elements of M^{eq} by identifying $I \in \mathcal{F}_M(\boldsymbol{\varphi})$ with C, where $C \in D/E_0$ is such that $I = C/E_1$. If $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ are two nice pairs for M then we say that $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ are *equivalent* if there exists a $\emptyset$-definable bijection between $\bigcup \mathcal{F}_M(\boldsymbol{\varphi})$ and $\bigcup \mathcal{F}_M(\boldsymbol{\psi})$.

Let Φ_M be a set consisting of the first representative, in some fixed ordering of L, of every equivalence class of nice pairs. Define $\mathcal{F}_M = \bigcup\{\mathcal{F}_M(\boldsymbol{\varphi}) : \boldsymbol{\varphi} \in \Phi_M\}$ For every indiscernible subset $I \subseteq M^{\text{eq}}$ and finite $A \subseteq M^{\text{eq}}$ define $I\text{-crd}_M(A)$ to be the least finite

$$J \subseteq I \text{ such that } 2 \cdot |J| < |I| \text{ and } I - J \text{ is indiscernible over } A \cup J,$$

if such J exists. It is not difficult to see that if the class of all finite subsets of I satisfying the above condition is nonempty then it is closed under intersection. If no such J exists we say that $I\text{-crd}_M(A)$ is undefined.

If $\boldsymbol{\varphi}$ is a nice pair for M and $I\text{-crd}_M(A)$ is defined for every $I \in \mathcal{F}_M(\boldsymbol{\varphi})$, then define $\boldsymbol{\varphi}\text{-crd}_M(A) = \bigcup\{I\text{-crd}_M(A) : I \in \mathcal{F}_M(\boldsymbol{\varphi})\}$. If $\boldsymbol{\varphi}\text{-crd}_M(A)$ is defined for all $\boldsymbol{\varphi} \in \Phi_M$, then define $\text{crd}_M(A) = \bigcup\{\boldsymbol{\varphi}\text{-crd}_M(A) : \boldsymbol{\varphi} \in \Phi_M\}$. $I\text{-crd}(A)$ is read "the I-coordinates of A" (and similarly for $\boldsymbol{\varphi}\text{-crd}(A)$) and $\text{crd}(A)$ is read "the coordinates of A".

Let's derive some properties about $\boldsymbol{\varphi}\text{-crd}_M$. Suppose that $A \subseteq M^{\text{eq}}$ is finite and $\boldsymbol{\varphi}\text{-crd}_M(A)$ is defined. If $I \in \mathcal{F}_M(\boldsymbol{\varphi})$ is infinite then it is not hard to see that I is strictly minimal with trivial geometry so by stability theory $I - \text{acl}(A)$ is indiscernible over $A \cup (\text{acl}(A) \cap I)$ and for no $C \subset \text{acl}(A) \cap I$ is it the case that $I - C$ is indiscernible over $A \cup C$; therefore $I\text{-crd}_M(A) = \text{acl}(A) \cap I$. If some member of $\mathcal{F}_M(\boldsymbol{\varphi})$ is infinite then all members of $\mathcal{F}_M(\boldsymbol{\varphi})$ are infinite and it follows that $\boldsymbol{\varphi}\text{-crd}_M(A) = \text{acl}(A) \cap \bigcup \mathcal{F}_M(\boldsymbol{\varphi})$, so $\boldsymbol{\varphi}\text{-crd}_M(A)$ is finite.

Now suppose that all members of $\mathcal{F}_M(\boldsymbol{\varphi})$ are finite and hence of the same cardinality. It is still the case that $\boldsymbol{\varphi}\text{-crd}_M(A)$ is finite and to show this it is enough to show that $\{I \in \mathcal{F}_M(\boldsymbol{\varphi}) : I\text{-crd}_M(A) \neq \emptyset\}$ is finite. For a contradiction suppose that this is not true. Since M admits quantifier elimination in a language with finite relational vocabulary and the cardinalities of A and all $I \in \mathcal{F}_M(\boldsymbol{\varphi})$ are bounded by some $k < \omega$ it follows from Ramsey's theorem that there are a formula $\psi(\bar{x}, \bar{y})$, a tuple $\bar{a} \in A$ and for every $i < \omega$, $I_i \in \mathcal{F}_M(\boldsymbol{\varphi})$ and $\bar{b}^i_1, \bar{b}^i_2 \in I_i$ such that $M \models \psi(\bar{a}, \bar{b}^i_1)$ and $M \models \neg\psi(\bar{a}, \bar{b}^i_2)$. $\mathcal{F}_M(\boldsymbol{\varphi})$ is mutually indiscernible so by by compactness it follows that if $B = \bigcup_{i<\omega}\big(\text{rng}(\bar{b}^i_1) \cup \text{rng}(\bar{b}^i_2)\big)$ then $|B| \leq \aleph_0$ and $|S_{(\psi,\bar{x})}(B, M)| > \aleph_0$ (where $S_{(\psi,\bar{x})}(B, M)$ is the notation from Section 3) which implies that M is unstable, a contradiction.

The above observations gives the following, which will frequently be used without being mentioned:

Fact 9.1 Suppose that $\boldsymbol{\varphi}$ is a nice pair for M, $A \subseteq M^{\text{eq}}$ is finite and $\boldsymbol{\varphi}$-$\text{crd}_M(A)$ is defined. Then $\boldsymbol{\varphi}$-$\text{crd}_M(A)$ is finite and if some member, or equivalently, all members of $\mathcal{F}_M(\boldsymbol{\varphi})$ are infinite, then $\boldsymbol{\varphi}$-$\text{crd}_M(A) = \text{acl}(A) \cap \bigcup \mathcal{F}_M(\boldsymbol{\varphi})$. It follows that $\boldsymbol{\varphi}$-$\text{crd}_M(A)$ is A-definable.

