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V-variable fractals: dimension results

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Abstract. The families of V-variable fractals for $V = 1, 2, 3, \ldots$, together with their natural probability distributions, interpolate between the corresponding families of random homogeneous fractals and of random recursive fractals. We investigate certain random $V \times V$ matrices associated with these fractals and use them to compute the almost sure Hausdorff dimension of V-variable fractals satisfying the uniform open set condition.

Keywords. V-variable fractals, Hausdorff dimension.

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1 Introduction

1.1 Overview

In this paper we begin the study of analysis on V-variable fractals by computing their Hausdorff dimension, under the assumption of an open set condition. In order to make the paper self-contained we include an informal discussion of some simple examples of V-variable fractals and their relationship to other notions of random fractals. For more details see [4].

The key idea is to code up relevant information as a product of random $V \times V$ matrices, one for each level in the construction of a generic V-variable fractal. The almost sure growth rate of the norms of these products can be obtained from a variant of the Furstenberg–Kesten theorem for products of random matrices. Necks are defined in Definition 5.3 and used in (5.11) and (5.16) to construct an appropriate comparison measure μ on a generic V-variable fractal, and then to bound the local mass growth rate of μ in (5.17) and (5.21).

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1.2 Background

Fractal sets generated by a single (contractive) iterated function system [IFS], and random fractal sets generated from a family of such IFSs together with a probability distribution on this family, are studied as mathematical models of disordered systems. In this latter setting the most commonly studied random fractals are random recursive fractals and random homogeneous fractals. In particular, Hausdorff, walk and spectral dimensions have been computed in special cases.

See [7, 8, 14] for general background on fractals and random fractals, including their applications, and see [18, 15, 6, 10, 11, 17, 12, 13, 1, 16, 21] and the references therein for the study of the various dimensions and other analytic properties of fractals.

The classes of V-variable fractals, together with their natural probability distributions, are defined in the next section. For $V = 1, 2, \ldots$, they interpolate between recursive and homogeneous fractals, and similarly for the random versions in each case. They have some initially surprising properties as noted in Section 1.4. In particular, they correspond to the elements of the attractor of a single deterministic IFS, operating not on $\mathbb{R}^n$ but on the metric space of V-tuples of compact subsets of $\mathbb{R}^n$. Their natural probability distribution can be obtained from the unit mass measure on this deterministic IFS, and so they can be rapidly generated by a "chaos game" or Monte Carlo Markov Chain [MCMC] algorithm.

1.3 Preliminary notation

Fix $(\boldsymbol{F}, P)$ where $\boldsymbol{F}$ is a collection of (contractive) IFSs operating on $\mathbb{R}^n$ and P is a probability distribution on $\boldsymbol{F}$.

From this data, as sketched in the model example in Section 2, one constructs a pair $(\mathcal{K}_\infty, \mathcal{R}_\infty)$ where $\mathcal{K}_\infty$ is the class of *recursive fractals* corresponding to $(\boldsymbol{F}, P)$ and $\mathcal{R}_\infty$ is a natural probability distribution on $\mathcal{K}_\infty$, see [6, 10, 17]. One also constructs a corresponding pair $(\mathcal{K}_1, \mathcal{R}_1)$ where $\mathcal{K}_1$ is the corresponding class of *homogeneous fractals* and $\mathcal{R}_1$ is the natural probability distribution on $\mathcal{K}_\infty$, see [12].

For each natural number V we also construct the family $\mathcal{K}_V$ of *V-variable fractals* together with a natural probability distribution $\mathcal{R}_V$ on $\mathcal{K}_V$. These families $(\mathcal{K}_V, \mathcal{R}_V)$ interpolate between the previous two classes $(\mathcal{K}_1, \mathcal{R}_1)$ (the $V = 1$ case) and $(\mathcal{K}_\infty, \mathcal{R}_\infty)$ (the limit case as $V \to \infty$). Each class of V-variable random fractals, with its probability distribution, has the surprising property that it can be obtained from the attractor of a *single* IFS operating on V-tuples of compact subsets of $\mathbb{R}^n$.

For the general development of V-variable fractals see [4], and for other examples and discussion see [3, 2].

1.4 Properties of V-variable fractals

We are interested in V-variable fractals for the following reasons, see [4, 3].

(i) Families of V-variable fractals, together with their probability distributions, interpolate between random homogeneous fractals and random recursive fractals.

(ii) Certain families of functions $(\Phi^{\boldsymbol{a}}, a \in \mathcal{A}_V, P_V)$ and $(\mathcal{F}^{\boldsymbol{a}}, a \in \mathcal{A}_V, P_V)$ together with the probability distribution P_V on $\mathcal{A}_V$ are IFSs with probabilities in the standard sense, the first acting on Ω^V and the other on $\mathcal{C}(\mathbb{R}^2)^V$. (See (2.1) and (2.2), or Section (4.1), for the relevant definitions.) Their measure attractors on Ω^V and $\mathcal{C}(\mathbb{R}^2)^V$ respectively, projected in any one of the V component directions, give the collection of random V-variable fractal trees and sets together with their natural probability distributions.

(iii) The chaos game for these IFSs can be used to generate a sample of V-variable fractals, whose empirical distribution approaches the probability distribution on V-variable fractals as the sample size approaches infinity.

(iv) Analogous results apply to V-variable fractal measures under weak local contractive conditions as opposed to strict global contractivity. Such conditions are natural, for example, in modelling stochastic processes where individual sample paths may be bounded, but there is no uniform bound. In this case one has an IFS operating on a non-locally compact state space. But the chaos game result can be extended to this setting.

(v) By taking large V the chaos game gives a fast forward process for the generation of a sample of fractals approximating random recursive fractals and their probability distribution. This is of practical interest, since normally one builds individual examples of random recursive fractals by a computationally expensive backward process.

2 Sierpinski gaskets, a model example

2.1 Recursive and homogeneous gaskets

Let $F = (\mathbb{R}^2; f_1, f_2, f_3)$ be the IFS consisting of three contraction maps on $\mathbb{R}^2$, each with contraction ratio $1/2$, and having fixed points that are the vertices of an equilateral triangle T of unit diameter. Let $G = (\mathbb{R}^2; f_1, f_2, f_3)$ be the IFS

consisting of three contraction maps, each with contraction ratio $1/3$, and having the same fixed points as the corresponding maps in F. The attractors of F and G are denoted by S_F and S_G respectively, see Figure 1.

Figure 1. Prefractal approximations to the attractors S_F and S_G respectively.

Consider $\boldsymbol{F} = \{F, G\}$, together with the probability distribution $P = \{1/2, 1/2\}$ on $\boldsymbol{F}$. Let Ω *denote the set of labelled trees* which are rooted, 3-branching, and infinite, where the label at each node is either F or G; see Section 3.2. A modified Sierpinski gaket K^ω can be generated from each $\omega \in \Omega$, see Figure 2.

Such fractals K^ω are examples of *recursive fractals*. If the nodes of ω at each fixed level are the same but may vary from level to level, then K^ω is called a *homogeneous fractal*.

It will also be convenient to consider finite, level k, labelled trees for each $k \geq 0$; see Figure 2. Such trees correspond to *level k prefractals*.

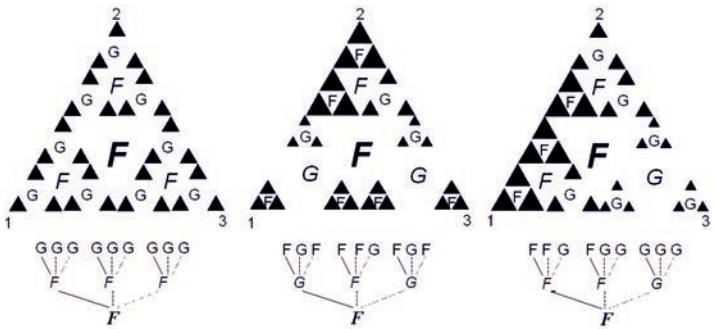

Figure 2. Level 3 recursive prefractals that are 1, 2 and 3-variable respectively, together with the corresponding finite labelled trees. Each vertex of the triangle T is labelled according to the function which fixes it. The labelling convention is clear: the 3 tree labels above each node when read from left to right correspond to applying the corresponding functions onto the bottom left, top and bottom right respectively of the relevant triangle.

Fractals such as S_F and S_G have the following properties:

(i) *Spatial self similarity*: loosely expressed, at each fixed "scale" the component parts are equivalent up to simple transformations, for example, translations in the case here.

(ii) *Scale self similarity*: the equivalence class at each scale is the same.

Homogeneous fractals have spatial self similarity but, generically, do not have scale self similarity. Recursive fractals generically have neither spatial nor scale self similarity. Both classes have statistical self similarity if one imposes suitable probability distributions on their construction.

If the labels of the tree ω are chosen in an iid manner according to some probability distribution P, except that in the homogeneous case all labels at each fixed level are same, then one obtains *random recursive fractals* and *random homogeneous fractals* respectively.

2.2 V-variable gaskets

Now assume that, at each level of $\omega \in \Omega$, the subtrees rooted at that level have the property that they belong to at most V distinct isomorphism classes. In this case, ω is said to be a *V-variable labelled tree*, and the fractal K^ω is said to be a *V-variable gasket* or *fractal*. Such fractals have a form of partial spatial self similarity. Similarly, define *V-variable finite labelled trees* and *V-variable prefractals* as in Figure 2. Notice that homogeneous fractals and 1-variable fractals are the same.

Let Ω_V *denote the class of V-variable trees* ω corresponding to $\boldsymbol{F}$, and let $\mathcal{K}_V$ *denote the corresponding class of V-variable fractals* K^ω.

