

Random iterations of homeomorphisms on the circle

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Dedicated to Professor Dmitrii S. Silvestrov on the occasion of his 70th Birthday

Abstract We study random independent and identically distributed iterations of functions from an iterated function system of homeomorphisms on the circle which is minimal. We show how such systems can be analyzed in terms of iterated function systems with probabilities which are non-expansive on average.

Keywords Markov chains, stationary distributions, minimal, iterated function systems, circle homeomorphisms, synchronization, random dynamical systems

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1 Introduction

We study iterations of a finite family of circle homeomorphisms. This topic has been studied already from a number of different points of view. One may, for example, take a purely deterministic approach and study the associated action of the *group*

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of circle homeomorphisms (the special case of the group of orientation preserving circle diffeomorphisms is treated in [12, 19, 13]). Or one may, as we will, take a probabilistic approach and investigate Markov chains generated by random independent and identically distributed (i.i.d.) iterations of functions from the family (such as in [16, 8, 21]).

We restrict our attention to families of functions which are forward minimal in the sense that for any two points on the circle, there are orbits from the first point arbitrary close to the second one using some concatenations of functions from the family. The set of distances which are preserved simultaneously by all maps allows us to distinguish between distinct types of ergodic behavior for such Markov chains.

By finding a topologically conjugate system which is non-expansive on average, under the additional assumption that the system of inverse maps is forward minimal, we prove limit theorems including almost sure synchronization of random trajectories (which is sometimes also referred to as Antonov’s theorem [1]) provided that the system is not topologically conjugate to a family containing only isometries, and uniqueness and fiberwise properties of stationary distributions.

In contrast to many previous authors we do not assume that all maps preserve orientation or, *a priori*, that the system of inverse maps is forward minimal (such as in [1, 12, 14, 19, 13, 21]) or contains at least one map which is minimal (as in [21]). Our setting is also studied in [17] (without any minimality condition), where a different approach is used and ideas of [3] are adapted which in turn are built on ideas of [15, 8]. See also [22]. One further precursor in a more specific setting is the work by Furstenberg [10] where the homeomorphisms are the projective actions of elements of $SL_2(\mathbb{R})$.

2 Random iterations

Let K be a compact topological space equipped with its Borel sets. We call a finite set $F = \{f_1, \dots, f_N\}$ of continuous functions $f_j: K \rightarrow K, j = 1, \dots, N$, an *iterated function system (IFS)*. If all maps f_j are homeomorphisms, as we will in general assume here, then we also consider the associate IFS $F^{-1} := \{f_1^{-1}, \dots, f_N^{-1}\}$ of the inverse maps.

We will discuss different points of view on random and deterministic iterations of functions from an IFS and recall some standard notations and facts.

Given $(I_n)_{n \geq 1}$ a stochastic sequence with values in $\{1, \dots, N\}$, for $x \in K$ define

$$Z_n^x := (f_{I_n} \circ \dots \circ f_{I_1})(x), \quad Z_0^x = x.$$

We may consider without loss of generality the (*a priori*) unspecified common domain of the random variables I_n as $\Sigma = \{1, \dots, N\}^{\mathbb{N}}$, equipped with a probability measure P defined on its Borel subsets, with I_n being defined as $I_n(\omega) = \omega_n$ for every $\omega = (\omega_1 \omega_2 \dots) \in \Sigma$ and $n \geq 1$.

We will later also consider the shift map $\sigma: \Sigma \rightarrow \Sigma$ defined by $\sigma(\omega_1 \omega_2 \dots) := (\omega_2 \omega_3 \dots)$.

For any $\omega = (\omega_1 \omega_2 \dots) \in \Sigma$, any $n \geq 0$ and any $x \in K$ we thus define $Z_n^x(\omega) = Z_n(x, \omega)$, where

$$Z_n(x, \omega) := (f_{\omega_n} \circ \dots \circ f_{\omega_1})(x), \quad Z_0(x, \omega) = x. \tag{1}$$

The sequence $(Z_n(x, \omega))_{n \geq 0}$ is called the *trajectory* corresponding to the *realization* ω of the random process $(Z_n^x)_{n \geq 0}$ starting at $x \in K$. It is common to also consider iterates in the reversed order and to define

$$\widehat{Z}_n(x, \omega) := (f_{\omega_1} \circ \dots \circ f_{\omega_n})(x), \quad \widehat{Z}_0(x, \omega) = x. \tag{2}$$

If F is an IFS of homeomorphisms, then we also consider the associate sequence $(Z_n^-(x, \omega))_{n \geq 0}$ defined by

$$Z_n^-(x, \omega) := (f_{\omega_n}^{-1} \circ \dots \circ f_{\omega_1}^{-1})(x), \quad Z_0^-(x, \omega) = x,$$

and the sequence $(\widehat{Z}_n^-(x, \omega))_{n \geq 1}$ defined by

$$\widehat{Z}_n^-(x, \omega) := (f_{\omega_1}^{-1} \circ \dots \circ f_{\omega_n}^{-1})(x), \quad \widehat{Z}_0^-(x, \omega) = x. \tag{3}$$

Note that for every $\omega \in \Sigma$ and $x \in K$ it holds

$$\widehat{Z}_n^-(x, \omega) = (f_{\omega_n} \circ \dots \circ f_{\omega_1})^{-1}(x) \quad \text{and} \quad \widehat{Z}_n(x, \omega) = (f_{\omega_1}^{-1} \circ \dots \circ f_{\omega_n}^{-1})^{-1}(x).$$

2.1 Iterated function systems with probabilities and Markov chains

Let $(I_n)_{n \geq 1}$ be i.i.d. variables. The probability measure P is then a Bernoulli measure determined by a probability vector $\mathbf{p} = (p_1, \dots, p_N)$. It then follows that $Z_n^x = Z_n(x, \cdot)$ defined in (1) and $\widehat{Z}_n^x = \widehat{Z}_n(x, \cdot)$ defined in (2) both have the same distribution for any fixed $n \geq 1$, and $(Z_n^x)_{n \geq 0}$ is a (time-homogeneous) Markov chain with transfer operator T defined for bounded measurable functions $h : K \rightarrow \mathbb{R}$ by

$$Th(x) := \sum_{j=1}^N p_j h(f_j(x)). \tag{4}$$

If \mathbf{p} is *non-degenerate*, that is, if $p_j > 0$ for every $j = 1, \dots, N$, then we call the pair (F, \mathbf{p}) an *IFS with probabilities*. The Markov chain $(Z_n^x)_{n \geq 0}$ is obtained by independent random iterations where in each iteration step the functions f_j are chosen with probability p_j .

Markov chains generated by IFSs with probabilities is a particular class of Markov chains that has received a considerable attention in recent years. The IFS terminology was coined by Barnsley and Demko [4].²

A Borel probability measure μ on K is an *invariant probability measure* for the IFS with probabilities (F, \mathbf{p}) if

$$T_*\mu = \mu, \quad \text{where} \quad T_*\mu(\cdot) = \sum_j p_j \mu(f_j^{-1}(\cdot)).$$

²A common abuse of notation is to use the term ‘‘IFS’’ for the Markov chain $(Z_n^x)_{n \geq 0}$ obtained from an IFS with probabilities. We here stress the deterministic nature of an IFS and the fact that an IFS can be used to build other objects like e.g. $(\widehat{Z}_n(x, \omega))_{n \geq 0}$. A common way to construct fractal sets is for example to regard them as sets of limit points for the latter sequence (assuming conditions such as, for example, contractivity ensuring the limit to exist).