We adopt the usual convention that if $N \preccurlyeq M$ then elements of N^{eq} are identified with elements of M^{eq} which, as equivalence classes, have nonempty intersection with N^k where $2k$ is the arity of the equivalence relation in question. From the definitions of nice pairs, equivalence between them and the previous fact we get:

Fact 9.2 If $N \preccurlyeq M$ then $\boldsymbol{\varphi}$ is a nice pair for N if and only if $\boldsymbol{\varphi}$ is a nice pair for M, so $\Phi_N = \Phi_M$. If $N \preccurlyeq M$, $A \subseteq N^{\text{eq}}$ is finite and $\boldsymbol{\varphi} \in \Phi_N$ then $\boldsymbol{\varphi}$-$\text{crd}_N(A)$ is defined if and only if $\boldsymbol{\varphi}$-$\text{crd}_M(A)$ is defined and if they are defined then $\boldsymbol{\varphi}$-$\text{crd}_N(A) = \boldsymbol{\varphi}$-$\text{crd}_M(A)$.

For any structure N and $0 < k < \omega$ let $N^{\text{eq}\restriction k}$ denote the set of all $a \in N^{\text{eq}}$ such that a is an equivalence class of a $\emptyset$-definable equivalence relation on N^l where $l \leq k$.

Since the idea with mutually indiscernible families defined by nice pairs is that M is built up from these (which is a special case of the idea that ω-categorical, ω-stable structures are built up from strictly minimal sets), it is natural to ask why it suffices to consider mutually indiscernible families $\mathcal{J}$ such that all members of $\mathcal{J}$ are subsets of $M^{\text{eq}\restriction 2}$. Here we try to explain why. Suppose that $\mathcal{I}$ is any mutually indiscernible family such that $\bigcup \mathcal{I}$ is $\emptyset$-definable and for some $a \in M$, I-$\text{crd}_M(\{a\})$ is defined and nonempty for all $I \in \mathcal{I}$ and let $\mathcal{I}$-$\text{crd}_M(\{a\}) = \bigcup \{I\text{-crd}_M(\{a\}) : I \in \mathcal{I}\}$. By a similar argument that lead to Fact 9.1, $\mathcal{I}$-$\text{crd}_M(\{a\})$ is finite and $\{a\}$-definable. It follows that for all a' with $tp(a') = tp(a)$ the same holds for $\mathcal{I}$-$\text{crd}(\{a'\})$. By the definition of I-crd_M and the assumption that $\mathcal{I}$ is mutually indiscernible it follows that there is b such that $tp(b) = tp(a)$ and $\mathcal{I}$-$\text{crd}_M(\{a\}) \cap \mathcal{I}$-$\text{crd}_M(\{b\})$ contains exactly one element. Let $D \subseteq M^2$ be the set of all realizations of $tp(a, b)$ and let E_1 be the $\emptyset$-definable equivalence relation on D defined by $E_1(x_1, x_2, y_1, y_2)$ if and only if

$$\mathcal{I}\text{-crd}_M(\{x_1\}) \cap \mathcal{I}\text{-crd}_M(\{x_2\}) = \mathcal{I}\text{-crd}_M(\{y_1\}) \cap \mathcal{I}\text{-crd}_M(\{y_2\}).$$

Then there is a $\emptyset$-definable bijection $f : \bigcup \mathcal{I} \to D/E_1$, so all $\emptyset$-definable relations between elements in $\bigcup \mathcal{I}$ are mirrored onto D/E_1. Therefore we loose nothing by considering the mutually indiscernible family $\big\{\{f(c) : c \in I\} : \ I \in \mathcal{I}\big\}$ instead of $\mathcal{I}$.

The next proposition collects the results that we will use from the theory of countable stable structures that admit elimination of quantifiers.

Proposition 9.3 (Lachlan) *Suppose that M is a possibly uncountable infinite L-structure which is stable and admits elimination of quantifiers.*

(i) *For every $\boldsymbol{\varphi} \in \Phi_M$ and $A \subseteq M$ with $|A| \leq n$, $\boldsymbol{\varphi}$-$\text{crd}_M(A)$ is defined, and this implies that for all $a \in M^{\text{eq}\restriction n}$, $\boldsymbol{\varphi}$-$\text{crd}_M(\{a\})$ is defined.*

(ii) *If* $\varphi \in \Phi_M$, $I \in \mathcal{F}_M(\varphi)$, $A \subseteq M$ *and* $|A| \leq n$ *then*

$$I\text{-crd}_M(A) = \bigcup \big\{ I\text{-crd}_M(\{a\}) : a \in A \big\}.$$

It follows that for all $c \in \bigcup \mathcal{F}_M$ *there is* $a \in M$ *such that* $c \in \text{crd}_M(\{a\})$.

(iii) *If* $c \in I \in \mathcal{F}_M$, $a \in M$ *and* $c \in \text{crd}_M(\{a\})$ *then* $\text{crd}_M(\{I\}) = \text{crd}_M(\{c\}) \subseteq \text{crd}_M(\{a\})$.

(iv) *The binary relation* $<_M$, *on* Φ_M, *defined by* $\psi <_M \varphi$ *if and only if*

$$\varphi\text{-crd}_M(\{I\}) \neq \emptyset \textit{ for some (or equivalently, all) } I \in \mathcal{F}_M(\psi),$$

is transitive and irreflexive and hence a partial order. We also have $\psi <_M \varphi$ *if and only if for some (or equivalently, all)* $a \in \bigcup \mathcal{F}_M(\psi)$, $\varphi\text{-crd}_M(\{a\}) \neq \emptyset$.

(v) *Suppose that* $\varphi \in \Phi_M$, $J \subseteq I \in \mathcal{F}_M(\varphi)$, $A \subseteq M$,

$$B = \bigcup \bigcup \{ \mathcal{F}_M(\psi) : \psi \in \Phi_M \textit{ and } \psi \not<_M \varphi \}$$

and $J \cap \text{crd}_M(\{a\}) = \emptyset$, *for all* $a \in A$. *Then* J *is indiscernible over* $A \cup B \cup \Big(\big(\bigcup \mathcal{F}_M(\varphi) \big) - J \Big)$.

(vi) *For every* $a \in M^{\text{eq}\restriction 2}$, $\big| \{ b \in M : \text{crd}_M(\{b\}) = \text{crd}_M(\{a\}) \} \big| \leq m_1(L, n)$.

