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The Ekström–Persson conjecture regarding random covering sets

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Abstract

We consider the Hausdorff dimension of random covering sets formed by balls with centres chosen independently at random according to an arbitrary Borel probability measure on $\mathbb{R}^d$ and radii given by a deterministic sequence tending to zero. We prove, for a certain parameter range, the conjecture by Ekström and Persson concerning the exact value of the dimension in the special case of radii $(n^{-\alpha})_{n=1}^{\infty}$. For balls with an arbitrary sequence of radii, we find sharp bounds for the dimension and show that the natural extension of the Ekström–Persson conjecture is not true in this case. Finally, we construct examples demonstrating that there does not exist a dimension formula involving only the lower and upper local dimensions of the measure and a critical parameter determined by the sequence of radii.

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1 | INTRODUCTION

The limsup set $E(A_n)$ of a sequence $(A_n)_{n=1}^{\infty}$ of subsets of some space X consists of those points of X which belongs to infinitely many of the sets A_n, that is,

$$E(A_n) := \limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k.$$

A random covering set is a limsup set where the selection of the sets A_n involves some randomness. In this paper, we consider the case where the random sets A_n are balls with centres chosen independently at random according to an arbitrary Borel probability measure on $\mathbb{R}^d$ and with radii given by a deterministic sequence tending to zero. The study of random covering sets may be traced back to a paper of Borel [3] in the late 1890's, where he stated that a given point of a circle belongs almost surely to infinitely many independently placed arcs provided the sum of their lengths is infinite. That article is the origin of a theorem nowadays known as the second Borel–Cantelli lemma. The dimensional properties of related limsup sets were implicitly studied by Besicovitch [2] and Eggleston [7] in the connection of Besicovitch–Eggleston sets concerning k-adic expansions of real numbers. This kind of limsup sets appear also in Diophantine approximation as demonstrated by the classical theorems of Khintchine [22] and Jarník [19].

In this paper, we concentrate on the Hausdorff dimension of random covering sets. For the closely related topic of shrinking target problem, we refer to [4, 5, 15, 16, 23, 24, 27, 35]. To our knowledge, the first result concerning the Hausdorff dimension of random covering sets is due to Jaffard. In [18, Theorem 2], he proved a ubiquity theorem from which it readily follows that the Hausdorff dimension of the set of points on the d-dimensional torus covered by infinitely many balls with radii $\underline{r} = (r_n)_{n=1}^{\infty}$ whose centres are placed independently according to the Lebesgue measure equals almost surely the critical exponent of convergence $s_2(\underline{r})$ (defined in Equation (2.1)) provided this number is at most d. This result was later reproved by Fan and Wu in [13] and by Durand in [6] using different methods. Observe that, in the special case $r_n = n^{-\alpha}$, one easily checks that $s_2(\underline{r}) = \frac{1}{\alpha}$. Various versions of ubiquity theorems, also known as mass transference principles (see [1, 5, 11, 18, 26]), have turned out to be useful tools in the study of random covering sets. In [20], an almost sure dimension formula for uniformly independently placed rectangles on the d-dimensional torus was derived under a monotonicity assumption on side lengths of the rectangles. The formula is as in Equation (2.1) with r_n replaced by the singular value function of a rectangle. Persson [31] proved an almost sure lower bound for uniformly independently placed open sets. For rectangles, this lower bound equals the value obtained in [20], proving that the monotonicity assumption made in that paper is not needed, since the upper bound proved in [20] is (trivially) valid for all realisations. Finally, in [14] Feng et al. derived an almost sure dimension formula in the case of independent random general Lebesgue measurable sets chosen according to a compactly supported Borel probability measure on a Riemannian manifold having an absolutely continuous component with respect to the Riemannian volume.

There are natural ways to continue the study of random covering sets: one may replace the underlying space, which is a Riemannian manifold in [14], by a metric space, one may study uniform coverings, one may investigate random covering sets generated by a dependent sequence of random sets or one may consider purely singular generating probability measures. The first line has been pursued in [8, 9, 11, 17, 21, 30]. For uniform coverings, see [25]. Fan et al. [12] and Liao and Seuret [28] studied the dependent case for balls with radii $(n^{-\alpha})_{n=1}^{\infty}$ in the setting, where the centres are distributed along an orbit of an expanding Markov map on the unit circle and the initial point of the orbit is chosen according to a Gibbs measure. Persson and Rams [33] investigated the similar setting for piecewise expanding maps on the unit interval. In these cases, the dimension formula depends on α and the multifractal spectrum of the Gibbs measure. Hu and Li [17] considered the setting with balls in metric spaces with a general sequence of radii $\underline{r}$ and centres chosen according to an exponentially mixing measure whose one-dimensional marginals are Ahlfors regular. In that case, the dimension is given by the critical exponent $s_2(\underline{r})$. The independent case with a purely singular generating probability measure was studied by Seuret [34] for balls with radii $(n^{-\alpha})_{n=1}^{\infty}$ and centres chosen independently according to a Gibbs measure on the symbolic space giving a result analogous to that of [12, 28].

In [10], Ekström and Persson studied the model with balls having radii $(n^{-\alpha})_{n=1}^{\infty}$ and centres placed independently at random according to a general Borel probability measure on $\mathbb{R}^d$. They proved that the Hausdorff dimension of the random covering set is almost surely bounded from below by $\frac{1}{\alpha} - \delta$ provided the upper Hausdorff dimension of the measure is larger than $\frac{1}{\alpha}$. Here, δ is the essential infimum of the differences of the upper and lower local dimensions in a certain range, see Equation (2.4). For a general α, they gave almost sure lower and upper bounds for the dimension depending on the fine and coarse multifractal spectra of the measure and stated a conjecture that the almost sure dimension is equal to the value of the increasing 1-Lipschitz hull of the fine multifractal spectrum at the point $\frac{1}{\alpha}$. In particular, if the upper Hausdorff dimension of the measure is larger than $\frac{1}{\alpha}$, it is easy to see that this value is equal to $\frac{1}{\alpha}$.

In this paper, we prove that the Ekström–Persson conjecture is true in the range where $\frac{1}{\alpha}$ is at most the upper Hausdorff dimension of the measure, that is, for almost all ω,

$$\dim_{\mathrm{H}} E(B(\omega_n, n^{-\alpha})) = \frac{1}{\alpha}$$

provided $\frac{1}{\alpha} \leqslant \overline{\dim}_{\mathrm{H}}\mu$ (see Theorem 2.3). We also derive a lower (and an upper) bound for the almost sure dimension which improves the one in Equation (2.3) and is valid for all sequences of radii, that is, for almost all ω,

$$s_2(\underline{r})\overline{\delta} \leqslant \dim_{\mathrm{H}} E(B(\omega_n, r_n)) \leqslant s_2(\underline{r})$$

provided $s_2(\underline{r}) < \overline{\dim}_{\mathrm{H}}\mu$, where $\overline{\delta}$ is the essential supremum of the ratio of the lower and upper local dimensions over a certain range (see (Theorem 2.5). We construct an example showing that the bounds in Theorem 2.5 are sharp and that the natural extension of the Ekström–Persson conjecture is not true for general sequences of radii (Example 7.2). Example 7.1 demonstrates that the dimension may be independent of the difference between the lower and upper local dimensions. Finally, combining Examples 7.1 and 7.2, we conclude that there is no dimension formula for general sequences of radii $\underline{r}$ which depends only on $s_2(\underline{r})$ and the lower and upper local dimensions of the generating measure.

The paper is organised as follows. In Section 2, we introduce some notation and state our main results. In Section 3, we state some preliminary lemmas needed in later sections. Section 4 is devoted to the proof of our main technical tool (Theorem 2.8) from which our main theorems (Theorems 2.3 and 2.5) follow. In Section 5, we prove Theorem 2.3 and, in Section 6, we prove Theorem 2.5. Finally, in Section 7, we construct examples (Examples 7.1 and 7.2) showing the sharpness of our results.

2 | NOTATION AND RESULTS

We denote by $\mathcal{P}(\mathbb{R}^d)$ the space of Borel probability measures on $\mathbb{R}^d$ and denote by $\mathrm{spt}\mu$ the support of a measure $\mu \in \mathcal{P}(\mathbb{R}^d)$. We write $B(x, r)$ for the open ball with centre $x \in \mathbb{R}^d$ and radius $r > 0$. We will use the notation $\underline{r} = (r_k)_{k=1}^{\infty}$ for sequences of positive numbers.

Remark 2.1.

(a) We note that our results remain unchanged if all the open balls are replaced by closed balls.
(b) In some articles mentioned in the Introduction, the generating measure is assumed to be compactly supported. We will also utilise tools stated only for compactly supported measures. However, our results are valid for all Borel probability measures μ, since the Hausdorff dimension of any set E may be approximated by the Hausdorff dimensions of sets $E \cap B(0, R)$ with R tending to infinity and we may write $\mu = \mu|_{B(0,R)} + \mu|_{\mathbb{R}^d \setminus B(0,R)}$, where $\mu|_A$ is the restriction of μ to a set A. For a precise statement, see [10, Lemma 9.3] (also stated as Lemma 3.5). Note that the upper bounds in Theorems 2.3 and 2.5 are trivial.

For a measure $\mu \in \mathcal{P}(\mathbb{R}^d)$, the lower and upper local dimensions of μ at a point x are defined by

$$\underline{\dim}_{\mathrm{loc}}\mu(x) := \liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \quad\text{and}\quad \overline{\dim}_{\mathrm{loc}}\mu(x) := \limsup_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}.$$

If these notions agree, we write $\dim_{\mathrm{loc}}\mu(x)$ for the common value. The lower and upper Hausdorff dimensions of μ are defined by

$$\underline{\dim}_{\mathrm{H}}\mu := \operatorname*{ess\,inf}_{x\sim\mu} \underline{\dim}_{\mathrm{loc}}\mu(x) \quad\text{and}\quad \overline{\dim}_{\mathrm{H}}\mu := \operatorname*{ess\,sup}_{x\sim\mu} \underline{\dim}_{\mathrm{loc}}\mu(x).$$

Again, if these notions agree, we denote the common value by $\dim_{\mathrm{H}}\mu$.

