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Random affine code tree fractals and Falconer–Sloan condition

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Abstract. We calculate the almost sure dimension for a general class of random affine code tree fractals in $\mathbb{R}^d$. The result is based on a probabilistic version of the Falconer–Sloan condition $C(s)$ introduced in Falconer and Sloan [Continuity of subadditive pressure for self-affine sets. *Real Anal. Exchange* **34** (2009), 413–427]. We verify that, in general, systems having a small number of maps do not satisfy condition $C(s)$. However, there exists a natural number n such that for typical systems the family of all iterates up to level n satisfies condition $C(s)$.

1. *Introduction*

In the investigation of dimensional properties of self-similar and self-conformal sets an important tool is the thermodynamic formalism. There is a natural way to attach a pressure function to a self-similar or self-conformal iterated function system and, for example, the Hausdorff dimension and multifractal spectrum can be calculated using the pressure. Since the pressure is defined by an additive potential function, there are many tools available for the purpose of analysing it.

In his famous theorem from 1988, Falconer [**5**] proved that the dimension of any typical self-affine set is equal to the unique zero of the pressure function under the assumption that the norms of the linear parts are less than $1/3$. Later, Solomyak [**20**] verified that $1/3$ can be replaced by $1/2$, which is the best possible bound; see [**18**]. The potential is defined by means of the singular value functions of the iterates of the linear parts and, contrary to the self-conformal setting, the potential ϕ is not additive. In the self-affine case ϕ is subadditive, guaranteeing the existence of the pressure and its unique zero. However, ϕ is not superadditive—not even in the weak sense that $\phi(n+m) \geq \phi(n) + \phi(m) - C$ for some constant C. In many cases this causes severe problems; see for example [**4**, **8**, **10–13**, **15**].

There are various ways to introduce randomness to the self-affine setting. In [**14**], Jordan *et al* considered a fixed affine iterated function system with a small random perturbation in translations at each step of the construction. When investigating random subsets of self-affine attractors, Falconer and Miao [**9**] selected at each step of the construction a random subfamily of the original function system independently. In both [**14**] and [**9**] there is total independence in both space, that is, between different nodes at a fixed construction level, and scale, meaning that once a node is chosen its descendants are chosen independently of the previous history. Such systems are called statistically self-affine, since the law controlling the construction is the same at every node. However, typical realizations are not self-affine. Inspired by the random V-variable fractals introduced by Barnsley *et al* in [**1**], a new class of random self-affine code tree fractals was proposed in [**13**]. In this class typical realizations mimic the self-affinity of deterministic iterated function systems. Moreover, the probability distributions have a certain independence only in scale and, therefore, typical realizations are locally random but globally nearly homogeneous. In particular, the attractor is a finite union of self-affine copies of sets with arbitrarily small diameter. Thus, typical realizations are close to deterministic self-affine sets. In a code tree fractal the linear parts of the iterated function system may depend on the construction step. For example, attractors of graph directed Markov systems generated by affine maps [**7**], or more generally sub-self-affine sets [**6**], are code tree fractals.

In this paper we generalize the dimension results in [**13**] concerning random affine code tree fractals. In [**13**], the existence of the pressure was proven under quite general conditions (see Theorem 3.1). However, when verifying the relation between the dimension and the zero of the pressure several additional assumptions were needed—the most restrictive one being that $d = 2$. The main cause for the extra assumptions was the non-superadditivity of the potential defining the pressure. In the self-affine setting various approaches have been introduced to overcome the problems caused by the non-superadditivity of the potential. These include the cone condition [**4**, **8**, **12**, **15**], irreducibility [**11**] and non-existence of parallelly mapped vectors [**13**]. In this paper we focus on a general condition (see Definition 2.1) introduced recently by Falconer and Sloan [**10**]. Under the Falconer–Sloan condition (for brevity, F–S condition), higher dimensional spaces can also be considered; see Theorem 3.2. The only additional assumption compared to Theorem 3.1 is that some iterates of the system satisfy the F–S condition with positive probability.

The F–S condition is related to a family of linear maps on $\mathbb{R}^d$. The condition is open in the sense that the set of families of linear maps satisfying it is open in any natural topology. In this paper we also address a problem proposed by Falconer concerning the genericity of the F–S condition. In $\mathbb{R}^2$, the F–S condition is easy to check but in higher dimensional spaces the question is more delicate. It turns out that a family of linear maps $\{S_i\}_{i=1}^k$ on $\mathbb{R}^d$

does not satisfy the F–S condition unless k is sufficiently large (see Remark 2.2(b))—the minimal value of k being much larger than d. However, in Corollary 2.7 we prove that there exists a natural number n depending only on d such that for any generic family $\{S_i\}_{i=1}^k$ the family $\{S_{i_1} \circ \cdots \circ S_{i_l} \mid i_j \in \{1, \ldots, k\}$ for $j = 1, \ldots, l$ and $1 \le l \le n\}$ satisfies the F–S condition. The set is generic in both the topological sense, that is, it is open and dense, and in the measure theoretic sense meaning that it has full Lebesgue measure. Theorem 2.6 provides an explicit criterion guaranteeing that a family $\{S_i\}_{i=1}^k$ belongs to the generic set. In Remark 2.8, we explain why the complement of this generic set is non-empty, that is, why Corollary 2.7 is not valid for all families.

In many problems related to self-affine iterated function systems it is sufficient to study iterates of the maps. This is also the case in Theorem 3.2. The applicability of the F–S condition is based on the fact that the upper bound n for the number of iterates needed in order that the family $\{S_{i_1} \circ \cdots \circ S_{i_l} \mid i_j \in \{1, \ldots, k\}$ for $j = 1, \ldots, l$ and $1 \le l \le n\}$ satisfies the F–S condition is a constant depending only on the dimension of the ambient space. In particular, Corollary 2.7 implies that typical systems satisfy the assumptions of Theorem 3.2.

The paper is organized as follows. In §2, we recall the Falconer–Sloan setting and prove that the F–S condition is valid for a family of iterates of a generic family (Corollary 2.7). Moreover, we give an explicit criterion implying that a family belongs to this generic set (Theorem 2.6). In §3, we recall the notation from [**13**] concerning random affine code tree fractals and prove that the dimension of a typical affine code tree fractal is given by the zero of the pressure (Theorem 3.2).

2. *Falconer–Sloan condition $C(s)$*

In this section we consider the genericity of the F–S condition introduced in [**10**] for the purpose of overcoming problems caused by the fact that in the self-affine setting the natural potential defining the pressure (for definition see (3.1)) is not supermultiplicative. Intuitively, the reason behind the applicability of the F–S condition is as follows: letting A and B be $d \times d$ matrices, the norm $\|AB\|$ may be much smaller than $\|A\| \cdot \|B\|$. This happens if the vector v which determines the norm of B is mapped by B onto an eigenspace of A which corresponds to some small eigenvalue of A. In the expression of the pressure (for $s = 1$), there is a sum of terms of the form $\|AB\|$. The F–S condition guarantees that $\|AB\|$ is not much less than $\|A\| \cdot \|B\|$ simultaneously for all pairs (A, B).

We begin by recalling the notion from [**10**]. For all $m \in \mathbb{N}$ with $0 \le m \le d$, we denote by Λ^m the mth exterior power of $\mathbb{R}^d$ with the convention $\Lambda^0 = \mathbb{R}$. An m-vector $\mathbf{v} \in \Lambda^m$ is *decomposable* if it can be written as $\mathbf{v} = v_1 \wedge \cdots \wedge v_m$ for some $v_1, \ldots, v_m \in \mathbb{R}^d$. Let Λ_0^m be the set of decomposable m-vectors. If $\{e_1, \ldots, e_d\}$ is a basis of $\mathbb{R}^d$, then $\{e_{i_1} \wedge \cdots \wedge e_{i_m} \mid 1 \le i_1 < \cdots < i_m \le d\}$ is a basis of Λ^m. Supposing that $\{e_1, \ldots, e_d\}$ is an orthonormal basis of $\mathbb{R}^d$, *the Hodge star operator* $* : \Lambda^m \to \Lambda^{d-m}$ is defined as the linear map satisfying

$$*(e_{i_1} \wedge \cdots \wedge e_{i_m}) = e_{j_1} \wedge \cdots \wedge e_{j_{d-m}}$$

for all $1 \le i_1 < \cdots < i_m \le d$, where $1 \le j_1 < \cdots < j_{d-m} \le d$ satisfy $\{i_1, \ldots, i_m\} \cup \{j_1, \ldots, j_{d-m}\} = \{1, \ldots, d\}$. Let $\omega = e_1 \wedge \cdots \wedge e_d$ be the normalized volume form on $\mathbb{R}^d$. Recall that Λ^d is one dimensional. We define the inner product $\langle \cdot \mid \cdot \rangle$ on Λ^m by the (implicit) formula

$$\langle \mathbf{v} \mid \mathbf{w} \rangle \omega = \mathbf{v} \wedge *\mathbf{w}.$$

Then the inner product is independent of the choice of the orthonormal basis $\{e_1, \ldots, e_d\}$ and, moreover, $\{e_{i_1} \wedge \cdots \wedge e_{i_m} \mid 1 \leq i_1 < \cdots < i_m \leq d\}$ becomes an orthonormal basis of Λ^m. Any linear map $S : \mathbb{R}^d \to \mathbb{R}^d$ induces a linear map $S : \Lambda^m \to \Lambda^m$ such that $S(v_1 \wedge \cdots \wedge v_m) = Sv_1 \wedge \cdots \wedge Sv_m$ for all $v_1 \wedge \cdots \wedge v_m \in \Lambda_0^m$.

Now we are ready to recall the definition of the condition $C(s)$ from [**10**]—first for integer parameters and after that for non-integral parameters s.

