# Partiell integration

Låt F och G vara deriverbara funktioner. Produktregeln ger

$$\frac{d}{dx}(F(x)G(x)) = F'(x)G(x) + F(x)G'(x)$$

Om nu bägge leden integreras får vi

$$F(x)G(x) = \int F'(x)G(x)\,dx + \int F(x)G'(x)\,dx$$

och

$$\boxed{\int F(x)G'(x)\,dx = F(x)G(x) - \int F'(x)G(x)\,dx.}$$

Den sista ekv. ger en metod, <u>partiell integration</u>, som ibland är användbar för att beräkna integraler.

<u>Ex.</u> $\int \ln x\,dx = \int \underbrace{(\ln x)}_{F(x)}\underbrace{\left(\frac{d}{dx}x\right)}_{G'(x)} dx$

$= \underbrace{(\ln x)}_{F}\underbrace{x}_{G} - \int \underbrace{\frac{1}{x}}_{F'}\cdot\underbrace{x}_{G}\,dx = x\ln x - \int 1\,dx =$

$= x\ln x - x + C$

Ex. $\int \underbrace{x}_{F} \underbrace{e^x}_{G'} dx = \underbrace{x}_{F} \underbrace{e^x}_{G} - \int \underbrace{1}_{F'} \cdot \underbrace{e^x}_{G} dx$

$= xe^x - e^x + C = e^x(x-1) + C.$

Ex. $\int \underbrace{x^2}_{F} \underbrace{\cos x}_{G'} dx =$

$= \underbrace{x^2}_{F} \underbrace{\sin x}_{G} - \int \underbrace{2x}_{F'} \underbrace{\sin x}_{G} dx$ (part. integr.)

$= x^2 \sin x - \int \underbrace{2x}_{U} \underbrace{\sin x}_{V'} dx$

$= x^2 \sin x - \left( \underbrace{2x}_{U} \underbrace{(-\cos x)}_{V} - \int \underbrace{2}_{U'} \underbrace{(-\cos x)}_{V} dx \right)$

$= x^2 \sin x + 2x \cos x + 2 \int \cos dx$

$= x^2 \sin x + 2x \cos x + 2 \sin x + C.$

Ex. $\int_{-\pi}^{\pi} x^2 \cos x \, dx = 2 \int_0^{\pi} x^2 \cos x \, dx$

(eftersom $x^2 \cos x$ är jämn)

$= 2 \left[ x^2 \sin x + 2x \cos x + 2 \sin x \right]_0^{\pi} = 2(2\pi(-1) - 0) = -4\pi.$

Övn./ex. Beräkna $\int_1^e x \ln x \, dx$.

Lösning. $\int \underbrace{x}_{G'} \underbrace{\ln x}_{F} \, dx = \boxed{\text{partiell integr.}}$

$$= \underbrace{\frac{x^2}{2}}_{G} \underbrace{\ln x}_{F} - \int \underbrace{\frac{x^2}{2}}_{G} \cdot \underbrace{\frac{1}{x}}_{F'} dx = \frac{x^2}{2}\ln x - \int \frac{x}{2} dx$$

$$= \frac{x^2}{2}\ln x - \frac{x^2}{4} + C .$$

Så $\int_1^e x \ln x \, dx = \left[ \frac{x^2}{2}\ln x - \frac{x^2}{4} \right]_1^e =$

$$= \frac{e^2}{2}\ln e - \frac{e^2}{4} - \left( \frac{1}{2}\ln 1 - \frac{1}{4} \right)$$

$$= \frac{e^2}{4} + \frac{1}{4} = \frac{e^2+1}{4} .$$

Övn./ex. Beräkna $\int x^2 \ln x \, dx$.

Lösning. $\int \underbrace{x^2}_{G'} \underbrace{\ln x}_{F} \, dx = \boxed{\text{partiell integr.}} = \underbrace{\frac{x^3}{3}}_{G} \underbrace{\ln x}_{F} -$

$$\int \underbrace{\frac{x^3}{3}}_{G} \cdot \underbrace{\frac{1}{x}}_{F'} dx = \frac{x^3}{3}\ln x - \int \frac{x^2}{3} dx =$$

$$= \frac{x^3}{3}\ln x - \frac{x^3}{9} + C .$$

Övn./Ex. Beräkna $\int_0^1 x^2 e^{-x}\,dx$.