If, in Proposition 9.3 we add the assumption that M is countable, then (i) - (vi) occur in (or easily follows from) results found in [25] or [28] (full proofs are given in [27]); (i) follows from Theorem 2.3 in [25]; (ii) is Lemma 5.1 in [25]; (iii) follows from Lemma 6.2 in [25]; (iv) follows from Lemma 6.2 in [25] (or Lemma 4.1 in [28]); (v) is mentioned after the proof of Lemma 4.2 in [28] (and can be obtained from Lemma 6.1 and Lemma 6.3 in [25]); (vi) is Lemma 6.4 in [25].

For uncountable M; if $(*)$ is one of (i) - (iv) or (vi), then $(*)$ follows by using Fact 9.2 and considering a countable elementary substructure of M which contains all the finitely many elements from M^{eq} that are mentioned in $(*)$; in the same way the conclusion of (v) follows since we can consider one formula at a time and a formula contains only finitely many parameters from $A \cup B \cup \Big(\big(\bigcup \mathcal{F}_M(\varphi) \big) - J \Big)$.

Here, we say that N is a *κ-saturated substructure of M*, where κ is a (possibly finite) cardinal, if for all $a \in M$ and $A \subseteq N$ with $|A| < \kappa$ there exists $b \in N$ such that, in M, $tp(b/A) = tp(a/A)$.

For every $\varphi \in \Phi_M$, let

$$s^n_\varphi = \max\{ |I\text{-crd}_M(A)| : I \in \mathcal{F}_M(\varphi), A \subseteq M, |A| \leq n \},$$

so s^n_φ is a finite number; in fact $s^n_\varphi \leq n \cdot |S_2(Th(M))|$, as follows from Proposition 9.3 (ii) and (7), below. We say that $\mathcal{S}$ is an *n-sufficient family of M* if $\mathcal{S} = \bigcup \{ \mathcal{S}(\varphi) : \varphi \in \Phi_M \}$, where for all $\varphi \in \Phi_M$,

(I) $\mathcal{S}(\varphi)$ is nonempty and every member of $\mathcal{S}(\varphi)$ is a nonempty subset of a member of $\mathcal{F}_M(\varphi)$,

(II) if $S \in \mathcal{S}(\boldsymbol{\varphi})$ then $|S| \geq s^n_{\boldsymbol{\varphi}}$, and

(III) if $I \in \mathcal{F}_M(\boldsymbol{\varphi})$ and $\mathrm{crd}_M(\{I\}) \subseteq \bigcup \mathcal{S}$, then there exists $S \subseteq I$ such that $S \in \mathcal{S}(\boldsymbol{\varphi})$.

We say that $\mathcal{S}$ is *nonredundant* if

(IV) $\mathrm{crd}_M(\{c\}) \subseteq \bigcup \mathcal{S}$ for every $c \in \bigcup \mathcal{S}$.

Theorem 9.4 *Suppose that $\mathcal{S}$ is an n-sufficient family and let*

$$N = \Big\{a \in M : \mathrm{crd}_M(\{a\}) \subseteq \bigcup \mathcal{S}\Big\}.$$

Then N is an n-saturated substructure of M and if $\mathcal{S}$ is nonredundant then $\bigcup \mathcal{S} = \big\{c \in \bigcup \mathcal{F}_M : \exists a \in N, c \in \mathrm{crd}_M(\{a\})\big\}$.

Remark 9.5 If $\boldsymbol{\varphi} \in \Phi_M$ is maximal (i.e. there is no $\boldsymbol{\psi} \in \Phi_M$ such that $\boldsymbol{\varphi} <_M \boldsymbol{\psi}$) then Proposition 9.3 (vi) implies that $|\mathcal{F}_M(\boldsymbol{\varphi})| \leq m_1(L, n)$. Let $\mathcal{S}$ be a nonredundant n-sufficient family of M. By Proposition 9.3 (iii),(iv) it follows that $\bigcup \mathcal{S}$ is finite if and only if every member of $\mathcal{S}$ is finite. Hence, Proposition 9.3 (vi) implies that if every member of $\mathcal{S}$ is finite then N in Theorem 9.4 is finite.

For simplicity of notation we will usually omit the subscript M in the next two proofs, so for example, we write $\mathrm{crd}(\ )$ instead of $\mathrm{crd}_M(\ )$. Types are always with respect to M.

Proof of Theorem 9.4. Let N be and $\mathcal{S}$ be as in Theorem 9.4. We prove (1) below, which implies that N is an n-saturated substructure of M. The involvement of the set C is not necessary here, but is used in the proof of Corollary 9.8.

(1) If $A, C \subseteq N$, $|A| < n$, $\mathrm{crd}(\{c\}) = \emptyset$ for all $c \in C$ and $a \in M$, then there is $a' \in M$ such that $tp(a'/A \cup C) = tp(a/A \cup C)$ and $\mathrm{crd}(\{a'\}) \subseteq \bigcup \mathcal{S}$.

Let r be the length of the longest chain in $(\Phi, <)$ and define Φ^i inductively, for $0 \leq i \leq r$, by $\Phi^0 = \emptyset$ and

$$\Phi^{i+1} = \Big\{\boldsymbol{\varphi} \in \Phi : \boldsymbol{\varphi} \text{ is maximal in } \Phi - \bigcup_{j \leq i} \Phi^j \text{ with respect to } <\Big\}.$$

Note that $\Phi = \bigcup_{i \leq r} \Phi^i$. For $0 \leq i \leq r$, let $\mathcal{F}^i = \bigcup\{\mathcal{F}(\boldsymbol{\varphi}) : \boldsymbol{\varphi} \in \Phi^i\}$ and $\mathcal{S}^i = \bigcup\{\mathcal{S}(\boldsymbol{\varphi}) : \boldsymbol{\varphi} \in \Phi^i\}$. Then $\mathcal{F} = \bigcup\{\mathcal{F}^i : i \leq r\}$ and $\mathcal{S} = \bigcup\{\mathcal{S}^i : i \leq r\}$.