Definition 2.1. Consider a family $\{S_i : \mathbb{R}^d \to \mathbb{R}^d\}_{i \in I}$ consisting of linear maps. Let $m \in \mathbb{N}$ with $0 \leq m \leq d$. The family $\{S_i\}_{i \in I}$ satisfies *condition* $C(m)$ if for all $0 \neq \mathbf{v}, \mathbf{w} \in \Lambda_0^m$ there is $i \in I$ such that $\langle S_i \mathbf{v} \mid \mathbf{w} \rangle \neq 0$. Let $0 < s < d$ be non-integral and let m be the integer part of s. The family $\{S_i\}_{i \in I}$ satisfies *condition* $C(s)$ if for all $0 \neq \mathbf{v}, \mathbf{w} \in \Lambda_0^m$ and $0 \neq \mathbf{v} \wedge v, \mathbf{w} \wedge w \in \Lambda_0^{m+1}$ there is $i \in I$ such that $\langle S_i \mathbf{v} \mid \mathbf{w} \rangle \neq 0$ and $\langle S_i(\mathbf{v} \wedge v) \mid \mathbf{w} \wedge w \rangle \neq 0$.

Remark 2.2. (a) The family $\{S_i\}_{i \in I}$ satisfies condition $C(m)$ if and only if for all $0 \neq \mathbf{v} \in \Lambda_0^m$ the set $\{S_i \mathbf{v} \mid i \in I\}$ spans Λ^m. Here the if part is clear, whereas the only if part involves a slight subtlety. Indeed, Definition 2.1 deals with decomposable vectors and Λ_0^m is not a vector space when $m \notin \{0, 1, d-1, d\}$. For the only if part, assume that there exists $0 \neq \mathbf{v} \in \Lambda_0^m$ such that the set $\{S_i \mathbf{v} \mid i \in I\}$ does not span Λ^m. Letting k be the maximal number of linearly independent vectors in $\{S_i \mathbf{v} \mid i \in I\}$, we have $k < \binom{d}{m} = \dim \Lambda^m$. Denote these vectors by $\mathbf{w}^1, \ldots, \mathbf{w}^k$ and consider $i = 1, \ldots, k$. Now $\mathbf{u} = u_1 \wedge \cdots \wedge u_m \in \Lambda_0^m$ is perpendicular to $\mathbf{w}^i = w_1^i \wedge \cdots \wedge w_m^i$ if and only if the vectors $P_i u_1, \ldots, P_i u_m$ are linearly dependent. Here P_i is the orthogonal projection onto the m-dimensional linear subspace spanned by $w_1^i, \ldots, w_m^i$. Using the notation B for the $m \times m$ matrix whose columns are the vectors $P_i u_1, \ldots, P_i u_m$ expressed in the basis $\{w_1^i, \ldots, w_m^i\}$, we observe that the vectors $P_i u_1, \ldots, P_i u_m$ are linearly dependent if and only if the determinant of B is zero. This implies the existence of a polynomial map $Q : \mathbb{R}^{d^m} \to \mathbb{R}$ such that $\langle \mathbf{u} \mid \mathbf{w}^i \rangle = 0$ if and only if $Q(u_1, \ldots, u_m) = 0$. This, in turn, gives that for all $i = 1, \ldots, k$ the set

$$M_i = \{(u_1, \ldots, u_m) \in \mathbb{R}^{d^m} \mid \langle \mathbf{u} \mid \mathbf{w}^i \rangle = 0\}$$

has codimension one and, clearly, $0 \in M_i$. Note that $\mathbf{u} = u_1 \wedge \cdots \wedge u_m = 0$ if and only if the vectors $u_1, \ldots, u_m$ are linearly dependent, that is, all the $m \times m$ minors are zero for the $d \times m$ matrix whose columns are the vectors $u_1, \ldots, u_m$. Since there are $\binom{d}{m}$ such minors and $k < \binom{d}{m}$, there exists $\overline{\mathbf{u}} = (\overline{u}_1, \ldots, \overline{u}_m) \in \bigcap_{i=1}^k M_i$ such that $\overline{\mathbf{u}} \neq 0$. In particular, $\langle \overline{\mathbf{u}} \mid \mathbf{w}^i \rangle = 0$ for all $i = 1, \ldots, k$. Therefore, condition $C(m)$ is not satisfied.

(b) From (a), we see that there must be at least $\binom{d}{m}$ maps in the family $\{S_i\}_{i \in I}$ for condition $C(m)$ to be satisfied. Note that when d is large and $1 < m < d-1$ the number $\binom{d}{m}$ is much larger than d.

(c) If $m < s < m+1$ and $\{S_i\}_{i \in I}$ satisfies condition $C(s)$, then it satisfies condition $C(t)$ for all $m \leq t \leq m+1$. In [**10**, Lemma 2.6], it is shown that the irreducibility condition used by Feng in [**11**] is (essentially) equivalent to the condition $C(1)$.

We proceed by introducing the notation needed for studying the validity of the F–S condition. Let $F, G : \mathbb{R}^d \to \mathbb{R}^d$ be linear mappings with d different real eigenvalues $\{\lambda_1, \ldots, \lambda_d\}$ and $\{t_1, \ldots, t_d\}$, respectively. Let $\{\hat{e}_1, \ldots, \hat{e}_d\}$ and $\{\tilde{e}_1, \ldots, \tilde{e}_d\}$ be the corresponding normalized eigenvectors. We assume that for all $k = 1, \ldots, d$,

$$\lambda_{i_1} \cdots \lambda_{i_k} \neq \lambda_{j_1} \cdots \lambda_{j_k} \quad \text{and} \quad t_{i_1} \cdots t_{i_k} \neq t_{j_1} \cdots t_{j_k} \quad \text{for all pairs} \\ (i_1, \ldots, i_k) \neq (j_1, \ldots, j_k). \tag{2.1}$$

Let $A = A(F, G) : \mathbb{R}^d \to \mathbb{R}^d$ be the linear map satisfying $\tilde{e}_i = A^{-1} e_i$, that is, $e_i = A\tilde{e}_i$ for all $1 \le i \le d$. Let $\mathcal{S}_k = \mathcal{S}_k(F, G)$ be the family of compositions of F and G up to level k, that is,

$$\mathcal{S}_k = \{T_1 \circ \cdots \circ T_j \mid 1 \le j \le k \text{ and } T_i \in \{F, G\}\ \forall 1 \le i \le j\}. \tag{2.2}$$

Using the eigenbasis $\{\hat{e}_1, \ldots, \hat{e}_d\}$ of F as the basis of A, we view A as a $d \times d$ matrix. Denote by $\mathcal{M}_d$ the class of $d \times d$ matrices whose minors are all non-zero.

With the above notation, we prove two lemmas.

LEMMA 2.3. *Let $1 \le m \le d$ and $A \in \mathcal{M}_d$ be as above. For all $1 \le i_1 < \cdots < i_m \le d$, write*

$$\hat{e}_{i_1} \wedge \cdots \wedge \hat{e}_{i_m} = \sum_{1 \le j_1 < \cdots < j_m \le d} c_{i_1 \cdots i_m}^{j_1 \cdots j_m} \, \tilde{e}_{j_1} \wedge \cdots \wedge \tilde{e}_{j_m}. \tag{2.3}$$

Then $c_{i_1 \cdots i_m}^{j_1 \cdots j_m} \neq 0$ for all $(i_1, \ldots, i_m)$ and $(j_1, \ldots, j_m)$.

Proof. We denote the set of all permutations of $(j_1, \ldots, j_m)$ by $\mathrm{Per}(j_1, \ldots, j_m)$ and write $\mathrm{sgn}(\sigma)$ for the sign of a permutation $\sigma \in \mathrm{Per}(j_1, \ldots, j_m)$. Since $\hat{e}_{i_l} = \sum_{j=1}^d A_{j i_l} \tilde{e}_j$ for all $1 \le l \le m$ and the wedge product is antisymmetric and multilinear, we have

$$\begin{aligned} \hat{e}_{i_1} \wedge \cdots \wedge \hat{e}_{i_m} &= \sum_{j_1=1}^d \cdots \sum_{j_m=1}^d A_{j_1 i_1} \cdots A_{j_m i_m} \tilde{e}_{j_1} \wedge \cdots \wedge \tilde{e}_{j_m} \\ &= \sum_{1 \le j_1 < \cdots < j_m \le d} \Bigg(\sum_{\sigma \in \mathrm{Per}(j_1, \ldots, j_m)} \mathrm{sgn}(\sigma) A_{\sigma_1 i_1} \cdots A_{\sigma_m i_m} \Bigg) \tilde{e}_{j_1} \wedge \cdots \wedge \tilde{e}_{j_m} \\ &= c_{i_1 \cdots i_m}^{j_1 \cdots j_m} \, \tilde{e}_{j_1} \wedge \cdots \wedge \tilde{e}_{j_m}. \end{aligned}$$

Thus, the coefficient $c_{i_1 \cdots i_m}^{j_1 \cdots j_m}$ is the minor of A determined by the columns $i_1, \ldots, i_m$ and rows $j_1, \ldots, j_m$ and, by the definition of $\mathcal{M}_d$, we have $c_{i_1 \cdots i_m}^{j_1 \cdots j_m} \neq 0$. □

For all $1 \le m \le d$, define

$$B_1 = \{\hat{e}_{i_1} \wedge \cdots \wedge \hat{e}_{i_m} \mid 1 \le i_1 < \cdots < i_m \le d\}$$

and

$$B_2 = \{\tilde{e}_{j_1} \wedge \cdots \wedge \tilde{e}_{j_m} \mid 1 \le j_1 < \cdots < j_m \le d\}.$$

Then B_1 and B_2 are bases of Λ^m. Furthermore, the elements of B_1 and B_2 are the eigenvectors of $F : \Lambda^m \to \Lambda^m$ and $G : \Lambda^m \to \Lambda^m$ with eigenvalues $\lambda_{i_1} \cdots \lambda_{i_m}$ and $t_{j_1} \cdots t_{j_m}$, respectively.