Lösn.

$$\int x^2 e^{-x}\,dx = \boxed{\text{partiell integr.}} =$$

$$= x^2(-e^{-x}) - \int 2x(-e^{-x})\,dx =$$

$$= -x^2e^{-x} + \int 2xe^{-x}\,dx = \boxed{\begin{array}{l}\text{partiell integration}\\ \text{på andra termen}\end{array}}$$

$$= -x^2e^{-x} + 2x(-e^{-x}) - \int 2(-e^{-x})\,dx$$

$$= -x^2e^{-x} - 2xe^{-x} + \int 2e^{-x}\,dx =$$

$$= -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C$$

$$= -e^{-x}(x^2+2x+2) + C.$$

Så $\int_0^1 x^2 e^{-x}\,dx = \left[-e^{-x}(x^2+2x+2)\right]_0^1 =$

$$-e^{-1}\cdot 5 - (-e^{-0}\cdot 2) = 2 - \frac{5}{e}.$$

Övn./Ex. (klurig!) Beräkna $\int e^x \cos x \, dx$.

Lösn. Låt $I = \int e^x \cos x \, dx$.

$$I = \int \underbrace{e^x}_{F} \underbrace{\cos x}_{G'} \, dx = \boxed{\text{partiell integr.}} =$$

$$= \underbrace{e^x}_{F} \underbrace{\sin x}_{G} - \int \underbrace{e^x}_{F'} \underbrace{\sin x}_{G} \, dx = \boxed{\begin{matrix}\text{partiell integr.} \\ U = e^x, \; V' = \sin x\end{matrix}}$$

$$= e^x \sin x - \left( \underbrace{e^x}_{U} \underbrace{(-\cos x)}_{V} - \int \underbrace{e^x}_{U'} \underbrace{(-\cos x)}_{V} \, dx \right)$$

$$= e^x \sin x + e^x \cos x - \int e^x \cos x \, dx$$

$$= e^x \sin x + e^x \cos x - I.$$

Detta medför att

$$2I = e^x (\sin x + \cos x), \text{ så}$$

$$\int e^x \cos x \, dx = I = \frac{1}{2} e^x (\sin x + \cos x) + C.$$

Övn./Ex. Beräkna följande integraler. (Alla hittills introducerade metoder kan vara användbara.)

a) $\int x^5 \sqrt{x^6+1}\, dx$

b) $\int \cos y \sin y \, dy$

c) $\int_0^{\frac{\pi}{2}} \sin^3 t \, dt$

d) $\int \frac{\ln x}{x^2}\, dx$

e) $\int \sqrt{x\sqrt{x\sqrt{x}}}\, dx$

f) $\int e^{-nu}\, du$

g) $\int \frac{3s+2}{1+s^2}\, ds$

h) $\int_0^1 \frac{x^3}{x^8+1}\, dx$ i) $\int \arcsin x \, dx$.

# Lösningar.

a) $\int x^5\sqrt{x^6+1}\,dx = \boxed{\begin{array}{l} t = x^6+1 \\ \frac{dt}{dx} = 6x^5 \\ dt = 6x^5\,dx \end{array}} =$

$$= \frac{1}{6}\int \underbrace{\sqrt{x^6+1}}_{t}\,\underbrace{6x^5\,dx}_{dt} = \frac{1}{6}\int\sqrt{t}\,dt$$

$$= \frac{1}{6}\left(\frac{2}{3}t^{\frac{3}{2}} + C\right) = \frac{t^{\frac{3}{2}}}{9} + \underbrace{\frac{C}{9}}_{=D}$$

$$= \frac{(x^6+1)^{\frac{3}{2}}}{9} + D\ .$$

b) $\int \cos y \sin y\,dy = \boxed{\begin{array}{l} t = \sin y \\ \frac{dt}{dy} = \cos y \\ dt = \cos y\,dy \end{array}}$

$$= \int t\,dt = \frac{t^2}{2} + C = \frac{\sin^2 y}{2} + C$$

(eller $\int \cos y \sin y\,dy = \int \frac{\sin(2y)}{2}\,dy$

$$= -\frac{\cos(2y)}{4} + C\ .\ )$$

c) $$\int_0^{\frac{\pi}{2}} \sin^3 t\, dt = \int_0^{\frac{\pi}{2}} (\sin^2 t) \sin t\, dt =$$