Definition 2.2. Let $\mu \in \mathcal{P}(\mathbb{R}^d)$ and consider the probability space $(\Omega, \mathbb{P})$, where $\Omega := \operatorname{spt}\mu^{\mathbb{N}}$ and $\mathbb{P} := \mu^{\mathbb{N}}$. Let $\underline{r} = (r_k)_{k=1}^{\infty}$ be a sequence of positive numbers. Given $\omega \in \Omega$, the *random covering set* generated by the sequence $\underline{r}$ is the limsup set

$$E_{\underline{r}}(\omega) := \limsup_{k\to\infty} B(\omega_k, r_k) = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} B(\omega_n, r_n).$$

In the special case $r_k = k^{-\alpha}$ for all $k \in \mathbb{N}$ and for some $\alpha > 0$, we write $E_\alpha(\omega)$ for the corresponding limsup set, that is,

$$E_\alpha(\omega) := \limsup_{k\to\infty} B(\omega_k, k^{-\alpha}).$$

By Kolmogorov's zero-one law, the quantity $\dim_{\mathrm{H}} E_{\underline{r}}(\omega)$ is constant $\mathbb{P}$-almost surely since $\{\omega \in \Omega \mid \dim_{\mathrm{H}} E_{\underline{r}}(\omega) \leqslant \beta\}$ is a tail event for all $\beta \geqslant 0$ (see, e.g., [21, Lemma 3.1]). Let $f_\mu(\underline{r})$ denote this almost sure value of $\dim_{\mathrm{H}} E_{\underline{r}}(\omega)$. We also write $f_\mu(\alpha)$ for the almost sure value of $\dim_{\mathrm{H}} E_\alpha(\omega)$. For a sequence $\underline{r}$ of positive numbers, set

$$s_1(\underline{r}) := \liminf_{k\to\infty} \frac{\log k}{-\log r_k},\ s_3(\underline{r}) := \limsup_{k\to\infty} \frac{\log k}{-\log r_k} \text{ and}$$

$$s_2(\underline{r}) := \inf\left\{ s > 0 \mid \sum_{n=1}^{\infty} r_n^s < \infty \right\} = \sup\left\{ s > 0 \mid \sum_{n=1}^{\infty} r_n^s = \infty \right\}. \tag{2.1}$$

We always have the inequalities

$$s_1(\underline{r}) \leqslant s_2(\underline{r}) \leqslant s_3(\underline{r}).$$

Furthermore, if $\underline{r}$ is decreasing, then $s_2(\underline{r}) = s_3(\underline{r})$. This follows from the well-known fact (due to Abel) that $\sum_{n=1}^{\infty} a_n < \infty$ implies $\lim_{n\to\infty} na_n = 0$ if $(a_n)_{n=1}^{\infty}$ is decreasing. Since the centres ω_k are independent identically distributed random variables, the quantity $f_\mu(\underline{r})$ remains unchanged if the sequence $\underline{r}$ is reordered. Thus, we may always assume that $\underline{r}$ is decreasing. The quantity $s_2(\underline{r})$ is an upper bound for $\dim_{\mathrm{H}} E_{\underline{r}}(\omega)$ for any realisation ω. This can easily be seen by observing that

$$E_{\underline{r}}(\omega) \subseteq \bigcup_{k=N}^{\infty} B(\omega_k, r_k)$$

for every $N \in \mathbb{N}$. In the special case where $\underline{r} = (k^{-\alpha})_{n=1}^{\infty}$, we have that

$$s_1(\underline{r}) = s_3(\underline{r}) = \frac{1}{\alpha} \tag{2.2}$$

and, thus, $\frac{1}{\alpha}$ is always an upper bound for $f_\mu(\alpha)$. The almost sure dimension $f_\mu(\alpha)$ has been studied by Ekström and Persson in [10, Theorem 2.1]. They proved the following result.

Theorem Ekström–Persson. *Let* $\mu \in \mathcal{P}(\mathbb{R}^d)$ *and* $\alpha > 0$. *If* $\frac{1}{\alpha} < \overline{\dim}_{\mathrm{H}}\mu$, *then*

$$f_\mu(\alpha) \geqslant \frac{1}{\alpha} - \delta, \tag{2.3}$$

where

$$\delta := \operatorname*{ess\,inf}_{\substack{x\sim\mu,\\ \underline{\dim}_{\mathrm{loc}}\mu(x) > 1/\alpha}} \left(\overline{\dim}_{\mathrm{loc}}\mu(x) - \underline{\dim}_{\mathrm{loc}}\mu(x)\right). \tag{2.4}$$

Ekström and Persson also obtained lower and upper bounds for $f_\mu(\alpha)$ when $\frac{1}{\alpha} > \overline{\dim}_{\mathrm{H}}\mu$, and they conjectured that the equality

$$f_\mu(\alpha) = \overline{F}_\mu\left(\frac{1}{\alpha}\right) \tag{2.5}$$

always holds, where F_μ is the fine multifractal spectrum defined by

$$F_\mu(s) := \dim_{\mathrm{H}}\{x \in \operatorname{spt}\mu \mid \underline{\dim}_{\mathrm{loc}}\mu(x) \leqslant s\}$$

and $\overline{F}_\mu$ denotes the increasing 1-Lipschitz hull of F_μ, that is,

$$\overline{F}_\mu(s) := \inf\{h(s) \mid h \geqslant F_\mu \text{ is increasing and 1-Lipschitz continuous}\}.$$

Our first main result is the following theorem, which verifies Equation (2.5) in the case where $\frac{1}{\alpha} \leqslant \overline{\dim}_{\mathrm{H}}\mu$.

Theorem 2.3. *Let $\mu \in \mathcal{P}(\mathbb{R}^d)$ and $\alpha > 0$. If $\frac{1}{\alpha} \leqslant \overline{\dim}_{\mathrm{H}}\mu$, then*

$$f_\mu(\alpha) = \frac{1}{\alpha}.$$

Since $E_\beta(\omega) \subset E_\alpha(\omega) \subset \mathrm{spt}\mu$ for $\beta > \alpha$, we immediately obtain the following corollary.

Corollary 2.4. *Let $\mu \in \mathcal{P}(\mathbb{R}^d)$ be such that $\dim_{\mathrm{H}}\mathrm{spt}\mu = \overline{\dim}_{\mathrm{H}}\mu$. Then for every $\alpha > 0$,*

$$f_\mu(\alpha) = \min\left\{\frac{1}{\alpha}, \dim_{\mathrm{H}}\mathrm{spt}\mu\right\}.$$

Our second main theorem provides bounds for $f_\mu(\underline{r})$ for general sequences $\underline{r}$.

Theorem 2.5. *Let $\mu \in \mathcal{P}(\mathbb{R}^d)$ and let $\underline{r} := (r_k)_{k=1}^\infty$ be a sequence of positive numbers tending to 0. If $s_2(\underline{r}) < \overline{\dim}_{\mathrm{H}}\mu$, then*

$$s_2(\underline{r})\overline{\delta} \leqslant f_\mu(\underline{r}) \leqslant s_2(\underline{r}),$$

where

$$\overline{\delta} := \operatorname*{ess\,sup}_{\substack{x\sim\mu,\\ \underline{\dim}_{\mathrm{loc}}\mu(x) > s_2(\underline{r})}} \frac{\underline{\dim}_{\mathrm{loc}}\mu(x)}{\overline{\dim}_{\mathrm{loc}}\mu(x)}.$$

Remark 2.6.

(a) One may apply the methods used in [10] to prove a counterpart of the lower bound (2.3) also for a general sequence $\underline{r}$. In order to do this, one has to replace $\frac{1}{\alpha}$ by $s_1(\underline{r})$ in some places and by $s_3(\underline{r})$ in other places, and this leads to the lower bound

$$f_\mu(\underline{r}) \geqslant s_1(\underline{r}) - \hat{\delta},$$

where

$$\hat{\delta} := \operatorname*{ess\,inf}_{\substack{x\sim\mu,\\ \underline{\dim}_{\mathrm{loc}}\mu(x) > s_3(\underline{r})}} \left(\overline{\dim}_{\mathrm{loc}}\mu(x) - \underline{\dim}_{\mathrm{loc}}\mu(x)\right).$$

Recall that, for a general decreasing sequence $\underline{r}$, the strict inequality $s_1(\underline{r}) < s_2(\underline{r})$ is possible, whilst $s_2(\underline{r}) = s_3(\underline{r})$ for all decreasing sequences.

b) The lower bound $s_2(\underline{r})\overline{\delta}$ obtained in Theorem 2.5 is always positive and larger than the quantity $s_2(\underline{r}) - \hat{\delta}$. Indeed, fix $\mu \in \mathcal{P}(\mathbb{R}^d)$ and a sequence $\underline{r}$ such that $s_2 := s_2(\underline{r}) < \overline{\dim}_{\mathrm{H}}\,\mu$. By the definition of $\overline{\delta}$, we have for μ-almost every x satisfying the property $\underline{\dim}_{\mathrm{loc}}\mu(x) > s_2$ that

$$\begin{aligned} s_2\overline{\delta} &\geqslant s_2\frac{\underline{\dim}_{\mathrm{loc}}\mu(x)}{\overline{\dim}_{\mathrm{loc}}\mu(x)} = s_2 - \frac{s_2}{\overline{\dim}_{\mathrm{loc}}\mu(x)}\left(\overline{\dim}_{\mathrm{loc}}\mu(x) - \underline{\dim}_{\mathrm{loc}}\mu(x)\right) \\ &\geqslant s_2 - \left(\overline{\dim}_{\mathrm{loc}}\mu(x) - \underline{\dim}_{\mathrm{loc}}\mu(x)\right), \end{aligned}$$

where the last inequality follows from the fact that $s_2 < \underline{\dim}_{\mathrm{loc}}\mu(x) \leqslant \overline{\dim}_{\mathrm{loc}}\mu(x)$. Since the above inequality holds true for μ-almost every x such that $\underline{\dim}_{\mathrm{loc}}\mu(x) > s_2$, we have (by the definition of $\hat{\delta}$) that $s_2\overline{\delta} \geqslant s_2 - \hat{\delta}$.