Remark 2.4. Let $a_1, \ldots, a_d \in \mathbb{R} \setminus \{0\}$ with $a_i \neq a_j$ for $i \neq j$ and let $v = (v_1, \ldots, v_d) \in \mathbb{R}^d$ with $v_i \neq 0$ for all $i = 1, \ldots, d$. Denoting by $(a_j)^i$ the ith power of a_j, it follows from the Vandermonde determinant formula that the vectors $\{((a_1)^i v_1, \ldots, (a_d)^i v_d) \mid i = k, \ldots, k + d - 1\}$ span $\mathbb{R}^d$ for all $k \in \mathbb{N}$. By induction, it is easy to see that the vectors

$$\{v^{i_j} = ((a_1)^{i_j} v_1, \ldots, (a_d)^{i_j} v_d) \mid j = 1, \ldots, d \text{ and } i_1 < \cdots < i_d\}$$

span $\mathbb{R}^d$. Indeed, the case $d = 1$ is obvious. Assuming that the claim is true for d, we show that the vectors $\{v^{i_1}, \ldots, v^{i_{d+1}}\}$ span $\mathbb{R}^{d+1}$. Suppose to the contrary that this is not the case, that is, there is j such that $v^{i_j} = \sum_{k \neq j} \alpha_k v^{i_k}$. For all $k = 1, \ldots, d + 1$, we denote by $\Pi_k : \mathbb{R}^{d+1} \to \mathbb{R}^d$ the projection which omits the kth coordinate. Fix $l \neq j$. Now the induction hypothesis implies that $\Pi_j v^{i_j} = \sum_{k \neq j} b_k \Pi_j v^{i_k}$ and $\Pi_l v^{i_j} = \sum_{k \neq l} c_k \Pi_l v^{i_k}$, where the coefficients b_k and c_k are unique. Since $a_l \neq a_j$, we have $b_k \neq c_k$ for some $k \neq j$. On the other hand, $\Pi_j v^{i_j} = \sum_{k \neq j} \alpha_k \Pi_j v^{i_k}$ and $\Pi_l v^{i_j} = \sum_{k \neq l} \alpha_k \Pi_l v^{i_k}$ and, therefore, $\alpha_k = b_k = c_k$ for all $k \neq j$, which is a contradiction.

LEMMA 2.5. *Let $0 \neq \mathbf{v} \in \Lambda^m$ and $n = \binom{d}{m}$. Then there are at most $n(n - 1)$ numbers $i \in \mathbb{N}$ with the property that at least one coordinate of $F^i \mathbf{v}$ with respect to the basis B_2 is equal to zero.*

Proof. Let $\mathbf{v} = (v_1, \ldots, v_n)$ be the coordinates of $\mathbf{v}$ with respect to the basis B_1 and let k be the number of non-zero coordinates. We denote by $V_{\mathbf{v}}$ the k-dimensional plane spanned by those basis vectors in B_1 that correspond to the non-zero coordinates of $\mathbf{v}$. Let $\gamma_1, \ldots, \gamma_n$ be the eigenvalues of $F : \Lambda^m \to \Lambda^m$. Observe that for the ith iterate F^i of F we have $F^i \mathbf{v} = (\gamma_1^i v_1, \ldots, \gamma_n^i v_n)$. Combining (2.1) with Remark 2.4 implies that the set $\{F^{i_1}\mathbf{v}, \ldots, F^{i_k}\mathbf{v}\}$ spans $V_{\mathbf{v}}$ for all natural numbers $i_1 < i_2 < \cdots < i_k$. For all $j = 1, \ldots, n$, let

$$W_j = \{\mathbf{w} \in \Lambda^m \mid \mathbf{w} = (w_1, \ldots, w_n) \text{ with respect to } B_2 \text{ and } w_j = 0\}.$$

Applying Lemma 2.3 gives for all $j = 1, \ldots, n$ and $1 \leq i_1 < \cdots < i_m \leq d$ that $\hat{e}_{i_1} \wedge \cdots \wedge \hat{e}_{i_m} \notin W_j$. Thus, the dimension of $V_{\mathbf{v}} \cap W_j$ is strictly less than k. We conclude that for all $j = 1, \ldots, n$, there are at most $k - 1$ indices i such that $F^i \mathbf{v} \in W_j$ and, therefore, there are at most $n(k - 1)$ indices i such that $F^i \mathbf{v} \in W_j$ for some $j = 1, \ldots, n$. Since this is true for all $1 \leq k \leq n$, the claim follows. □

Now we are ready to prove our main theorem in this section. For this purpose, set $n_0 = \max_{0 \leq m \leq d} \binom{d}{m}$. After proving Corollary 2.7, we discuss the criterion which is based on the following theorem and gives a sufficient condition for the validity of the F–S condition (see Remark 2.8).

THEOREM 2.6. *Let F and G be as in (2.1) and assume that $A = A(F, G) \in \mathcal{M}_d$. Then the family $\mathcal{S}_{2n_0^2}$ defined in (2.2) satisfies the condition $C(s)$ for all $0 \leq s \leq d$.*

Proof. By Remark 2.2(c), it is enough to prove that the family $\mathcal{S}_{2n_0^2}$ satisfies the condition $C(s)$ for non-integral s. Letting m be the integer part of s, set $n_1 = \binom{d}{m}$ and $n_2 = \binom{d}{m+1}$ and define $M = n_1(n_1 - 1) + n_2(n_2 - 1) + 1$ and $N = n_1 + n_2 - 1$. Let $0 \neq \mathbf{v}, \mathbf{w} \in \Lambda^m$ and $0 \neq \mathbf{u}, \mathbf{z} \in \Lambda^{m+1}$. By applying Lemma 2.5 to the iterates $F^i \mathbf{v}$ and $F^i \mathbf{u}$, where $1 \leq i \leq M$, we deduce that there exists $1 \le i_0 \le M$ such that all coordinates of the iterates $F^{i_0}\mathbf{v}$ and $F^{i_0}\mathbf{u}$ with respect to the basis B_2 are non-zero. Furthermore, from Remark 2.4, we see that for all $j_1 < \cdots < j_{n_1}$ the vectors $G^{j_1}(F^{i_0}\mathbf{v}), \ldots, G^{j_{n_1}}(F^{i_0}\mathbf{v})$ span Λ^m. Hence, there are at least $N - n_1 + 1$ indices $j = 1, \ldots, N$ such that the points $G^j(F^{i_0}\mathbf{v})$ do not belong to the orthogonal complement $\mathbf{w}^\perp$ of $\mathbf{w}$. A similar argument implies that among these $N - n_1 + 1$ indices there exists j_0 such that $G^{j_0}(F^{i_0}\mathbf{u}) \notin \mathbf{z}^\perp$ and, therefore,

$$\langle G^{j_0} F^{i_0}\mathbf{v} \mid \mathbf{w}\rangle \neq 0 \quad \text{and} \quad \langle G^{j_0} F^{i_0}\mathbf{u} \mid \mathbf{z}\rangle \neq 0,$$

implying that $\mathcal{S}_{M+N}$ satisfies $C(s)$. Since $M + N \le 2n_0^2$, this completes the proof of the claim. □

Let $k \in \mathbb{N}$. We identify the space of families $\mathcal{F} = \{S_i : \mathbb{R}^d \to \mathbb{R}^d\}_{i=1}^k$ of linear maps with $\mathbb{R}^{d^2k}$. For $\mathcal{F} \in \mathbb{R}^{d^2k}$, define

$$\mathcal{S}_l(\mathcal{F}) = \{S_{i_1} \circ \cdots \circ S_{i_j} \mid 1 \le j \le l \text{ and } S_{i_m} \in \mathcal{F}\ \forall 1 \le m \le j\}.$$

With this notation, we have the following consequence of Theorem 2.6.

COROLLARY 2.7. *Letting $k \ge 2$ be a natural number, the set*

$$\mathcal{C} = \{\mathcal{F} \in \mathbb{R}^{d^2k} \mid \mathcal{S}_{2n_0^2}(\mathcal{F}) \text{ satisfies } C(s)\ \forall 0 \le s \le d\}$$

is open, dense and has full Lebesgue measure. More precisely, $\mathbb{R}^{d^2k} \setminus \mathcal{C}$ is contained in a finite union of $(d^2k - 1)$-dimensional algebraic varieties.

Proof. We start with an easy observation: assuming that $\mathcal{F} \subset \mathcal{G}$ are families of linear maps on $\mathbb{R}^d$ and $\mathcal{F}$ satisfies condition $C(s)$, then $\mathcal{G}$ satisfies it too. Thus, it is enough to prove the claim in the case $k = 2$. The set of $d \times d$ matrices with a fixed non-zero minor is a $(d^2 - 1)$-dimensional algebraic variety. Since the number of minors is finite, the set $\mathbb{R}^{d^2} \setminus \mathcal{M}_d$ can be represented as a finite union of $(d^2 - 1)$-dimensional algebraic varieties, implying that $\mathcal{M}_d \subset \mathbb{R}^{d^2}$ is open, dense and has full Lebesgue measure. Moreover, note that the set of pairs (F, G) of linear maps having d real eigenvalues and not satisfying (2.1) is a finite union of $(2d^2 - 1)$-dimensional algebraic varieties. Thus, the set of pairs (F, G) satisfying the assumptions of Theorem 2.6 is open and has positive Lebesgue measure. For the purpose of verifying that $\mathcal{C}$ is dense and has full Lebesgue measure, we need to extend our argument to the case where F and G are allowed to have complex eigenvalues satisfying (2.1).

Recall that if $\lambda = re^{i\theta}$ is a complex eigenvalue of F, also $\overline{\lambda} = re^{-i\theta}$ is an eigenvalue of F and there is a two-dimensional invariant subspace $V \subset \mathbb{R}^d$ where F acts as the rotation by angle θ composed with scaling by r. Let $e_1, e_2 \in \mathbb{R}^d$ be such that V is spanned by e_1 and e_2 and let e_3 be an eigenvector of F corresponding to a real eigenvalue t. Then $e_3 \wedge e_1$ and $e_3 \wedge e_2$ span an eigenspace of F on Λ^2 corresponding to the eigenvalue $t\lambda$. If ρ is another complex eigenvalue of F and e_4 and e_5 span the corresponding eigenspace, then $e_1 \wedge e_2$ and $e_4 \wedge e_5$ are eigenvectors of F on Λ^2 with eigenvalues $\lambda\overline{\lambda}$ and $\rho\overline{\rho}$, respectively. The four-dimensional subspace spanned by $\{e_1 \wedge e_4, e_1 \wedge e_5, e_2 \wedge e_4, e_2 \wedge e_5\}$ is divided into two invariant two-dimensional subspaces corresponding to the complex eigenvalues $\lambda\rho$ and $\lambda\overline{\rho}$. By (2.1), the numbers $\lambda\overline{\lambda}$, $\rho\overline{\rho}$, $\lambda\rho$ and $\lambda\overline{\rho}$ are different. In this way we find a basis of Λ^m consisting of eigenvectors of F. Since the Vandermonde determinant formula applies also for complex entries, Theorem 2.6 is valid for an open dense set of pairs of linear maps (F, G) having full Lebesgue measure. This completes the proof. $\square$

Remark 2.8. (a) Let $\mathcal{F} = \{T_i : \mathbb{R}^d \to \mathbb{R}^d\}_{i=1}^m$ be an iterated function system consisting of affine mappings $T_i(x) = S_i(x) + a_i$. When considering the validity of the F–S condition, the translation parts a_i play no role. From Theorem 2.6 and Corollary 2.7, we conclude that if there are $i \neq j$ such that the eigenvalues of S_i and S_j satisfy (2.1) and the eigenvectors of S_i are mapped to those of S_j by some $A \in \mathcal{M}_d$, then $\mathcal{S}_{2n_0^2}(\mathcal{F})$ satisfies the condition $C(s)$ for all $0 \leq s \leq d$.