As is illustrated in Figure 3, there are natural maps Φ^a from V-tuples of infinite, respectively level k, labelled trees $(\omega_1, \ldots, \omega_V)$ to V-tuples of infinite, respectively level $k+1$, labelled trees $(\omega'_1, \ldots, \omega'_V)$, respectively. That is,

$$(\omega'_1, \ldots, \omega'_V) = \Phi^a(\omega_1, \ldots, \omega_V). \tag{2.1}$$

The label at the base or level 0 node of each ω'_v is determined by Φ^a and is either F or G. The three subtrees of each ω'_v rooted at level one are also determined by Φ^a and taken from the set $\{\omega_1, \ldots, \omega_V\}$, possibly with repetition.

The maps Φ^a are described by $V \times (M+1)$ arrays a, where $M = 3$ and $V = 4$ in Figure 3. In general, M is the maximum cardinality of the set of functions in each IFS from the family $\boldsymbol{F}$.

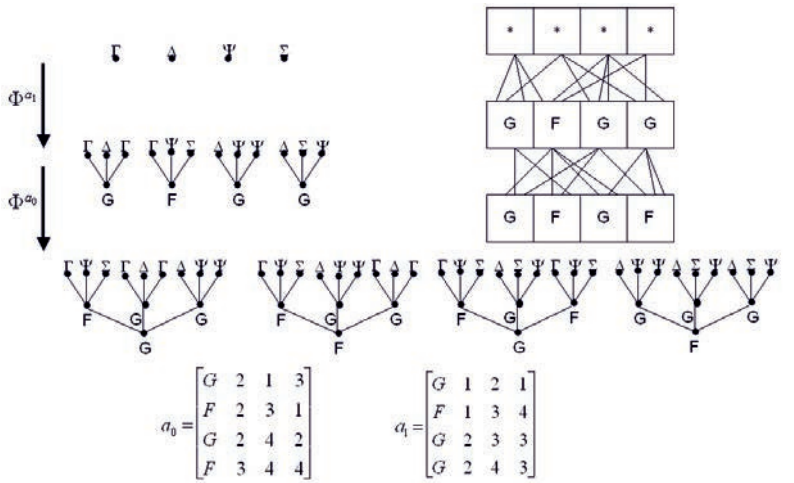

Figure 3. A $V = 4$ example. The map Φ^{a_0} is described by the array a_0. It is also described by the labels in the level 0 (bottom) boxes, and the network of lines between these boxes and the level 1 (middle) boxes.

For example, row 3 of a_0 is $G, 2, 4, 2$. It contains the information that the third component of $\Phi^{a_0}(\omega_1, \omega_2, \omega_3, \omega_4)$ is the tree whose root node is labelled G, and whose three subtrees, rooted at the next level, are ω_2, ω_4 and ω_2 respectively. This information is also provided by the facts that box 3 at level 0 contains the symbol G, and the three lines from this box in the order left to right connect to boxes 2, 4 and 2 at level 1.

Similar remarks apply to Φ^{a_1}.

Corresponding to the family $\boldsymbol{F}$ let

$$\begin{aligned} \mathcal{A}_V &= \text{the set of all such arrays (indices) } a, \\ \mathcal{A}_V^\infty &= \{a_0 a_1 \dots a_k \dots : a_i \in \mathcal{A}_V \text{ for all } i\}. \end{aligned} \tag{2.2}$$

The maps $\mathcal{F}^a$ act on V-tuples of compact subsets of $\mathbb{R}^n$ in an analogous manner, see Figure 4.

If $\boldsymbol{a} = a_0 a_1 \dots a_k \dots \in \mathcal{A}_V^\infty$, then

$$\begin{aligned} (\omega_1^{\boldsymbol{a}}, \dots, \omega_V^{\boldsymbol{a}}) &:= \lim_{k\to\infty} \Phi^{a_0} \circ \cdots \circ \Phi^{a_{k-1}}(\omega_1^0, \dots, \omega_V^0), \\ (K_1^{\boldsymbol{a}}, \dots, K_V^{\boldsymbol{a}}) &:= \lim_{k\to\infty} \mathcal{F}^{a_0} \circ \cdots \circ \mathcal{F}^{a_{k-1}}(K_1^0, \dots, K_V^0). \end{aligned} \tag{2.3}$$

The limits exist and are independent of $(\omega_1^0, \dots, \omega_V^0)$ and $(K_1^0, \dots, K_V^0)$ respectively. Up to level k, the labelled trees $(\omega_1^{\boldsymbol{a}}, \dots, \omega_V^{\boldsymbol{a}})$ depend only on the maps $\Phi^{a_0}, \dots, \Phi^{a_{k-1}}$, see Figure 4 where $k = 2$.

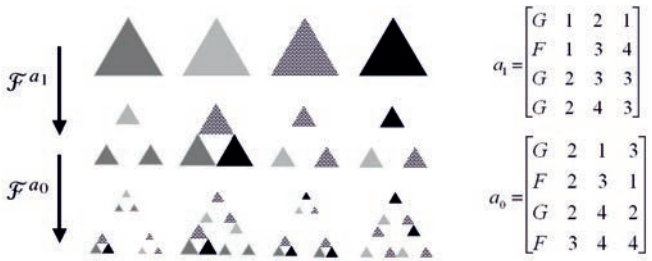

Figure 4. The map $\mathcal{F}^{a_1}$ followed by $\mathcal{F}^{a_0}$ act on $\mathcal{C}(\mathbb{R}^2)^4$, the space of 4-tuples of compact subsets of $\mathbb{R}^2$, in a manner which is encoded in the corresponding arrays a_0 and a_1. For example, row 3 of a_0 is $G, 2, 4, 2$ and contains the information that, for any (K_1, K_2, K_3, K_4), component 3 of $\mathcal{F}(K_1, K_2, K_3, K_4)$ will be $g_1(K_2) \cup g_2(K_4) \cup g_3(K_2)$. Note a_0 and a_1 are as in Figure 3.

2.3 Symbolic representation

There are two ways of representing V-variable fractals.

First, any V-variable fractal K can be represented by a labelled tree $\omega \in \Omega$ as in Figure 2. This does not utilise the V-variability.

Second, and more useful here, the fractal sets $K_v^{\boldsymbol{a}}$ in (2.3) are also described by the infinite sequence $\boldsymbol{a} = a_0 a_1 \dots a_k \dots$, often called an *address* for the V-tuple $K^{\boldsymbol{a}}$. The map $\boldsymbol{a} \mapsto K^{\boldsymbol{a}}$ is typically many-to-one.

The labelled trees $\omega_v^{\boldsymbol{a}}$ and the fractal sets $K_v^{\boldsymbol{a}}$ are V-variable, and every V-variable tree and fractal set can be obtained in this manner. Not only is $(\omega_1^{\boldsymbol{a}}, \dots, \omega_V^{\boldsymbol{a}})$ a V-tuple of V-variable trees, but it satisfies the stronger condition that, for each k, there are at most V isomorphism classes of trees rooted at level k, chosen from *all* the $\omega_1^{\boldsymbol{a}}, \dots, \omega_V^{\boldsymbol{a}}$ taken together. A similar remark applies to $(K_1^{\boldsymbol{a}}, \dots, K_V^{\boldsymbol{a}})$.

2.4 Random V-variable gaskets

So far we have not required a probability distribution on the class Ω_V of V-variable trees or on the class $\mathcal{K}_V$ of V-variable fractals. However, there is a natural probability distribution P_V on $\mathcal{A}_V$ that is inherited from the probability distribution P on the family of IFSs $\boldsymbol{F} = \{F, G\}$. This induces a probability distribution on $\mathcal{A}_V^\infty$, thence from (2.3) on $(\Omega_V)^V \subset \Omega^V$ and $(\mathcal{K}_V)^V$, and thence by projection on Ω_V and $\mathcal{K}_V$.

More precisely, all components of $a \in \mathcal{A}_V$ are chosen independently; those in the first column from $\boldsymbol{F}$ according to P and the remaining entries from $\{1, \dots, V\}$ according to the uniform distribution. The resulting probability distribution P_V on $\mathcal{A}_V$ induces a probability distribution on $\mathcal{A}_V^\infty$, obtained by choosing the ele-

ments a_k of the sequence $\boldsymbol{a} = a_0 a_1 \dots a_k \dots$ in an iid manner according to P_V. The mapping $\mathcal{A}_V^\infty \to \Omega^V$ given by (2.3) now induces a probability distribution on $(\Omega_V)^V \subset \Omega^V$. The projected distribution on Ω_V in any coordinate direction is independent of the coordinate direction, and similarly for $\mathcal{K}_V$. The probability distribution on Ω_1 obtained in this manner is the same as the random homogeneous distribution. The distribution on Ω_V ($\subset \Omega$) converges to the random recursive distribution on Ω as $V \to \infty$. See [4] for details.

2.5 Flow matrices

Recall that the three fixed points of f_1, f_2 and f_3 are also the fixed points of g_1, g_2 and g_3; namely, the vertices of an equilateral triangle T of unit diameter. Suppose $K = K^\omega$ is a recursive Sierpinski triangle with labelled tree $\omega \in \Omega$. The 3^k scaled triangles in the k-level prefractal approximation provide an efficient cover of K for large k, see Figure 2 where $k = 3$, and Figure 4 where $k = 2$. In order to study the Hausdorff measure $\mathcal{H}^\alpha(K)$, it is natural to consider

$$S(\omega, k, \alpha) := \sum_{\{\boldsymbol{m} \in \omega : |\boldsymbol{m}| = k\}} |T_{\boldsymbol{m}}|^\alpha.$$

Here, on the right side, we are summing the diameters to the power α of the 3^k triangles in the level k prefractal for K, see also (4.4).