By induction on i we will prove that, for every $i \leq r$,

(2) if $A, C \subseteq N$, $|A| < n$, $\mathrm{crd}(\{c\}) = \emptyset$ for all $c \in C$ and $a \in M$, then there is $a' \in M$ such that $tp(a'/A \cup C) = tp(a/A \cup C)$ and for every $\boldsymbol{\varphi} \in \bigcup_{j \leq i} \Phi^j$, $\boldsymbol{\varphi}$-$\mathrm{crd}(\{a'\}) \subseteq \bigcup \mathcal{S}(\boldsymbol{\varphi})$.

Clearly (1) follows from (2) with $i = r$. For $i = 0$, (2) is trivially satisfied. Suppose that (2) holds for i. We will show that it holds for $i + 1$.

Assume that $A, C \subseteq N$, $|A| < n$, $\mathrm{crd}(\{c\}) = \emptyset$ for all $c \in C$ and $a \in M$. By the induction hypothesis there exists $a'' \in M$ such that

(3) $tp(a''/A \cup C) = tp(a/A \cup C)$ and φ-crd$(\{a''\}) \subseteq \bigcup \mathcal{S}(\varphi)$ for every $\varphi \in \bigcup_{j \leq i} \Phi^j$.

If crd$(\{a''\}) \cap \bigcup \mathcal{F}^{i+1} \subseteq$ crd(A) then, since crd$(A) \subseteq \bigcup \mathcal{S}$, it follows that crd$(\{a''\}) \cap \bigcup \mathcal{F}^{i+1} \subseteq \mathcal{S}^{i+1}$, so by (3) we can let $a' = a''$.

Now suppose that for some $k > 0$

$$\big(\mathrm{crd}(\{a''\}) \cap \bigcup \mathcal{F}_{i+1}\big) - \mathrm{crd}(A) = \{b_1, \ldots, b_k\}$$

where $b_j \neq b_l$ if $j \neq l$. First we prove that

(4) for every $I \in \mathcal{F}^{i+1}$ such that I-crd$(\{a''\}) \neq \emptyset$ there exists $S \subseteq I$ such that $S \in \mathcal{S}^{i+1}$.

Suppose that $I \in \mathcal{F}^{i+1}$ and that I-crd$(\{a''\}) \neq \emptyset$. Then $I \in \mathcal{F}(\psi)$ for some $\psi \in \Phi^{i+1}$. If $i = 0$ then ψ is maximal so crd$(\{I\}) = \emptyset$ and by (III) (in the definition of n-sufficient family) there exists $S \subseteq I$ such that $S \in \mathcal{S}(\psi) \subseteq \mathcal{S}^{i+1}$.

Now suppose that $i > 0$. Let $\varphi \in \Phi$ be such that $\psi < \varphi$. Then $\varphi \in \bigcup_{j \leq i} \Phi^j$. Proposition 9.3 (iii) gives

(5) φ-crd$(\{I\}) \subseteq \varphi$-crd$(\{a''\})$.

By (3), φ-crd$(\{a''\}) \subseteq \bigcup \mathcal{S}(\varphi)$ and by (5),

(6) φ-crd$(\{I\}) \subseteq \bigcup \mathcal{S}(\varphi)$.

Since (6) holds for all $\varphi \in \Phi$ such that $\psi < \varphi$ it follows that crd$(\{I\}) \subseteq \bigcup \mathcal{S}$. Now an application of (III) gives us $S \subseteq I$ such that $S \in \mathcal{S}(\psi) \subseteq \mathcal{S}^{i+1}$. This completes the proof of (4).

By (II), (4) and the assumption that $|A| < n$ it follows that there are distinct $c_1, \ldots, c_k \in \bigcup \mathcal{S}^{i+1} -$ crd(A) such that for every $1 \leq j \leq k$ and $I \in \mathcal{F}^{i+1}$, $c_j \in I$ if and only if $b_j \in I$. Let

$$B = \mathrm{crd}\big(A \cup \{a''\}\big) \cap \bigcup_{j \leq i} \mathcal{F}^j,$$
$$B^* = \mathrm{crd}(A) \cap \bigcup \mathcal{F}^{i+1}.$$

Then, by Proposition 9.3 (v),

$$tp(c_1, \ldots, c_k/A \cup B \cup B^* \cup C) \ = \ tp(b_1, \ldots, b_k/A \cup B \cup B^* \cup C).$$

If we choose a' such that

$$tp(a', c_1, \ldots, c_k/A \cup B \cup B^* \cup C) \ = \ tp(a'', b_1, \ldots, b_k/A \cup B \cup B^* \cup C),$$

then

$$\mathrm{crd}(\{a'\}) \cap \bigcup_{j \leq i+1} \mathcal{F}^j \subseteq \ \{c_1, \ldots, c_k\} \cup B \cup B^* \subseteq \bigcup_{j \leq i+1} \mathcal{S}^j.$$

and it follows that (2) holds for $i + 1$. This completes the proof that N is an n-saturated substructure of M.

Now we prove the second assertion of the theorem. Suppose that $\mathcal{S}$ is nonredundant. By the definition of N we clearly have

$$\{c \in \mathcal{F} :\ \exists a \in N,\ c \in \mathrm{crd}(\{a\})\} \subseteq \bigcup \mathcal{S},$$

so we only prove the inclusion '$\supseteq$'.

Suppose that $c \in \bigcup \mathcal{S}$. By Proposition 9.3 (ii) there is $a \in M$ such that $c \in \mathrm{crd}(\{a\})$, and by Proposition 9.3 (iii), $\mathrm{crd}(\{c\}) \subseteq \mathrm{crd}(\{a\})$. Since $\mathcal{S}$ is nonredundant, $\mathrm{crd}(\{c\}) \subseteq \bigcup \mathcal{S}$ so by the definition of an n-sufficient family it follows that (1) holds with $\{c\}$ in place of A and the proof is the same as before. In other words, there exists $a' \in M$ such that $tp(a', c) = tp(a, c)$ and $\mathrm{crd}(\{a'\}) \subseteq \bigcup \mathcal{S}$, so in particular $c \in \mathrm{crd}(\{a'\})$ and $a' \in N$. $\square$

If $A \subseteq M$ and $A = \{a \in M : \mathrm{crd}_M(\{a\}) \subseteq \bigcup \mathcal{S}\}$, where $\mathcal{S}$ is an n-sufficient family, then we say that A is *coordinatized by* $\mathcal{S}$.