(b) Let $\mathcal{F}$ be as in remark (a). If $\mathcal{F}$ is not irreducible, that is, if there exists a non-trivial proper subspace $V \subset \mathbb{R}^d$ satisfying $S_i(V) \subset V$ for all $i = 1, \ldots, m$, then by Remark 2.2(a) the family $\mathcal{S}_N(\mathcal{F})$ does not satisfy the condition $C(s)$ for any $0 < s < d$ and for any $N \in \mathbb{N}$.

3. *Random affine code tree fractals*

In this section we consider the Falconer–Sloan setting for a class of random affine code tree fractals introduced in [**13**], which are locally random but globally nearly homogeneous. It turns out that the earlier results in [**13**] can be improved under a probabilistic version of the condition $C(s)$. We begin by recalling the notation from [**13**].

Let $\mathcal{F} = \{F^\lambda = \{f_1^\lambda, \ldots, f_{M_\lambda}^\lambda\} \mid \lambda \in \Lambda\}$ be a family of iterated function systems on $\mathbb{R}^d$. Here the index set Λ is a topological space. Assume that for all $i = 1, \ldots, M_\lambda$ the maps $f_i^\lambda : \mathbb{R}^d \to \mathbb{R}^d$ are affine, that is, $f_i^\lambda(x) = T_i^\lambda(x) + a_i^\lambda$, where T_i^λ is a non-singular linear mapping and $a_i^\lambda \in \mathbb{R}^d$. We consider the case where the norms and the numbers of the maps are uniformly bounded, meaning that

$$\sup_{\lambda\in\Lambda, i=1,\ldots,M_\lambda} \|T_i^\lambda\| < 1, \qquad \sup_{\lambda\in\Lambda, i=1,\ldots,M_\lambda} |a_i^\lambda| < \infty \quad \text{and} \quad M = \sup_{\lambda\in\Lambda} M_\lambda < \infty.$$

Identifying F^λ with an element of $\mathbb{R}^{(d^2+d)M_\lambda}$ gives $\mathcal{F} \subset \bigcup_{i=1}^M \mathbb{R}^{(d^2+d)i}$, where the union is disjoint. We equip $\bigcup_{i=1}^M \mathbb{R}^{(d^2+d)i}$ with the natural topology and assume that $\lambda \mapsto F^\lambda$ is a Borel map. Similarly, the linear parts T_i^λ are embedded in $\mathbb{R}^{d^2 M_\lambda}$.

We continue by introducing the concept of a code tree, which is a modification of the standard tree construction of the attractor of an iterated function system. Indeed, instead of using the same family of maps at each construction step, different families with different numbers of maps are allowed in a code tree. Setting $I = \{1, \ldots, M\}$, the length of a word $\tau \in I^k$ is $|\tau| = k$. Consider a function $\omega : \bigcup_{k=0}^\infty I^k \to \Lambda$, where $I^0 = \{\emptyset\}$. We associate to ω a natural tree rooted at $\emptyset$ as follows: let $\Sigma_*^\omega \subset \bigcup_{k=0}^\infty I^k$ be the unique set satisfying the following conditions:

- $\emptyset \in \Sigma_*^\omega$;
- if $i_1 \cdots i_k \in \Sigma_*^\omega$ and $\omega(i_1 \cdots i_k) = \lambda$, then $i_1 \cdots i_k l \in \Sigma_*^\omega$ if and only if $l \leq M_\lambda$;
- if $i_1 \cdots i_k \notin \Sigma_*^\omega$, then for all l we have $i_1 \cdots i_k l \notin \Sigma_*^\omega$.

The function ω restricted to Σ_*^ω is called an *$\mathcal{F}$-valued code tree* and the set of all $\mathcal{F}$-valued code trees is denoted by Ω. Note that in a code tree the vertex $i_1 \cdots i_k$ may be identified with the function system $F^{\omega(i_1\cdots i_k)}$ and, moreover, the edge connecting $i_1 \cdots i_k$

to $i_1 \cdots i_k l$ may be identified with the map $f_l^{\omega(i_1\cdots i_k)}$. *A sub code tree* of a code tree ω is the restriction of ω to a subset $B \subset \Sigma_*^\omega$, where B is rooted at some vertex $i_1 \cdots i_k \in \Sigma_*^\omega$ and B contains all descendants of $i_1 \cdots i_k$ which belong to Σ_*^ω. We endow Ω with the topology generated by the sets

$$\left\{\omega \in \Omega \,\middle|\, \Sigma_*^\omega \cap \bigcup_{j=0}^{k} I^j = J \text{ and } \omega(\mathbf{i}) \in U_{\mathbf{i}}\ \forall \mathbf{i} \in J\right\},$$

where $k \in \mathbb{N}$, $U_{\mathbf{i}} \subset \Lambda$ is open for all $\mathbf{i} \in J$ and $J \subset \bigcup_{j=0}^k I^j$ is a tree rooted at $\emptyset$ and having all leaves in I^k. With this topology, functions ω_1 and ω_2 are 'close' to each other if their supports $\Sigma_*^{\omega_1}$ and $\Sigma_*^{\omega_2}$ agree up to the level k and the values $\omega_1(\mathbf{i})$ and $\omega_2(\mathbf{i})$ are 'close' to each other for all words $\mathbf{i}$ with $|\mathbf{i}| \le k$.

We equip $I^{\mathbb{N}}$ with the product topology. For each code tree $\omega \in \Omega$, define

$$\Sigma^\omega = \{\mathbf{i} = i_1 i_2 \cdots \in I^{\mathbb{N}} \mid i_1 \cdots i_n \in \Sigma_*^\omega\ \forall n \in \mathbb{N}\}.$$

Then Σ^ω is compact. For all $k \in \mathbb{N}$ and $\mathbf{i} \in \Sigma^\omega \cup \bigcup_{j=k}^\infty I^j$, let $\mathbf{i}_k = i_1 \cdots i_k$ be the initial word of $\mathbf{i}$ with length k. We use the following type of natural abbreviations for compositions:

$$f_{\mathbf{i}_k}^\omega = f_{i_1}^{\omega(\emptyset)} \circ f_{i_2}^{\omega(i_1)} \circ \cdots \circ f_{i_k}^{\omega(i_1\cdots i_{k-1})} \quad \text{and} \quad T_{\mathbf{i}_k}^\omega = T_{i_1}^{\omega(\emptyset)} T_{i_2}^{\omega(i_1)} \cdots T_{i_k}^{\omega(i_1\cdots i_{k-1})}.$$

Observe that, by the definition of the topology on Ω, the maps $\omega \mapsto f_{\mathbf{i}_k}^\omega$ and $\omega \mapsto T_{\mathbf{i}_k}^\omega$ are Borel measurable. The code tree fractal corresponding to $\omega \in \Omega$ is $A^\omega = \{Z^\omega(\mathbf{i}) \mid \mathbf{i} \in \Sigma^\omega\}$, where $Z^\omega(\mathbf{i}) = \lim_{k\to\infty} f_{\mathbf{i}_k}^\omega(0)$. Note that the attractor A^ω is well defined since the maps f_i^λ are uniformly contracting and the translation vectors a_i^λ belong to a bounded set. For $k \in \mathbb{N}$, $\omega \in \Omega$ and $\mathbf{i} \in \Sigma^\omega$, the *cylinder of length k determined by* $\mathbf{i}$ is

$$[\mathbf{i}_k] = \{\mathbf{j} \in \Sigma^\omega \mid j_l = i_l\ \forall l = 1, \ldots, k\}.$$

Next we introduce the concept of a neck level, which is an essential feature of our model. The existence of neck levels guarantees that in our setting the attractor is globally nearly homogeneous. In fact, if $N_m \in \mathbb{N}$ is a neck level of ω, then all the sub code trees of ω rooted at vertices $\mathbf{i} \in \Sigma_*^\omega$ with $|\mathbf{i}| = N_m$ are identical. In particular, the attractor A^ω is a finite union of affine copies of the attractor of the common sub code tree. Neck levels play an important role in the study of V-variable fractals; see for example [**1–3**].

A neck list $N = (N_m)_{m\in\mathbb{N}}$ is an increasing sequence of natural numbers. Let $\widetilde{\Omega}$ be the set of $(\omega, N) \in \Omega \times \mathbb{N}^{\mathbb{N}}$ satisfying:

- $N_m < N_{m+1}$ for all $m \in \mathbb{N}$; and
- if $\mathbf{i}_{N_m}\mathbf{j}_l, \mathbf{i}'_{N_m} \in \Sigma_*^\omega$, then $\mathbf{i}'_{N_m}\mathbf{j}_l \in \Sigma_*^\omega$ and $\omega(\mathbf{i}_{N_m}\mathbf{j}_l) = \omega(\mathbf{i}'_{N_m}\mathbf{j}_l)$.