(c) In [32, Theorem 2.8], Persson obtained an alternative lower bound for $f_\mu(\alpha)$. More precisely, he showed that if $\frac{1}{\alpha} \leqslant s(\mu)$, then

$$f_\mu(\alpha) \geqslant \frac{1}{\alpha}\frac{\overline{\dim}_{\mathrm{H}}\mu}{s(\mu)},$$

where

$$s(\mu) = \lim_{\varepsilon\to 0+} \sup\{t \mid G_\mu(t) \geqslant t - \varepsilon\}$$

and $G_\mu(t)$ denotes the upper coarse spectrum of μ. By Theorem 2.3, this is not always optimal. It is an interesting open question whether a similar result could be proved for general sequences, that is, whether

$$f_\mu(\underline{r}) \geqslant s_2(\underline{r})\frac{\overline{\dim}_{\mathrm{H}}\mu}{s(\mu)} \tag{2.6}$$

holds for all $\underline{r}$ with $s_2(\underline{r}) \leqslant s(\mu)$. If this indeed were the case, it would be interesting to compare it to the lower bound in Theorem 2.5. Recall that $G_\mu(t)$ is defined by the formula

$$G_\mu(t) := \lim_{\varepsilon\to 0}\limsup_{r\to 0}\frac{\log(N(t+\varepsilon,r) - N(t-\varepsilon,r))}{-\log r},$$

where $N(t,r)$ is the number of r-adic cubes Q with $\mu(Q) \geqslant r^t$. Fix a sequence $\underline{r}$ and numbers s and u with $s_2(\underline{r}) < s < u < 1$. Let μ_1 be the measure constructed in Example 7.1 supported on [0,1]. Since μ_1 is homogeneous in space, essentially all r-adic cubes with positive μ_1-measure have almost equal measure. Further, for all $t \in [s,u]$, there are infinitely many r with $\mu_1(Q)$ comparable to r^t for essentially all r-adic cubes Q with nonzero measure, which implies that $N(t+\varepsilon,r) - N(t-\varepsilon,r)$ is comparable to r^{-t}. Thus, $G_{\mu_1}(t) = t$ for all $t \in [s,u]$. Let μ_2 be the natural probability measure on the standard Cantor set with dimension s included in [2,3], that is, μ_2 is like μ_1 with $u = s$. Set $\mu := \frac{1}{2}(\mu_1 + \mu_2)$. Then, $G_\mu(t) = t$ for all $t \in [s,u]$, $\overline{\dim}_{\mathrm{H}}\mu = s$, $s(\mu) \geqslant u$ and $\overline{\delta} = 1$. Therefore, Theorem 2.5 gives better lower bound than Equation (2.6) for μ. We do not know if there exists a measure for which the lower bound in Equation (2.6) is larger than the one in Theorem 2.5.

By noting that $\overline{\delta} = 1$ in Theorem 2.5, if $\dim_{\mathrm{loc}}\mu(x)$ exists and is larger than $s_2(\underline{r})$ in a set of positive measure, we obtain the following corollary.

Corollary 2.7. *Suppose that $\mu \in \mathcal{P}(\mathbb{R}^d)$ is such that the local dimension $\dim_{\mathrm{loc}}\mu(x)$ exists in a set of positive μ-measure. Then, for any sequence $\underline{r}$ satisfying $s_2(\underline{r}) < \operatorname{ess\,sup}_{x\sim\mu}\dim_{\mathrm{loc}}\mu(x)$, we have that*

$$f_\mu(\underline{r}) = s_2(\underline{r}).$$

In the proof of our main technical result (Theorem 2.8), we will make use of potential theoretic arguments. The t-potential of a measure $\mu \in \mathcal{P}(\mathbb{R}^d)$ at a point x is defined by

$$\phi^t_\mu(x) := \int |x-y|^{-t}\, d\mu(y)$$

and the t-energy of μ is

$$I_t(\mu) := \int \phi^t_\mu(x)\, d\mu(x) = \int\int |x-y|^{-t}\, d\mu(y)\, d\mu(x).$$

The mutual t-energy of two measures $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$ is defined by

$$J_t(\mu,\nu) := \int\int |x-y|^{-t}\, d\mu(y)\, d\nu(x).$$

We also define the t-capacity of a set $A \subseteq \mathbb{R}^d$ by

$$\mathrm{Cap}_t(A) := \sup\{I_t(\mu)^{-1} \mid \mu \in \mathcal{P}(\mathbb{R}^d), \mathrm{spt}\mu \subset A\}.$$

It is well known that if $\mathrm{Cap}_t(A) > 0$, then $\dim_{\mathrm{H}} A \geqslant t$ (see, e.g., [29, Chapter 8]). A measure $\mu \in \mathcal{P}(\mathbb{R}^d)$ is (C,s)-Frostman if $\mu(B(x,r)) \leqslant Cr^s$ for every $x \in \mathbb{R}^d$ and $r > 0$. If there is no need to emphasise the constant C, we will just say that μ is s-Frostman. It is well known that $I_t(\mu) < \infty$ for all $t < s$ provided μ is s-Frostman (see [29, p. 109]). Given $\mu \in \mathcal{P}(\mathbb{R}^d)$ and $A \subseteq \mathbb{R}^d$, we write μ_A for the normalised restriction of μ to A, that is,

$$\mu_A(B) = \mu(A)^{-1}\mu_{|_A}(B) = \mu(A)^{-1}\mu(A \cap B)$$

for $B \subseteq \mathbb{R}^d$ with the interpretation that μ_A is the zero measure whenever $\mu_{|_A}$ is.

The proofs of Theorems 2.3 and 2.5 are based on the following theorem, proved in Section 4.

Theorem 2.8. *Let $\mu \in \mathcal{P}(\mathbb{R}^d)$ be s-Frostman. Let $(r_k)_{k=1}^\infty$ be a decreasing sequence of positive numbers tending to zero. Let $0 < t < s$. Suppose that there exists a bounded sequence of positive numbers $(b_k)_{k=1}^\infty$ such that*

$$\mu\left(\left\{x \in \mathrm{spt}\mu \mid \sum_{k=1}^\infty \chi_{A^t_k}(x) b_k = \infty\right\}\right) = 1, \tag{2.7}$$

where $A^t_k := \{x \in \mathrm{spt}\mu \mid I_t(\mu_{B(x,r_k)}) \leqslant b_k^{-1}\}$ and χ_B is the characteristic function of a set B. Then $f_\mu(\underline{r}) \geqslant t$.

3 | PRELIMINARY LEMMAS

In this section, we state and prove some lemmas needed in the proof of Theorem 2.8.

Lemma 3.1. *Let $\mu \in \mathcal{P}(\mathbb{R}^d)$ and let $\varphi \geqslant 0$ be a Borel function. Then*

$$\int \varphi \, d\mu = \int_0^\infty \mu(\{z \mid \varphi(z) > t\}) \, dt.$$

Proof. See [29, Theorem 1.15]. In that theorem, it is assumed that $\varphi(z) \geqslant t$, but the same proof works also in the case $\varphi(z) > t$. □

Lemma 3.2. *Assume that $\mu \in \mathcal{P}(\mathbb{R}^d)$ is (C, s)-Frostman. Let $0 < t < s$, $z_0 \in \operatorname{spt}\mu$ and $r_0 > 0$. Then, the following inequalities are valid. For every $y \in \mathbb{R}^d$,*

$$\phi^t_{\mu|_{B(z_0,r_0)}}(y) \leqslant \frac{C^{\frac{t}{s}} s}{s-t} \mu(B(z_0, r_0))^{1-\frac{t}{s}} \leqslant \frac{Cs}{s-t} r_0^{s-t} \text{ and}$$

$$I_t(\mu_{B(z_0,r_0)}) \leqslant \frac{C^{\frac{t}{s}} s}{s-t} \mu(B(z_0, r_0))^{-\frac{t}{s}}.$$

Proof. The second inequality on the first line is trivial, since μ is (C, s)-Frostman. To prove the first inequality, write $B_0 := B(z_0, r_0)$ and let

$$\rho_0 := \sup\{0 < \rho \leqslant r_0 \mid C\rho^s \leqslant \mu(B_0)\}.$$

Since $0 < \mu(B_0) \leqslant Cr_0^s$ and the map $\rho \mapsto C\rho^s$ is continuous, $\rho_0 \in]0, r_0]$ and $\rho_0 = (C^{-1}\mu(B_0))^{\frac{1}{s}}$. Fix $y \in \mathbb{R}^d$. By Lemma 3.1 and a change of variables,

$$\begin{aligned}
\phi^t_{\mu|_{B_0}}(y) &= \int_{B_0} |x-y|^{-t} \, d\mu(x) = \int_0^\infty \mu(\{x \in B_0 \mid |x-y|^{-t} > \gamma\}) \, d\gamma \\
&= \int_0^\infty \mu(B_0 \cap B(y, \gamma^{-\frac{1}{t}})) \, d\gamma = t \int_0^\infty \mu(B_0 \cap B(y, a)) a^{-(t+1)} \, da \\
&\leqslant t \left(\int_0^{\rho_0} C a^{s-t-1} \, da + \int_{\rho_0}^\infty \mu(B_0) a^{-(t+1)} \, da \right) \leqslant t \left(\frac{C}{s-t} \rho_0^{s-t} + \mu(B_0) \frac{\rho_0^{-t}}{t} \right) \\
&= \frac{Ct}{s-t} \left(C^{-1}\mu(B_0)\right)^{\frac{s-t}{s}} + \mu(B_0) \left(\left(C^{-1}\mu(B_0)\right)^{\frac{1}{s}} \right)^{-t} \\
&= \left(\frac{Ct}{s-t} C^{\frac{t-s}{s}} + C^{\frac{t}{s}} \right) \mu(B_0)^{1-\frac{t}{s}} = \frac{C^{\frac{t}{s}} s}{s-t} \mu(B_0)^{1-\frac{t}{s}},
\end{aligned}$$

which proves the first inequality. The last inequality follows by multiplying this pointwise estimate by $\mu(B_0)^{-1}$ and integrating with respect to the probability measure μ_{B_0}. □

Lemma 3.3. *Let $(\xi_n)_{n=1}^\infty$ be a sequence of independent random variables. Then*

$$\liminf_{n \to \infty} \xi_n \leqslant \liminf_{n \to \infty} \mathbb{E}(\xi_n)$$

almost surely.