Such a measure μ is also called a *stationary distribution* for the corresponding Markov chain, since if X is a μ -distributed random variable, independent of $(I_n)_{n \geq 0}$ then $(Z_n^X)_{n \geq 0}$ will be a stationary stochastic sequence.

Remark 1. By continuity of all functions $f_j, j = 1, \dots, N$, it follows that $(Z_n^X)_{n \geq 0}$ has the *weak Feller property*, that is, T maps the space of real valued continuous functions on K to itself. It is well known that Markov chains with the weak Feller property have at least one stationary distribution, see for example [18]. Hence, any IFS with probabilities (F, \mathbf{p}) has at least one invariant probability measure.

Remark 2. Another formalism (which will not be used here) for analyzing stochastic sequences related to an IFS with probabilities is the one of a (deterministic) step skew product map $(\omega, x) \mapsto (\sigma(\omega), f_{\omega_1}(x))$ with the shift map $\sigma: \Sigma \rightarrow \Sigma$ in the base and locally constant fiber maps. The Bernoulli measure is a σ -invariant measure in the base. Invariant measures (and hence stationary distributions) are closely related to measures which are invariant for the step skew product (see, for example, [23, Chapter 5]).

Given a positive integer n , define by $T^n = T \circ \dots \circ T$ and $T_*^n = T_* \circ \dots \circ T_*$ (each n times) the concatenations of T and T_* , respectively. We call a stationary distribution μ for $(Z_n^x)_{n \geq 0}$ *attractive* if for any $x \in K$ we have $T_*^n \delta_x \rightarrow \mu$ as $n \rightarrow \infty$ in the weak* topology, where δ_x denotes the Dirac measure concentrated in $x \in K$. In other words, for any continuous $h: K \rightarrow \mathbb{R}$ and for any $x \in K$ we have

$$\lim_{n \rightarrow \infty} T^n h(x) = \int h d\mu. \tag{5}$$

An attractive stationary distribution is uniquely stationary.

Let ρ be some metric on K . We say that an IFS with probabilities (F, \mathbf{p}) is *contractive on average* with respect to ρ if for any $x, y \in \mathbb{S}^1$ we have

$$\sum_{j=1}^N p_j \rho(f_j(x), f_j(y)) \leq c \rho(x, y), \tag{6}$$

for some constant $c < 1$ and *non-expansive on average* if (6) holds for some constant $c \leq 1$.

Remark 3. It is well known that a Markov chain $(Z_n^x)_{n \geq 0}$ generated by an IFS with probabilities (F, \mathbf{p}) which is contractive on average has an attractive (and hence unique) stationary distribution. More generally the distribution of Z_n^x then converges (in the weak* topology) to the stationary distribution with an exponential rate that can be quantified for example by the Wasserstein metric, see e.g. [20].

Far less is known for non-expansive systems. The theory for Markov chains generated by non-expansive systems can be regarded as belonging to the realm of Markov chains where $\{T^n h\}$ is equicontinuous for any continuous $h: K \rightarrow \mathbb{R}$, or “stochastically stable” Markov chains (see [18] for a survey).

The Markov chain $(Z_n^x)_{n \geq 0}$ is *topologically recurrent* if for any open set $O \subset K$ and any $x \in K$ we have

$$P(Z_n^x \in O \text{ for some } n) > 0.$$

In the present paper we are going to study a special class of topologically recurrent Feller continuous Markov chains generated by IFSs with probabilities of homeomorphisms on the circle. The topology of the circle and the hence implied monotonicity of the maps play a crucial role for our results.

3 IFSs with homeomorphisms on the circle

From now on we will always assume $K = \mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ to be the unit circle and consider an IFS $F = \{f_j\}_{j=1}^N$ of homeomorphisms $f_j: \mathbb{S}^1 \rightarrow \mathbb{S}^1$. Let $d(x, y) := \min\{|y - x|, 1 - |y - x|\}$ be the standard metric on \mathbb{S}^1 .

3.1 Deterministic iterations and simultaneously preserved distances

An IFS $F = \{f_j\}_{j=1}^N$ is *forward minimal* if for any open set $O \subset K$ and any $x \in K$ there exist some $n \geq 0$ and some $\omega \in \Sigma$ such that

$$Z_n(x, \omega) \in O.$$

In other words, for a forward minimal IFS it is possible to go from any point x arbitrarily close to any point y by applying some concatenations of functions in the IFS. We say that the IFS $F = \{f_j\}_{j=1}^N$ of homeomorphisms f_j is *backward minimal* if the IFS $\{f_j^{-1}\}_{j=1}^N$ is forward minimal.

Remark 4. Note that F is forward (backward) minimal if and only if for every nonempty closed set $A \subset \mathbb{S}^1$ satisfying $f_j(A) \subset A$ ($f_j^{-1}(A) \subset A$) for every j we have $A = \mathbb{S}^1$.

Note that not every forward minimal IFS is automatically backward minimal if $N > 1$ (see [5] for a discussion and counterexamples). By [5, Corollary E], an IFS is both forward and backward minimal if and only if there exists an $\omega \in \Omega$ such that $(Z_n(x, \omega))_{n \geq 0}$ is dense, for any $x \in \mathbb{S}^1$. (By forward minimality this property trivially holds for some fixed $x \in \mathbb{S}^1$, but the choice of ω might depend on $x \in \mathbb{S}^1$.) A simple sufficient condition for an IFS of circle homeomorphisms to be both forward and backward minimal is that at least one of the maps has a dense orbit. A class of IFSs which are forward and backward minimal (so-called expanding-contracting blenders) but without a map with a dense orbit can be found in [9, Section 8.1].

The following is somehow related to the study of the well-known concept of *rotation numbers* of orientation-preserving circle homeomorphisms which was introduced by Poincaré and which provides an invariant to (almost completely) characterize topologically conjugacy.³ Rotation numbers are also important when studying an IFS (which can be considered as a special group action) of orientation-preserving circle homeomorphisms. The surveys [12, 19] review these facts, see also [13].

Here we deal with a more general class of IFSs in which not necessarily all maps preserve orientation.

³The rotation number $r(f)$ of a circle homeomorphism f is rational if, and only if, f has a periodic orbit. If $r(f)$ is irrational then f is semi-conjugate to a rotation by angle $r(f)$ and, in particular, this semi-conjugacy is a conjugacy if f is minimal.

Given F and a metric ρ on \mathbb{S}^1 , let $L = L(F, \rho)$ defined by

$$L := \left\{ s \in [0, 1/2] : \rho(x, y) = s \text{ implies that } \rho(f_j(x), f_j(y)) = s \right. \\ \left. \text{for any } j = 1, \dots, N \text{ and } (x, y) \in \mathbb{S}^1 \times \mathbb{S}^1 \right\} \tag{7}$$

be the set of ρ -distances which simultaneously are preserved by all maps in F .