The first condition means that N is a neck list and the second condition guarantees that the sub code trees rooted at a certain neck level are identical. *A shift* $\Xi : \widetilde{\Omega} \to \widetilde{\Omega}$ is defined by means of neck levels, that is, $\Xi(\omega, N) = (\hat{\omega}, \hat{N})$, where $\hat{N}_m = N_{m+1} - N_1$ and $\hat{\omega}(\mathbf{j}_l) = \omega(\mathbf{i}_{N_1}\mathbf{j}_l)$ for all $m, l \in \mathbb{N}$. We denote the elements of $\widetilde{\Omega}$ by $\tilde{\omega}$ and, for all $i \in \mathbb{N}$, we write $N_i(\tilde{\omega}) = N_i$ for the projection of $\tilde{\omega} = (\omega, N)$ onto the ith coordinate of N. Moreover, on $\widetilde{\Omega}$ we use the topology generated by the *cylinders*

$$[(\omega, N)_m] = \{(\hat{\omega}, \hat{N}) \in \widetilde{\Omega} \mid \hat{N}_i = N_i\ \forall i \le m \text{ and } \hat{\omega}(\tau) = \omega(\tau)\ \forall \tau \text{ with } |\tau| < N_m\}.$$

For any function ϕ of ω, we use the notation $\phi(\tilde{\omega})$ to view ϕ as a function of $\tilde{\omega}$. Finally, for all $n < m \in \mathbb{N} \cup \{0\}$, let

$$\Sigma_*^{\tilde{\omega}}(n, m) = \{i_{N_n+1} \cdots i_{N_m} \mid \mathbf{i}_{N_n} i_{N_n+1} \cdots i_{N_m} \in \Sigma_*^{\tilde{\omega}}\},$$

where $N_0 = 0$.

For the purpose of defining the pressure, we proceed by recalling the notation from [**5**]. Let $T : \mathbb{R}^d \to \mathbb{R}^d$ be a non-singular linear mapping and let

$$0 < \sigma_d \le \sigma_{d-1} \le \cdots \le \sigma_2 \le \sigma_1 = \|T\|$$

be the singular values of T, that is, the lengths of the semi-axes of the ellipsoid $T(B(0, 1))$, where $B(x, \rho) \subset \mathbb{R}^d$ is the closed ball with radius $\rho > 0$ centred at $x \in \mathbb{R}^d$. We define the *singular value function* by

$$\Phi^s(T) = \begin{cases} \sigma_1\sigma_2 \cdots \sigma_{m-1}\sigma_m^{s-m+1} & \text{if } 0 \le s \le d, \\ \sigma_1\sigma_2 \cdots \sigma_{d-1}\sigma_d^{s-d+1} & \text{if } s > d, \end{cases}$$

where m is the integer such that $m - 1 \le s < m$. The singular value function is submultiplicative, that is,

$$\Phi^s(TU) \le \Phi^s(T)\Phi^s(U)$$

for all linear maps $T, U : \mathbb{R}^d \to \mathbb{R}^d$. For further properties of the singular value function, see for example [**5**]. We assume that there exist $\underline{\sigma}, \overline{\sigma} \in (0, 1)$ such that

$$0 < \underline{\sigma} \le \sigma_d(T_i^\lambda) \le \sigma_1(T_i^\lambda) \le \overline{\sigma} < 1$$

for all $\lambda \in \Lambda$ and for all $i = 1, \ldots, M_\lambda$. Note that, whilst the condition $\overline{\sigma} < 1$ follows from the uniform contractivity assumption, the existence of $\underline{\sigma} > 0$ is an additional assumption.

For all $k \in \mathbb{N}$ and $s \ge 0$, let

$$S^{\tilde{\omega}}(k, s) = \sum_{\mathbf{i}_k \in \Sigma_*^{\tilde{\omega}}} \Phi^s(T_{\mathbf{i}_k}^{\tilde{\omega}}).$$

The *pressure* is defined as follows:

$$p^{\tilde{\omega}}(s) = \lim_{k \to \infty} \frac{\log S^{\tilde{\omega}}(k, s)}{k} \tag{3.1}$$

provided that the limit exists. Since $T \mapsto \Phi^s(T)$ is a continuous function, the map $\tilde{\omega} \mapsto p^{\tilde{\omega}}(s)$ is Borel measurable.

According to the following theorem, the pressure exists and has a unique zero for typical random affine code tree fractals.

THEOREM 3.1. *Assume that P is an ergodic Ξ-invariant Borel probability measure on $\widetilde{\Omega}$ such that $\int_{\widetilde{\Omega}} N_1(\tilde{\omega})\, dP(\tilde{\omega}) < \infty$. Then, for P-almost all $\tilde{\omega} \in \widetilde{\Omega}$, the pressure $p^{\tilde{\omega}}(s)$ exists for all $s \in [0, \infty[$. Furthermore, $p^{\tilde{\omega}}$ is strictly decreasing and there exists a unique s_0 such that $p^{\tilde{\omega}}(s_0) = 0$ for P-almost all $\tilde{\omega} \in \widetilde{\Omega}$.*

Proof. See [**13**, Theorem 4.3]. □

In [**13**, Remark 2.1], it was shown that any compact subset of the attractor of an iterated function system is a code tree fractal and, in particular, any sub-self-affine set is a code tree fractal. While verifying this, one ends up studying subsystems of the original iterated function system. For example, suppose that $F^1 = \{f_1, f_2, f_3\}$ and let $F^2 = \{f_1, f_2\}$ and $F^3 = \{f_2, f_3\}$. When changing the translation vector of the second map in F^2, one needs to modify also the translation vector of the first map in F^3, since these maps are the same. Therefore, it is useful to allow identifications of translation vectors between different families. For this purpose, we equip the set $\widehat{\Lambda} = \{(\lambda, i) \mid \lambda \in \Lambda, i = 1, \ldots, M_\lambda\}$ with an equivalence relation $\sim$ satisfying the following assumptions:

- the cardinality $\mathcal{A}$ of the set of equivalence classes $\mathbf{a} := \widehat{\Lambda}/\sim$ is finite;
- for every $\lambda \in \Lambda$, we have $(\lambda, i) \sim (\lambda, j)$ if and only if $i = j$; and
- the equivalence classes, regarded as subsets of Λ, are Borel sets.

The notation $\mathbf{a}$ for the set of equivalence classes refers to the fact that some translation vectors of the maps f_i^λ are identified even though the maps are not. The second condition means that different translation vectors inside a system F^λ are never identified. The first condition allows us to view the set of equivalence classes $\mathbf{a}$ as an element of $\mathbb{R}^{d\mathcal{A}}$. From now on, we will write $A_{\mathbf{a}}^{\tilde{\omega}}$ for the attractor of a code tree $\tilde{\omega}$ to emphasize that it depends on the set of equivalence classes of translation vectors $\mathbf{a}$.

Now we are ready to state our main theorem in this section. Generalizing the earlier results in [**13**], we prove that, under the assumptions of Theorem 3.1, for random affine code tree fractals the Hausdorff, packing and box counting dimensions, denoted by $\dim_{\mathrm{H}}$, $\dim_{\mathrm{p}}$ and $\dim_{\mathrm{B}}$, respectively, are almost surely equal to the unique zero of the pressure provided that a probabilistic version of the F–S condition is satisfied. We denote by s_0 the unique zero of the pressure given by Theorem 3.1.

THEOREM 3.2. *Assume that $0 < \underline{\sigma} \le \overline{\sigma} < \frac{1}{2}$. Let P be an ergodic Ξ-invariant Borel probability measure on $\widetilde{\Omega}$ such that $\int_{\widetilde{\Omega}} N_1(\tilde{\omega})\, dP(\tilde{\omega}) < \infty$. Suppose that for all $0 < s < d$*

$$P\{\tilde{\omega} \in \widetilde{\Omega} \mid \{T_{\mathbf{j}}^{\tilde{\omega}} \mid \mathbf{j} = \mathbf{i}_l,\ 1 \le l \le N_1 \text{ and } \mathbf{i}_{N_1} \in \Sigma_*^{\tilde{\omega}}(0, 1)\} \text{ satisfies condition } C(s)\} > 0. \tag{3.2}$$

Then, for P-almost all $\tilde{\omega} \in \widetilde{\Omega}$,

$$\dim_{\mathrm{H}}(A_{\mathbf{a}}^{\tilde{\omega}}) = \dim_{\mathrm{p}}(A_{\mathbf{a}}^{\tilde{\omega}}) = \dim_{\mathrm{B}}(A_{\mathbf{a}}^{\tilde{\omega}}) = \min\{s_0, d\}$$

for $\mathcal{L}^{d\mathcal{A}}$-almost all $\mathbf{a} \in \mathbb{R}^{d\mathcal{A}}$.

Remark 3.3. (a) In [**13**, Theorem 5.1], a special case of Theorem 3.2 was proven under substantially stronger assumptions. First of all, [**13**, Theorem 5.1] deals only with the planar case $d = 2$. Moreover, instead of (3.2) the following non-existence of parallelly mapped vectors is assumed:

$$P\{\tilde{\omega} \in \widetilde{\Omega} \mid \exists v \in \mathbb{R}^2 \setminus \{0\} \text{ such that } T_{\mathbf{i}_{N_1}}^{\tilde{\omega}}(v) \text{ are parallel } \forall \mathbf{i}_{N_1} \in \Sigma_*^{\tilde{\omega}}(0, 1)\} < 1. \tag{3.3}$$

Observe that in the case $d = 2$ the condition $C(s)$ is equivalent to the condition $C(1)$ for all $0 < s < 2$. Furthermore, for a family $\{S_i\}_{i=1}^k$ condition $C(1)$ means that for all vectors $v, w \in \mathbb{R}^2 \setminus \{0\}$ there exists i such that $\langle S_i v \mid w\rangle \ne 0$. Therefore, condition (3.3) implies condition (3.2) in the case $d = 2$. Condition (3.2) is weaker than condition (3.3), since in the former one all iterates up to level N_1 are considered whilst in the second one only iterates at level N_1 play a role. In [**13**, Theorem 5.1], there are also technical conditions concerning the measure P which are not needed here. As explained in [**13**], the upper bound $\frac{1}{2}$ for $\overline{\sigma}$ is optimal in Theorem 3.2.

(b) The map $N_1(\tilde{\omega})$ is Borel measurable as a projection. Since $\tilde{\omega} \mapsto T_{\mathbf{j}}^{\tilde{\omega}}$ is a Borel map for all finite words $\mathbf{j}$ and the set of families of linear maps satisfying condition $C(s)$ is open, the set in (3.2) is a Borel set.

Before the proof of Theorem 3.2, we present an example which demonstrates how certain random V-variable and random graph directed systems fit in our framework.