Proof. [8, Lemma 3.6]. □

Lemma 3.4. *Let $X \subset \mathbb{R}^d$ be compact and let $\nu \in \mathcal{P}(X)$. Suppose that $(A_n)_{n=1}^{\infty}$ is a sequence of Borel subsets of X with $\lim_{n\to\infty} \nu(A_n) = 1$. Then, $\lim_{n\to\infty} \nu_{A_n} = \nu$ in the weak-$*$ topology and*

$$\lim_{n\to\infty} I_t(\nu_{A_n}) = I_t(\nu)$$

for every $t > 0$.

Proof. [14, Lemma 3.5]. □

We finish this section with a simplified version of a lemma proved in [10].

Lemma 3.5. *Let $\mu \ll \nu$ be Borel probability measures on a separable metric space X such that the density $\frac{d\mu}{d\nu}$ is bounded. Then $f_\mu(\underline{r}) \leqslant f_\nu(\underline{r})$ for every sequence $\underline{r}$.*

Proof. [10, Lemma 9.3]. □

4 | PROOF OF THEOREM 2.8

In the proof of Theorem 2.8, we will make use of the following deterministic result.

Lemma 4.1. *Let ν be a finite Borel measure on a compact metric space X and let $(\varphi_n)_{n=1}^{\infty}$ be a sequence of nonnegative continuous functions on X with the property that $\lim_{n\to\infty} \varphi_n\, d\nu = \nu$ in the weak-$*$ topology and $\liminf_{n\to\infty} I_t(\varphi_n \rho\, d\nu) \leqslant I_t(\rho\, d\nu)$ whenever ρ is a product of finitely many of the functions $\{\varphi_n\}_{n\in\mathbb{N}}$. Then for every $t > 0$,*

$$\mathrm{Cap}_t\left(\mathrm{spt}\nu \cap \limsup_{n\to\infty}(\mathrm{spt}\varphi_n)\right) \geqslant \frac{\nu(X)^2}{I_t(\nu)}.$$

Proof. [8, Lemma 1.4]. □

We are now ready to prove Theorem 2.8.

Proof of Theorem 2.8. In the proof, we are going to use Lemmas 3.4 and 4.1, which involve compactly supported measures. If the support of μ is not compact, we may consider $\eta := \mu_{B(0,R)}$ for an arbitrary $R > 0$ such that $\mu(B(0,R)) > 0$. Then, η is a compactly supported Borel probability measure which is (C, s)-Frostman with some constant C different from the one of μ. Note that the value of C plays no role in the statement of Theorem 2.8. We may choose R such that the boundary of $B(0,R)$ has zero μ-measure. Thus, for η-almost all $x \in \mathrm{spt}\eta$, we have that $\eta_{B(x,r_k)} = \mu_{B(x,r_k)}$ for all large enough $k \in \mathbb{N}$. Therefore, η satisfies the assumptions of Theorem 2.8. Since η is absolutely continuous with respect to μ with bounded density, Lemma 3.5 implies that the lower bound obtained for $f_\eta(\underline{r})$ is also a lower bound for $f_\mu(\underline{r})$. Thus, we may assume that $\mathrm{spt}\mu$ is compact.

Fix $0 < t < s$ and a sequence $(b_k)_{k=1}^{\infty}$ such that Equation (2.7) holds. Our aim is to show that $f_\mu(\underline{r}) \geqslant t$. Since $t > 0$ is fixed, we will denote the sets A_k^t from the statement simply by A_k. By our assumption,

$$\sum_{k=1}^{\infty} b_k \chi_{A_k}(x) = \infty$$

for μ-almost every x. Hence, we can choose sequences $(M_n)_{n=1}^{\infty}$ and $(N_n)_{n=1}^{\infty}$ of natural numbers such that $M_n < N_n < M_{n+1}$ for every $n \in \mathbb{N}$,

$$r_{M_n} < \frac{4^{-n}}{2} \text{ and } \mu(F_n) \geqslant 1 - 2^{-n}, \tag{4.1}$$

where

$$F_n := \left\{ x \in \operatorname{spt}\mu \mid \sum_{k=M_n}^{N_n} \chi_{A_k}(x) b_k \geqslant 2^n \right\}.$$

Clearly, F_n is a Borel set. We now establish some notation. Set

$$\Sigma_n := \{\mathbf{i} = (i_{M_n}, \dots, i_{N_n}) \in \{0,1\}^{N_n - M_n + 1}\}.$$

For $\mathbf{i} = (i_{M_n}, \dots, i_{N_n}) \in \Sigma_n$, let

$$A_{n,\mathbf{i}} := \left(\bigcap_{\substack{j=M_n \\ i_j=1}}^{N_n} A_j \right) \setminus \left(\bigcup_{\substack{j=M_n \\ i_j=0}}^{N_n} A_j \right) \text{ and } c_{n,\mathbf{i}} := \sum_{\substack{j=M_n \\ i_j=1}}^{N_n} b_j.$$

Set $\Sigma_n' := \{\mathbf{i} \in \Sigma_n \mid c_{n,\mathbf{i}} \geqslant 2^n\}$. Note that

$$\bigcup_{k=M_n}^{N_n} A_k = \bigcup_{\mathbf{i} \in \Sigma_n} A_{n,\mathbf{i}}$$

with the right-hand side union disjoint, and

$$F_n = \bigcup_{\mathbf{i} \in \Sigma_n'} A_{n,\mathbf{i}}. \tag{4.2}$$

We will now consider the Borel probability measures

$$\mu_n := \mu(F_n)^{-1} \sum_{\mathbf{i} \in \Sigma_n'} \mu(A_{n,\mathbf{i}}) a_{n,\mathbf{i}} \sum_{j=M_n}^{N_n} b_j \, \mu|_{A_j \cap A_{n,\mathbf{i}}}, \tag{4.3}$$

where

$$a_{n,\mathbf{i}} := \left(\sum_{j=M_n}^{N_n} b_j \mu(A_j \cap A_{n,\mathbf{i}}) \right)^{-1} \tag{4.4}$$

provided that $\mu(A_{n,\mathbf{i}}) > 0$. Observe that

$$\sum_{j=M_n}^{N_n} b_j\, \mu(A_j \cap A_{n,\mathbf{i}}) = \int_{A_{n,\mathbf{i}}} \sum_{j=M_n}^{N_n} b_j\, \chi_{A_j}(x)\, d\mu(x) = c_{n,\mathbf{i}}\, \mu(A_{n,\mathbf{i}}),$$

hence

$$a_{n,\mathbf{i}} = (c_{n,\mathbf{i}}\mu(A_{n,\mathbf{i}}))^{-1} \tag{4.5}$$

for $\mu(A_{n,\mathbf{i}}) > 0$. Thus, for any Borel set $B \subset \mathbb{R}^d$, we have that

$$\begin{aligned}
\mu_n(B) &= \mu(F_n)^{-1} \sum_{\mathbf{i}\in\Sigma_n'} \mu(A_{n,\mathbf{i}})a_{n,\mathbf{i}} \sum_{j=M_n}^{N_n} b_j\, \mu(A_j \cap A_{n,\mathbf{i}} \cap B) \\
&= \mu(F_n)^{-1} \sum_{\mathbf{i}\in\Sigma_n'} \mu(A_{n,\mathbf{i}})a_{n,\mathbf{i}} \int_{A_{n,\mathbf{i}}\cap B} \sum_{j=M_n}^{N_n} b_j\, \chi_{A_j}(x)\, d\mu(x) \\
&= \mu(F_n)^{-1} \sum_{\mathbf{i}\in\Sigma_n'} \mu(A_{n,\mathbf{i}})a_{n,\mathbf{i}}\, c_{n,\mathbf{i}}\, \mu(A_{n,\mathbf{i}} \cap B) \\
&= \mu(F_n)^{-1} \sum_{\mathbf{i}\in\Sigma_n'} \mu(A_{n,\mathbf{i}} \cap B) = \mu(F_n)^{-1}\mu(F_n \cap B).
\end{aligned}$$

Hence

$$\mu_n = \mu_{F_n}. \tag{4.6}$$

Since $\lim_{n\to\infty} \mu(F_n) = 1$, Lemma 3.4 implies that $\mu_n \xrightarrow[n\to\infty]{w\text{-}*} \mu$ and $I_t(\mu_n) \xrightarrow[n\to\infty]{} I_t(\mu)$ (recall that μ is s-Frostman, hence $I_t(\mu) < \infty$).