As in Figure 5, let $(K'_1, \dots, K'_V) = \mathcal{F}^a(K_1, \dots, K_V)$, where $(K_1, \dots, K_V)$ has labelled trees $\boldsymbol{\omega} = (\omega_1, \dots, \omega_V)$ and $(K'_1, \dots, K'_V)$ has labelled trees $\boldsymbol{\omega}' = (\omega'_1, \dots, \omega'_V)$. Setting $\boldsymbol{S}(\boldsymbol{\omega}, k, \alpha) = (S(\omega_1, k, \alpha), \dots, S(\omega_V, k, \alpha))$, one can see by examining Figure 5 that

$$\boldsymbol{S}(\boldsymbol{\omega}', k, \alpha) = M^a(\alpha) \boldsymbol{S}(\boldsymbol{\omega}, k-1, \alpha), \tag{2.4}$$

where, in general, $M^a(\alpha)$ is the $V \times V$ *flow matrix* constructed from a as in Definition 4.2. See also Proposition 4.1. If $\boldsymbol{\omega}^{\boldsymbol{a}} = (\omega_1^{\boldsymbol{a}}, \dots, \omega_V^{\boldsymbol{a}})$ and $(K_1^{\boldsymbol{a}}, \dots, K_V^{\boldsymbol{a}})$ are as in (2.3), it follows on iterating (2.4) that

$$\boldsymbol{S}(\boldsymbol{\omega}^{\boldsymbol{a}}, k, \alpha) = M^{a_0} \circ \cdots \circ M^{a_{k-1}} \mathbf{1}.$$

The vector $\mathbf{1}$ is a vector of units, see also Proposition 4.3.

For a matrix M, let $\|M\|$ be obtained by summing the absolute values of all components. It is now plausible that there is a unique d such that $\|M^{a_0} \circ \cdots \circ M^{a_{k-1}}\|$ should diverge exponentially fast to $+\infty$ a.s. for $\alpha < d$, decay exponentially fast to 0 a.s. for $\alpha > d$, and that this d should be the a.s. Hausdorff dimension of $K_1^{\boldsymbol{a}}, \dots, K_V^{\boldsymbol{a}}$. We show this is the case in the main theorem, see Sections 4 and 5. In Section 6 we compute some examples.

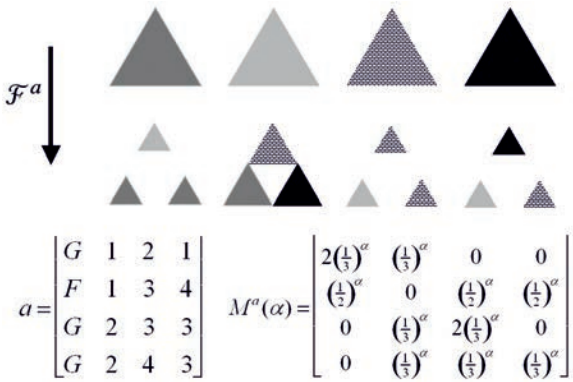

Figure 5. The first row represents fractals $(K_1, \ldots, K_4)$, or prefractals of level $\geq k-1$, corresponding to labelled trees $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_4)$. The second row represents $(K'_1, \ldots, K'_4) = \mathcal{F}^a(K_1, \ldots, K_4)$ with labelled trees $\boldsymbol{\omega}' = (\omega'_1, \ldots, \omega'_4)$. If the vth component of $\boldsymbol{S}(\boldsymbol{\omega}, k-1, \alpha)$ is the $k-1$ level approximation to $\mathcal{H}^\alpha(K_v)$, then, in the case here, $S(\omega'_3, k, \alpha) = \left(\frac{1}{3}\right)^\alpha S(\omega_2, k-1, \alpha) + 2\left(\frac{1}{3}\right)^\alpha S(\omega_3, k-1, \alpha)$. Similarly, $\boldsymbol{S}(\boldsymbol{\omega}', k, \alpha) = M^a(\alpha)\boldsymbol{S}(\boldsymbol{\omega}, k-1, \alpha)$.

Results for walk and spectral dimensions of V-variable fractals, and their properties, have been obtained in work by Uta Freiberg, Ben Hambly and Hutchinson.

3 Notation and assumptions

The *diameter* of a set A is denoted by $|A|$. The *Hausdorff measure* of A in the dimension α is denoted by $\mathcal{H}^\alpha(A)$. The *Hausdorff dimension* of A is denoted by $\dim_{\mathcal{H}}(A)$.

3.1 A family of contractive IFSs

Fix a family $\boldsymbol{F} = \{F^\lambda : \lambda \in \Lambda\}$ of IFSs acting on the space $\mathbb{R}^n$, where we have $F^\lambda = (\mathbb{R}^n; f_1^\lambda, \ldots, f_{M_\lambda}^\lambda)$, each $f_m^\lambda : \mathbb{R}^n \to \mathbb{R}^n$ with Lipschitz constant $r_m^\lambda < 1$, and Λ may be infinite. Fix a probability distribution P on Λ, or equivalently on $\boldsymbol{F}$.

Assume

$$0 < r_{\min} \leq r_m^\lambda \leq r_{\max} < 1, \quad M := \max_{\lambda \in \Lambda} M_\lambda < \infty. \tag{3.1}$$

We usually further assume the $f_m^\lambda : \mathbb{R}^n \to \mathbb{R}^n$ are *similitudes*. That is,

$$|f_m^\lambda(x) - f_m^\lambda(y)| = r_m^\lambda |x - y| \quad \forall x, y \in \mathbb{R}^n. \tag{3.2}$$

In this case we will also assume that $\boldsymbol{F}$ satisfies a *uniform open set condition*. That is, there exists a non-empty open set $\mathcal{O}$ such that

$$\bigcup_{m=1}^{M_\lambda} f_m^\lambda(\mathcal{O}) \subset \mathcal{O}, \quad f_m^\lambda(\mathcal{O}) \cap f_n^\lambda(\mathcal{O}) = \emptyset \text{ if } m \neq n \text{ and } \lambda \in \Lambda. \tag{3.3}$$

We note explicitly when (3.2) and (3.3) apply.

3.2 Labelled trees

We characterise the *set* Ω *of labelled trees* ω, corresponding to the family $\boldsymbol{F}$ of IFSs, as follows:

An (unlabelled) *tree* is a set ω of finite sequences $\boldsymbol{m} = m_1 \dots m_k$ where each $m_j \in \{1, \dots, M\}$, together with the empty sequence $\emptyset$, and which is closed under the operation of taking initial segments. Sequences $\boldsymbol{m} \in \omega$ are called *nodes*. (Note that the branching number at each node is $\leq M$.) The *level* $|\boldsymbol{m}|$ of the node $\boldsymbol{m} = m_1 \dots m_k$ is defined to be $|\boldsymbol{m}| = k$, while $|\emptyset| = 0$.

Some, but not all, such trees, correspond to one or more labelled trees. More precisely:

A *labelled tree*, corresponding to the family of IFSs $\boldsymbol{F}$, is a tree ω together with a map from the set of nodes of ω into Λ. This map is also denoted by ω. If $\boldsymbol{m} \in \omega$ and $\omega(\boldsymbol{m}) = \lambda$, then we require that the nodes of ω whose immediate predecessor is $\boldsymbol{m}$ are precisely $\boldsymbol{m}1, \dots, \boldsymbol{m}M_\lambda$. The edges rooted at a node correspond in a natural way to the functions in the IFS associated to that node.

Let Ω *denote the set of all labelled trees* which correspond to $\boldsymbol{F}$. We frequently refer to a labelled tree simply as a tree.

A V*-variable labelled tree* $\omega \in \Omega$ is a labelled tree such that, for each level k, there are at most V non-isomorphic subtrees of ω rooted at level k. Let $\Omega_V \subset \Omega$ *denote the set of all* V*-variable labelled trees* which correspond to $\boldsymbol{F}$.

3.3 Approximating fractal sets

For $E \subset \mathbb{R}^n$, $\omega \in \Omega$ and nodes $\boldsymbol{m} = m_1 \dots m_k \in \omega$, define

$$E_\emptyset = E^0 = E, \quad E_{m_1 \dots m_k} = f_{m_1}^{\omega(\emptyset)} \circ f_{m_2}^{\omega(m_1)} \circ \cdots \circ f_{m_k}^{\omega(m_1 \dots m_{k-1})}(E),$$
$$E^k = \bigcup_{|\boldsymbol{m}|=k} E_{m_1 \dots m_k}. \tag{3.4}$$

When we need to make the dependence on ω explicit, we will write $E_{\boldsymbol{m}}(\omega)$ or $E^k(\omega)$, for example.