Example 3.4. Let Λ be a finite set of directed labelled multigraphs $\lambda = (W, E^\lambda, \mathcal{F}^\lambda)$, where $W = \{1, 2, \ldots, V\}$ is the common finite set of vertices for all $\lambda \in \Lambda$, E^λ is a finite set of directed edges and, for each directed edge $e \in E^\lambda$, there is an associated map $\phi_e^\lambda \in \mathcal{F}^\lambda$ which is a contraction on $\mathbb{R}^d$. For all edges e, we denote by $i(e)$ and $t(e)$ the initial and terminal vertices of e, respectively.

Recall that in the general setting of graph directed systems (see for example [**17**]), for each vertex $v \in W$, there is an associated metric space X_v and, for each edge $e \in E^\lambda$, the associated map is $\phi_e^\lambda : X_{t(e)} \to X_{i(e)}$. Here we make the simplifying assumption that $X_v = \mathbb{R}^d$ for all $v \in V$. Let

$$M = \max_{\substack{v \in W \\ \lambda \in \Lambda}} \#\{e \in E^\lambda \mid i(e) = v\}$$

be the maximum number of maps within any fixed graph $\lambda \in \Lambda$ with the same range. Recall that in a deterministic graph directed system there is only one graph λ and the composition $\phi_{e_1} \circ \phi_{e_2}$ is allowed provided that $t(e_1) = i(e_2)$. In some random graph directed models (see for example [**19**]) the graph λ is fixed and the maps ϕ_e are random, whereas in our model the graphs are allowed to be random as well.

Fix a probability measure μ on Λ and set $\mathcal{G} = \Lambda^{\{0\}\cup\mathbb{N}}$. Let $\mu^\infty = \mu^{\{0\}\cup\mathbb{N}}$ be the product measure on $\mathcal{G}$ and let $\sigma : \mathcal{G} \to \mathcal{G}$,

$$\sigma(g_0 g_1 \cdots) = g_1 g_2 \cdots \quad \text{for all } \mathbf{g} = g_0 g_1 \cdots \in \mathcal{G},$$

be the left shift. To all $\mathbf{g} \in \mathcal{G}$, we associate a V-tuple of code trees $\omega = (\omega_1, \ldots, \omega_V)$ as follows: for all $\lambda \in \Lambda$ and $v \in \{1, 2, \ldots, V\}$, let $\mathcal{F}_v^\lambda = \{\phi_e^\lambda \mid e \in E^\lambda \text{ and } i(e) = v\}$ be the iterated function system consisting of those maps in λ whose ranges correspond to the vertex v. We write $I = \{1, \ldots, M\}$ and rename the edges with $i(e) = v$ as $e_1, \ldots, e_m$. Observe that m may depend on $v \in W$ and $\lambda \in \Lambda$. The definition of M implies that $m \leq M$. For all $v \in W$, set $\omega_v(\emptyset) = \mathcal{F}_v^{g_0}$. Now we proceed inductively. Assuming that $\omega_v(i_1 \cdots i_n) = \mathcal{F}_w^{g_n} = \{\phi_{e_1}^{g_n}, \ldots, \phi_{e_m}^{g_n}\}$ for some $w \in W$, define $\omega_v(i_1 \cdots i_n i_{n+1}) = \mathcal{F}_{t(e_{i_{n+1}})}^{g_{n+1}}$ for $i_{n+1} = 1, \ldots, m$. Observe that every $\mathbf{g} \in \mathcal{G}$ defines a sequence of graphs, which, in turn, determines a sequence of ordered walks starting from v. The code tree fractal corresponding to ω_v is the set of the limit points of the set of maps associated to all infinite paths starting from v. This code tree fractal is the vth component in the graph directed set corresponding to the infinite sequence $\mathbf{g}$.

A V-tuple ω of code trees defines a V-tuple of code tree fractals $\bar{A}^\omega = (A_1^\omega, \ldots, A_V^\omega)$ componentwise as described at the beginning of this section. Note that for fixed $\mathbf{g} \in \mathcal{G}$, any sub code tree rooted at level n is determined by the code of its top node. Since this code is an element of the set $\{\mathcal{F}_k^{g_n}\}_{k=1}^V$, there are at most V distinct code trees at a fixed level. By definition, this means that $\omega = (\omega_1, \ldots, \omega_V)$ and the corresponding code tree fractals, $\{A_v^\omega \mid v \in W\}$, are V-variable.

In order to apply Theorem 3.2 to the above system, we need some further assumptions. Suppose that $\phi_e^\lambda(x) = T_e^\lambda(x) + a_e^\lambda$ is a non-singular affine map on $\mathbb{R}^d$ with singular values uniformly bounded from below by $\underline{\sigma} > 0$ and from above by $\overline{\sigma} < \frac{1}{2}$ for all $\lambda \in \Lambda$ and $e \in E^\lambda$. We equip the set $\widehat{\Lambda} = \{(\lambda, e) \mid \lambda \in \Lambda \text{ and } e \in E^\lambda\}$ with the trivial equivalence relation, that is, $(\lambda, e) \sim (\lambda', e')$ if $(\lambda, e) = (\lambda', e')$. Then the set of equivalence classes $\mathbf{a} = \widehat{\Lambda}/\sim$ may be identified with the collection of all translation vectors. Since Λ is finite and the number of edges is bounded, the number $\mathcal{A}$ of equivalence classes in $\mathbf{a}$ is finite and, therefore, $\mathbf{a} \in \mathbb{R}^{d\mathcal{A}}$. To ensure that the V-tuple of code trees corresponding to $\mathbf{g}$ has no 'dying' branches and, in particular, defines a V-tuple of non-empty code tree fractals, we assume that in μ-almost all graphs $\lambda \in \Lambda$ every vertex is an initial vertex of some edge, that is,

$$\mu\{\lambda \in \Lambda \mid \forall v \in W\ \exists e \in E^\lambda \text{ with } i(e) = v\} = 1.$$

In addition to the above assumptions, the existence of neck levels needs to be guaranteed. Recall that at a neck level all the sub code trees are identical. Such levels exist provided that there is a vertex $v_0 \in W$ such that the μ-measure of the set of graphs $\lambda \in \Lambda$ all of whose edges have terminal vertex equal to v_0 is positive. Hence, we assume that there exists a vertex $v_0 \in W$ such that $\mu(\Lambda_{\text{preneck}}) > 0$, where

$$\Lambda_{\text{preneck}} = \{\lambda \in \Lambda \mid t(e) = v_0\ \forall e \in E^\lambda\}.$$

We emphasize that this is a natural assumption for a collection of random graphs. For example, it is satisfied if the random graphs are constructed as follows: first choose for each $v \in W$ the number of edges with initial vertex equal to v. Then for each edge choose the terminal vertex independently according to a probability vector $(p_1, \ldots, p_V)$ with $p_{v_0} > 0$. We first define auxiliary neck levels inductively as follows: set

$$\tilde{N}_1(\mathbf{g}) = \min\{n \geq 0 \mid t(e) = v_0\ \forall e \in E^{g_n}\} + 1$$

and define

$$\tilde{N}_{k+1}(\mathbf{g}) = \min\{n \geq \tilde{N}_k(\mathbf{g}) \mid t(e) = v_0\ \forall e \in E^{g_n}\} + 1.$$

This sequence is well defined for μ^∞-almost all $\mathbf{g} \in \mathcal{G}$ since the distances $\tilde{N}_{k+1} - \tilde{N}_k$ form a sequence of independent geometrically distributed random variables and, therefore, for the expectation we have

$$\int \tilde{N}_k(\mathbf{g})\, d\mu^\infty(\mathbf{g}) = k \int \tilde{N}_1(\mathbf{g})\, d\mu^\infty(\mathbf{g}) < \infty \tag{3.4}$$

for all $k \in \mathbb{N}$. The neck list is defined by $N_k = \tilde{N}_{2n_0^2 k}$ for all $k \in \mathbb{N}$, where n_0 is as in Theorem 2.6.

Observe that the existence of neck levels implies that $A_{v_0}^\omega$ is a finite union of affine copies of the attractor determined by the common sub code tree at the first neck level N_1. Since all the sub code trees at this level are identical, all the components of the V-tuble attractor $\bar{A}^\omega$ are finite unions of affine images of the same fixed set. Thus, the dimensions of the components of $\bar{A}^{\omega}$ are equal to that of $A^{\omega}_{v_0}$. For the purpose of calculating the almost sure dimension value of $A^{\omega}_{v_0}$, we apply Theorem 3.2.

We proceed by verifying that the assumptions of Theorem 3.2 are satisfied. Since we attached to almost every code tree ω_{v_0} a unique neck list, we may identify $\widetilde{\Omega}$ with the space of all code trees ω_{v_0}. Moreover, the product measure μ^{∞} determines a mixing, thereby ergodic, Ξ-invariant measure P on $\widetilde{\Omega}$. Now (3.4) and the definition of N_1 imply that $\int N_1(\tilde{\omega}_{v_0})\, dP(\tilde{\omega}_{v_0}) < \infty$.

Finally, we have to ensure that the F–S condition (3.2) is valid. Intuitively, this is achieved if we assume that there are many allowed sequences of edges with initial and terminal vertices equal to v_0 such that the associated maps satisfy the assumptions of Theorem 2.6. More precisely, we suppose that there exists $l \in \mathbb{N}$ such that

$$\begin{aligned}&\mu^l\{(\lambda_1, \ldots, \lambda_l) \in \Lambda^l \mid \lambda_j \notin \Lambda_{\text{preneck}} \text{ for } j = 1, \ldots, l-1,\ \lambda_l \in \Lambda_{\text{preneck}},\\ &\quad \exists e^{\lambda_1}_{i_1} \cdots e^{\lambda_l}_{i_l} \text{ and } e^{\lambda_1}_{j_1} \cdots e^{\lambda_l}_{j_l} \text{ with } i(e^{\lambda_1}_{i_1}) = i(e^{\lambda_1}_{j_1}) = t(e^{\lambda_l}_{i_l}) = t(e^{\lambda_l}_{j_l}) = v_0 \text{ and}\\ &\quad F := T^{\lambda_1}_{e_{i_1}} \cdots T^{\lambda_l}_{e_{i_l}} \text{ and } G := T^{\lambda_1}_{e_{j_1}} \cdots T^{\lambda_l}_{e_{j_l}} \text{ satisfy the assumptions of Theorem 2.6}\}\\ &\quad > 0. \end{aligned} \tag{3.5}$$

Since we use the product measure μ^{∞} on $\mathcal{G}$, there is positive probability that the same pair of maps (F, G) appears successively $2n_0^2$ times. Therefore, from Theorem 2.6 we see that the condition (3.2) is satisfied. Observe that the condition (3.5) is satisfied with $l = 1$ if there are maps ϕ^{λ}_{e} and $\phi^{\lambda}_{e'}$ as in Theorem 2.6 with $i(e) = i(e') = t(e) = t(e') = v_0$ and $\lambda \in \Lambda_{\text{preneck}}$ is chosen with positive probability. This, in turn, is true for typical families by Corollary 2.7.