For all $n \in \mathbb{N}$, define a Borel function $E_n : \mathbb{R}^d \to \mathbb{R}$ by

$$E_n(x) := \mu(F_n)^{-1} \sum_{\mathbf{i}\in\Sigma_n'} \mu(A_{n,\mathbf{i}})a_{n,\mathbf{i}} \sum_{j=M_n}^{N_n} b_j \int_{A_j\cap A_{n,\mathbf{i}}} \frac{\chi_{B(x,r_j)}(z)}{\mu(B(z,r_j))}\, d\mu(z). \tag{4.7}$$

Using the fact $\chi_{B(x,r)}(z) = \chi_{B(z,r)}(x)$ and Fubini's theorem, we conclude that

$$\int E_n(x)\, d\mu(x) = 1,$$

hence Markov's inequality yields

$$\mu(\{x \in \operatorname{spt}\mu \mid E_n(x) \geqslant n^2\}) \leqslant n^{-2}.$$

Fix $N \in \mathbb{N}$ large enough so that $\mu(G) > 0$, where

$$G := \bigcap_{n=N}^{\infty} \{x \in \operatorname{spt}\mu \mid E_n(x) < n^2\}. \tag{4.8}$$

Clearly, G is a Borel set. Such an N exists since

$$\mu(\mathbb{R}^d \setminus G) \leqslant \sum_{n=N}^{\infty} n^{-2}.$$

By the Lebesgue's density theorem,

$$\lim_{r\downarrow 0} \frac{\mu(G\cap B(x,r))}{\mu(B(x,r))} = 1$$

for μ-almost every $x \in G$. Thus, we can choose a decreasing sequence $(\varepsilon_n)_{n=1}^{\infty}$ of positive numbers tending to 0 such that

$$\mu(H_n) \geqslant \Big(1-\frac{1}{n}\Big)\mu(G), \tag{4.9}$$

where

$$H_n := \left\{ x\in G \mid \frac{\mu(G\cap B(x,r))}{\mu(B(x,r))} \geqslant 1-\frac{1}{n} \text{ for all } 0<r<\varepsilon_n \right\}. \tag{4.10}$$

For all $n \in \mathbb{N}$, we write

$$q_n := \min\left\{ n, \max\{k\in\mathbb{N} \mid \varepsilon_k > r_{M_n}\}\right\}.$$

Then

$$\lim_{n\to\infty} q_n = \infty \text{ and } r_k < \varepsilon_{q_n} \text{ for all } k \in \{M_n,\dots,N_n\}. \tag{4.11}$$

Our goal is to construct, for almost every $\omega \in \Omega$, a measure supported on $E_{\underline{r}}(\omega)$ having finite t-energy. To this end, for each $\omega\in\Omega$ and $k\in\mathbb{N}$, write $B_k^\omega := B(\omega_k, r_k)$ and let $U_k^\omega \subset B_k^\omega$ be the smallest closed ball centred at ω_k such that $\mu(G\cap U_k^\omega) \geqslant \frac{\mu(G\cap B_k^\omega)}{2}$. Since B_k^ω is open, $\operatorname{dist}(U_k^\omega, \mathbb{R}^d \setminus B_k^\omega) > 0$. Let $r(\omega_k)$ be the radius of U_k^ω. Note that $\lim_{\delta\to 0}\mu(G\cap B(y,\rho-\delta)) = \mu(G\cap B(y,\rho))$ for all $y\in\mathbb{R}^d$ and $\rho>0$. Fix $y\in\mathbb{R}^d$, $\varepsilon>0$ and $k\in\mathbb{N}$. Let $\lim_{i\to\infty} y_i = y$ and denote by $\overline{B}(z,\rho)$ the closed ball centred at z with radius ρ. There exists $\delta>0$ such that, for all large $i\in\mathbb{N}$, we have by the definition of $r(y)$ that

$$\begin{aligned}\mu(G\cap \overline{B}(y_i, r(y)-2\varepsilon)) \leqslant \mu(G\cap \overline{B}(y,r(y)-\varepsilon)) &< \frac{\mu(G\cap B(y,r_k-\delta))}{2}\\ &\leqslant \frac{\mu(G\cap B(y_i,r_k))}{2}.\end{aligned}$$

Thus, $r(y_i) \geqslant r(y)-2\varepsilon$ for all large $i\in\mathbb{N}$, which implies that r is lower semicontinuous. Therefore, $(\omega,x)\mapsto \chi_{B_k^\omega}(x)$ and $(\omega,x)\mapsto \chi_{U_k^\omega}(x) = \chi_{\overline{B}(\omega_k, r(\omega_k))}(x)$ are Borel maps and we may choose a continuous function $\tilde{\psi}_k^\omega$ such that

$$\chi_{U_k^\omega} \leqslant \tilde{\psi}_k^\omega \leqslant \chi_{B_k^\omega} \tag{4.12}$$

and $(\omega, x) \mapsto \tilde{\psi}_k^\omega(x)$ is a Borel map. For example, we may interpolate linearly between the boundaries of U_k^ω and B_k^ω. Finally, for all $k \in \mathbb{N}$ and $\omega \in \Omega$ such that $\mu(G \cap B_k^\omega) > 0$, let

$$\psi_k^\omega := c_k^\omega \frac{\tilde{\psi}_k^\omega}{\mu(G \cap B_k^\omega)}, \tag{4.13}$$

where $c_k^\omega \leqslant 2$ is such that $\psi_k^\omega \, d\mu_{|_G}$ is a Borel probability measure. Note that $\omega \mapsto c_k^\omega$ is a Borel function.

We will now consider the Borel measures $\varphi_n^\omega \, d\mu_{|_G}$, where

$$\varphi_n^\omega := \mu(F_n)^{-1} \sum_{\mathbf{i} \in \Sigma_n'} \mu(A_{n,\mathbf{i}}) a_{n,\mathbf{i}} \sum_{k=M_n}^{N_n} b_k \, \chi_{A_k \cap A_{n,\mathbf{i}} \cap H_{q_n}}(\omega_k) \psi_k^\omega. \tag{4.14}$$

We require the following lemma. □

Sublemma 4.2. *Let*

$$\mu_n^\omega := \mu(F_n)^{-1} \sum_{\mathbf{i} \in \Sigma_n'} \mu(A_{n,\mathbf{i}}) a_{n,\mathbf{i}} \sum_{k=M_n}^{N_n} b_k \, \chi_{A_k \cap A_{n,\mathbf{i}} \cap H_{q_n}}(\omega_k) \delta_{\omega_k},$$

where δ_{ω_k} is the Dirac measure at ω_k. Then, $\mu_n^\omega \xrightarrow[n\to\infty]{w\text{-}*} \mu_{|_G}$ almost surely.

Proof. Let $h \in C(\mathbb{R}^d)$ with compact support. Since $\lim_{n\to\infty} \mu(F_n) = 1$ by Equation (4.1) and $H_{q_n} \cap F_n \subset G$ by Equation(4.10), we have that $\lim_{n\to\infty} \mu(H_{q_n} \cap F_n) = \mu(G)$ by Equations (4.9) and (4.11). Recalling that $\mu_n = \mu_{F_n}$ (see Equation (4.6)) and writing $\nu(h) := \int h \, d\nu$ for a Borel measure ν, we obtain

$$\begin{aligned}
\mathbb{E}(\mu_n^\omega(h)) &= \mu(F_n)^{-1} \sum_{\mathbf{i} \in \Sigma_n'} \mu(A_{n,\mathbf{i}}) a_{n,\mathbf{i}} \sum_{k=M_n}^{N_n} b_k \int_{A_k \cap A_{n,\mathbf{i}} \cap H_{q_n}} h(\omega_k) \, d\mu(\omega_k) \\
&= \int_{H_{q_n}} d\mu_n = \mu(F_n)^{-1} \int_{H_{q_n}} h \, d\mu_{|_{F_n}} = \mu(F_n)^{-1} \int h \, d\mu_{|_{H_{q_n} \cap F_n}} \\
&\xrightarrow[n\to\infty]{} \mu_{|_G}(h).
\end{aligned}$$

Set $b := \sup_{k \in \mathbb{N}} b_k$. Recalling Equation (4.2), disjointedness of the sets $A_{n,\mathbf{i}}$, Equation (4.5), the definition of Σ_n', Equations (4.3) and (4.6), we estimate the variance

$$\begin{aligned}
&\mathrm{Var}(\mu_n^\omega(h)) \\
&= \mathrm{Var}\left(\mu(F_n)^{-1} \sum_{k=M_n}^{N_n} b_k \sum_{\mathbf{i} \in \Sigma_n'} \mu(A_{n,\mathbf{i}}) a_{n,\mathbf{i}} \chi_{A_k \cap A_{n,\mathbf{i}} \cap H_{q_n}}(\omega_k) h(\omega_k) \right)
\end{aligned}$$

$$
\begin{aligned}
&= \mu(F_n)^{-2} \sum_{k=M_n}^{N_n} b_k^2 \operatorname{Var}\left(\sum_{\mathbf{i}\in\Sigma_n'} \mu(A_{n,\mathbf{i}}) a_{n,\mathbf{i}} \chi_{A_k\cap A_{n,\mathbf{i}}\cap H_{q_n}}(\omega_k) h(\omega_k) \right) \\
&\leqslant \|h\|_\infty^2 \mu(F_n)^{-2} \sum_{k=M_n}^{N_n} b_k^2 \,\mathbb{E}\left(\left(\sum_{\mathbf{i}\in\Sigma_n'} \mu(A_{n,\mathbf{i}}) a_{n,\mathbf{i}} \chi_{A_k\cap A_{n,\mathbf{i}}}(\omega_k) \right)^2 \right) \\
&= \|h\|_\infty^2 \mu(F_n)^{-2} \sum_{k=M_n}^{N_n} b_k^2 \sum_{\mathbf{j}\in\Sigma_n'} \int_{A_{n,\mathbf{j}}} \left(\sum_{\mathbf{i}\in\Sigma_n'} \mu(A_{n,\mathbf{i}}) a_{n,\mathbf{i}} \chi_{A_k\cap A_{n,\mathbf{i}}}(\omega_k) \right)^2 d\mu(\omega_k) \\
&= \|h\|_\infty^2 \mu(F_n)^{-2} \sum_{k=M_n}^{N_n} b_k^2 \sum_{\mathbf{j}\in\Sigma_n'} \int_{A_{n,\mathbf{j}}} \mu(A_{n,\mathbf{j}})^2 a_{n,\mathbf{j}}^2 \chi_{A_k\cap A_{n,\mathbf{j}}}(\omega_k)\, d\mu(\omega_k) \\
&= \|h\|_\infty^2 \mu(F_n)^{-2} \sum_{\mathbf{j}\in\Sigma_n'} \mu(A_{n,\mathbf{j}})^2 a_{n,\mathbf{j}}^2 \sum_{k=M_n}^{N_n} b_k^2\, \mu(A_k \cap A_{n,\mathbf{j}}) \\
&= \|h\|_\infty^2 \mu(F_n)^{-2} \sum_{\mathbf{j}\in\Sigma_n'} \mu(A_{n,\mathbf{j}}) c_{n,\mathbf{j}}^{-1} a_{n,\mathbf{j}} \sum_{k=M_n}^{N_n} b_k^2\, \mu(A_k \cap A_{n,\mathbf{j}}) \\
&\leqslant \|h\|_\infty^2 b 2^{-n} \mu(F_n)^{-2} \sum_{\mathbf{j}\in\Sigma_n'} \mu(A_{n,\mathbf{j}}) a_{n,\mathbf{j}} \sum_{k=M_n}^{N_n} b_k\, \mu(A_k \cap A_{n,\mathbf{j}}) \\
&= \|h\|_\infty^2 b 2^{-n} \mu(F_n)^{-1}.
\end{aligned}
$$