For $\mathcal{O}$ as in (3.3) we have

$$\begin{gathered}\mathcal{O} \supset \mathcal{O}_{m_1} \supset \cdots \supset \mathcal{O}_{m_1 \ldots m_k} \supset \cdots, \quad \overline{\mathcal{O}} \supset \overline{\mathcal{O}}_{m_1} \supset \cdots \supset \overline{\mathcal{O}}_{m_1 \ldots m_k} \supset \cdots, \\ \mathcal{O}_{\boldsymbol{m}} \cap \mathcal{O}_{\boldsymbol{n}} = \emptyset \quad \text{if } |\boldsymbol{m}| = |\boldsymbol{n}|, \ \boldsymbol{m} \neq \boldsymbol{n}, \\ \mathcal{O} \supset \mathcal{O}^1 \supset \cdots \supset \mathcal{O}^k \supset \cdots, \quad \overline{\mathcal{O}} \supset \overline{\mathcal{O}}^1 \supset \cdots \supset \overline{\mathcal{O}}^k \supset \cdots, \\ K^\omega := \bigcap_{k \geq 0} \overline{\mathcal{O}}^k(\omega).\end{gathered} \tag{3.5}$$

The set K^ω *is the fractal set corresponding to* ω. Even if the open set condition does not apply, we obtain the same set K^ω by replacing $\overline{\mathcal{O}}$ by any non-empty compact set E for which

$$\bigcup_{m=1}^{M_\lambda} f_m^\lambda(E) \subset E \quad \forall \lambda \in \Lambda. \tag{3.6}$$

The set $\mathcal{K}_V$ *of* V*-variable fractals sets* is defined by

$$\mathcal{K}_V = \{K^\omega : \omega \in \Omega_V\}.$$

4 The Hausdorff dimension of V-variable fractals

4.1 The maps Φ^a and $\mathcal{F}^a$

(See the examples in Figures 3, 4, 5 and 6.) The index set $\mathcal{A}_V$ is defined to be the set of all arrays a of the form

$$a = \begin{bmatrix} I^a(1) & J^a(1,1) & \ldots & J^a(1,M) \\ \vdots & \vdots & \ddots & \vdots \\ I^a(V) & J^a(V,1) & \ldots & J^a(V,M) \end{bmatrix}, \tag{4.1}$$

where $I^a(v) \in \Lambda$, M is as in (3.1), $J^a(v,m) \in \{1, \ldots, V\}$ if $1 \leq m \leq M_{I^a(v)}$ and $J^a(v,m) = 0$ if $M_{I^a(v)} < m \leq M$. See also [4, Section 5.3].

The probability distribution P_V on $\mathcal{A}_V$ is obtained as follows. The elements in the first column are chosen independently according to P. The elements $J^a(v,m)$ for $1 \leq m \leq M_{I^a(v)}$ are chosen independently of one another and of elements in the first column, according to the uniform distribution on $\{1, \ldots, V\}$. Any remaining elements $J^a(v,m)$ for $m > M_{I^a(v)}$ are set equal to 0.

The map $\Phi_V^a : \Omega^V \to \Omega^V$ is defined by requiring, for any $(\omega_1, \ldots, \omega_V) \in \Omega^V$, that the vth component of $\Phi^a(\omega_1, \ldots, \omega_V) \in \Omega^V$ is the labelled tree whose base node is labelled $F^{I^a(v)}$, and whose mth subtree rooted at level one is $\omega_{J^a(v,m)}$ for $1 \le m \le M_{I^a(v)}$. Any zeros at the end of each row of a are markers so that all rows have equal length but otherwise play no role.

Similarly, for any V-tuple of sets $(K_1, \ldots, K_V)$, the map $\mathcal{F}^{a_V} : \mathcal{C}(\mathbb{R}^n) \to \mathbb{R}^n)$ is defined by

$$\mathcal{F}^a(K_1, \ldots, K_V) = \left(\bigcup_{m=1}^{M_{I^a(1)}} f_m^{I^a(1)}(K_{J^a(1,m)}), \ldots, \bigcup_{m=1}^{M_{I^a(V)}} f_m^{I^a(V)}(K_{J^a(V,m)}) \right). \tag{4.2}$$

For $\boldsymbol{a} = a_0 \ldots a_{k-1} \ldots \in \mathcal{A}_V^\infty$, assuming (3.6) for some compact E, we define

$$\begin{aligned} (\omega_1^{\boldsymbol{a}}, \ldots, \omega_V^{\boldsymbol{a}}) &:= \lim_{k\to\infty} \Phi^{a_0} \circ \cdots \circ \Phi^{a_{k-1}}(\omega_1^0, \ldots, \omega_V^0), \\ (K_1^{\boldsymbol{a}}, \ldots, K_V^{\boldsymbol{a}}) &:= \lim_{k\to\infty} \mathcal{F}^{a_0} \circ \cdots \circ \mathcal{F}^{a_{k-1}}(K_1^0, \ldots, K_V^0). \end{aligned} \tag{4.3}$$

The limits exist, and are independent of $(\omega_1^0, \ldots, \omega_V^0)$ and $(K_1^0, \ldots, K_V^0)$, respectively.

If $v \in \{1, \ldots, V\}$, then $\omega_v^{\boldsymbol{a}} \in \Omega_V$, and every V-variable labelled tree can be obtained in this manner. However, if Ω_V^* is the set of V-tuples of labelled trees obtained in this manner, then $\Omega_V^* \subsetneq (\Omega_V)^V$.

The probability distribution P_V^∞ on $\mathcal{A}_V^\infty$ is defined by selecting the a_k in an iid manner according to P_V. This induces a probability distribution on $\Omega_V^* \subset (\Omega_V)^V$ via (4.3), and thence a *probability distribution ρ_V on Ω_V* by projecting in any of the V coordinate directions. Similarly one obtains *a probability distribution $\mathcal{R}_V$ on $\mathcal{K}_V$*. Both ρ_V and $\mathcal{R}_V$ are independent of choice of projection direction, although the initial distributions are not product distributions. See [4].

4.2 Approximating Hausdorff measure

Assume the open set condition (3.3), and without loss of generality assume that $|\mathcal{O}| = 1$. (See Figure 2 and take $\mathcal{O}$ to be the interior of the triangle T.)

Suppose $\omega \in \Omega$ is a labelled tree with labels from $\boldsymbol{F}$. Keeping in mind (3.5), we think of the collection of sets $\{\overline{\mathcal{O}}_{\boldsymbol{m}} : \boldsymbol{m} \in \omega, |\boldsymbol{m}| = k\}$ as an "efficient" cover of K^ω for large k. In order to compute the Hausdorff dimension of K^ω, we consider

the following quantities for $\alpha > 0$:

$$\begin{aligned} r_{\emptyset}(\omega) &:= |\overline{\mathcal{O}}| = 1, \\ r_{m_1\ldots m_k}(\omega) &:= |\overline{\mathcal{O}}_{m_1\ldots m_k}| \\ &= r_{m_1}^{\omega(\emptyset)} \cdot \ldots \cdot r_{m_k}^{\omega(m_1\ldots m_{k-1})} \quad \text{for } m_1 \ldots m_k \in \omega, \\ S(\omega, k, \alpha) &:= \sum_{\{\boldsymbol{m}\in\omega:|\boldsymbol{m}|=k\}} |\overline{\mathcal{O}}_{\boldsymbol{m}}|^{\alpha} \\ &= \sum_{\{\boldsymbol{m}\in\omega:|\boldsymbol{m}|=k\}} (r_{\boldsymbol{m}}(\omega))^{\alpha}, \quad \text{noting } S(\omega, 0, \alpha) = 1. \end{aligned} \tag{4.4}$$

We are interested in the behaviour of $S(\omega, k, \alpha)$ as $k \to \infty$ since, for large k, it is an approximation to the Hausdorff measure $\mathcal{H}^{\alpha}(K^{\omega})$.

4.3 Flow matrices

(See the example in Figure 5.) As remarked in the introduction, there are two ways of representing a V-variable fractal K. One can either use the corresponding labelled tree $\omega \in \Omega$, or one can use a sequence $\boldsymbol{a} \in (\mathcal{A}_V)^{\infty}$ as in (4.3) which generates a V-tuple containing K as a component. We use the latter in order to study $S(\omega, k, \alpha)$.

For $(\omega_1, \ldots, \omega_V) \in \Omega^V$ define the following vector of real numbers:

$$\boldsymbol{S}\big((\omega_1, \ldots, \omega_V), k, \alpha\big) = (S(\omega_1, k, \alpha), \ldots, S(\omega_V, k, \alpha)). \tag{4.5}$$

Then $\boldsymbol{S}\big((\omega_1, \ldots, \omega_V), 0, \alpha\big) = \mathbf{1}$, the V-vector whose components all equal 1.

Proposition 4.1. *If $(\omega_1, \ldots, \omega_V) \in \Omega^V$, and $(\omega'_1, \ldots, \omega'_V) = \Phi^a(\omega_1, \ldots, \omega_V)$, then*

$$S(\omega'_v, k, \alpha) = \sum_{w=1}^{V} \left(\sum_{\{m: J^a(v,m)=w\}} \left(r_m^{I^a(v)}\right)^{\alpha} \right) S(\omega_w, k-1, \alpha).$$

Proof. For any $\omega \in \Omega$ it follows from (4.4) that

$$r_{m_1\ldots m_k}(\omega) = r_{m_1}^{\omega(\emptyset)} r_{m_2\ldots m_k}(\omega^{(m_1)}),$$

where $\omega^{(m_1)}$ is the labelled subtree of ω rooted at node m_1 at level one. From the definition of $\Phi^a(\omega_1, \ldots, \omega_V)$ in Section 4.1, it follows that

$$r_{m_1\ldots m_k}(\omega'_v) = r_{m_1}^{I^a(v)} r_{m_2\ldots m_k}(\omega_{J^a(v,m_1)}).$$

Hence, from (4.4) and the above

$$\begin{aligned} S(\omega'_v, k, \alpha) &= \sum_{m_1 \ldots m_k \in \omega'_v} \left(r_{m_1}^{I^a(v)}\right)^\alpha \left(r_{m_2 \ldots m_k}(\omega_{J^a(v,m_1)})\right)^\alpha \\ &= \sum_{m_1=1}^{M_{I^a(v)}} \left(\left(r_{m_1}^{I^a(v)}\right)^\alpha \sum_{m_2 \ldots m_k \in \omega_{J^a(v,m_1)}} \left(r_{m_2 \ldots m_k}(\omega_{J^a(v,m_1)})\right)^\alpha \right) \\ &= \sum_{w=1}^{V} \sum_{\{m_1 : J^a(v,m_1)=w\}} \left(\left(r_{m_1}^{I^a(v)}\right)^\alpha \sum_{m_2 \ldots m_k \in \omega_w} \left(r_{m_2 \ldots m_k}(\omega_w)\right)^\alpha \right) \\ &= \sum_{w=1}^{V} \left(\sum_{\{m : J^a(v,m)=w\}} \left(r_m^{I^a(v)}\right)^\alpha \right) S(\omega_w, k-1, \alpha). \end{aligned}$$ □

Motivated by this we make the following definition.