Theorem 9.6 *Let $m = 2 \cdot \max\{n, s_2(L), m_1(L, n)\} + 2$. If N is an m-saturated substructure of M then N is coordinatized by a nonredundant n-sufficient family of M.*

Proof. Let $Q = \bigcup\{\mathrm{crd}(A) :\ A \subseteq N \text{ and } |A| \leq n\}$ and for every $\boldsymbol{\varphi} \in \Phi$, let

$$\mathcal{S}(\boldsymbol{\varphi}) = \big\{I \cap Q :\ I \in \mathcal{F}(\boldsymbol{\varphi}) \text{ and } I \cap Q \neq \emptyset\big\}.$$

We will show that $\mathcal{S} = \bigcup\{\mathcal{S}(\boldsymbol{\varphi}) :\ \boldsymbol{\varphi} \in \Phi\}$ is a nonredundant n-sufficient family such that N is coordinatized by $\mathcal{S}$. First note that since every type in $S_2(Th(M))$ is principal we have $|S_2(Th(M))| \leq s_2(L)$. Also observe that if $a_1, a_2, b_1, b_2 \in M$, $\boldsymbol{\theta} \in \Phi$ and

$$\big|\boldsymbol{\theta}\text{-crd}(\{a_1\}) \cap \boldsymbol{\theta}\text{-crd}(\{a_2\})\big| \neq \big|\boldsymbol{\theta}\text{-crd}(\{b_1\}) \cap \boldsymbol{\theta}\text{-crd}(\{b_2\})\big|$$

then $tp(a_1, a_2) \neq tp(b_1, b_2)$. Therefore, since $\mathcal{F}(\boldsymbol{\theta})$ is a mutually indiscernible family and $|I| > 2 \cdot I\text{-crd}(\{a\})$ for all $I \in \mathcal{F}(\boldsymbol{\theta})$ and $a \in M$, it follows that

(7) for all $\boldsymbol{\theta} \in \Phi$ and $a \in M$, $|\boldsymbol{\theta}\text{-crd}(\{a\})| \leq |S_2(Th(M))|$.

From the definition of $\mathcal{S}$, Proposition 9.3 (ii) and the assumption that N is 1-saturated (because $0 < m$) it follows that (I) is satisfied.

Next we verify that (II) is satisfied. Suppose that $S \in \mathcal{S}(\boldsymbol{\varphi})$. Then there exists $I \in \mathcal{F}(\boldsymbol{\varphi})$ such that $S = I \cap Q \neq \emptyset$. Hence, by the definition of $\mathcal{S}(\boldsymbol{\varphi})$ and Proposition 9.3 (ii), $I\text{-crd}(\{a\}) \neq \emptyset$ for some $a \in N$. Let $c \in I\text{-crd}(\{a\})$. Since N is m-saturated and $m > 2 \cdot |S_2(Th(M))|$ it follows from (7) that every element in $\boldsymbol{\varphi}\text{-crd}(\{a\})$ is represented by a pair in N^2. In particular c is represented by a pair $(c_1, c_2) \in N^2$. Since $\mathcal{F}(\boldsymbol{\varphi})$ is mutually indiscernible there are $b_1, \dots, b_n \in M$ such that $I\text{-crd}(\{a\}) \subseteq I\text{-crd}(\{b_1, \dots, b_n\})$ and $|I\text{-crd}(\{b_1, \dots, b_n\})| = s^n_{\boldsymbol{\varphi}}$. Since N is m-saturated and $m \geq n + 2$ there are $b'_1, \dots, b'_n \in N$ such that $tp(b'_1, \dots, b'_n, c_1, c_2) = tp(b_1, \dots, b_n, c_1, c_2)$ which implies $|I\text{-crd}(\{b'_1, \dots, b'_n\})| = s^n_{\boldsymbol{\varphi}}$. Hence $|S| = |I \cap Q| \geq s^n_{\boldsymbol{\varphi}}$, so S satisfies (II).

Now we prove that $\mathcal{S}$ satisfies (III). Suppose that $\boldsymbol{\varphi} \in \Phi$, $I \in \mathcal{F}(\boldsymbol{\varphi})$ and $\mathrm{crd}(\{I\}) \subseteq \bigcup \mathcal{S}$. First suppose that $\boldsymbol{\varphi}$ is maximal. Then $\mathrm{crd}(\{I'\}) = \emptyset$ for all $I' \in \mathcal{S}(\boldsymbol{\varphi})$ so it follows from Proposition 9.3 (vi) that $|\mathcal{S}(\boldsymbol{\varphi})| \leq m_1(L, n)$. The assumption that N is m-saturated (together with $2 \cdot m_1(L, n) < m$) now implies that every $I' \in \mathcal{S}(\boldsymbol{\psi})$ is represented by a pair from N^2. In particular, I is represented by a pair $(a_1, a_2) \in N^2$. Then I-$\mathrm{crd}(\{a_1, a_2\}) \neq \emptyset$ so by Proposition 9.3 (ii), for $i = 1$ or $i = 2$, I-$\mathrm{crd}(\{a_i\}) \neq \emptyset$. Hence $I \cap Q \neq \emptyset$ so $I \cap Q \in \mathcal{S}(\boldsymbol{\varphi})$.