Remark 5. Note that since all maps of the IFS are homeomorphisms it follows that for every $x, y \in \mathbb{S}^1$ with $\rho(x, y) \in L(F, \rho)$ we have

$$\rho(x, y) = \rho(f_j(x), f_j(y)) = \rho(f_j^{-1}(x), f_j^{-1}(y)) \quad \text{for all } j = 1, \dots, N,$$

and thus $L(F, \rho) = L(F^{-1}, \rho)$. Moreover, note that by continuity of the maps of the IFS, the set L is closed.

We have the following dichotomy.

Lemma 1. *If $L = L(F, \rho)$ is finite, then*

$$L = \left\{ 0, \frac{1}{k}, \frac{2}{k}, \dots, \frac{\lfloor k/2 \rfloor}{k} \right\},$$

for some $k \geq 1$.

If $L = L(F, \rho)$ is infinite, then $L = [0, 1/2]$. All IFS maps are then isometries (with respect to ρ).

Proof. Consider the operation $\oplus : L \times L \rightarrow \mathbb{S}^1$ defined by

$$s_1 \oplus s_2 := \min\{s_1 + s_2, 1 - s_1 - s_2\}.$$

Note that L is closed under this operation, that is, $\oplus : (L \times L) \rightarrow L$. Indeed, given $s_1, s_2 \in L$, if $x, z \in \mathbb{S}^1$ are such that $\rho(x, z) = s_1 \oplus s_2$, then there is a point $y \in \mathbb{S}^1$ such that $\rho(x, y) = s_1$ and $\rho(y, z) = s_2$. Thus, we have $\rho(f_j(x), f_j(y)) = s_1$ and $\rho(f_j(y), f_j(z)) = s_2$ for every $j = 1, \dots, N$. Since all maps f_j are homeomorphisms, it follows that $\rho(x, z) = \rho(f_j(x), f_j(z))$ for all $j = 1, \dots, N$ and hence $s_1 \oplus s_2 \in L$.

It follows that if L is finite (and nontrivial) then the smallest positive element of L must be a rational number of the form $1/k$ for some integer $k > 1$ and hence L must have the given form.

If L is infinite, then $L = [0, 1/2]$, since L has then arbitrary small positive elements and must therefore be a dense, and by continuity of all maps in F , also a closed subset of $[0, 1/2]$. All IFS maps are then isometries. □

Remark 6. If $L(F, d)$ is finite and $1/k$ is its smallest positive element, then the IFS $\tilde{F} = \{\tilde{f}_j\}$ with maps $\tilde{f}_j(x) = k(f_j(x/k) \bmod 1/k)$, $j = 1, \dots, N$, satisfies $L(\tilde{F}, d) = \{0\}$. Thus, we can describe the dynamical properties of an IFS with the set of preserved distances $L(F, d)$ being finite in terms of the dynamics of an IFS with no positive preserved distances. Observe that each of the maps f_j is semiconjugate with \tilde{f}_j by means of the map $\pi : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ defined by $\pi(x) = kx \bmod 1$, that is, we have $\pi \circ f_j = \tilde{f}_j \circ \pi$.

Intuitively we may in all cases regard the infimum of all positive elements of $L = L(F, d)$ as the “common prime period” of all maps, where the case when L is infinite corresponds to a degenerated case. As mentioned above, for orientation-preserving homeomorphisms this number can be compared with the rotation number functions in [12, 19, 13].

3.2 Random iterations

First, recall the following well-known fact about forward minimal IFSs with probabilities on \mathbb{S}^1 (compare also [19, Lemma 2.3.14]). We say that a measure μ has *full support* if the support of μ is \mathbb{S}^1 .

Lemma 2. *Let (F, \mathbf{p}) be an IFS with probabilities of homeomorphisms on \mathbb{S}^1 and μ_+ be an invariant probability measure for (F, \mathbf{p}) . If F is forward minimal then μ_+ is nonatomic and has full support.*

Proof. By contradiction, suppose that μ_+ is atomic. Let $x \in \mathbb{S}^1$ be a point of maximal positive μ_+ -mass. By invariance of μ_+ , we obtain

$$\mu_+(\{x\}) = \sum_{j=1}^N p_j \mu_+(\{f_j^{-1}(x)\})$$

and hence, since we assume that \mathbf{p} is non-degenerate, we have $\mu_+(\{f_j^{-1}(x)\}) = \mu_+(\{x\})$ for every j . Hence, we obtain that the (nonempty) set

$$A := \{y \in \mathbb{S}^1 : \mu_+(\{y\}) = \mu_+(\{x\})\}$$

satisfies $f_j^{-1}(A) \subset A$ for every j . Since μ_+ is finite, A is finite (and, in particular, closed). Hence, since every f_j^{-1} is bijective, we in fact have $f_j^{-1}(A) = A$ and $f_j(A) = A$ for every j . Assuming that F is either backward minimal or forward minimal, we hence obtain $A = \mathbb{S}^1$, which is a contradiction. Hence μ_+ is nonatomic.

An analogous argument shows that μ_+ has full support. Indeed, let the (closed) set $A = \text{supp } \mu_+$ denote the support of μ_+ . By invariance of μ_+ , for every j we have $\mu_+(f_j^{-1}(A)) = \mu_+(A) = 1$ which implies $A \subset f_j^{-1}(A)$, i.e. $f_j(A) \subset A$ for every j , so if (F, \mathbf{p}) is forward minimal, then μ_+ has full support. \square

We say that a probability measure μ on \mathbb{S}^1 is *s-invariant* for $s \in [0, 1]$ if $(R_s)_* \mu = \mu$, where $R_s(x) = (x + s) \bmod 1$. Analogously, we say that an \mathbb{S}^1 -valued random variable X is *s-invariant* if its distribution is *s-invariant*, in which case X and $R_s(X)$ have the same distribution.

Lemma 3. *Let (F, \mathbf{p}) be an IFS with probabilities of homeomorphisms on \mathbb{S}^1 which is forward minimal. Then any invariant probability measure for (F, \mathbf{p}) is s-invariant for any $s \in L(F, d)$.*

Proof. Let μ be an invariant probability measure for (F, \mathbf{p}) . Let $s \in L(F, d)$. Consider an arbitrary interval I of length s satisfying $\mu(I) \geq \mu(I')$ for all other intervals I' of length s . By invariance of μ we have $\mu(I) = \sum_j p_j \mu(f_j^{-1}(I))$. Hence, since \mathbf{p} is non-degenerate, it follows that $\mu(I) = \mu(f_j^{-1}(I))$ for every j .

Since I is of length $s \in L(F, d) = L(F^{-1}, d)$, the interval $f_j^{-1}(I)$ is also of length s for any j . More generally, the μ -measure of the image of I under arbitrary finite concatenations of functions from F^{-1} is an interval of length s and of measure $\mu(I)$. By forward minimality and continuity of the maps in F it therefore follows that all intervals of length s have the same μ -measure equal to $\mu(I)$.