Definition 4.2. The $V \times V$ *flow matrix* $M^a = M^a(\alpha)$ for $a \in \mathcal{A}_V$ is defined by

$$M^a_{vw} = \sum_{\{m : J^a(v,m)=w\}} \left(r_m^{I^a(v)}\right)^\alpha, \quad 1 \le v, w \le V.$$

Flow matrices are the main book keeping tool for tracking the size of covers of V-variable fractals.

Proposition 4.3. *Suppose* $\boldsymbol{a} \in \mathcal{A}_V^\infty$, *and* $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_V) = \boldsymbol{\omega}^{\boldsymbol{a}} \in \Omega^V$ *has address* $\boldsymbol{a}$ *as in* (4.3). *Then,*

$$\boldsymbol{S}(\boldsymbol{\omega}^{\boldsymbol{a}}, k, \alpha) = M^{a_0} \circ \cdots \circ M^{a_{k-1}} \mathbf{1},$$

i.e.

$$S(\omega_v^{\boldsymbol{a}}, k, \alpha) = \sum_{w=1}^{V} \left(M^{a_0} \circ \cdots \circ M^{a_{k-1}}\right)_{vw} \quad \text{for } 1 \le v \le V.$$

Proof. From Proposition 4.1 and Definition 4.2, for *any* $\boldsymbol{\omega} \in \Omega^V$,

$$\boldsymbol{S}\left(\Phi^a(\boldsymbol{\omega}), k, \alpha\right) = M^a \boldsymbol{S}(\boldsymbol{\omega}, k-1, \alpha).$$

Hence,

$$\begin{aligned} \boldsymbol{S}\left(\Phi^{a_0} \circ \cdots \circ \Phi^{a_{k-1}}(\boldsymbol{\omega}), k, \alpha\right) &= M^{a_0} \circ \cdots \circ M^{a_{k-1}} \boldsymbol{S}(\boldsymbol{\omega}, 0, \alpha) \\ &= M^{a_0} \circ \cdots \circ M^{a_{k-1}} \mathbf{1}. \end{aligned}$$

Since $\boldsymbol{\omega}^{\boldsymbol{a}}$ is of the form $\Phi^{a_0} \circ \cdots \circ \Phi^{a_{k-1}}(\boldsymbol{\omega})$ for some $\boldsymbol{\omega}$, the result follows. □

For fixed $\boldsymbol{a} = a_0 \dots a_{k-1} \dots$ and $\boldsymbol{\omega}$ as in Proposition 4.3, we often write

$$S_v(k, \alpha) = S(\omega_v, k, \alpha). \tag{4.6}$$

Note that $S_v(k, \alpha)$ depends only on $a_0, \dots, a_{k-1}$, and not on a_j for $j \geq k$.

4.4 Computing the Hausdorff dimension

If A is a matrix, we define the norm $\|A\| := \sum_{v,w} |A_{vw}|$. It is easily checked that this norm is submultiplicative, $\|AB\| \leq \|A\| \cdot \|B\|$.

Main Theorem. *Fix $\boldsymbol{F} = \{F^\lambda : \lambda \in \Lambda\}$, a probability distribution P on Λ, an integer $V \geq 1$ and a real number $\alpha > 0$. Under the assumption* (3.1)*,*

$$\begin{aligned}\gamma(\alpha) &:= \lim_{k\to\infty} \mathbb{E}\, \frac{1}{k} \log \|M^{a_0}(\alpha) \cdot \dots \cdot M^{a_{k-1}}(\alpha)\| \\ &= \lim_{k\to\infty} \frac{1}{k} \log \|M^{a_0}(\alpha) \cdot \dots \cdot M^{a_{k-1}}(\alpha)\| \quad \text{a.s.}\end{aligned}$$

In particular, the second limit is independent of $\boldsymbol{a} = a_0 \dots a_{k-1} \dots$ for P_V^∞ a.e. $\boldsymbol{a} \in \mathcal{A}_V^\infty$. The function $\gamma(\alpha)$ is monotonically decreasing in α, and there is a unique d such that $\gamma(d) = 0$.

Assuming (3.2)*, and the open set condition* (3.3)*, $\dim_{\mathcal{H}}(K^\omega) = d$ for ρ_V a.e. $\omega \in \Omega_V$.*

This theorem is proved in Section 5. For an example see Section 6.

Remark 4.4. The limit $\gamma(\alpha)$ is sometimes called a "Lyapunov exponent", since $\|M^{a_0}(\alpha) \cdot \dots \cdot M^{a_{k-1}}(\alpha)\|$ grows like $e^{k\gamma(\alpha)}$ as $k \to \infty$.

The fact $\gamma(\alpha)$ exists and is independent of $\boldsymbol{a}$, for a.e. $\boldsymbol{a} \in \mathcal{A}_V^\infty$, is a consequence of the version of the Furstenberg–Kesten theorem [9] in [5, Theorem C, p. 72]. Individual terms $\frac{1}{k} \log(M^{a_0}(\alpha) \cdot \dots \cdot M^{a_{k-1}}(\alpha))_{vw}$ will not normally converge as $k \to \infty$. In particular, for fixed v and w, $(M^{a_0}(\alpha) \cdot \dots \cdot M^{a_{k-1}}(\alpha))_{vw} = 0$ infinitely often a.s. In fact, if $J^{a_{k-1}}(u, m) \neq w$ for all u and m, which happens with positive probability, then all entries in the w column of $M^{a_{k-1}}(\alpha)$ are zero, and hence all entries in the w column of $M^{a_0}(\alpha) \cdot \dots \cdot M^{a_{k-1}}(\alpha)$ are also zero. However, $\lim_{k\to\infty} \frac{1}{k} \log S_v(k, \alpha)$ exists a.s. and equals $\gamma(\alpha)$, for every $v \in \{1, \dots, V\}$, see Lemma 5.5.

Remark 4.5. Assuming the open set condition it follows, for random homogeneous fractals K^ω (the $V = 1$ case), that

$$\dim_{\mathcal{H}}(K^\omega) \text{ is the unique } \alpha \text{ for which } \mathbb{E}_\lambda \log \sum_{m=1}^{M_\lambda} (r_m^\lambda)^\alpha = 0, \text{ for a.e. } \omega. \tag{4.7}$$

See [12] for the direct computation of the dimension in some particular cases. For random recursive fractals K^ω, the "$V \to \infty$" case,

$$\dim_{\mathcal{H}}(K^\omega) \text{ is the unique } \alpha \text{ for which } \mathbb{E}_\lambda \sum_{m=1}^{M_\lambda} (r_m^\lambda)^\alpha = 1, \text{ for a.e. } \omega. \tag{4.8}$$

See [6, 17, 10, 11].

5 Proof of the Main Theorem

The proof is broken into a number of lemmas.

Assumptions

We continue the assumptions from the beginning of Section 3, and the notation from Sections 3 and 4. The integer $V \geq 1$ is fixed and α is non-negative.

Define

$$R_{\min}(\alpha) = \inf_\lambda \sum_{m=1}^{M_\lambda} (r_m^\lambda)^\alpha, \quad R_{\max}(\alpha) = \sup_\lambda \sum_{m=1}^{M_\lambda} (r_m^\lambda)^\alpha. \tag{5.1}$$

The sequence $\boldsymbol{a} = a_0 \dots a_{k-1} \dots \in \mathcal{A}_V$ is chosen according to P_V^∞ as in Section 4.1. The V-tuple of fractal sets corresponding to $\boldsymbol{a}$ is denoted $\boldsymbol{K}^{\boldsymbol{a}}$, and the corresponding V-tuple of labelled trees is $\boldsymbol{\omega}^{\boldsymbol{a}}$. Let

$$\begin{aligned} \boldsymbol{K}^{\boldsymbol{a}} &= (K_1^{\boldsymbol{a}}, \dots, K_V^{\boldsymbol{a}}) = (K_1, \dots, K_V), \\ \boldsymbol{\omega}^{\boldsymbol{a}} &= (\omega_1^{\boldsymbol{a}}, \dots, \omega_V^{\boldsymbol{a}}) = (\omega_1, \dots, \omega_V). \end{aligned} \tag{5.2}$$

From the next lemma, it follows there is a unique d such that as $k \to \infty$ the product $\|M^{a_0}(\alpha) \cdot \ldots \cdot M^{a_{k-1}}(\alpha)\|$ grows exponentially fast to ∞ if $\alpha < d$, and decays exponentially fast to 0 if $\alpha > d$. This does not imply convergence to a non-zero limit or boundedness of $\|M^{a_0}(d) \cdot \ldots \cdot M^{a_{k-1}}(d)\|$, but it does imply that any infinite growth, or decay to zero, should be slower than exponential.

Lemma 5.1. *The limit*

$$\lim_{k\to\infty} \mathbb{E}\, \frac{1}{k} \log \|M^{a_0}(\alpha) \cdot \ldots \cdot M^{a_{k-1}}(\alpha)\| =: \gamma(\alpha)$$

exists. In addition,

$$\lim_{k\to\infty} \frac{1}{k} \log \|M^{a_0}(\alpha) \cdot \ldots \cdot M^{a_{k-1}}(\alpha)\| = \gamma(\alpha) \quad a.s.,$$

and in particular is a.s. independent of $\boldsymbol{a}$*. Moreover,*

$$\log R_{\min}(\alpha) \le \gamma(\alpha) \le \log R_{\max}(\alpha).$$

The function γ *is strictly decreasing, Lipschitz, has derivative in the interval* $[\log r_{\min}, \log r_{\max}]$*,* $\gamma(0) > 0$ *and* $\gamma(\alpha) \to -\infty$ *as* $\alpha \to \infty$*. In particular, there is a unique* d *such that* $\gamma(d) = 0$*.*

Proof. The first two claims hold for some $\gamma(\alpha)$ with $-\infty \le \gamma(\alpha) < \infty$. This follows from the version of the Furstenberg–Kesten theorem in [5, Theorem C, p. 72] since the a_k are chosen in an iid manner.