For the proof of Theorem 3.2, we need the following notation and auxiliary results.

Definition 3.5. Let $c > 0$ and $0 < s < d$. We say that a family of non-singular linear mappings $\{S_j : \mathbb{R}^d \to \mathbb{R}^d\}_{j=1}^{k}$ is (c, s)-*full* if

$$\sum_{j=1}^{k} \Phi^s(U S_j V) \geq c\Phi^s(U)\Phi^s(V)$$

for all non-singular linear mappings $U, V : \mathbb{R}^d \to \mathbb{R}^d$.

In Lemmas 3.6 and 3.7, we explore consequences of the probabilistic version of the F–S condition (3.2).

LEMMA 3.6. *Assuming that the condition (3.2) is satisfied, there exists $c > 0$ such that*

$$\varrho = P\big\{\tilde{\omega} \in \widetilde{\Omega} | \{T^{\tilde{\omega}}_{\mathbf{i}_{N_1}}\}_{\mathbf{i}_{N_1} \in \Sigma^{\tilde{\omega}}_{*}(0,1)} \textit{ is } (c, s)\textit{-full}\big\} > 0.$$

Proof. Since the set of (c, s)-full families is a Borel set, the set in the definition of ϱ is a Borel set. Let $U, V : \mathbb{R}^d \to \mathbb{R}^d$ be non-singular linear maps. Suppose that

$$\mathcal{F} = \{T^{\tilde{\omega}}_{\mathbf{j}} \mid \mathbf{j} = \mathbf{i}_l,\ 1 \leq l \leq N_1 \text{ and } \mathbf{i}_{N_1} \in \Sigma^{\tilde{\omega}}_{*}(0, 1)\}$$

satisfies the condition $C(s)$. By the proof of [**10**, Proposition 2.1] (see also [**10**, Corollary 2.2]), there exists $\mathbf{j}$ such that

$$\Phi^s(U T^{\tilde{\omega}}_{\mathbf{j}} V) \geq C(\mathcal{F})\Phi^s(U)\Phi^s(V), \tag{3.6}$$

where the constant $C(\mathcal{F})$ is independent of U and V. Observe that $C(\mathcal{F})$ depends on s but it is an interpolation of the constants obtained by replacing s by m and $m+1$, where m is the integer part of s (recall Remark 2.2). Let $\bar{\mathbf{i}}_{N_1} \in \Sigma_*^{\omega}(0,1)$ be such that $\mathbf{j} = \bar{\mathbf{i}}_{|\mathbf{j}|}$. Writing $T_{\bar{\mathbf{i}}_{N_1}}^{\tilde{\omega}} = T_{\mathbf{j}}^{\tilde{\omega}} T_{i_{|\mathbf{j}|+1}}^{\tilde{\omega}(i_{|\mathbf{j}|})} \cdots T_{i_{N_1}}^{\tilde{\omega}(i_{N_1-1})}$ and applying (3.6) gives

$$\Phi^s(UT_{\bar{\mathbf{i}}_{N_1}}^{\tilde{\omega}} V) \geq \underline{\sigma}^{N_1-|\mathbf{j}|}\Phi^s(UT_{\mathbf{j}}V) \geq C(\mathcal{F})\underline{\sigma}^{N_1}\Phi^s(U)\Phi^s(V).$$

This implies that

$$\sum_{\mathbf{i}_{N_1}\in\Sigma_*^{\tilde{\omega}}(0,1)} \Phi^s(UT_{\mathbf{i}_{N_1}}^{\tilde{\omega}} V) \geq C(\mathcal{F})\underline{\sigma}^{N_1}\Phi^s(U)\Phi^s(V) \tag{3.7}$$

for all linear mappings $U, V : \mathbb{R}^d \to \mathbb{R}^d$. From (3.2), we conclude that there exists $c > 0$ such that

$$P\{\tilde{\omega} \in \widetilde{\Omega} \mid C(\mathcal{F})\underline{\sigma}^{N_1} > c\} > 0,$$

giving the claim. □

In the following lemma we denote by $\#A$ the number of elements in a set A.

LEMMA 3.7. *Assume that the condition (3.2) is satisfied and let ϱ and c be as in Lemma 3.6. Define for all $n, m \in \mathbb{N}$*

$$E^{\tilde{\omega}}(n, n+m) = \#\{n < j \leq n+m \mid \{T_{\mathbf{i}_{N_1}}^{\Xi^{j-1}(\tilde{\omega})}\} \text{ is } (c,s)\text{-full}\}$$

and suppose that P is Ξ-invariant and ergodic. Then, for P-almost all $\tilde{\omega} \in \widetilde{\Omega}$, the following is true: for all $\varepsilon > 0$ there exists $n_1(\tilde{\omega}, \varepsilon) > 0$ such that for all $n > n_1(\tilde{\omega}, \varepsilon)$ we have

$$E^{\tilde{\omega}}(n, n + \lceil \varepsilon n \rceil) \geq 1,$$

where $\lceil x \rceil$ is the smallest integer m with $x \leq m$.

Proof. Let χ be the characteristic function of the set $\{\tilde{\omega} \in \widetilde{\Omega} \mid \{T_{\mathbf{i}_{N_1}}^{\tilde{\omega}}\}$ is (c,s)-full$\}$. Since

$$E^{\tilde{\omega}}(0,n) = \sum_{j=0}^{n-1} \chi(\Xi^j(\tilde{\omega})),$$

we obtain from the Birkhoff ergodic theorem that for P-almost all $\tilde{\omega} \in \widetilde{\Omega}$

$$\lim_{n\to\infty} \frac{E^{\tilde{\omega}}(0,n)}{n} = \int_{\widetilde{\Omega}} \chi(\tilde{\omega})\, dP(\tilde{\omega}) = \varrho. \tag{3.8}$$

Fix $\tilde{\omega} \in \widetilde{\Omega}$ satisfying (3.8) and let $\varepsilon > 0$. Defining $0 < \tilde{\varepsilon} = (\varrho\varepsilon n - 1)/((\varepsilon+2)n) < \varrho$ for sufficiently large n, there exists $n_1(\tilde{\omega}, \varepsilon) > 0$ such that for all $n > n_1(\tilde{\omega}, \varepsilon)$ and for all $m \geq 0$ we have

$$(\varrho - \tilde{\varepsilon})(n+m) < E^{\tilde{\omega}}(0, n+m) < (\varrho + \tilde{\varepsilon})(n+m)$$

and, therefore,

$$E^{\tilde{\omega}}(n, n+m) = E^{\tilde{\omega}}(0, n+m) - E^{\tilde{\omega}}(0,n) > (\varrho - \tilde{\varepsilon})m - 2\tilde{\varepsilon}n.$$

Finally, taking $m \geq \varepsilon n$ gives $(\varrho - \tilde{\varepsilon})m - 2\tilde{\varepsilon}n \geq 1$, which implies that $E^{\tilde{\omega}}(n, n+m) \geq 1$. In particular, $E^{\tilde{\omega}}(n, n + \lceil \varepsilon n \rceil) \geq 1$. □

LEMMA 3.8. *Under the assumptions of Theorem 3.1, we have for P-almost all $\tilde{\omega} \in \widetilde{\Omega}$ that*

$$\lim_{n\to\infty} \frac{N_{n+\lceil \varepsilon n\rceil}(\tilde{\omega}) - N_{n-1}(\tilde{\omega})}{N_n(\tilde{\omega})} = \varepsilon$$

for all $\varepsilon > 0$.

Proof. Since $N_n(\tilde{\omega}) = \sum_{j=0}^{n-1} N_1(\Xi^j(\tilde{\omega}))$, the Birkhoff ergodic theorem implies that for P-almost all $\tilde{\omega} \in \widetilde{\Omega}$

$$\lim_{n\to\infty} \frac{N_n(\tilde{\omega})}{n} = \int_{\tilde{\Omega}} N_1(\tilde{\omega})\, dP(\tilde{\omega}) = b < \infty.$$

Now, for any typical $\tilde{\omega}$, we have

$$\lim_{n\to\infty} \frac{N_{n+\lceil \varepsilon n\rceil}(\tilde{\omega})}{N_n(\tilde{\omega})} = \lim_{n\to\infty} \frac{N_{n+\lceil \varepsilon n\rceil}(\tilde{\omega})}{n+\lceil \varepsilon n\rceil} \cdot \frac{n+\lceil \varepsilon n\rceil}{n} \cdot \frac{n}{N_n(\tilde{\omega})} = b(1+\varepsilon)\frac{1}{b} = 1+\varepsilon$$

and similarly we see that $\lim_{n\to\infty} (N_{n-1}(\tilde{\omega})/N_n(\tilde{\omega})) = 1$. Therefore,

$$\lim_{n\to\infty} \frac{N_{n+\lceil \varepsilon n\rceil}(\tilde{\omega}) - N_{n-1}(\tilde{\omega})}{N_n(\tilde{\omega})} = \varepsilon. \qquad \square$$

Now we are ready to prove Theorem 3.2.