This property implies that μ is s -invariant. Indeed, consider an arbitrary interval (c, d) in \mathbb{S}^1 , where $d = R_\alpha(c)$, for some $0 < \alpha \leq 1/2$. If α is larger than s then

$$\begin{aligned} \mu((c, d)) &= \mu((c, R_s(c))) + \mu((R_s(c), d)) = \mu((d, R_s(d))) + \mu((R_s(c), d)) \\ &= \mu((R_s(c), R_s(d))). \end{aligned}$$

Otherwise, if α is smaller than or equal to s , then

$$\begin{aligned} \mu((c, d)) + \mu((d, R_s(c))) &= \mu((c, R_s(c))) = \mu(I) = \mu((d, R_s(d))) \\ &= \mu((d, R_s(c))) + \mu((R_s(c), R_s(d))), \end{aligned}$$

which also implies $\mu((c, d)) = \mu((R_s(c), R_s(d)))$. □

Given a measurable transformation $\Phi : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ and a probability measure μ , we denote by $\Phi_*\mu$ the *pushforward* of μ defined by $\Phi_*\mu(E) = \mu(\Phi^{-1}(E))$ for each Borel set E of \mathbb{S}^1 .

Remark 7. Recall that if μ is nonatomic (i.e. continuous) and fully supported Borel measure on \mathbb{S}^1 then its distribution function defines a homeomorphism $\Phi : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ and $\Phi_*^{-1}\mu = \mu_{\text{Leb}}$.

We state a preliminary result.⁴

Proposition 1. *Let (F, \mathbf{p}) be an IFS with probabilities of homeomorphisms on \mathbb{S}^1 which is backward minimal. Let μ_- be an invariant measure for (F^{-1}, \mathbf{p}) and let $\Phi_- : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ be defined by $\Phi_-(x) := \mu_-([0, x])$. Then*

$$\rho(x, y) := \min\{\mu_-([x, y]), \mu_-([y, x])\}$$

is a metric on \mathbb{S}^1 and (F, \mathbf{p}) is non-expansive on average with respect to ρ .

The IFS $G = \{g_j\}_{j=1}^N$ given by the maps $g_j := \Phi_- \circ f_j \circ \Phi_-^{-1}$, $j = 1, \dots, N$, with probabilities \mathbf{p} is non-expansive on average with respect to d and we have $L(G, d) = L(F, \rho)$.

Proof. Let (F, \mathbf{p}) be an IFS with probabilities of homeomorphisms on \mathbb{S}^1 which is backward minimal. Let $\Phi_-(x) = \mu_-([0, x])$, where μ_- is an invariant probability measure for (F^{-1}, \mathbf{p}) , and define

$$\rho(x, y) := \min\{\mu_-([x, y]), \mu_-([y, x])\}.$$

Clearly, $L(G, d) = L(F, \rho)$. By Lemma 2 applied to (F^{-1}, \mathbf{p}) , μ_- is nonatomic and has full support and hence we have $\rho(x, y) \geq 0$ and $\rho(x, y) = 0$ if and only if

⁴The main idea is well known (see, for example, [16, p. 118] and [13], where the authors also consider a measurable bijection analogous to the here defined conjugation map Φ_-).

$x = y$. Moreover, clearly $\rho(x, y) = \rho(y, x)$ and $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$. Hence, ρ defines a metric on \mathbb{S}^1 . The definition of ρ and the invariance of μ_- together imply

$$\begin{aligned} \sum_{j=1}^N p_j \rho(f_j(x), f_j(y)) &= \sum_{j=1}^N p_j \min\{\mu_-([f_j(x), f_j(y)]), \mu_-([f_j(y), f_j(x)])\} \\ &= \sum_{j=1}^N p_j \min\{\mu_-(f_j([x, y])), \mu_-(f_j([y, x]))\} \\ &\leq \min\left\{\sum_{j=1}^N p_j \mu_-(f_j([x, y])), \sum_{j=1}^N p_j \mu_-(f_j([y, x]))\right\} \\ &= \min\{\mu_-([x, y]), \mu_-([y, x])\} = \rho(x, y), \end{aligned}$$

which proves that (F, \mathbf{p}) is non-expansive on average with respect to ρ . □

The following result can be regarded as the heart of the paper.⁵

Theorem 1. *Let (F, \mathbf{p}) be an IFS with probabilities of homeomorphisms on \mathbb{S}^1 which is forward minimal and non-expansive on average with respect to some metric ρ . Then $\rho(Z_n^x, Z_n^y)$ converges almost surely to an L -valued random variable for any $x, y \in \mathbb{S}^1$, where $L = L(F, \rho)$.*

As an immediate corollary of Proposition 1 and Theorem 1 we get the following result. This type of result is usually referred to as Antonov’s theorem (see [2], where all maps in the IFS are assumed to preserve orientation, see also [13, 14]). Also in our generality, the present corollary is not new and follows (although not explicitly stated) from results by Malicet [17] who studied an even more general setting (without assuming minimality).

Corollary 1. *Let (F, \mathbf{p}) be an IFS with probabilities of homeomorphisms on \mathbb{S}^1 which is forward and backward minimal. Then exactly one of the following cases occurs:*

- 1) (synchronization) *For any $x, y \in \mathbb{S}^1$ and almost every $\omega \in \Sigma$ we have $d(Z_n(x, \omega), Z_n(y, \omega)) \rightarrow 0$ as $n \rightarrow \infty$.*
- 2) (factorization) *There exists a positive integer $k \geq 2$ and a homeomorphism $\Psi : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ of order k (that is, $\Psi^k = id$) which commutes with all f_j . Moreover, there is a naturally associated IFS $\check{F} = \{\check{f}_j\}$ where each map \check{f}_j is a topological factor⁶ (with a common factoring map) of the corresponding map f_j of F such that (F, \mathbf{p}) has the synchronization property claimed in item 1).*
- 3) (invariance) *All maps f_j are conjugate (with a common conjugation map) to an isometry (with respect to d). There exists a probability measure which is*

⁵A similar statement (without proof and stated for systems where all homeomorphisms preserve orientation) can be found for example in [13].

⁶We call a map $g : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ a topological factor of $f : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ if there exists a continuous surjective map $\pi : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ such that $\pi \circ f = g \circ \pi$.

invariant for all maps $f_j, j = 1, \dots, N$, and hence also uniquely invariant for (F, \mathbf{p}) .

Proof. Apply Proposition 1 to (F, \mathbf{p}) and consider the homeomorphism $\Phi_- : \mathbb{S}^1 \rightarrow \mathbb{S}^1$, and the metric ρ such that (F, \mathbf{p}) is non-expansive on average with respect to ρ . Consider the IFS (G, \mathbf{p}) , conjugate to (F, \mathbf{p}) through the conjugating map Φ_- , which is non-expansive on average with respect to d and recall $L = L(F, \rho) = L(G, d)$. We consider three cases:

Case $L = \{0\}$. By Theorem 1, we have $\rho(Z_n^x, Z_n^y) \rightarrow 0$ a.s. and thus $d(Z_n^x, Z_n^y) \rightarrow 0$ a.s., proving item 1).

Case L finite and nontrivial. By Lemma 1, $L(G, d) = \{0, 1/k, \dots, \lfloor k/2 \rfloor/k\}$ for some $k \geq 2$. By Remark 6 applied to (G, d) , with $\check{f}_j(x) = \check{g}_j(x) := k(g_j(x/k) \bmod 1/k)$ we have $\check{f}_j \circ \Psi = \Psi \circ f_j$, where $\Psi = \pi \circ \Phi_-^{-1}$ with $\pi(x) = kx \bmod 1$, and the IFS (\check{F}, \mathbf{p}) satisfies $L(\check{F}, d) = L(\check{G}, d) = \{0\}$.