Suppose A and B are square matrices of the same size with non-negative entries. Assume

$$\alpha_1 \le \sum_j A_{ij} \le \alpha_2, \quad \beta_1 \le \sum_j B_{ij} \le \beta_2 \quad \text{for all } i.$$

Let $C = AB$. Since $\sum_j C_{ij} = \sum_k A_{ik}\big(\sum_j B_{kj}\big)$,

$$\alpha_1\beta_1 \le \sum_j C_{ij} \le \alpha_2\beta_2 \quad \text{for all } i.$$

In particular, from Definition 4.2 and (5.1),

$$R^k_{\min}(\alpha) \le \sum_w \big(M^{a_0}(\alpha) \cdot \ldots \cdot M^{a_{k-1}}(\alpha)\big)_{vw} \le R^k_{\max}(\alpha) \text{ for all } v,$$

and so

$$VR^k_{\min}(\alpha) \le \|M^{a_0}(\alpha) \cdot \ldots \cdot M^{a_{k-1}}(\alpha)\| \le VR^k_{\max}(\alpha).$$

Taking logs of both sides, and letting $k \to \infty$, gives the third claim in the lemma.

From Definition 4.2, if $0 \le \alpha < \beta$,

$$\begin{aligned} r^{k(\beta-\alpha)}_{\min}\|M^{a_0}(\alpha) \cdot \ldots \cdot M^{a_{k-1}}(\alpha)\| &\le \|M^{a_0}(\beta) \cdot \ldots \cdot M^{a_{k-1}}(\beta)\| \\ &\le r^{k(\beta-\alpha)}_{\max}\|M^{a_0}(\alpha) \cdot \ldots \cdot M^{a_{k-1}}(\alpha)\|. \end{aligned}$$

Taking logs and letting $k \to \infty$,

$$(\beta - \alpha)\log r_{\min} \le \gamma(\beta) - \gamma(\alpha) \le (\beta - \alpha)\log r_{\max}.$$

Hence γ is Lipschitz, differentiable a.e., monotonically decreasing to $-\infty$, and has derivative in the range $\log r_{\min}$ to $\log r_{\max}$.

Since

$$\gamma(0) \in [\log R_{\min}(0),\ \log R_{\max}(0)] = [\log(\min_\lambda M_\lambda),\ \log M = \log(\max_\lambda M_\lambda)],$$

it follows that $\gamma(0) > 0$, and so there exists a unique d such that $\gamma(d) = 0$. □

Remark 5.2. Bob Scealy [19] has pointed out that one can avoid using the Furstenberg–Kesten theorem and instead use the fact that one has a renewal process, where the renewals are the occurrence of a neck. He has used this idea in his investigation of V-variable fractal graphs.

In the next following lemmas, the notion of a "neck" $a \in \mathcal{A}_V$ will play an important role, see Figure 6.

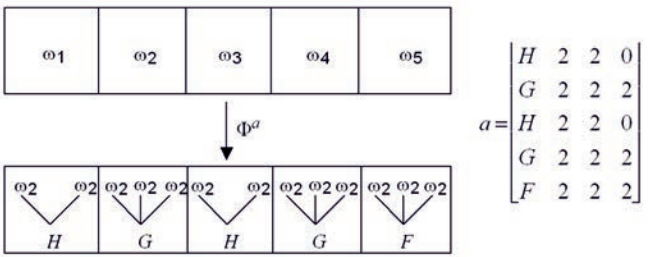

Figure 6. A neck a with IFSs F, G and H, and with $J^a(v,m) = 2$ or 0 for all v and m.

Definition 5.3. An element $a \in \mathcal{A}_V$ is called a *neck* if all $J^a(v,m)$ are equal for $v \in \{1,\dots,V\}$ and if $m \in \{1,\dots,M_{I^a}(v)\}$.

A neck occurs at level k in $\boldsymbol{a} = a_0 \dots a_{k-1} \dots \in \mathcal{A}_V^\infty$, or more simply we say that k *is a neck*, if a_{k-1} is a neck.

Remark 5.4. An element a chosen according to P_V is a neck with probability at least V^{1-MV}. It follows that necks in a sequence $\boldsymbol{a} \in \mathcal{A}_V^\infty$ occur infinitely often a.s.

If a neck occurs in $\boldsymbol{a}$ at level k, then all subtrees of $(\omega_1^{\boldsymbol{a}},\dots,\omega_V^{\boldsymbol{a}})$ rooted at level k are equal. That is, if $|\boldsymbol{m}| = |\boldsymbol{m}'| = k$ and $v, v' \in \{1,\dots,V\}$, then $\boldsymbol{mn} \in \omega_v$ iff $\boldsymbol{m}'\boldsymbol{n} \in \omega_{v'}$, and in this case $\omega_v(\boldsymbol{mn}) = \omega_{v'}(\boldsymbol{m}'\boldsymbol{n})$.

For the following lemma recall that $S_v(k,\alpha)$ is the sum of the elements in the vth row of $M^{a_0}(\alpha)\cdot\ldots\cdot M^{a_{k-1}}(\alpha)$. From (4.4) we have that $S(\omega,k,\alpha)$, for large k, is an approximation to the Hausdorff measure $\mathcal{H}^\alpha(K^\omega)$. See also Remark 4.4.

Lemma 5.5. *For $\boldsymbol{a} \in \mathcal{A}_V^\infty$ and $\boldsymbol{\omega} = \boldsymbol{\omega}^{\boldsymbol{a}} = (\omega_1,\dots,\omega_V)$,*

$$\lim_{k\to\infty} \frac{1}{k} \log S_v(k,\alpha) = \gamma(\alpha) \quad \text{a.s.} \tag{5.3}$$

Proof. Let $S_v(k,\alpha) = S(\omega_v, k, \alpha)$, as in (4.6).

Suppose the address sequence $\boldsymbol{a}$ has a *neck* at level p, where $J^{a_{p-1}}(v,m) = u$ for all $v \in \{1,\dots,V\}$ and $m \in \{1,\dots,M_{I^{a_{p-1}}(v)}\}$. It follows that all columns of $M^{a_{p-1}}$ are zero, except for the uth column, and hence the same is true for $M^{a_0} \circ \cdots \circ M^{a_{p-1}}$.

Suppose A and B are $V \times V$ matrices such that all columns of A are zero except for the uth column, which we denote by $\boldsymbol{a}$. Let $\boldsymbol{b}$ be the uth row of B. Then

$$AB = [\, b_1\boldsymbol{a}\ b_2\boldsymbol{a}\ \dots\ b_V\boldsymbol{a}\,].$$

It follows that

$$\sum_w (AB)_{vw} = a_v \sum_w b_w = \left(\sum_w A_{vw}\right) \sum_w b_w.$$

In particular, the second factor is independent of v.

Apply this to

$$S_v(k,\alpha) = \sum_w \left(M^{a_0} \circ \cdots \circ M^{a_{k-1}}\right)_{vw}, \tag{5.4}$$

see Proposition 4.3 and (4.6), with

$$A = M^{a_0} \circ \cdots \circ M^{a_{p-1}}, \quad B = M^{a_p} \circ \cdots \circ M^{a_{k-1}}.$$

It follows that

$$S_v(k,\alpha) = S_v(p,\alpha)\, g(k,\alpha), \tag{5.5}$$

where $g(k,\alpha)$ is independent of v, and where $S_v(p,\alpha) > 0$ for all v.

From (5.4), summing over v,

$$\|M^{a_0} \circ \cdots \circ M^{a_{k-1}}\| = \sum_v S_v(k,\alpha) = g(k,\alpha) \sum_v S_v(p,\alpha).$$

Hence from Lemma 5.1,

$$\gamma(\alpha) = \lim_{k\to\infty} \frac{1}{k} \log \|M^{a_0}(\alpha) \cdot \ldots \cdot M^{a_{k-1}}(\alpha)\| = \lim_{k\to\infty} \log \frac{1}{k} g(k,\alpha) \quad \text{a.s.}$$

Going back to (5.5), since $S_v(p,\alpha) > 0$, it follows that

$$\lim_{k\to\infty} \frac{1}{k} \log S_v(k,\alpha) = \gamma(\alpha) \quad \text{a.s.} \qquad \square$$

Lemma 5.6. *If $\alpha > d$ with d as in Lemma 5.1, then we have $\mathcal{H}^\alpha(K_v^{\boldsymbol{a}}) = 0$ for $v \in \{1,\dots,V\}$ and for a.e. $\boldsymbol{a}$. In particular,* $\dim_{\mathcal{H}}(K_v^{\boldsymbol{a}}) \le d$.

Proof. We usually drop the reference to $\boldsymbol{a}$ and write K_v for $K_v^{\boldsymbol{a}}$, and ω_v for $\omega_v^{\boldsymbol{a}}$.

Let E be any set such that $K_1 \cup \cdots \cup K_V \subset E$, and without loss of generality suppose $|E| = 1$. For $v \in \{1, \ldots, V\}$ and $\boldsymbol{m} = m_1 \ldots m_k \in \omega_v$, let

$$E_{v;m_1 \ldots m_k} = f_{m_1}^{\omega_v(\emptyset)} \circ f_{m_2}^{\omega_v(m_1)} \circ \cdots \circ f_{m_k}^{\omega_v(m_1 \ldots m_{k-1})}(E),$$

as in (3.4). Then

$$K_v \subset \bigcup_{\{\boldsymbol{m} \in \omega_v : |\boldsymbol{m}| = k\}} E_{v;\boldsymbol{m}},$$

and

$$S_v(k, \alpha) = \sum_{\{\boldsymbol{m} \in \omega : |\boldsymbol{m}| = k\}} |E_{v;\boldsymbol{m}}|^{\alpha},$$

as in (4.6) and (4.4).