Thus,

$$
\sum_{n=1}^{\infty} \operatorname{Var}(\mu_n^\omega(h)) < \infty.
$$

Given $\delta > 0$, we have by Chebyshev's inequality that

$$
\sum_{n=1}^{\infty} \mathbb{P}\left\{ |\mu_n^\omega(h) - \mu_{|_G}(h)| \geqslant \delta \right\} \leqslant \sum_{n=1}^{\infty} \delta^{-2} \operatorname{Var}(\mu_n^\omega(h)) < \infty.
$$

Hence, by Borel–Cantelli lemma,

$$
\limsup_{n\to\infty} |\mu_n^\omega(h) - \mu_{|_G}(h)| \leqslant \delta
$$

almost surely and, thus, $\lim_{n\to\infty} \mu_n^\omega(h) = \mu_{|_G}(h)$ almost surely. Since $C_0(\mathbb{R}^d)$ is separable, $\mu_n^\omega \xrightarrow[n\to\infty]{w\text{-}*} \mu_{|_G}$ almost surely. This completes the proof of Lemma 4.2. □

Proof of Theorem 2.8 continued. Since $\mu_n^\omega \xrightarrow[n\to\infty]{w\text{-}*} \mu_{|_G}$ almost surely, $\lim_{k\to\infty} r_k = 0$ and $\psi_k^\omega\, d\mu_{|_G}$ is a probability measure for $\omega_k \in H_{q_n}$ (see Equation (4.13)), we have that $\varphi_n^\omega\, d\mu_{|_G} \xrightarrow[n\to\infty]{w\text{-}*} \mu_{|_G}$

almost surely. Let $\rho \in C_0(\mathbb{R}^d)$ be nonnegative. In order to apply Lemma 4.1, we will estimate the t-energies of the random measures $\varphi_n^\omega \rho \, d\mu_{|_G}$.

Observe first that

$$\mathbb{E}\Big(I_t\left(\varphi_n^\omega \rho \, d\mu_{|_G}\right)\Big) = S_1 + S_2, \tag{4.15}$$

where

$$\begin{aligned} S_1 &:= \mu(F_n)^{-2} \sum_{k=M_n}^{N_n} b_k^2 \sum_{\mathbf{i},\mathbf{j} \in \Sigma_n'} \mu(A_{n,\mathbf{i}})\mu(A_{n,\mathbf{j}}) a_{n,\mathbf{i}} a_{n,\mathbf{j}} \\ &\qquad \times \mathbb{E}\Big(\chi_{A_k \cap A_{n,\mathbf{i}} \cap H_{q_n}}(\omega_k)\chi_{A_k \cap A_{n,\mathbf{j}} \cap H_{q_n}}(\omega_k) I_t\left(\psi_k^\omega \rho \, d\mu_{|_G}\right)\Big) \text{ and} \\ S_2 &:= \mu(F_n)^{-2} \sum_{\substack{l,m=M_n \\ l \neq m}}^{N_n} b_l \, b_m \sum_{\mathbf{i},\mathbf{j} \in \Sigma_n'} \mu(A_{n,\mathbf{i}})\mu(A_{n,\mathbf{j}}) a_{n,\mathbf{i}} a_{n,\mathbf{j}} \\ &\qquad \times \mathbb{E}\Big(\chi_{A_l \cap A_{n,\mathbf{i}} \cap H_{q_n}}(\omega_l)\chi_{A_m \cap A_{n,\mathbf{j}} \cap H_{q_n}}(\omega_m) J_t\left(\psi_l^\omega \rho \, d\mu_{|_G}, \psi_m^\omega \rho \, d\mu_{|_G}\right)\Big). \end{aligned}$$

We first estimate S_1. By Equation (4.13), we have for $\omega_k \in A_k \cap H_{q_n}$ that

$$I_t\left(\psi_k^\omega \rho \, d\mu_{|_G}\right) \leqslant 4\|\rho\|_\infty^2 I_t\left(\frac{\chi_{B_k^\omega}}{\mu\left(G \cap B_k^\omega\right)} \, d\mu_{|_G}\right) \leqslant 4\|\rho\|_\infty^2 \left(1 - \frac{1}{q_n}\right)^{-2} b_k^{-1}.$$

Thus, using the fact that the sets $A_{n,\mathbf{i}}$ are disjoint, Equations (4.5), (4.4), and (4.2), we obtain

$$\begin{aligned} S_1 &\leqslant 4\|\rho\|_\infty^2 \left(1 - \frac{1}{q_n}\right)^{-2} \mu(F_n)^{-2} \sum_{k=M_n}^{N_n} b_k^2 \sum_{\mathbf{i},\mathbf{j} \in \Sigma_n'} \mu(A_{n,\mathbf{i}})\mu(A_{n,\mathbf{j}}) a_{n,\mathbf{i}} a_{n,\mathbf{j}} \\ &\quad \times \mathbb{E}\Big(\chi_{A_k \cap A_{n,\mathbf{i}}}(\omega_k)\chi_{A_k \cap A_{n,\mathbf{j}}}(\omega_k) b_k^{-1}\Big) \\ &= 4\|\rho\|_\infty^2 \left(1 - \frac{1}{q_n}\right)^{-2} \mu(F_n)^{-2} \sum_{k=M_n}^{N_n} b_k \sum_{\mathbf{i} \in \Sigma_n'} \mu(A_{n,\mathbf{i}})^2 a_{n,\mathbf{i}}^2 \, \mu\big(A_k \cap A_{n,\mathbf{i}}\big) \\ &= 4\|\rho\|_\infty^2 \left(1 - \frac{1}{q_n}\right)^{-2} \mu(F_n)^{-2} \sum_{\mathbf{i} \in \Sigma_n'} \mu(A_{n,\mathbf{i}}) c_{n,\mathbf{i}}^{-1} a_{n,\mathbf{i}} \sum_{k=M_n}^{N_n} b_k \, \mu\big(A_k \cap A_{n,\mathbf{i}}\big) \\ &\leqslant 4\|\rho\|_\infty^2 \left(1 - \frac{1}{q_n}\right)^{-2} \mu(F_n)^{-1} 2^{-n} \xrightarrow[n \to \infty]{} 0. \end{aligned}$$

To estimate S_2, let $\delta > 0$ and let $n \in \mathbb{N}$ be large enough so that

$$\rho(x)\rho(y) \leqslant \rho(u)\rho(v) + \delta \tag{4.16}$$

whenever $x, y, u, v \in X$ are such that $\max\{|x-u|, |y-v|\} \leqslant r_{M_n}$. Let $\beta := 2^{-n}$ so that $r_{M_n} < \frac{\beta^2}{2}$ (recall Equation (4.1)). We now write

$$\mathbb{E}\Big(\chi_{A_l \cap A_{n,\mathbf{i}} \cap H_{q_n}}(\omega_l)\chi_{A_m \cap A_{n,\mathbf{j}} \cap H_{q_n}}(\omega_m) J_t\left(\psi_l^\omega \rho\, d\mu_{|_G}, \psi_m^\omega \rho\, d\mu_{|_G}\right)\Big) = E_1(l,m,\mathbf{i},\mathbf{j}) + E_2(l,m,\mathbf{i},\mathbf{j}),$$

where

$$E_1(l,m,\mathbf{i},\mathbf{j}) := \int\int_{|\omega_l - \omega_m| \geqslant \beta} \chi_{A_l \cap A_{n,\mathbf{i}} \cap H_{q_n}}(\omega_l)\chi_{A_m \cap A_{n,\mathbf{j}} \cap H_{q_n}}(\omega_m) \times J_t(\psi_l^\omega \rho\, d\mu_{|_G}, \psi_m^\omega \rho\, d\mu_{|_G})\, d\mu(\omega_l)\, d\mu(\omega_m) \text{ and} \tag{4.17}$$

$$E_2(l,m,\mathbf{i},\mathbf{j}) := \int\int_{|\omega_l - \omega_m| < \beta} \chi_{A_l \cap A_{n,\mathbf{i}} \cap H_{q_n}}(\omega_l)\chi_{A_m \cap A_{n,\mathbf{j}} \cap H_{q_n}}(\omega_m) \times J_t(\psi_l^\omega \rho\, d\mu_{|_G}, \psi_m^\omega \rho\, d\mu_{|_G})\, d\mu(\omega_l)\, d\mu(\omega_m). \tag{4.18}$$