Since by Lemma 3 we have $\Phi_-^{-1}(R_{1/k}(x)) = (\Phi_-^{-1}(x) + 1/k) \bmod 1$, it follows that

$$\begin{aligned} \Psi(R_{1/k}(x)) &= (\pi \circ \Phi_-^{-1})(R_{1/k}(x)) \\ &= \pi((\Phi_-^{-1}(x) + 1/k) \bmod 1) = (\pi \circ \Phi_-^{-1})(x) = \Psi(x), \end{aligned}$$

and thus Ψ is an order k homeomorphism having the claimed properties, proving item 2).

Case L infinite. By Lemma 1, we have $L(G, d) = [0, 1/2]$. All maps in G are thus isometries (with respect to d) and hence simultaneously preserve the Lebesgue measure. The measure $\mu_+ := (\Phi_-^{-1})_* \mu_{Leb}$ is invariant for all maps of F , and by Lemma 3 uniquely invariant for (F, \mathbf{p}) , proving item 3). \square

Remark 8. IFSs with nontrivial L can be regarded as degenerated systems. For a typical system satisfying the conditions of Theorem 1 we thus have that $\rho(Z_n^x, Z_n^y) \rightarrow 0$ as $n \rightarrow \infty$ a.s. for any $x, y \in \mathbb{S}^1$. Using techniques from [17, Theorem D] it seems plausible that it should be possible to prove that convergence is exponential (see also [16]), and that (F, \mathbf{p}) is contractive on average with respect to some metric in this case.

Proof of Theorem 1. Let (F, \mathbf{p}) be a forward minimal IFS which is non-expansive on average with respect to ρ . Let \mathcal{F}_n be the sigma field generated by I_1, \dots, I_n . Fix $x, y \in \mathbb{S}^1$. Note that Z_n^x and Z_n^y are both measurable with respect to \mathcal{F}_n and

$$\begin{aligned} \mathbb{E}(\rho(Z_{n+1}^x, Z_{n+1}^y) | \mathcal{F}_n) &= \mathbb{E}(\rho(f_{I_{n+1}}(Z_n^x), f_{I_{n+1}}(Z_n^y)) | \mathcal{F}_n) \\ &= \sum_{j=1}^N p_j \rho(f_j(Z_n^x), f_j(Z_n^y)) \leq \rho(Z_n^x, Z_n^y), \end{aligned}$$

so the stochastic sequence $(\rho(Z_n^x, Z_n^y))_{n \geq 0}$ is a bounded super-martingale with respect to the filtration $\{\mathcal{F}_n\}$. By the Martingale convergence theorem it follows that $\rho(Z_n^x, Z_n^y) \xrightarrow{\text{a.s.}} \xi$ as $n \rightarrow \infty$ for some random variable $\xi = \xi^{x,y}$.

Let $L = L(F, \rho)$. We will now show that ξ is L -valued a.s., that is, we will show that the distance between any two points $a, b \in \mathbb{S}^1$ with $\rho(a, b) = \xi(\omega)$ is preserved by all the maps in F for P a.a. $\omega \in \Sigma$.

We will show that any two points $a, b \in \mathbb{S}^1$ with $\rho(a, b) = \xi(\omega)$ can simultaneously be (almost) reached by $\{Z_n(x, \omega), Z_n(y, \omega)\}$ followed by an application of an arbitrary map for infinitely many n and that this leads to a contradiction if the distance between some points with distance $\xi(\omega)$ is not preserved by all maps in F for a typical ω .

Let us first prove the following claim that for any z and any index j , any open set in \mathbb{S}^1 will be visited followed by an application of the map f_j infinitely many times by trajectories $(Z_n(z, \omega))_{n \geq 0}$ corresponding to typical realizations ω .

Claim 1.1. *For any $z \in \mathbb{S}^1$ and any open set $O \subset \mathbb{S}^1$ and any $j \in \{1, \dots, N\}$ we have*

$$P(\Omega) = 1, \quad \text{where} \quad \Omega := \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \{\omega : Z_n(z, \omega) \in O, \omega_{n+1} = j\}.$$

Proof. Let $z \in \mathbb{S}^1$. Consider an open set $O \subset \mathbb{S}^1$ and an index $j \in \{1, \dots, N\}$. By forward minimality, for every $q \in \mathbb{S}^1$ there exists some positive integer n_q and some $c_q > 0$, such that

$$P(Z_{n_q}^q \in O, I_{n_q+1} = j) = P(Z_{n_q}^q \in O)P(I_{n_q+1} = j) > c_q > 0. \tag{8}$$

Considering the left hand side expression in (8) as a function of q , by continuity (recall the weak Feller property) one concludes that there exists an open set O_q containing q and some positive integer n_q and some $c'_q > 0$

$$P(Z_{n_q}^z \in O, I_{n_q+1} = j) > c'_q > 0$$

for any $z \in O_q$. Thus, by compactness, there exists a positive integer N such that

$$\inf_{q \in \mathbb{S}^1} P(Z_n^q \in O, I_{n+1} = j \text{ for some } n < N) =: s > 0.$$

Let

$$\begin{aligned} A_m &:= \{\omega : Z_n(z, \omega) \in O, \omega_{n+1} = j, \text{ for some } n \in \{mN, \dots, (m+1)N - 1\}\} \\ &= \{\omega : Z_{n-mN}(Z_{mN}(z, \omega), \sigma^{mN}(\omega)) \in O, \omega_{n+1} = j, \\ &\quad \text{for some } n \in \{mN, \dots, (m+1)N - 1\}\}. \end{aligned}$$

For any $m \geq 1$ we have

$$\begin{aligned} P(A_m) &\geq \inf_{q \in \mathbb{S}^1} P(\{\omega : Z_{n-mN}(q, \sigma^{mN}(\omega)) \in O, \omega_{n+1} = j, \\ &\quad \text{for some } n \in \{mN, \dots, (m+1)N - 1\}\}) \\ &= \inf_{q \in \mathbb{S}^1} P(Z_n^q \in O, I_{n+1} = j \text{ for some } n < N) = s > 0 \end{aligned}$$

and hence $P(A_m^c) \leq 1 - s$. More generally, we can similarly show that

$$P\left(\bigcap_{m=j}^k A_m^c\right) \leq (1 - s)^{k-j},$$

for any $j < k$, which implies

$$P\left(\bigcap_{j=1}^{\infty} \bigcup_{m=j}^{\infty} A_m\right) = 1 - P\left(\bigcup_{j=1}^{\infty} \bigcap_{m=j}^{\infty} A_m^c\right) = 1.$$

This implies the assertion. □

We can now choose Ω with $P(\Omega) = 1$ such that for any $\omega \in \Omega$, for any *a priori* fixed index j , the trajectory $(Z_n(x, \omega))_{n \geq 0}$ visits infinitely many times any open interval followed by an application of f_j . Indeed, let $\{a_k\}_k$ be a dense set in \mathbb{S}^1 and for every index pair $(k, \ell) \in \mathbb{N}^2$ let $\Omega_{k,\ell}^j$ be the set provided by the Claim for the point x , an index j , and the open set $O_{k,\ell} = (a_k - 1/\ell, a_k + 1/\ell)$. Let

$$\Omega := \bigcap_{j=1}^N \bigcap_{k \in \mathbb{N}} \bigcap_{\ell \in \mathbb{N}} \Omega_{k,\ell}^j$$

and note that $P(\Omega) = 1$.