Since $\gamma(\alpha) < 0$, it follows from (5.3) that

$$\lim_{k \to \infty} S_v(k, \alpha) = \lim_{k \to \infty} e^{k(\frac{1}{k} \log S_v(k, \alpha))} = 0 \quad \text{a.s.} \tag{5.6}$$

Hence $\mathcal{H}^{\alpha}(K_v) = 0$ a.s. □

Lemma 5.7. *Assume $\boldsymbol{F}$ satisfies the open set condition. If $\alpha < d$, where d is as in Lemma 5.1, then we have $\mathcal{H}^{\alpha}(K_v^{\boldsymbol{a}}) > 0$ a.s. for $1 \le v \le V$. In particular,* $\dim_{\mathcal{H}}(K_v^{\boldsymbol{a}}) \ge d$ *a.s.*

Proof. Suppose $\alpha < d$. As before, $K_v = K_v^{\boldsymbol{a}}$ and $\omega_v = \omega_v^{\boldsymbol{a}}$.

For a.e. $\boldsymbol{a}$ and each $1 \le v \le V$, we construct a unit mass measure μ on K_v such that for some c,

$$\mu(B_r(x)) \le c r^{\alpha} \quad \text{if } r > 0,\ x \in \mathbb{R}^n. \tag{5.7}$$

It then follows by the mass distribution principle [7, p. 60] that $\mathcal{H}^{\alpha}(K_v) > 0$, and so $\dim(K_v) \ge d$.

A. *Properties of Necks.* For $\boldsymbol{a} \in \mathcal{A}_V^{\infty}$ and $k \ge 0$, let $n^{\boldsymbol{a}}(k) = n(k)$ denote the first level $\ge k$ at which a neck occurs.

Then we claim

$$\forall \epsilon > 0\ \exists N > 0 \text{ such that } \forall k \big(n(k) - k \le N + \epsilon k\big) \text{ a.s.}, \tag{5.8}$$

where N will depend on $\boldsymbol{a}$.

To see this, fix $\epsilon > 0$ and $k > 0$, and let

$$E_k = \{\boldsymbol{a} : n^{\boldsymbol{a}}(k) - k > \epsilon k\}.$$

It follows from Remark 5.4 that

$$P_V^{\infty}(E_k) \le \left(1 - V^{1-MV}\right)^{\epsilon k}.$$

Since $\sum_{k\ge 1} P_V^{\infty}(E_k) < \infty$, it follows from the Borel–Cantelli lemma that, with probability one, E_k occurs for only finitely many k, and so $n(k) - k \le \epsilon k$ for all k sufficiently large. Hence for some N depending on $\boldsymbol{a}$ and ϵ, $n(k) - k \le N + \epsilon k$ for all k. This proves (5.8).

B. *Construction of ν and μ.* Suppose $\boldsymbol{a}$ is as in (5.8) and consider the tree $\boldsymbol{\omega}^{\boldsymbol{a}} = (\omega_1, \ldots, \omega_V)$. For fixed v, a unit measure ν will first be constructed on the *set $\widetilde{\omega}_v$ of infinite paths through ω_v*.

For $\boldsymbol{m} \in \omega_v$ the corresponding *cylinder set*, a subset of $\widetilde{\omega}_v$, is defined by

$$[\boldsymbol{m}] = \{\boldsymbol{p} \in \widetilde{\omega}_v : \boldsymbol{m} \le \boldsymbol{p}\},$$

where $\boldsymbol{m} \le \boldsymbol{p}$ means that $\boldsymbol{m}$ is an initial segment of $\boldsymbol{p}$.

The weight function w is defined on cylinder sets by

$$w([\boldsymbol{m}]) = r^{\alpha}_{v;\boldsymbol{m}} := \left(r^{\omega_v(\emptyset)}_{m_1} \cdot \ldots \cdot r^{\omega_v(m_1 \ldots m_{k-1})}_{m_k}\right)^{\alpha}. \tag{5.9}$$

We define a unit mass measure ν on $\widetilde{\omega}_v$ by setting, if $\boldsymbol{m} \in \omega_v$ and $|\boldsymbol{m}| = k$ is a neck,

$$\nu([\boldsymbol{m}]) = \frac{w([\boldsymbol{m}])}{\sum\{w([\boldsymbol{m}']) : \boldsymbol{m}' \in \omega_v, |\boldsymbol{m}'| = k\}}. \tag{5.10}$$

The expression for $\nu([\boldsymbol{m}])$ in case $|\boldsymbol{m}|$ is not a neck can be found in (5.15).

In order to show this does define a (unit mass) measure on $\widetilde{\omega}_v$, first recall that $\boldsymbol{a}$ has necks of arbitrarily large size. We will prove that if $k \le j$ are both necks, $\boldsymbol{m} \in \omega_v$ and $|\boldsymbol{m}| = k$, then ν satisfies the consistency condition

$$\nu([\boldsymbol{m}]) = \sum\{\nu([\boldsymbol{n}]) : \boldsymbol{m} \le \boldsymbol{n} \in \omega_v, |\boldsymbol{n}| = j\}. \tag{5.11}$$

Here, and elsewhere, $\boldsymbol{m} \le \boldsymbol{n}$ means $\boldsymbol{m}$ is an initial segment of the *finite* sequence $\boldsymbol{n}$.

Note that $[\boldsymbol{m}] = \bigcup\{[\boldsymbol{n}] : \boldsymbol{m} \le \boldsymbol{n} \in \omega_v, |\boldsymbol{n}| = j\}$, and this is a union of disjoint sets. It follows from (5.11) that ν extends to a unit mass measure on the σ-algebra of subsets of $\widetilde{\omega}_v$ generated by the cylinder sets $[\boldsymbol{m}]$ for which $|\boldsymbol{m}|$ is a neck. This is just the σ-algebra generated by all cylinder sets, i.e. the class of Borel sets.

In order to prove (5.11), note that if $\boldsymbol{m} \in \omega_v$, $|\boldsymbol{m}| = k$ where k is a neck for $\boldsymbol{a}$, and $\boldsymbol{ms} \in \omega_v$, then it follows from (5.9) and Remark 5.4 that

$$r^{\alpha}_{v;\boldsymbol{ms}} = \theta^{\alpha}(\boldsymbol{s})\, r^{\alpha}_{v;\boldsymbol{m}}, \tag{5.12}$$

where $\theta(\boldsymbol{s})$ does not depend on either $\boldsymbol{m}$ or v. Suppose now that $j \geq k$. Then from (5.9) and (5.12),

$$\begin{aligned}\sum\{w([\boldsymbol{n}]) : \boldsymbol{m} \leq \boldsymbol{n} \in \omega_v,\ |\boldsymbol{n}| = j\}& \\ = r^{\alpha}_{v;\boldsymbol{m}} \sum\{\theta^{\alpha}(\boldsymbol{s}) : \boldsymbol{m}\boldsymbol{s} \in \omega_v,\ |\boldsymbol{s}| = j-k\}& \\ =: \lambda(k,j,\alpha)\, r^{\alpha}_{v;\boldsymbol{m}} = \lambda(k,j,\alpha)\, w([\boldsymbol{m}]),&\end{aligned} \tag{5.13}$$

where λ does not depend on $\boldsymbol{m}$ or v. Replacing $\boldsymbol{m}$ by $\boldsymbol{m}'$ and $\boldsymbol{n}$ by $\boldsymbol{n}'$, and summing also over $\boldsymbol{m}'$,

$$\begin{aligned}\sum\{w([\boldsymbol{n}']) : \boldsymbol{n}' \in \omega_v,\ |\boldsymbol{n}'| = j\}& \\ = \lambda(k,j,\alpha) \sum\{w([\boldsymbol{m}']) : \boldsymbol{m}' \in \omega_v,\ |\boldsymbol{m}'| = k\}.&\end{aligned} \tag{5.14}$$

Dividing (5.13) by (5.14) and using (5.10) gives (5.11).

If $\boldsymbol{m} \in \omega_v$ with $|\boldsymbol{m}| = k$, not necessarily a neck, and $j \geq n(k)$, then

$$\begin{aligned}\nu([\boldsymbol{m}]) &= \sum\{\nu([\boldsymbol{n}]) : \boldsymbol{m} \leq \boldsymbol{n} \in \omega_v, |\boldsymbol{n}| = n(k)\} \\ &= \frac{\sum\{w([\boldsymbol{n}]) : \boldsymbol{m} \leq \boldsymbol{n} \in \omega_v,\ |\boldsymbol{n}| = n(k)\}}{\sum\{w([\boldsymbol{n}']) : \boldsymbol{n}' \in \omega_v,\ |\boldsymbol{n}'| = n(k)\}} \\ &= \frac{\sum\{r^{\alpha}_{v;\boldsymbol{n}} : \boldsymbol{m} \leq \boldsymbol{n} \in \omega_v,\ |\boldsymbol{n}| = n(k)\}}{\sum\{r^{\alpha}_{v;\boldsymbol{n}'} : \boldsymbol{n}' \in \omega_v,\ |\boldsymbol{n}'| = n(k)\}} \\ &= \frac{\sum\{r^{\alpha}_{v;\boldsymbol{p}} : \boldsymbol{m} \leq \boldsymbol{p} \in \omega_v,\ |\boldsymbol{p}| = j\}}{\sum\{r^{\alpha}_{v;\boldsymbol{p}'} : \boldsymbol{p}' \in \omega_v,\ |\boldsymbol{p}'| = j\}},\end{aligned} \tag{5.15}$$

using (5.12) in the final equality.