Proof of Theorem 3.2. In [**13**, (5.20)], it is proven that under the assumptions of Theorem 3.1 we have $\overline{\dim}_{\mathrm{B}}(A_{\mathbf{a}}^{\tilde{\omega}}) \le \min\{s_0, d\}$ for P-almost all $\tilde{\omega} \in \widetilde{\Omega}$. Here $\overline{\dim}_{\mathrm{B}}$ is the upper box counting dimension. Note that the assumption $d = 2$ is not needed in the proof of [**13**, (5.20)]. Since always $\dim_{\mathrm{H}} \le \dim_{\mathrm{p}} \le \overline{\dim}_{\mathrm{B}}$ (see for example [**7**, (3.17) and (3.29)]), it is sufficient to verify that

$$\dim_{\mathrm{H}}(A_{\mathbf{a}}^{\tilde{\omega}}) \ge \min\{s_0, d\} \tag{3.9}$$

for P-almost all $\tilde{\omega} \in \widetilde{\Omega}$. Let $s < \min\{s_0, d\}$. In the proof of [**13**, Theorem 3.2], it is shown that (3.9) follows provided that for P-almost all $\tilde{\omega} \in \widetilde{\Omega}$ there exist a probability measure $\mu^{\tilde{\omega}}$ on $\Sigma^{\tilde{\omega}}$ and a constant $D(\tilde{\omega}) > 0$ such that

$$\mu^{\tilde{\omega}}([\mathbf{i}_l]) \le D(\tilde{\omega})\Phi^s(T_{\mathbf{i}_l}^{\tilde{\omega}}) \tag{3.10}$$

for all $\mathbf{i} \in \Sigma^{\tilde{\omega}}$ and $l \in \mathbb{N}$.

For the purpose of verifying (3.10), we define for all $\tilde{\omega} \in \widetilde{\Omega}$ and $m \in \mathbb{N}$

$$\mu_m^{\tilde{\omega}} = \frac{\sum_{\mathbf{i}_{N_m} \in \Sigma_*^{\tilde{\omega}}(0,m)} \Phi^s(T_{\mathbf{i}_{N_m}}^{\tilde{\omega}})\delta_{\mathbf{i}_{N_m}}}{\sum_{\mathbf{i}_{N_m} \in \Sigma_*^{\tilde{\omega}}(0,m)} \Phi^s(T_{\mathbf{i}_{N_m}}^{\tilde{\omega}})}, \tag{3.11}$$

where $\delta_{\mathbf{i}_{N_m}}$ is the Dirac measure at some fixed point of the cylinder $[\mathbf{i}_{N_m}]$. The choice of the cylinder point plays no role in what follows. Since $\Sigma^{\tilde{\omega}}$ is compact, the sequence $(\mu_m^{\tilde{\omega}})_{m\in\mathbb{N}}$ has a weak*-converging subsequence with a limit measure $\mu^{\tilde{\omega}}$. We proceed by showing that $\mu^{\tilde{\omega}}$ satisfies (3.10).

By Lemma 3.8, the following is true for P-almost all $\tilde{\omega} \in \widetilde{\Omega}$: for all $\varepsilon > 0$, there exists $n_2(\tilde{\omega}, \varepsilon) > 0$ such that for all $n > n_2(\tilde{\omega}, \varepsilon)$,

$$N_{n+\lceil \varepsilon n\rceil}(\tilde{\omega}) - N_{n-1}(\tilde{\omega}) < 2\varepsilon N_n(\tilde{\omega}). \tag{3.12}$$

Furthermore, it follows from the definition of the pressure that for P-almost all $\tilde{\omega} \in \widetilde{\Omega}$ there exists for all $\varepsilon > 0$ a number $n_3(\tilde{\omega}, \varepsilon) > 0$ such that for all $n > n_3(\tilde{\omega}, \varepsilon)$, we have

$$e^{(p^{\tilde{\omega}}(s)-\varepsilon)N_n(\tilde{\omega})} < \sum_{\mathbf{i}_{N_n} \in \Sigma_*^{\tilde{\omega}}(0,n)} \Phi^s(T^{\tilde{\omega}}_{\mathbf{i}_{N_n}}) < e^{(p^{\tilde{\omega}}(s)+\varepsilon)N_n(\tilde{\omega})}. \tag{3.13}$$

Let $\varepsilon > 0$. Consider $\tilde{\omega} \in \widetilde{\Omega}$ satisfying Lemma 3.7, (3.12) and (3.13) and set $n_0(\tilde{\omega}, \varepsilon) = \max\{n_1(\tilde{\omega}, \varepsilon), n_2(\tilde{\omega}, \varepsilon), n_3(\tilde{\omega}, \varepsilon)\}$. For all $\mathbf{i}_l \in \Sigma_*^{\tilde{\omega}}$ with $l > N_{n_0(\tilde{\omega},\varepsilon)}$, there exists $n > n_0(\tilde{\omega}, \varepsilon)$ such that $N_{n-1} < l \le N_n$. Now Lemma 3.7 implies the existence of $1 \le k \le \lceil \varepsilon n \rceil$ such that $\{T^{\Xi^{n+k-1}(\tilde{\omega})}_{\mathbf{j}_{N_1}}\}$ is (c, s)-full. Let m be a natural number with $m > \varepsilon n$. In the remaining part of the proof we use the following abbreviations: $\sum_{\mathbf{j}} = \sum_{\mathbf{j}:\mathbf{i}_l\mathbf{j} \in \Sigma_*^{\tilde{\omega}}(0,n+k-1)}$, $\sum_{N_1} = \sum_{\mathbf{j}_{N_1} \in \Sigma_*^{\tilde{\omega}}(n+k-1,n+k)}$, $\sum_{N_{m-k}} = \sum_{\mathbf{k}_{N_{m-k}} \in \Sigma_*^{\tilde{\omega}}(n+k,n+m)}$ and $\sum_{N_{n+k-1}} = \sum_{\mathbf{i}_{N_{n+k-1}} \in \Sigma_*^{\tilde{\omega}}(0,n+k-1)}$, and denote by $T^{\tilde{\omega}}_{(\mathbf{i}_l)\mathbf{j}}$ the last $|\mathbf{j}|$ maps of $T^{\tilde{\omega}}_{\mathbf{i}_l\mathbf{j}}$. Using the definition of $\mu^{\tilde{\omega}}_{n+m}$, applying the submultiplicativity of Φ^s in the numerator and utilizing the (c, s)-fullness in the denominator, we obtain

$$\begin{aligned}
\mu^{\tilde{\omega}}_{n+m}([\mathbf{i}_l]) &= \frac{\sum_{\mathbf{j}} \sum_{N_1} \sum_{N_{m-k}} \Phi^s(T^{\tilde{\omega}}_{\mathbf{i}_l\mathbf{j}} T^{\Xi^{n+k-1}(\tilde{\omega})}_{\mathbf{j}_{N_1}} T^{\Xi^{n+k}(\tilde{\omega})}_{\mathbf{k}_{N_{m-k}}})}{\sum_{N_{n+k-1}} \sum_{N_1} \sum_{N_{m-k}} \Phi^s(T^{\tilde{\omega}}_{\mathbf{i}_{N_{n+k-1}}} T^{\Xi^{n+k-1}(\tilde{\omega})}_{\mathbf{j}_{N_1}} T^{\Xi^{n+k}(\tilde{\omega})}_{\mathbf{k}_{N_{m-k}}})} \\
&\le \frac{\Phi^s(T^{\tilde{\omega}}_{\mathbf{i}_l}) \sum_{\mathbf{j}} \sum_{N_1} \sum_{N_{m-k}} \Phi^s(T^{\tilde{\omega}}_{(\mathbf{i}_l)\mathbf{j}}) \Phi^s(T^{\Xi^{n+k-1}(\tilde{\omega})}_{\mathbf{j}_{N_1}}) \Phi^s(T^{\Xi^{n+k}(\tilde{\omega})}_{\mathbf{k}_{N_{m-k}}})}{c \sum_{N_{n+k-1}} \sum_{N_{m-k}} \Phi^s(T^{\tilde{\omega}}_{\mathbf{i}_{N_{n+k-1}}}) \Phi^s(T^{\Xi^{n+k}(\tilde{\omega})}_{\mathbf{k}_{N_{m-k}}})} \\
&= \frac{\Phi^s(T^{\tilde{\omega}}_{\mathbf{i}_l}) \sum_{\mathbf{j}} \sum_{N_1} \Phi^s(T^{\tilde{\omega}}_{(\mathbf{i}_l)\mathbf{j}}) \Phi^s(T^{\Xi^{n+k-1}(\tilde{\omega})}_{\mathbf{j}_{N_1}})}{c \sum_{N_{n+k-1}} \Phi^s(T^{\tilde{\omega}}_{\mathbf{i}_{N_{n+k-1}}})}.
\end{aligned}$$

Recall that in every family there are at most M maps, $\Phi^s(T_j) \le 1$ for all j and $k \le \lceil \varepsilon n \rceil$, and suppose that $\varepsilon < p^{\tilde{\omega}}(s)$. Applying (3.12) in the numerator and (3.13) in the denominator, we obtain for all $l > N_{n_0(\tilde{\omega},\varepsilon)}$ that

$$\mu^{\tilde{\omega}}_{n+m}([\mathbf{i}_l]) \le \frac{\Phi^s(T^{\tilde{\omega}}_{\mathbf{i}_l}) M^{N_n(\tilde{\omega}) - N_{n-1}(\tilde{\omega}) + N_{n+\lceil \varepsilon n \rceil}(\tilde{\omega}) - N_n(\tilde{\omega})}}{c e^{(p^{\tilde{\omega}}(s)-\varepsilon)N_{n+k-1}(\tilde{\omega})}} \le \frac{\Phi^s(T^{\tilde{\omega}}_{\mathbf{i}_l}) M^{2\varepsilon N_n(\tilde{\omega})}}{c e^{(p^{\tilde{\omega}}(s)-\varepsilon)N_n(\tilde{\omega})}}.$$

Taking ε so small that $M^{2\varepsilon} < e^{p^{\tilde{\omega}}(s)-\varepsilon}$, we set

$$D(\tilde{\omega}) = \max\left\{c^{-1}, \max_{l \le N_{n_0(\tilde{\omega},\varepsilon)}} \left\{ \frac{\mu^{\tilde{\omega}}[\mathbf{i}_l]}{\Phi^s(T^{\tilde{\omega}}_{\mathbf{i}_l})} \right\}\right\}.$$

Then for all $l > 0$ we have

$$\mu^{\tilde{\omega}}_{n+m}([\mathbf{i}_l]) \le D(\tilde{\omega}) \Phi^s(T^{\tilde{\omega}}_{\mathbf{i}_l}).$$

Letting m tend to infinity and recalling that cylinders are open, we obtain (3.10) from the Portmanteau theorem [**16**, Theorem 17.20]. □

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