By the above, without loss of generality, we can also assume that Ω is such that for every $\omega \in \Omega$ we have $\rho(Z_n(x, \omega), Z_n(y, \omega)) \rightarrow \xi(\omega)$ as $n \rightarrow \infty$.

Fix $\omega \in \Omega$. Let a, b, c be points in \mathbb{S}^1 , with $\rho(a, b) = \rho(a, c) = \xi(\omega)$, where b is obtained from a by a clockwise rotation and c is obtained from a by a counterclockwise rotation. Note that if $0 < \xi(\omega) < 1/2$ then the points a, b, c will be distinct, and otherwise $b = c$. By definition of Ω we know that if O_a is an open set containing a , O_b is an open set containing b , and O_c is an open set containing c then there are infinitely many n such that $Z_n(x, \omega) \in O_a$ and either $Z_n(y, \omega) \in O_b$ or $Z_n(y, \omega) \in O_c$. We say that a is *clockwise nice* if for arbitrarily small open sets O_a and O_b containing a and b , respectively either $Z_n(x, \omega) \in O_a$ and $Z_n(y, \omega) \in O_b$ simultaneously or $Z_n(y, \omega) \in O_a$ and $Z_n(x, \omega) \in O_b$ simultaneously for infinitely many n , and *counterclockwise nice* if for arbitrarily small open sets O_a and O_b containing a and b , respectively either $Z_n(x, \omega) \in O_a$ and $Z_n(y, \omega) \in O_c$ simultaneously or $Z_n(y, \omega) \in O_a$ and $Z_n(x, \omega) \in O_c$ simultaneously for infinitely many n . We call a *nice* if a is both clockwise nice and counterclockwise nice.

Claim 1.2. *Any $a \in \mathbb{S}^1$ is nice.*

Proof. We first prove that there exist both clockwise nice and counterclockwise nice points. Indeed, by definition of Ω , any $a \in \mathbb{S}^1$ is either clockwise nice, counterclockwise nice, or nice. By contradiction, suppose that all points $a \in \mathbb{S}^1$ are only clockwise nice (the case that all points are only counterclockwise nice is analogous). Then, in particular, a given point a and the point c obtained from a counterclockwise rotation of a would both be only clockwise nice. But c being clockwise nice would imply that a is counterclockwise nice, contradiction.

Thus, there exist points of either type which are arbitrarily close to each other. Hence, there exists at least one point in \mathbb{S}^1 which is nice.

By definition of Ω it follows that nice points are mapped to nice points by all maps, so by forward minimality it follows that every point in \mathbb{S}^1 is nice. \square

Let us now prove that the distance between any two points $a, b \in \mathbb{S}^1$ with $\rho(a, b) = \xi(\omega)$ is preserved by all the maps in F . Arguing by contradiction, suppose that $\xi(\omega) \notin L$, and consider an interval $[a, b]$ with $\rho(a, b) = \xi(\omega)$ such that for some $j \in \{1, \dots, N\}$ we have

$$\rho(a, b) \neq \rho(f_j(a), f_j(b)).$$

By continuity of f_j , there exist open intervals O_a and O_b containing a and b , respectively and some positive number ε such that for any $a' \in O_a$ and any $b' \in O_b$ we have

$$|\rho(a', b') - \rho(f_j(a'), f_j(b'))| > \varepsilon.$$

By choice of Ω and the fact that a is nice, there exist arbitrary large integers n such that either $Z_n(x, \omega) \in O_a$, and $Z_n(y, \omega) \in O_b$ simultaneously or $Z_n(y, \omega) \in O_a$, and $Z_n(x, \omega) \in O_b$ simultaneously and $I_{n+1}(\omega) = \omega_{n+1} = j$. Hence

$$|\rho(Z_n(x, \omega), Z_n(y, \omega)) - \rho(Z_{n+1}(x, \omega), Z_{n+1}(y, \omega))| > \varepsilon,$$

contradicting the assumption that $\omega \in \Omega$.

This completes the proof that for any $x, y \in \mathbb{S}^1$, $\rho(Z_n^x, Z_n^y)$ converges almost surely to an L -valued random variable. \square

The following result about uniqueness of invariant probability measures is not new and was, to the best of our knowledge, first proved in [17]. A simple direct proof based on equicontinuity was recently presented in [22]. Note that equicontinuity of $\{T^n h\}$, where $T^n h(x) = \int h(Z_n(x, \omega)) dP(\omega)$ for any Lipschitz continuous function $h: \mathbb{S}^1 \rightarrow \mathbb{R}$, follows trivially from Proposition 1. Indeed, if ρ is the metric of Proposition 1, then $\int \rho(Z_n(x, \omega), Z_n(y, \omega)) dP(\omega) \leq \rho(x, y)$. For completeness we will show that uniqueness of invariant probability measures is also a very simple consequence of Theorem 1.

Corollary 2. Any IFS (F, \mathbf{p}) with probabilities of homeomorphisms on \mathbb{S}^1 which is forward and backward minimal has a unique invariant probability measure μ_+ .

Proof. Let μ_- be an invariant probability measure for (F^{-1}, \mathbf{p}) and define the metric ρ by $\rho(x, y) := \min\{\mu_-([x, y]), \mu_-([y, x])\}$. By Proposition 1, the IFS $G = \{g_j\}_j$ defined by $g_j := \Phi_- \circ f_j \circ \Phi_-^{-1}$, where $\Phi_-(x) = \mu_-([0, x])$, with probabilities \mathbf{p} is non-expansive on average with respect to d and we have $L := L(G, d) = L(F, \rho)$.

By Theorem 1, with Z_n as in (1) and $W_n := \Phi_- \circ Z_n \circ \Phi_-^{-1}$, we have that $d(W_n^x, W_n^y)$ converges almost surely to an L -valued random variable as $n \rightarrow \infty$, for any $x, y \in \mathbb{S}^1$.

We are now going to show that there is a unique invariant probability measure ν_+ for the IFS (G, \mathbf{p}) . This will imply that $\mu_+ := (\Phi_-^{-1})_* \nu_+$ is the unique invariant probability measure for (F, \mathbf{p}) .

Let us divide the proof into cases:

Case $L = \{0\}$. Consider first the (generic) case $L = \{0\}$. Thus, $d(W_n^x, W_n^y) \rightarrow 0$ as $n \rightarrow \infty$ a.s. for any $x, y \in \mathbb{S}^1$. Let ν_+ be an invariant probability measure for (F, \mathbf{p}) , that is, a stationary distribution for $(W_n^x)_{n \geq 0}$ (recall Remark 1). For any $x, y \in \mathbb{S}^1$ and for any continuous $h: \mathbb{S}^1 \rightarrow \mathbb{R}$, by Lebesgue’s dominated convergence theorem

$$T^n h(x) - T^n h(y) = \int_{\Sigma} h(W_n(x, \omega)) dP(\omega) - \int_{\Sigma} h(W_n(y, \omega)) dP(\omega) \rightarrow 0$$

as $n \rightarrow \infty$, and thus by invariance of ν_+ we have

$$\left| T^n h(x) - \int h d\nu_+ \right| = \left| T^n h(x) - \int T^n h d\nu_+ \right| \leq \int |T^n h(x) - T^n h(y)| d\nu_+(y)$$

and by Lebesgue’s dominated convergence theorem the latter tends to 0 as $n \rightarrow \infty$. This implies that ν_+ must be attractive and thus unique (recall Remark 3).