Define the map $\widetilde{\tau} : \widetilde{\omega}_v \to K_v$ by

$$\widetilde{\tau}(p_1 p_2 \ldots p_k \ldots) = \lim_{k\to\infty} f^{\omega_v(\emptyset)}_{p_1} \circ f^{\omega_v(p_1)}_{p_2} \circ \cdots \circ f^{\omega_v(p_1\ldots p_{k-1})}_{p_k}(x_0) \in K_v,$$

and note that the limit does not depend on x_0.

The measure ν on $\widetilde{\omega}_v$ projects to the unit mass measure μ on K_v, defined

$$\mu(A) = \nu\{\boldsymbol{p} \in \widetilde{\omega}_v : \widetilde{\tau}(\boldsymbol{p}) \in A\} \tag{5.16}$$

for A a Borel subset of K_v.

C. *An upper estimate for* ν. Again assume $\boldsymbol{a}$ satisfies (5.8), and consider the corresponding tree $\boldsymbol{\omega}^{\boldsymbol{a}} = (\omega_1, \ldots, \omega_V)$. Fix v. We show for $\boldsymbol{m} = m_1 \ldots m_k \in \omega_v$ that

$$\nu([\boldsymbol{m}]) \leq c_1 r^{\alpha}_{v;\boldsymbol{m}}, \tag{5.17}$$

for some constant c_1 depending on $\boldsymbol{a}$ but not on $\boldsymbol{m}$.

From (5.15),

$$\nu([\boldsymbol{m}]) = \frac{\sum\{r^{\alpha}_{v;\boldsymbol{n}} : \boldsymbol{m} \le \boldsymbol{n} \in \omega_v,\ |\boldsymbol{n}| = n(k)\}}{\sum\{r^{\alpha}_{v;\boldsymbol{n}'} : \boldsymbol{n}' \in \omega_v,\ |\boldsymbol{n}'| = n(k)\}} \le \frac{r^{\alpha}_{v;\boldsymbol{m}} M^{n(k)-k}}{S_v(n(k),\alpha)}. \tag{5.18}$$

To establish this inequality use (5.9) with $\boldsymbol{m}$ there replaced by $\boldsymbol{n}$, note that each $r_j^{\alpha} \le 1$ and the branching number of ω_v is bounded by M, and use the expression in (4.4) for $S_v(n(k),\alpha) = S(\omega_v, n(k), \alpha)$.

From (5.8), for any $\epsilon > 0$, there exists $N(\epsilon)$ such that

$$M^{n(k)-k} \le M^{N(\epsilon)} M^{k\epsilon} \tag{5.19}$$

for all k.

From (5.3), since $\alpha < d$ and so $\gamma(\alpha) > 0$, there exists $c_2 \in \mathbb{R}$ such that for all j,

$$\log S_v(j,\alpha) \ge c_2 + \frac{j}{2}\gamma(\alpha).$$

Hence,

$$S_v(n(k),\alpha) \ge c_3 e^{\frac{1}{2}n(k)\gamma(\alpha)} \ge c_3 e^{\frac{1}{2}k\gamma(\alpha)} \tag{5.20}$$

for some $c_3 > 0$ and all k.

Choose $\epsilon \le \frac{1}{2}\gamma(\alpha)/\log M$ so that $e^{\frac{1}{2}k\gamma(\alpha)} \ge M^{k\epsilon}$, and then choose $N = N(\epsilon)$. Dividing (5.19) by (5.20), and using (5.18), gives (5.17) with $c_1 = M^N/c_3$.

D. *The estimate for* μ. Fix $\boldsymbol{a} \in \mathcal{A}_V^{\infty}$ satisfying (5.8), in which case (5.17) holds. Assume the open set condition (3.3) holds with the open set $\mathcal{O}$.

Fix $x \in \mathbb{R}^n$ and $r > 0$. With μ the measure on K_v as in (5.16), we show by a standard argument, see [15, p. 737] or [7, p. 131], that

$$\mu(B_r(x)) \le c\, r^{\alpha}. \tag{5.21}$$

Here c is independent of x and r, and $B_r(x)$ is the open ball of radius r centred at x.

First note that, for each infinite sequence $\boldsymbol{p} = m_1 m_2 \ldots m_k \ldots \in \widetilde{\omega}_v = \widetilde{\omega}_v^{\boldsymbol{a}}$, there is a least k such that

$$r_{\min} r \le r_{v;m_1\ldots m_k} < r. \tag{5.22}$$

Let $Q(r) = Q^{\boldsymbol{a}}(r)$ be the set of all such $\boldsymbol{m} = m_1 \ldots m_k$. The sets $\mathcal{O}_{v;\boldsymbol{m}}$ for $\boldsymbol{m} \in Q(r)$ are disjoint from (3.5) and the definition of $Q(r)$, although the $|\boldsymbol{m}|$ are not necessarily equal. Let $Q(x,r) = Q^{\boldsymbol{a}}(x,r)$ be the set of $\boldsymbol{m} \in Q(r)$ such that $\mathcal{O}_{v;\boldsymbol{m}}$ meets $B_r(x)$.

Choose a_1 and a_2 so that $\mathcal{O}$ contains an open ball of radius a_1, and is contained in an open ball of radius a_2. Then the sets $\mathcal{O}_{v;\boldsymbol{m}}$ each contain a ball of radius $a_1 r_{v;\boldsymbol{m}}$, and hence of radius $a_1 r_{\min} r$, and they are contained in a ball of radius $a_2 r_{v;\boldsymbol{m}}$, and hence of radius $a_2 r$. It follows by a volume comparison that if $q = q(x,r)$ is the cardinality of $Q(x,r)$, since the inner balls are disjoint and are subsets of $B_{(1+2a_2)r}(x)$, that

$$q(a_1 r_{\min} r)^n \le (1+2a_2)^n r^n,$$

and so q is bounded independently of x and r.

Hence,

$$\begin{aligned}
\mu(B_r(x)) &= \mu(B_r(x) \cap K_v) \\
&= \nu\big(\{\boldsymbol{p} : \tau(\boldsymbol{p}) \in B_r(x) \cap K_v\}\big) \\
&\le \nu\left(\bigcup\{[\boldsymbol{m}] : \boldsymbol{m} \in Q(x,r)\}\right) \\
&= \sum\{\nu([\boldsymbol{m}]) : \boldsymbol{m} \in Q(x,r)\} \\
&\le c_1 \sum\{r_{v;\boldsymbol{m}}^{\alpha} : \boldsymbol{m} \in Q(x,r)\} \\
&\le c_1 q r^{\alpha},
\end{aligned}$$

using the definition of $Q(x,r)$, the disjointedness of the $\boldsymbol{m} \in Q(x,r) \subset Q(r)$, the estimate (5.17) and (5.22). This establishes (5.7) and hence the lemma. □

6 Examples

For the model problem in the introduction, with F and G each chosen with probability $1/2$, it follows from (4.7) that, for random homogeneous Sierpinski triangles, the dimension is $d(1) = 2\log 3/(\log 2 + \log 3) \approx 1.226$. For the corresponding random recursive case, from (4.8) the dimension is the solution d of $\frac{1}{2}3\left(\frac{1}{2}\right)^d + \frac{1}{2}3\left(\frac{1}{3}\right)^d = 1$, i.e. $d(\infty) \approx 1.262$. For $V > 1$ we used Maple 10 to compute the values of $\gamma_V(\alpha)$ shown in Figure 7. The computed graphs for $V > 1$ are concave up, although this does not show on the scale used.

Subsequent calculation similarly gave (V, d) for $1 \le V \le 20$ as follows:

1, 1.2262	2, 1.2402	3, 1.2463	4, 1.2500	5, 1.2524
6, 1.2538	7, 1.2549	8, 1.2557	9, 1.2565	10, 1.2570
11, 1.2576	12, 1.2580	13, 1.2583	14, 1.2585	15, 1.2588
16, 1.2590	17, 1.2592	18, 1.2594	19, 1.2596	20, 1.2597

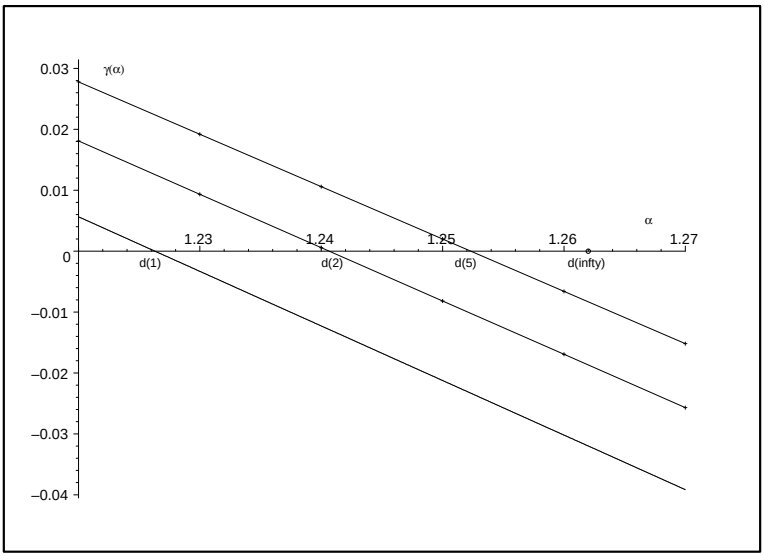

Figure 7. Graphs of $\gamma_V(\alpha) = \gamma(\alpha)$ for $V = 1, 2, 5$ respectively from left to right. Here $\boldsymbol{F} = \{F, G\}$ and $P = \{1/2, 1/2\}$ as in the introduction.

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