Diagnostiskt prov 5 – lösningsförslag

1. $\int \underbrace{x^4}_{G'} \underbrace{\ln x}_{F}\, dx = \boxed{\text{partiell integr.}} =$

$$\underbrace{\frac{x^5}{5}}_{G} \underbrace{\ln x}_{F} - \int \underbrace{\frac{x^5}{5}}_{G} \cdot \underbrace{\frac{1}{x}}_{F'}\, dx =$$

$$\frac{x^5 \ln x}{5} - \int \frac{x^4}{5}\, dx = \frac{x^5 \ln x}{5} - \frac{x^5}{25} + C.$$

$$\int \underbrace{x}_{F} \underbrace{e^{4x}}_{G'}\, dx = \boxed{\text{partiell integr.}} = \underbrace{x}_{F} \underbrace{\frac{e^{4x}}{4}}_{G} - \int \underbrace{1}_{F'} \cdot \underbrace{\frac{e^{4x}}{4}}_{G}\, dx$$

$$= \frac{x e^{4x}}{4} - \frac{e^{4x}}{16} + C.$$

$$\int \frac{5x}{\sqrt{1-x^2}}\, dx = \boxed{\begin{array}{l} t = 1-x^2 \\ \frac{dt}{dx} = -2x \\ dt = -2x\,dx \end{array}} = -\frac{5}{2} \int \frac{1}{\sqrt{t}}\, dt$$

$$= -5\sqrt{t} + C = \boxed{\text{åter-subst}} = -5\sqrt{1-x^2} + C.$$

$$\int \frac{2}{\sqrt{4-x^2}}\,dx = \int \frac{2}{\sqrt{4(1-(\frac{x}{2})^2)}}\,dx =$$

$$= \int \frac{1}{\sqrt{1-(\frac{x}{2})^2}}\,dx = 2\arcsin\frac{x}{2} + C.$$

(Man kan också använda den inversa subst. $x = \sin t$.)

$$\int \frac{3}{4+x^2}\,dx = \int \frac{3}{4(1+(\frac{x}{2})^2)}\,dx =$$

$$= \frac{3}{4}\cdot 2\arctan\frac{x}{2} + C = \frac{3}{2}\arctan\frac{x}{2} + C.$$

$$\int \frac{7}{(a+x)^3}\,dx = -\frac{7}{2(a+x)^2} + C.$$

$$\int \frac{14}{a+x}\,dx = 14\ln|a+x| + C.$$

$$\int \frac{5}{4-x^2}\,dx = -5\int \frac{1}{x^2-4}\,dx =$$

$$= -5\int \frac{1}{(x-2)(x+2)}\,dx.$$

Partialbråksuppdelning:

$$\frac{1}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2}$$

$$\Rightarrow 1 = A(x+2) + B(x-2)$$
$$= (A+B)x + 2A - 2B$$

$$\Rightarrow \begin{cases} A + B = 0 \\ 2A - 2B = 1 \end{cases} \Rightarrow \begin{cases} A = \frac{1}{4} \\ B = -\frac{1}{4} \end{cases}$$

Detta ger $\int \frac{5}{4-x^2}\,dx = -5\int \frac{1}{(x-2)(x+2)}\,dx$

$$= -5\int \frac{1}{4(x-2)}\,dx + 5\int \frac{1}{4(x+2)}\,dx$$

$$= -\frac{5}{4}\ln|x-2| + \frac{5}{4}\ln|x+2| + C$$

$$= \frac{5}{4}\ln\left|\frac{x+2}{x-2}\right| + C.$$

Vi har

$$\cos(x+y) = \cos x \cos y - \sin x \sin y, \text{ så}$$

$$\begin{aligned}\cos(2x) &= \cos^2 x - \sin^2 x \\ &= \cos^2 x - (1 - \cos^2 x) \\ &= 2\cos^2 x - 1\end{aligned}$$

$$\Longrightarrow \quad \cos^2 x = \frac{1 + \cos(2x)}{2}$$

Så $\int \cos^2 x \, dx = \int \left(\frac{1}{2} + \frac{1}{2}\cos(2x)\right) dx$

$$= \frac{x}{2} + \frac{1}{4}\sin(2x) + C.$$

$$\int \cos^2 x \sin x \, dx = \boxed{\begin{aligned} t &= \cos x \\ \frac{dt}{dx} &= -\sin x \\ dt &= -\sin x \, dx \end{aligned}}$$

$$= -\int t^2 \, dt = -\frac{t^3}{3} + C = \boxed{\text{åter-subst.}}$$

$$= -\frac{\cos^3 x}{3} + C.$$

Integranden är $\dfrac{1}{x^4-1} = \dfrac{1}{(x^2-1)(x^2+1)}$

$= \dfrac{1}{(x-1)(x+1)(x^2+1)}$, där nämnaren nu bara har förstagradsfaktorer eller andragradsfaktorer utan reella nollställen.