Case $L = \{0, 1/k, \dots, \lfloor k/2 \rfloor/k\}$ for some $k \geq 2$. By Lemma 3 all invariant probability measures for (G, \mathbf{p}) are $1/k$ -invariant. By contradiction, suppose that there are two distinct invariant probability measures ν_+^1 and ν_+^2 for (G, \mathbf{p}) . Hence, if X and Y are two random variables with distribution ν_+^1 and ν_+^2 respectively, independent of $\{I_n\}$, then $W_n^X \bmod 1/k$, and $W_n^Y \bmod 1/k$ will also have distinct distributions for any fixed $n \geq 0$, by $1/k$ -invariance of ν_+^1 and ν_+^2 . The latter is however impossible since the IFS $\tilde{G} = \{\tilde{g}_j\}$ defined by $\tilde{g}_j(x) = k(g_j(x/k) \bmod 1/k)$, $j = 1, \dots, N$, satisfies $L(\tilde{G}, d) = \{0\}$ (recall Remark 6) and therefore the distribution of $W_n^X \bmod 1/k$ converges to the same limit as the limiting distribution of $W_n^Y \bmod 1/k$, as $n \rightarrow \infty$. The invariant probability measure, ν_+ , is therefore unique.

Case $L = [0, 1/2]$. In this case, by Lemma 1 all maps in G are isometries (with respect to d). By Lemma 3, any invariant probability measure is s -invariant for any $s \in [0, 1/2]$, which implies that ν_+ must be the Lebesgue measure. \square

By applying Breiman’s ergodic theorem for Feller chains with a unique stationary distribution starting at a point (see, for example, [6] or [18]), we get the following result. Let δ_x denote the Dirac measure concentrated in the point $x \in \mathbb{S}^1$, and let

$$\mu_n^x(\omega) = \frac{1}{n} \sum_{k=0}^{n-1} \delta_{Z_k(x, \omega)},$$

denote the empirical distribution along the trajectory starting at $x \in \mathbb{S}^1$ determined by $\omega \in \Sigma$ at time $n - 1$.

Corollary 3. *Let (F, \mathbf{p}) be an IFS with probabilities of homeomorphisms on \mathbb{S}^1 which is forward and backward minimal and let μ_+ denote its unique invariant probability measure. Then $\mu_n^x(\omega)$ converges to μ_+ (in the weak* sense) P a.s. for any $x \in \mathbb{S}^1$.*

Remark 9. Corollary 3 slightly generalizes [21, Proposition 16] where a direct proof is given and the additional hypotheses that all maps in the IFS preserve orientation and that one map is minimal are assumed.

Let \xrightarrow{d} denote convergence in distribution. We are now ready to state our first result about invariant measures/stationary distributions for the IFS with probabilities generated by the inverse maps.

Proposition 2. *Let (F, \mathbf{p}) be an IFS with probabilities of homeomorphisms on \mathbb{S}^1 which is forward minimal and non-expansive on average with respect to d . Assume that some map f_j is not an isometry (with respect to d). Then $L(F, d) = \{0, 1/k, \dots, \lfloor k/2 \rfloor/k\}$ for some $k \geq 1$ and for any $1/k$ -invariant nonatomic and fully supported random variable X on \mathbb{S}^1 , independent of $(I_n)_{n \geq 0}$ we have*

$$\widehat{Z}_n^-(X, \omega) \xrightarrow{d} \widehat{Z}^-(\omega)$$

as $n \rightarrow \infty$ for P a.a. $\omega \in \Sigma$, where $\widehat{Z}^-(\omega)$ is a random variable with distribution

$$\mu_\omega^- = \frac{1}{k} \sum_{i=0}^{k-1} \delta_{\frac{1}{k}(i + \widehat{Z}^-(\omega))}$$

for some random variable $\widehat{Z}^-: \Sigma \rightarrow \mathbb{S}^1$ and $\mu_\omega^- = (f_{\omega_1}^{-1})_* \mu_{\sigma(\omega)}^-$ for P a.a. $\omega \in \Sigma$. Thus, the measure μ_- given by

$$\mu_- := \int \mu_\omega^- dP(\omega)$$

is the unique invariant probability measure for (F^{-1}, \mathbf{p}) .

Proof. Let $L = L(F, d)$. By Lemma 1 together with our hypotheses, we have $L = \{0, 1/k, \dots, \lfloor k/2 \rfloor/k\}$ for some $k \geq 1$. Hence, if $d(x, y) = s \in L$, then

$$d(f_j(x), f_j(y)) = s$$

for all $j = 1, \dots, N$ and thus

$$d(Z_n(x, \omega), Z_n(y, \omega)) = s$$

for any $\omega \in \Sigma$ and $n \geq 0$.

Let us denote by Z_n and \widehat{Z}_n^- the sequences defined in (1) and (3), respectively. By Theorem 1 we have that $d(Z_n^x, Z_n^y)$ converges almost surely to an L -valued random variable as $n \rightarrow \infty$.

Given $\omega \in \Sigma$, let

$$\widehat{Z}^-(\omega) := k \sup\{y: |Z_n([0, y], \omega)| \rightarrow 0, \text{ as } n \rightarrow \infty\},$$

where $|\cdot|$ denotes the length of an interval and where we use the notation

$$Z_n([0, y], \omega) := \{Z_n(z, \omega): z \in [0, y]\}.$$

Note that $y \mapsto |Z_n([0, y], \omega)|$ is an increasing function, for each fixed n and ω . Further, $|Z_n([0, y], \omega)|$ converges to an element in $\{0, 1/k, \dots, 1\}$ as $n \rightarrow \infty$, for any $y \in \mathbb{S}^1$ for P a.a. $\omega \in \Sigma$. Indeed, this follows from the fact that $d(Z_n^x, Z_n^y)$

converges to an element of L and the fact that $x \mapsto Z_n^x$ is a random homeomorphism. So $\widehat{Z}^- : \Sigma \rightarrow \mathbb{S}^1$ is a well-defined random variable.

Let m be an arbitrary $1/k$ -invariant nonatomic probability measure fully supported on \mathbb{S}^1 . Note that if I is an interval of length i/k , then $m(I) = i/k$ for any $0 \leq i \leq k$. If $x \notin \{\widehat{Z}^-(\omega)/k, (\widehat{Z}^-(\omega) + 1)/k, \dots, (\widehat{Z}^-(\omega) + (k - 1))/k\}$, then

$$\begin{aligned} m(\{y \in \mathbb{S}^1 : \widehat{Z}_n^-(y, \omega) \leq x\}) &= m(\{y \in \mathbb{S}^1 : \widehat{Z}_n^-(y, \omega) \in [0, x]\}) \\ &= m(\{y \in \mathbb{S}^1 : Z_n(\widehat{Z}_n^-(y, \omega), \omega) \in Z_n([0, x], \omega)\}) \\ &= m(\{y \in \mathbb{S}^1 : y \in Z_n([0, x], \omega)\}) \\ &\rightarrow \begin{cases} 0 & \text{if } x < \frac{\widehat{Z}^-(\omega)}{k}, \\ \frac{i}{k} & \text{if } \frac{\widehat{Z}^-(\omega) + (i - 1)}{k} < x < \frac{\widehat{Z}^-(\omega) + i}{k}, \quad 1 \leq i \leq k - 1, \\ 1 & \text{if } x > \frac{\widehat{Z}^-(\omega) + (k - 1)}{k} \end{cases} \end{aligned}$$