$$= \int_0^{\frac{\pi}{2}} (1-\cos^2 t) \sin t\, dt = \boxed{\begin{array}{l} x = \cos t \\ \frac{dx}{dt} = -\sin t \\ dx = -\sin t\, dt \end{array}}$$

$$= -\int_{\cos 0}^{\cos\frac{\pi}{2}} (1-x^2)\, dx = -\left[x - \frac{x^3}{3}\right]_1^0$$

$$= -\left(0 - 0 - \left(1 - \frac{1}{3}\right)\right) = \frac{2}{3}.$$

d) $$\int \frac{\ln x}{x^2}\, dx = \int \underbrace{\frac{1}{x^2}}_{G'} \underbrace{\ln x}_{F}\, dx = \boxed{\text{partiell integr.}}$$

$$= \underbrace{-\frac{1}{3x^3}}_{G} \underbrace{\ln x}_{F} - \int \underbrace{\left(-\frac{1}{3x^3}\right)}_{G} \underbrace{\frac{1}{x}}_{F'}\, dx$$

$$= -\frac{\ln x}{3x^3} + \int \frac{1}{3x^4}\, dx = -\frac{\ln x}{3x^3} - \frac{1}{15x^5} + C.$$

e) Vi har $\sqrt{x\sqrt{x\sqrt{x}}} = \left(x\left(x^1 \cdot x^{\frac{1}{2}}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}} =$

$$= \left(x\left(x^{\frac{3}{2}}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}} = \left(x^{1+\frac{3}{4}}\right)^{\frac{1}{2}} = \left(x^{\frac{7}{4}}\right)^{\frac{1}{2}} = x^{\frac{7}{8}},$$

så $\int \sqrt{x\sqrt{x\sqrt{x}}}\,dx = \int x^{\frac{7}{8}}\,dx =$

$$\frac{x^{\frac{7}{8}+1}}{\left(\frac{7}{8}+1\right)} + C = \frac{8x^{\frac{15}{8}}}{15} + C.$$

f) $\int e^{-nu}\,du = -\frac{e^{-nu}}{n} + C.$

g) $\int \frac{3s+2}{1+s^2}\,ds = \int \left(\frac{3s}{1+s^2} + \frac{2}{1+s^2}\right) ds$

$$= 3\int \frac{s}{1+s^2}\,ds + 2\int \frac{1}{1+s^2}\,ds$$

$$= 3\left(\frac{1}{2}\ln(1+s^2) + C_1\right) + 2(\arctan s + C_2)$$

$$= \frac{3}{2}\ln(1+s^2) + 2\arctan s + \underbrace{3C_1 + 2C_2}_{= C}$$

$$= \frac{3}{2}\ln(1+s^2) + 2\arctan s + C.$$

h) $\int_0^1 \frac{x^3}{x^8+1}\,dx = \boxed{\begin{array}{l} u = x^4 \\ \frac{du}{dx} = 4x^3 \\ du = 4x^3\,dx \end{array}}$

$= \frac{1}{4}\int_{0^4}^{1^4} \frac{1}{u^2+1}\,du = \frac{1}{4}\Big[\arctan u\Big]_0^1$

$= \frac{1}{4}(\arctan 1 - \arctan 0) = \frac{1}{4}\left(\frac{\pi}{4} - 0\right)$

$= \frac{\pi}{8}.$

i) $\int \arcsin x\,dx = \int \underbrace{1}_{G'} \cdot \underbrace{\arcsin x}_{F}\,dx = \boxed{\text{parti. int.}}$

$= \underbrace{x}_{G}\,\underbrace{\arcsin x}_{F} - \int \underbrace{x}_{G} \cdot \underbrace{\frac{1}{\sqrt{1-x^2}}}_{F'}\,dx =$

$= x \arcsin x - \int \frac{x}{\sqrt{1-x^2}}\,dx = \boxed{\begin{array}{l} t = 1-x^2 \\ \frac{dt}{dx} = -2x \\ dt = -2x\,dx \end{array}}$

$= x \arcsin x + \frac{1}{2}\int \frac{1}{\sqrt{t}}\,dt$

$= x \arcsin x + \frac{1}{2}\left(2\sqrt{t} + C\right) = \boxed{\text{återsubstitution}}$

$= x \arcsin x + \sqrt{1-x^2} + D$ (där $D = \frac{C}{2}$).