Now suppose that $\boldsymbol{\varphi}$ is not maximal. For $\boldsymbol{\psi}, \boldsymbol{\chi} \in \Phi$ we write $\boldsymbol{\psi} <^1 \boldsymbol{\chi}$ if $\boldsymbol{\psi} < \boldsymbol{\chi}$ and there does not exist $\boldsymbol{\theta} \in \Phi$ such that $\boldsymbol{\psi} < \boldsymbol{\theta} < \boldsymbol{\chi}$. Let

$$\Psi = \bigcup \{\boldsymbol{\psi} \in \Phi : \boldsymbol{\varphi} <^1 \boldsymbol{\psi}\}.$$

By Proposition 9.3 (iv), (v), $\bigcup \{\mathcal{F}(\boldsymbol{\psi}) : \boldsymbol{\psi} \in \Psi\}$ is a mutually indiscernible family so by using Proposition 9.3 (ii), (iii) and arguing as we did to get (7), it follows that

$$(8) \quad \left| \bigcup \{\boldsymbol{\psi}\text{-}\mathrm{crd}(\{I\}) : \boldsymbol{\psi} \in \Psi\} \right| \leq |S_2(Th(M))|.$$

By Proposition 9.3 (ii), the definition of $\mathcal{S}(\boldsymbol{\psi})$ and the assumption that $\mathrm{crd}(\{I\}) \subseteq \bigcup \mathcal{S}$ it follows that for every $c \in \mathrm{crd}(\{I\})$, there is $a_c \in N$ such that $c \in \mathrm{crd}(\{a_c\})$. Therefore, since N is m-saturated and $2\,|S_2(Th(M))| < m$ it follows from (7) that every element in $\mathrm{crd}(\{I\})$ is represented by a pair from N^2. Now (8), $2\,|S_2(Th(M))| + 2 \leq m$ and the m-saturation of N implies that there exists $(a_1, a_2) \in N^2$ and $I' \in M^{\mathrm{eq}\restriction 2}$ such that (a_1, a_2) is a representative of I' and, as elements of M^{eq}, I and I' realize the same type over $\bigcup \{\boldsymbol{\psi}\text{-}\mathrm{crd}(\{I\}) : \boldsymbol{\psi} \in \Psi\}$. It is easy to see (using Proposition 9.3 (iv)) that for $I_1, I_2 \in \mathcal{F}(\boldsymbol{\varphi})$, $\mathrm{crd}(\{I_1\}) = \mathrm{crd}(\{I_2\})$ if and only if $\boldsymbol{\psi}$-$\mathrm{crd}(\{I_1\}) = \boldsymbol{\psi}$-$\mathrm{crd}(\{I_2\})$ for all $\boldsymbol{\psi} \in \Psi$. Hence $\mathrm{crd}(\{I\}) = \mathrm{crd}(\{I'\})$.

Suppose that $\boldsymbol{\varphi} = (\varphi_0, \varphi_1)$ and that E_0 and D are as in the definition of a nice pair for M. Then the relation

$$\begin{aligned} R(x_1, x_2, y_1, y_2) \quad \Leftrightarrow \quad & (x_1, x_2), (y_1, y_2) \in D \\ & \wedge\ \mathrm{crd}(\{(x_1, x_2)/E_0\}) = \mathrm{crd}(\{(y_1, y_2)/E_0\}). \end{aligned}$$

is $\emptyset$-definable. By Proposition 9.3 (vi) and the definition of m we have

$$2 \cdot \left| \{I'' \in \mathcal{F}(\boldsymbol{\varphi}) : \mathrm{crd}(\{I''\}) = \mathrm{crd}(\{I'\})\} \right| < m.$$

From $(a_1, a_2) \in N^2$, the assumption that N is m-saturated, and the fact that R is $\emptyset$-definable it now follows that every $I'' \in \mathcal{F}(\boldsymbol{\varphi})$ such that $\mathrm{crd}(\{I''\}) = \mathrm{crd}(\{I'\})$ has a representative in N^2. In particular, I has a representative $(b_1, b_2) \in N^2$ so I-$\mathrm{crd}(\{b_1, b_2\}) \neq \emptyset$. Hence $|I \cap Q| \neq \emptyset$ so $I \cap Q \in \mathcal{S}(\boldsymbol{\varphi})$ and the proof that $\mathcal{S}$ satisfies (III) is completed. From the definition of $\mathcal{S}$ and Proposition 9.3 (iii) it is easy to see that (IV) holds so $\mathcal{S}$ is nonredundant

It remains to show that $N = \{a \in M : \mathrm{crd}(\{a\}) \subseteq \bigcup \mathcal{S}\}$. By the definition of $\mathcal{S}$ we immediately have '$\subseteq$', so we only prove '$\supseteq$'. The argument is similar to the proof that $\mathcal{S}$ satisfies (III), and we only indicate the main steps.

Suppose that $a \in M$ and $\mathrm{crd}(\{a\}) \subseteq \bigcup \mathcal{S}$. We need to show that $a \in N$. Let Ψ be the set of $\boldsymbol{\varphi} \in \Phi$ such that $\boldsymbol{\varphi}$-$\mathrm{crd}(\{a\}) \neq \emptyset$ and for every $\boldsymbol{\psi} \in \Phi$, if $\boldsymbol{\psi} < \boldsymbol{\varphi}$ then

$\boldsymbol{\psi}$-crd$(\{a\}) = \emptyset$. By Proposition 9.3 (iv),(v), $\bigcup\{\mathcal{F}(\boldsymbol{\psi}) : \boldsymbol{\psi} \in \Psi\}$ is a mutually indiscernible family and therefore

$$2 \cdot \left|\bigcup\{\boldsymbol{\psi}\text{-crd}(\{a\}) : \boldsymbol{\psi} \in \Psi\}\right| < m.$$

Similarly as in the verification of (III) we can derive, using $\text{crd}(\{a\}) \subseteq \bigcup \mathcal{S}$ and the m-saturation of N, that every element in $\bigcup\{\boldsymbol{\psi}\text{-crd}(\{a\}) : \boldsymbol{\psi} \in \Psi\}$ is represented by a pair from N^2. By the m-saturation of N there is $a' \in N$ such that a' has the same type over $\bigcup\{\boldsymbol{\psi}\text{-crd}(\{a\}) : \boldsymbol{\psi} \in \Psi\}$ as a and hence $\boldsymbol{\psi}\text{-crd}(\{a'\}) = \boldsymbol{\psi}\text{-crd}(\{a\})$ for all $\boldsymbol{\psi} \in \Psi$. Then, in fact, $\text{crd}(\{a'\}) = \text{crd}(\{a\})$ and by Proposition 9.3 (vi), $2 \cdot |\{a'' \in M : \text{crd}(\{a''\}) = \text{crd}(\{a'\})\}| < m$. From the assumption that N is m-saturated it now follows that $a \in N$. $\square$

When proving our final results a variant of the Tarski-Vaught test, adapted to L^n, will be used. It can be proved in the same way as the original Tarski-Vaught test; we only need to observe that if φ is an L^n-formula then every subformula of φ is in L^n.