We begin by estimating $E_1(l,m,\mathbf{i},\mathbf{j})$ for $l \neq m$ (or more precisely, the contribution of the E_1-terms to S_2). Observe that if $|\omega_l - \omega_m| \geqslant \beta$, then

$$|x-y| \geqslant (1-\beta)|\omega_l - \omega_m|$$

for $x \in B_l^\omega$ and $y \in B_m^\omega$. Thus, recalling Equations (4.13), (4.16), the facts that $H_{q_n} \subset G$ and $\psi_k^\omega\, d\mu_{|_G}$ is a probability measure and Equation (4.6), we obtain

$$\begin{aligned}
&\mu(F_n)^{-2} \sum_{\substack{l,m=M_n \\ l\neq m}}^{N_n} b_l\, b_m \sum_{\mathbf{i},\mathbf{j}\in\Sigma_n'} \mu(A_{n,\mathbf{i}})\mu(A_{n,\mathbf{j}}) a_{n,\mathbf{i}} a_{n,\mathbf{j}} E_1(l,m,\mathbf{i},\mathbf{j}) \\
&\leqslant (1-\beta)^{-t}\mu(F_n)^{-2} \sum_{\substack{l,m=M_n \\ l\neq m}}^{N_n} b_l\, b_m \sum_{\mathbf{i},\mathbf{j}\in\Sigma_n'} \mu(A_{n,\mathbf{i}})\mu(A_{n,\mathbf{j}}) a_{n,\mathbf{i}} a_{n,\mathbf{j}} \\
&\quad \times \int\int_{|\omega_l-\omega_m|\geqslant\beta} \chi_{A_l \cap A_{n,\mathbf{i}} \cap H_{q_n}}(\omega_l)\chi_{A_m \cap A_{n,\mathbf{j}} \cap H_{q_n}}(\omega_m)|\omega_l-\omega_m|^{-t} \\
&\quad \times (\rho(\omega_l)\rho(\omega_m)+\delta) \int\int \psi_l^\omega(x)\psi_m^\omega(y)\, d\mu_{|_G}(x)\, d\mu_{|_G}(y)\, d\mu(\omega_l)\, d\mu(\omega_m) \\
&\leqslant (1-\beta)^{-t}\mu(F_n)^{-2} \sum_{l,m=M_n}^{N_n} b_l\, b_m \sum_{\mathbf{i},\mathbf{j}\in\Sigma_n'} \mu(A_{n,\mathbf{i}})\mu(A_{n,\mathbf{j}}) a_{n,\mathbf{i}} a_{n,\mathbf{j}} \\
&\quad \times \int_G\int_G \chi_{A_l\cap A_{n,\mathbf{i}}}(\omega_l)\chi_{A_m\cap A_{n,\mathbf{j}}}(\omega_m)|\omega_l-\omega_m|^{-t}(\rho(\omega_l)\rho(\omega_m)+\delta)\, d\mu(\omega_l)\, d\mu(\omega_m) \\
&= (1-\beta)^{-t}\mu(F_n)^{-2}\int_G\int_G |z-v|^{-t}(\rho(z)\rho(v)+\delta)\, d\mu_{|_{F_n}}(z)\, d\mu_{|_{F_n}}(v) \\
&\leqslant (1-\beta)^{-t}\mu(F_n)^{-2}(I_t(\rho\, d\,\mu_{|_G}) + \delta I_t(\mu_{|_G})).
\end{aligned} \tag{4.19}$$

Next, we estimate the E_2-terms. To this end, for $x \in G$, let

$$\widetilde{E}_n(x) := \mu(F_n)^{-1} \sum_{l=M_n}^{N_n} b_l \sum_{\mathbf{i}\in\Sigma_n'} \mu(A_{n,\mathbf{i}}) a_{n,\mathbf{i}} \int_{A_l\cap A_{n,\mathbf{i}}\cap H_{q_n}} \psi_l^\omega(x)\, d\mu(\omega_l).$$

Note that $\widetilde{E}_n(x)$ is a Borel function. Recalling Equations (4.13), (4.10), (4.7), and (4.8) and the fact $\chi_{B(\omega_l,r_l)}(x) = \chi_{B(x,r_l)(\omega_l)}$, we estimate, for $x \in G$ and $n \geqslant N$,

$$\begin{aligned}\widetilde{E}_n(x) &\leqslant 2\left(1-\frac{1}{q_n}\right)^{-1} \mu(F_n)^{-1} \sum_{l=M_n}^{N_n} b_l \sum_{\mathbf{i}\in\Sigma_n'} \mu(A_{n,\mathbf{i}}) a_{n,\mathbf{i}} \int_{A_l\cap A_{n,\mathbf{i}}\cap H_{q_n}} \frac{\chi_{B_l^\omega}(x)}{\mu(B_l^\omega)}\, d\mu(\omega_l)\\ &\leqslant 2\left(1-\frac{1}{q_n}\right)^{-1} E_n(x) \leqslant 2\left(1-\frac{1}{q_n}\right)^{-1} n^2. \qquad (4.20)\end{aligned}$$

Observe that if $|\omega_l - \omega_m| < \beta$, $x \in B_l^\omega$ and $y \in B_m^\omega$, then $|x-y| < 2\beta$, since $r_{M_n} < \beta^2/2$ and the sequence $(r_k)_{k=1}^\infty$ is decreasing. Thus, by Equation (4.20) and Lemma 3.2,

$$\begin{aligned}&\mu(F_n)^{-2} \sum_{\substack{l,m=M_n\\ l\neq m}}^{N_n} b_l\, b_m \sum_{\mathbf{i},\mathbf{j}\in\Sigma_n'} \mu(A_{n,\mathbf{i}})\mu(A_{n,\mathbf{j}}) a_{n,\mathbf{i}} a_{n,\mathbf{j}} E_2(l,m,\mathbf{i},\mathbf{j})\\ &\leqslant \|\rho\|_\infty^2 \mu(F_n)^{-2} \sum_{\substack{l,m=M_n\\ l\neq m}}^{N_n} b_l\, b_m \sum_{\mathbf{i},\mathbf{j}\in\Sigma_n'} \mu(A_{n,\mathbf{i}})\mu(A_{n,\mathbf{j}}) a_{n,\mathbf{i}} a_{n,\mathbf{j}}\\ &\quad\times \int\int_{|\omega_l-\omega_m|<\beta} \chi_{A_l\cap A_{n,\mathbf{i}}\cap H_{q_n}}(\omega_l)\chi_{A_m\cap A_{n,\mathbf{j}}\cap H_{q_n}}(\omega_m)\\ &\quad\times \int\int_{|x-y|<2\beta} |x-y|^{-t}\psi_l^\omega(x)\psi_m^\omega(y)\, d\mu_{|_G}(x)\, d\mu_{|_G}(y)\, d\mu(\omega_l)\, d\mu(\omega_m)\\ &\leqslant \|\rho\|_\infty^2 \int\int_{|x-y|<2\beta} |x-y|^{-t}\widetilde{E}_n(x)\widetilde{E}_n(y)\, d\mu_{|_G}(x)\, d\mu_{|_G}(y)\\ &\leqslant 4\|\rho\|_\infty^2 \left(1-\frac{1}{q_n}\right)^{-2} n^4 \int\int_{|x-y|<2\beta} |x-y|^{-t}\, d\mu(x)\, d\mu(y)\\ &\leqslant C_{s,t}\|\rho\|_\infty^2 n^4 \beta^{s-t}. \qquad (4.21)\end{aligned}$$

Recalling that $\beta = 2^{-n}$, we obtain by combining the estimates from Equations (4.19), (4.21), and (4.1) that

$$S_2 \leqslant (1-2^{-n})^{-t}\mu(F_n)^{-2}(I_t(\rho\, d\,\mu_{|_G}) + \delta I_t(\mu_{|_G})) + C_{s,t}\|\rho\|_\infty^2 n^4\, 2^{-n(s-t)}.$$

Letting $\delta \to 0$ (and thus also $n \to \infty$) yields

$$\limsup_{n\to\infty} \mathbb{E}\Big(I_t\left(\varphi_n^\omega \rho\, d\mu_{|_G}\right)\Big) \leqslant I_t(\rho\, d\mu_{|_G}). \qquad (4.22)$$

Thus, by Lemma 3.3,

$$\liminf_{n\to\infty} I_t\Big(\varphi_n^\omega \rho\, d\mu_{|_G}\Big) \leqslant I_t(\rho\, d\mu_{|_G})$$

almost surely. Since $\rho \in C_0(\mathbb{R}^d)$ was arbitrary, we obtain that

$$\mathbb{P}(\Omega_0) := \mathbb{P}\Big(\Big\{\liminf_{n\to\infty} I_t\Big(\varphi_n^\omega \rho_i\, d\mu_{|_G}\Big) \leqslant I_t(\rho_i\, d\mu_{|_G}) \text{ for all } i \in \mathbb{N}\Big\}\Big) = 1,$$

where $\{\rho_i\}_{i\in\mathbb{N}}$ is a dense subset of the separable space $C_0(\mathbb{R}^d)$ with the property that $\rho_j + q \in \{\rho_i\}_{i\in\mathbb{N}}$ for every $j \in \mathbb{N}$ and for every nonnegative $q \in \mathbb{Q}$. Let $\omega \in \Omega_0$, $\varepsilon_0 > 0$ and let ρ be any finite product of the functions $\{\varphi_n^\omega\}_{n\in\mathbb{N}}$. Then, there is $i \in \mathbb{N}$ such that $\rho_i \geqslant \rho$ and $\|\rho - \rho_i\|_\infty < \varepsilon_0$. Thus,

$$\begin{aligned}\liminf_{n\to\infty} I_t\Big(\varphi_n^\omega \rho\, d\mu_{|_G}\Big) &\leqslant \liminf_{n\to\infty} I_t\Big(\varphi_n^\omega \rho_i\, d\mu_{|_G}\Big) \leqslant I_t(\rho_i\, d\mu_{|_G}) \\ &\leqslant I_t(\rho\, d\mu_{|_G}) + \big(2\varepsilon_0\|\rho\|_\infty + \varepsilon_0^2\big) I_t(\mu_{|_G}).\end{aligned}$$

Since $\varepsilon_0 > 0$ was arbitrary and $I_t(\mu_{|_G}) < \infty$, we obtain that

$$\liminf_{n\to\infty} I_t\Big(\varphi_n^\omega \rho\, d\mu_{|_G}\Big) \leqslant I_t(\rho\, d\mu_{|_G}).$$

Since $\omega \in \Omega_0$ was arbitrary, we have that the assumptions of Lemma 4.1 are satisfied almost surely. By Equations (4.14) and (4.13), we have that $\operatorname{spt}\varphi_n^\omega \subset \bigcup_{k=M_n}^{N_n} B(\omega_k, r_k)$, implying that $E_{\underline{r}}(\omega) \supset \limsup_{n\to\infty}(\operatorname{spt}\varphi_n^\omega)$. Hence, by Lemma 4.1,

$$\operatorname{Cap}_t\Big(E_{\underline{r}}(\omega)\Big) \geqslant \frac{\mu(G)^2}{I_t(\mu_{|_G})} > 0$$

almost surely, which implies that $\dim_{\mathrm{H}}\Big(E_{\underline{r}}(\omega)\Big) \geqslant t$ almost surely. □

The following corollary of Theorem 2.8 will be convenient for us in the proofs of Theorems 2.3 and 2.5.