Partialbråksuppdelning:

$$\frac{1}{(x-1)(x+1)(x^2+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}$$

$$\Longrightarrow 1 = A(x+1)(x^2+1) + B(x-1)(x^2+1) + (Cx+D)(x-1)(x+1)$$

$$= (A+B+C)x^3 + (A-B+D)x^2 + (A+B-C)x + (A-B-D)$$

$$\Longrightarrow \begin{cases} A+B+C=0 \\ A-B+D=0 \\ A+B-C=0 \\ A-B-D=1 \end{cases} \Longrightarrow \begin{cases} A=\frac{1}{4} \\ B=-\frac{1}{4} \\ C=0 \\ D=-\frac{1}{2} \end{cases}$$

Sammantaget ger detta:

$$\int \frac{1}{x^4-1}\,dx =$$

$$\int \frac{1}{4(x-1)}\,dx - \int \frac{1}{4(x+1)}\,dx - \int \frac{1}{2(x^2+1)}\,dx$$

$$= \frac{1}{4}\ln|x-1| - \frac{1}{4}\ln|x+1| - \frac{1}{2}\arctan x + C$$

$$= \frac{1}{4}\ln\left|\frac{x-1}{x+1}\right| - \frac{1}{2}\arctan x + C.$$

$$\int \frac{x^3}{x^4-1}\,dx = \frac{1}{4}\ln|x^4-1| + C.$$

Integranden är $\dfrac{x^3}{(x^2+1)^2}$ där nämnaren är en produkt av samma (andragrads-) faktor x^2+1 som saknar reella nollställen. (och täljaren har lägre grad än nämnaren).

Partialbråksuppdelning:

$$\frac{x^3}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$$

$$\Rightarrow x^3 = (Ax+B)(x^2+1) + Cx + D$$

$$= Ax^3 + Bx^2 + (A+C)x + (B+D)$$

$$\Rightarrow \begin{cases} A = 1 \\ B = 0 \\ A+C = 0 \\ B+D = 0 \end{cases} \Rightarrow \begin{cases} A = 1 \\ B = 0 \\ C = -1 \\ D = 0 \end{cases}$$

Sammantaget ger detta

$$\int \frac{x^3}{(x^2+1)^2}\,dx = \int \frac{x}{x^2+1}\,dx - \int \frac{x}{(x^2+1)^2}\,dx$$

där $\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\ln(x^2+1) + C.$

och $\int \frac{x}{(x^2+1)^2}\,dx = \boxed{\begin{array}{l} t = x^2+1 \\ \frac{dt}{dx} = 2x \\ dt = 2x\,dx \end{array}} = \int \frac{1}{2t^2}\,dt$

$$= -\frac{1}{2t} + C = -\frac{1}{2(x^2+1)} + C.$$

$$\int \frac{x}{x^4+2}\,dx = \boxed{\begin{aligned} t &= x^2 \\ \frac{dt}{dx} &= 2x \\ dt &= 2x\,dx \end{aligned}} = \int \frac{1}{2(t^2+2)}\,dt$$

$$= \int \frac{1}{4\left(\left(\frac{t}{\sqrt{2}}\right)^2+1\right)}\,dt = \frac{1}{4}\sqrt{2}\arctan\left(\frac{t}{\sqrt{2}}\right) + C$$

$$= \frac{\sqrt{2}}{4}\arctan\left(\frac{x^2}{\sqrt{2}}\right) + C.$$

$$\int \arccos x\,dx = \int \underbrace{1}_{G'}\cdot\underbrace{\arccos x}_{F}\,dx = \boxed{\text{partiell integr.}}$$

$$= \underbrace{x}_{G}\,\underbrace{\arccos x}_{F} - \int \underbrace{x}_{G}\underbrace{\left(-\frac{1}{\sqrt{1-x^2}}\right)}_{F'}dx$$

$$= x\arccos x + \int \frac{x}{\sqrt{1-x^2}}\,dx$$

$$= x\arccos x - \sqrt{1-x^2} + C$$

$$\int \underbrace{x}_{G'} \underbrace{\arccos x}_{F} \, dx = \boxed{\begin{matrix}\text{part.}\\ \text{integr.}\end{matrix}} =$$

$$= \underbrace{\frac{x^2}{2}}_{G} \underbrace{\arccos x}_{F} - \int \underbrace{\frac{x^2}{2}}_{G} \underbrace{\left(-\frac{1}{\sqrt{1-x^2}}\right)}_{F'} dx$$

$$= \frac{x^2 \arccos x}{2} + \int \frac{x^2}{2\sqrt{1-x^2}} \, dx, \text{ där}$$

$$\int \frac{x^2}{2\sqrt{1-x^2}} \, dx = \boxed{\begin{matrix} x = \sin t \\ \frac{dx}{dt} = \cos t \\ dx = \cos t \, dt \end{matrix}} =$$

$$= \int \frac{\sin^2 t}{2\sqrt{1-\sin^2 t}} \cos t \, dt = \int \frac{\sin^2 t}{2\sqrt{\cos^2 t}} \cos t \, dt$$

$$= \int \frac{1}{2} \sin^2 t \, dt = \boxed{\begin{matrix}\text{på liknande}\\ \text{sätt som i}\\ \text{en tidigare integral}\end{matrix}} =$$

$$= \int \frac{1}{2}\left(\frac{1}{2} - \frac{1}{2}\cos(2t)\right) dx$$

$$= \frac{t}{4} - \frac{1}{8}\sin(2t) + C = \boxed{\begin{matrix}\text{återsubst.}\\ x = \sin t\\ t = \arcsin x\end{matrix}}$$

$$= \frac{\arcsin x}{4} - \frac{1}{8}\sin(2 \arcsin x) + C.$$