Lemma 9.7 (Tarski-Vaught test for L^n) *Suppose that the vocabulary of L contains no function symbols of arity greater than $n-1$ and let M be an L-structure. For any subset N of M, N is an L^n-elementary substructure of M if and only if for any $\varphi(y, \bar{x}) \in L^n$ and any $\bar{a} \in N$ (where $|\bar{a}| = |\bar{x}|$), if $M \models \exists y \varphi(y, \bar{a})$ then there exists $b \in N$ such that $M \models \varphi(b, \bar{a})$.*

Now we are ready to derive consequences of the last two theorems which applies directly to complete L^n-theories. The fact that the languages in Theorem 9.4 and Theorem 9.6 where relational and that we want the next corollary to hold also when constant symbols are allowed is not a problem but makes the statement more complicated. To simplify it a little bit let us introduce the following notation. If M is an L-structure where L contains no function symbols, then let M_R be the relational structure with the same universe as M in which all relation symbols of L are interpreted in the same way as in M and in which, for every constant symbol c of L, a new unary relation symbol P_c is interpreted as $\{c^M\}$ (where c^M is the interpretation of c in M).

Corollary 9.8 *Suppose that the vocabulary of L is finite and contains no function symbols and that n is greater than or equal to the arity of every relation symbol in the vocabulary. Let T be a complete L^n-theory such that $S_n^n(T)$ is finite and suppose that $\mathcal{A}$ is an L^n-amalgamation class such that $\mathcal{A}$ is stable in L^n, every structure in $\mathcal{A}$ is a model of T and every (or equivalently, some) limit of $\mathcal{A}$ is infinite.*

(i) *Suppose that M is an elementary extension of a limit of $\mathcal{A}$ and let M^* be the expansion of M_R obtained by adding an n-ary relation symbol for every type in $S_n(Th(M_R))$. If $\mathcal{S}$ is an n-sufficient family of M^* and N is the substructure of M which is coordinatized (in M^*) by $\mathcal{S}$ then $N \preccurlyeq_{L^n} M$.*

(ii) *There is $m < \omega$, effectively computed from $|S_n^n(T)|$, such that if M is an elementary extension of a limit of $\mathcal{A}$ and $N \preccurlyeq_{L^n} M$ is $(L^n, m, \mathcal{A})$-saturated then N is coordinatized by a nonredundant n-sufficient family of M^*, where M^* is the expansion obtained from M_R by adding a new relation symbol for every type in $S_n(Th(M_R))$.*

Proof. Let M, M^* and $\mathcal{S}$ be as in (i) and suppose that N is the substructure that is coordinatized by $\mathcal{S}$. Observe that since M is an elementary extension of a limit of $\mathcal{A}$ and $S_n^n(T)$ is finite it follows from Lemma 6.4 that $S_n(Th(M_R))$ is finite and hence M^* admits elimination of quantifiers and the language of M^* is finite and relational. By the proof of Proposition 6.11 M and hence M^* are stable. Let C be the set of interpretations in M of all constant symbols in L. It is not hard to see that $\mathrm{crd}_{M^*}(\{c\}) = \emptyset$ for every $c \in C$. Since N is coordinatized by $\mathcal{S}$ in M^* it follows that $C \subseteq N$ and (1) in the proof of Theorem 9.4 holds with M replaced by M^*. This means that if $A \subseteq N$, $|A| \leq n-1$ and $a \in M$ then there is $a' \in N$ such that, in M, $tp(a'/A \cup C) = tp(a/A \cup C)$. In particular, by Lemma 9.7, we have $N \preccurlyeq_{L^n} M$.

Now we prove (ii). Let V be the vocabulary of L. Without loss of generality we may assume that there are at most $|S_n^n(T)|$ constant symbols, and for every $1 \leq i \leq n$, at most $|S_n^n(T)|$ i-ary relation symbols, because otherwise T would say that two symbols have the same same interpretation so could discard one of them. Let a vocabulary V$'$ consist precisely of all relation symbols from V, $|S_n^n(T)|$ new unary relation symbols (to replace the constant symbols) and $|S_n^n(T)|$ new n-ary relation symbols and let L' be the language over V. As was mentioned in the beginning of this section we can compute numbers $m_0(L', n)$ and $m_1(L', n)$ such that Proposition 9.3 hold for any infinite stable L'-structure which admits elimination of quantifiers. We may assume that if every symbol in the vocabulary of a (relational) language L_1 occurs in a (relational) language L_2 then $m_i(L_1, n) \leq m_i(L_2, n)$ for $i = 1, 2$. Then, for every language L^*, the vocabulary of which is included in the vocabulary of L', Theorem 9.6 holds for infinite stable L^*-structures that admit elimination of quantifiers if we let $m = 2 \cdot \max\{n, s_2(L'), m_1(L', n)\} + 2$.

Suppose that M is an elementary extension of a limit of $\mathcal{A}$ and that $N \preccurlyeq_{L^n} M$ is $(L^n, m, \mathcal{A})$-saturated. Let M^* be an expansion of M_R as described above and let N^* is the substructure of M^* with the same universe as N. We may assume that the language of M^* is included in L'. By the same argument as in the proof of (i) M^* admits elimination of quantifiers and is stable. Let N^* be the substructure of M^* which has the same universe as N. Since M, being a limit of $\mathcal{A}$, satisfies (i) and (ii) in Proposition 6.3 and N is $(L^n, m, \mathcal{A})$-saturated it follows that N^* is an m-saturated substructure of M^*. By Theorem 9.6 there exists a nonredundant n-sufficient family $\mathcal{S}$ (of M^*) such that $N = \{a \in M : \mathrm{crd}_{M^*}(\{a\}) \subseteq \bigcup \mathcal{S}\}$. $\square$