as $n \rightarrow \infty$ for P a.a. $\omega \in \Sigma$. Thus, if X is an m -distributed random variable on \mathbb{S}^1 , independent of $(I_n)_{n \geq 0}$ and $\widehat{Z}^-(\omega)$ has distribution

$$\mu_\omega^- = \frac{1}{k} \sum_{i=0}^{k-1} \delta_{(i+\widehat{Z}^-(\omega))/k} \quad \text{for } P \text{ a.a. } \omega \in \Sigma$$

then $m(\widehat{Z}_n^-(X, \omega) \leq x) \rightarrow m(\widehat{Z}^-(\omega) \leq x)$ as $n \rightarrow \infty$ if x is a continuity point of the cumulative distribution function of $\widehat{Z}^-(\omega)$ (for P a.a. $\omega \in \Sigma$). Thus, $\widehat{Z}_n^-(X, \omega)$ converges in distribution to $\widehat{Z}^-(\omega)$ as $n \rightarrow \infty$ for P a.a. $\omega \in \Sigma$. By taking limits in the equality $\widehat{Z}_n^-(X, \omega) = f_{\omega_1}^{-1}(\widehat{Z}_{n-1}^-(X, \sigma(\omega)))$, it therefore follows that

$$\widehat{Z}^-(\omega) = f_{\omega_1}^{-1}(\widehat{Z}^-(\sigma\omega)),$$

for P a.a. ω . Thus if μ_ω^- denotes the distribution of $\widehat{Z}^-(\omega)$, then

$$\mu_\omega^- = (f_{\omega_1}^{-1})_* \mu_{\sigma(\omega)}^-.$$

for P a.a. ω .

By integrating both sides of this equality with respect to P (recall that P is a Bernoulli measure determined by a probability vector $\mathbf{p} = (p_1, \dots, p_N)$) we thus obtain that

$$\begin{aligned} \mu_- &:= \int \mu_\omega^- dP(\omega) = \sum_{j=1}^N \int_{\omega:\omega_1=j} (f_{\omega_1}^{-1})_* \mu_{\sigma(\omega)}^- dP(\omega) \\ &= \sum_{j=1}^N \int_{\omega:\omega_1=j} (f_j^{-1})_* \mu_{\sigma(\omega)}^- dP(\omega) \\ &= \sum_{j=1}^N \int_{\omega:\omega_1=j} (f_j^{-1})_* \mu_- dP(\omega) \end{aligned}$$

$$= \sum_{j=1}^N p_j (f_j^{-1})_* \mu_-$$

and μ_- is therefore invariant for (F^{-1}, \mathbf{p}) . Since by construction μ_ω is independent of X , it follows that μ_- is indeed uniquely invariant. \square

Using Propositions 1 and 2 we get the following corollary.

Corollary 4. *Let (F, \mathbf{p}) be an IFS with probabilities of homeomorphisms on \mathbb{S}^1 which is forward and backward minimal. Assume that not all maps in F are conjugate (with a common conjugation map) to an isometry (with respect to d). Let μ_- be an invariant probability measure for (F^{-1}, \mathbf{p}) , and let k be the largest integer such that μ_- is $1/k$ -invariant. Conclusion: if X is a μ_- -distributed random variable, independent of $(I_n)_{n \geq 0}$, then*

$$\widehat{Z}_n^-(X, \omega) \xrightarrow{d} \widehat{Z}^-(\omega), \tag{9}$$

as $n \rightarrow \infty$ for P a.a. $\omega \in \Sigma$, where $\widehat{Z}^-(\omega)$ is a random variable with distribution μ_ω , uniformly distributed on k distinct points, and satisfying $\mu_\omega^- = (f_{\omega_1}^{-1})_* \mu_{\sigma(\omega)}^-$ for P a.a. $\omega \in \Sigma$. It therefore follows that μ_- is unique and given by $\mu_- = \int \mu_\omega^- dP(\omega)$.

Remark 10. Convergence in (9) also follows from Furstenbergs martingale argument [11], but here we say more about the limit: The limit is $1/k$ -invariant and independent of X (this implies that μ_- is uniquely invariant) and the limiting fiber measures μ_ω are uniform and supported on sets of size k .

Proof. Let μ_- be an invariant probability measure for (F^{-1}, \mathbf{p}) . By Proposition 1, the IFS $G = \{g_j\}_j$ defined by $g_j := \Phi_- \circ f_j \circ \Phi_-^{-1}$, where $\Phi_-(x) = \mu_-([0, x])$, with probabilities \mathbf{p} satisfies the hypotheses of Proposition 2. Note that F is forward minimal if, and only if, G is. Let $L(G, d)$ be the corresponding set of simultaneously preserved distances. By Lemma 1, we have $L(G, d) = \{0, 1/k, \dots, \lfloor k/2 \rfloor/k\}$ for some $k \geq 1$.

Let us denote by $(\widehat{W}_n^-)_{n \geq 0}$ the sequence for the IFS G which is analogously defined as in (3) for the IFS F . Since F and G are conjugate by means of Φ_- , it is easy to check that

$$\widehat{Z}_n^- = \Phi_-^{-1} \circ \widehat{W}_n^- \circ \Phi_-.$$

Note that if X is a μ_- -distributed random variable, independent of $(I_n)_{n \geq 0}$, then $Y := \Phi_-(X)$ is distributed according to the Lebesgue measure on \mathbb{S}^1 . Hence, in particular, it follows that Y is a $1/k$ -invariant, nonatomic, and fully supported random variable on \mathbb{S}^1 .

By Proposition 2, it therefore follows that

$$\widehat{W}_n^-(\Phi_-(X), \omega) = \Phi_-(\widehat{Z}_n^-(X, \omega)) \xrightarrow{d} \widehat{W}^-(\omega)$$

as $n \rightarrow \infty$ for P a.a. $\omega \in \Sigma$, where $\widehat{W}^-(\omega)$ is a random variable with distribution

$$v_\omega^- = \frac{1}{k} \sum_{i=0}^{k-1} \delta_{\frac{1}{k}(i + \widehat{W}^-(\omega))}$$

for some random variable $\widehat{W}^-: \Sigma \rightarrow \mathbb{S}^1$, that is, we have

$$\widehat{Z}_n^-(X, \omega) \xrightarrow{d} \Phi_-^{-1}(\widehat{W}^-(\omega))$$

as $n \rightarrow \infty$ for P a.a. $\omega \in \Sigma$. Thus, if we define $\widehat{Z}^-(\omega) := \Phi_-^{-1}(\widehat{W}^-(\omega))$, then this random variable has distribution $\mu_\omega(\cdot) = \nu_\omega(\Phi_-(\cdot))$. Moreover, the measure μ_- given by

$$\mu_- := \int \mu_\omega^- dP(\omega)$$

is the unique invariant probability measure for (F^{-1}, p) . \square

Remark 11. If (F, p) is both forward and backward minimal, then by applying the above Corollary to (F^{-1}, p) we obtain an alternative proof of uniqueness for μ_+ under these assumptions.

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