Remark 9.9 Let us relate the last corollary to the context of Chapter I. Suppose that T is a complete L^n-theory that satisfies the conditions in Theorem 4.1, so T^{c}

exists. Let M be any model of T^{c} and let M^* be obtained from M like in Corollary 9.8. For every n-sufficient family, $\mathcal{S}$, of M^*, any substructure of M which is coordinatized by an n-sufficient family of M^* is an L^n-elementary substructure of M. Moreover, every $N \preccurlyeq_{L^n} M$ which is (L^n, m, ∞)-saturated (where m is the number from Corollary 9.8) is coordinatized by a nonredundant n-sufficient family of M^*. If T has the L^n-amalgamation property (and not just the (L^n, ∞)-amalgamation property) then every (L^n, m)-saturated $N \models T$ can be L^n-elementarily embedded into some $M \models T^{\mathrm{c}}$ and is coordinatized by a nonredundant n-sufficient family of M^* like above.

References

[1] J. Baldwin, *Fundamentals of Stability Theory*, Springer-Verlag (1988).

[2] J. Baldwin, O. Lessmann, *Amalgamation properties and finite models in L^n-theories*, preprint.

[3] D. Berthier, *Stability of non-model-complete theories; products, groups*, J. London Math. Soc., 11 (1975), 453-464.

[4] A. E. Brouwer, A. M. Cohen, A. Neumaier, *Distance Regular Graphs*, Springer-Verlag (1989).

[5] C. C. Chang, H. J. Kiesler, *Model Theory*, North-Holland, Amsterdam (1992).

[6] G. Cherlin, L. Harrington, A. H. Lachlan, *$\aleph_0$-categorical, $\aleph_0$-stable structures*, Annals of Pure and Applied Logic, 28 (1985), 103-135.

[7] G. Cherlin, E. Hrushovski, *Finite structures with few types*, preprint.

[8] G. Cherlin, A. H. Lachlan, *Stable finitely homogeneous structures*, Trans. American Math. Soc., 296 (1986), 815-850.

[9] A. Dawar, *Types and indiscernibles in finite models*, in J. Makowsky and E.V. Ravve, editors, Logic Colloquium '95, Lecture Notes in Logic, vol 11, 51-65, Springer-Verlag (1998).

[10] A. Dawar, S. Lindell, S. Weinstein, *Infinitary Logic and Inductive Definability over Finite Structures*, Information and Computation, 119 (1995), 160-175.

[11] M. Djordjević, *Finite Variable Logic, Stability and Finite Models*, Filosofie Licentiatavhandling, U.U.D.M. Report 1999:7, Uppsala.

[12] P. Eklof, E. Fisher, *The elementary theory of abelian groups*, Annals of Mathematical Logic, 4 (1972), 115-171.

[13] R. Fraïssé, *Sur l'extension aux relations de quelques propriétés des ordres*, Ann. Sci. École Norm. Sup., 71 (1954), 363-388.

[14] M. Grohe, *Large finite structures with few L^k-types*, Proceedings of the 23rd International Symposium on Mathematical Foundations of Computer Science, Lecture Notes in Computer Science 1450, Springer-Verlag (1998).

[15] S. Hedman, *Finitary axiomatizations of categorical theories*, Ph. D. thesis, University of Illinois at Chicago (1999).

[16] W. A. Hodges, *Model theory*, Cambridge University Press (1993).

[17] A. J. Hoffman, R. R. Singleton, *On Moore graphs with diameters 2 and 3*, IBM J. Res. Develop. 4 (1960), 497-504.

[18] T. Hyttinen, *Stability and General Logics*, Mathematical Logic Quarterly, 45 (1999), 219-240.

[19] T. Hyttinen, *On Stability in Finite Models*, Archive for Mathematical Logic, 39 (2000), 89-102.

[20] T. Hyttinen, *Forking in Finite Models*, preprint.

[21] T. Hyttinen, *On canonical first-order theories and quantifier elimination*, preprint.

[22] N. Immerman, *Upper and lower bounds for first-order expressibility*, Journal of Computer and Systems Sciences, 25 (1982), 76-98.

[23] W. M. Kantor, M. W. Liebeck, H. D. Macpherson, *$\aleph_0$-categorical structures smoothly approximated by finite substructures*, Proc. London Math. Soc., 59 (1989), 439-463.

[24] B. Kim, A. Pillay, *From stability to simplicity*, Bulletin of Symbolic Logic, vol 4 (1998), 17-36.

[25] J. F. Knight, A. H. Lachlan, *Shrinking, Stretching, and Codes for Homogeneous Structures*, in J. T. Baldwin, editor, Classification Theory; Proceedings, Chicago 1985, Springer Lecture Notes in Mathematics 1292, Springer-Verlag (1987).

[26] A. H. Lachlan, *Finite homogeneous simple digraphs*, in J. Stern, editor, Proceedings of the Herbrand Symposium, Logic Colloquium '81, North-Holland, Amsterdam (1982), 189-208.

[27] A. H. Lachlan, *On countable stable structures which are homogeneous for a finite relational language*, Israel Journal of Mathematics, 49 (1984), 69-153.

[28] A. H. Lachlan, *Stable finitely homogeneous structures: a survey*, in B. T. Hart, A. H. Lachlan, M. A. Valeriote, editors, Algebraic Model Theory, Kluwer Academic Publishers (1997), 145-159.

[29] B. Poizat, *Deux ou trois choses que je sais de L_n*, J. Symbolic Logic, 47 (1982), 641-658.

[30] S. Roman, *Advanced Linear Algebra*, Springer (1992).

[31] S. Shelah, *Stability, the f.c.p. and superstability; model theoretic properties of formulas in first-order theories*, Annals of Mathematical Logic, 3 (1971), 271-362.

[32] S. Shelah, *The lazy model-theoretician's guide to stability*, Logique et analyse, 18 (1975), 235-308.

[33] S. Shelah, *Classification Theory*, North-Holland, Amsterdam (1990).

[34] B. Zilber, *Uncountably Categorical Structures*, AMS translations of mathematical monographs, vol 117 (1993).

Index of notation and terminology