Corollary 4.3. *Let $\mu \in \mathcal{P}(\mathbb{R}^d)$ be s-Frostman. Let $(r_k)_{k=1}^\infty$ be a decreasing sequence of positive numbers tending to zero. For $u > 0$, let*

$$\widehat{A}_k^u := \big\{x \in \operatorname{spt}\mu \mid \mu(B(x, r_k)) \geqslant r_k^u\big\}. \tag{4.23}$$

If for some $u > 0$ and $0 < t < s$ it is true that

$$\mu\left(\left\{x \in \operatorname{spt}\mu \mid \sum_{k=1}^\infty \chi_{\widehat{A}_k^u}(x) r_k^{\frac{tu}{s}} = \infty\right\}\right) = 1, \tag{4.24}$$

then $f_\mu(\underline{r}) \geqslant t$.

Proof. Suppose that μ is (C, s)-Frostman. For $x \in \widehat{A}_k^u$, Lemma 3.2 yields

$$I_t(\mu_{B(x,r_k)}) \leqslant \frac{C^{\frac{t}{s}} s}{s-t}\mu(B(x, r_k))^{-\frac{t}{s}} \leqslant \frac{C^{\frac{t}{s}} s}{s-t} r_k^{-\frac{tu}{s}},$$

hence the assumptions of Theorem 2.8 are satisfied with

$$b_k = \left(\frac{C^{\frac{t}{s}} s}{s-t}\right)^{-1} r_k^{\frac{tu}{s}}.$$

Thus, $f_\mu(\underline{r}) \geqslant t$. □

5 | PROOF OF THEOREM 2.3

To prove Theorem 2.3, we need the following lemma.

Lemma 5.1. *Let $\mu \in \mathcal{P}(\mathbb{R}^d)$, $\alpha > 0$ and let $s \geqslant \overline{\dim}_{\mathrm{H}}\,\mu$. Write $r_k := k^{-\alpha}$ for all $k \in \mathbb{N}$. Then for every $\varepsilon > 0$ and $t < \frac{1}{\alpha}$, we have that*

$$\mu\left(\left\{x \in \operatorname{spt}\mu \mid \sum_{k=1}^{\infty} r_k^t \chi_{\widehat{A}_k^{s+\varepsilon}}(x) = \infty\right\}\right) = 1.$$

Proof. Since $s \geqslant \overline{\dim}_{\mathrm{H}}\,\mu$, we have that $\underline{\dim}_{\mathrm{loc}}\mu(x) \leqslant s$ for μ-almost every $x \in X$. Fix such a point x. For every $\varepsilon > 0$, define the set

$$A^{s+\varepsilon}(x) := \{k \in \mathbb{N} \mid \mu(B(x, r_k)) \geqslant r_k^{s+\varepsilon}\}.$$

Since $r_k = k^{-\alpha}$, it is true that

$$\underline{\dim}_{\mathrm{loc}}\mu(x) = \liminf_{k\to\infty} \frac{\log \mu(B(x, r_k))}{\log r_k}.$$

Thus, the number of elements in $A^{s+\varepsilon}(x)$ is infinite for every $\varepsilon > 0$. Observe now that if $k \in A^{s+\varepsilon}(x)$ and $l < k$ is such that $l \notin A^{s+2\varepsilon}(x)$, then

$$k^{-\alpha(s+\varepsilon)} = r_k^{s+\varepsilon} \leqslant \mu(B(x, r_k)) \leqslant \mu(B(x, r_l)) < r_l^{s+2\varepsilon} = l^{-\alpha(s+2\varepsilon)}.$$

Therefore, $k > l^\gamma$ where $\gamma := 1 + \frac{\varepsilon}{s+\varepsilon} > 1$. Hence, we find arbitrarily large natural numbers k such that $\{\lceil k^{1/\gamma}\rceil, \lceil k^{1/\gamma}\rceil + 1, \dots, k\} \subset A^{s+2\varepsilon}(x)$. Thus, for any $t < \frac{1}{\alpha}$,

$$\sum_{k=1}^{\infty} \chi_{\widehat{A}_k^{s+2\varepsilon}}(x) r_k^t = \sum_{k \in A^{s+2\varepsilon}(x)} r_k^t \geqslant \liminf_{k\to\infty} \sum_{j=\lceil k^{1/\gamma}\rceil}^{k} r_j^t \geqslant \liminf_{k\to\infty} \sum_{j=\frac{k}{2}}^{k} r_j^t$$

$$\geqslant \liminf_{k\to\infty} \frac{k}{2} k^{-\alpha t} = \infty,$$

since $1 - \alpha t > 0$. Since $\varepsilon > 0$ was arbitrary, the claim follows. □

Proof of Theorem 2.3. Assume first that $\frac{1}{\alpha} < \overline{\dim}_{\mathrm{H}}\,\mu$. Fix $0 < t < \frac{1}{\alpha}$. Our aim is to show that $f_\mu(\alpha) \geqslant t$. Write $s := \overline{\dim}_{\mathrm{H}}\,\mu$ and choose $\varepsilon > 0$ small enough so that $\frac{1}{\alpha} < s - \varepsilon$ and

$$t\,\frac{s+\varepsilon}{s-\varepsilon} < \frac{1}{\alpha}. \tag{5.1}$$

By definition of s, we can choose $C > 0$ such that $\mu(D) > 0$, where

$$D := \{x \in \operatorname{spt}\mu \mid \mu(B(x,r)) \leqslant C r^{s-\varepsilon} \text{ for all } r > 0\}.$$

Applying Lemma 3.5 with $\mu = \mu_D$ and $\nu = \mu$, we conclude that it suffices to show that $f_{\mu_D}(\alpha) \geqslant t$. Observe that there exists $C' > 0$ such that

$$\mu_D(B(x,r)) \leqslant C' r^{s-\varepsilon} \tag{5.2}$$

for every $x \in \mathbb{R}^d$ and every $r > 0$. By Lebesgue's density theorem, we have that

$$\underline{\dim}_{\mathrm{loc}}\mu_D(x) = \underline{\dim}_{\mathrm{loc}}\mu(x)$$

for μ_D-almost every $x \in \mathbb{R}^d$ and, thus, $\overline{\dim}_{\mathrm{H}}\,\mu_D \leqslant s$. Using Equation (5.1) and Lemma 5.1 for μ_D, we conclude that

$$\sum_{k=1}^{\infty} r_k^{t\frac{s+\varepsilon}{s-\varepsilon}} \chi_{\widehat{A}_k^{s+\varepsilon}}(x) = \infty$$

for μ_D-almost every $x \in \mathbb{R}^d$. Applying Corollary 4.3 for μ_D with $u = s + \varepsilon$ and s replaced by $s - \varepsilon$ (see Equation (5.2)) yields

$$f_{\mu_D}(\alpha) \geqslant t.$$

The claim is true also in the case $\frac{1}{\alpha} = \overline{\dim}_{\mathrm{H}}\,\mu$ since $E_\beta(\omega) \subset E_\alpha(\omega)$ for $\frac{1}{\beta} < \frac{1}{\alpha}$ and $f_\mu(\alpha) \leqslant \frac{1}{\alpha}$. □

6 | PROOF OF THEOREM 2.5

Proof of Theorem 2.5. Let $0 < \varepsilon < \overline{\delta}$. By the definition of $\overline{\delta}$, there exists a Borel set $H_0 \subset \operatorname{spt}\mu$ such that $\mu(H_0) > 0$ and, for every $x \in H_0$,

$$\frac{\underline{\dim}_{\mathrm{loc}}\mu(x)}{\overline{\dim}_{\mathrm{loc}}\mu(x)} > \overline{\delta} - \varepsilon \text{ and } \underline{\dim}_{\mathrm{loc}}\mu(x) > s_2(\underline{r}).$$

For every small enough $\gamma > 0$, there exists a Borel set $H_1 \subseteq H_0$ with $\mu(H_1) > 0$ such that $\underline{\dim}_{\mathrm{loc}}\mu(x) > s_2(\underline{r}) + \gamma$ for every $x \in H_1$. Further, for some $s \geqslant s_2(\underline{r})$, we find a Borel set $H_2 \subseteq H_1$ with $\mu(H_2) > 0$ such that $\underline{\dim}_{\mathrm{loc}}\mu(x) \in [s+\gamma, s+2\gamma]$ for all $x \in H_2$. Then, for every $x \in H_2$, we have that

$$s + \gamma \leqslant \underline{\dim}_{\mathrm{loc}}\mu(x) \leqslant \overline{\dim}_{\mathrm{loc}}\mu(x) \leqslant \frac{\underline{\dim}_{\mathrm{loc}}\mu(x)}{\overline{\delta} - \varepsilon} \leqslant \frac{s + 2\gamma}{\overline{\delta} - \varepsilon}.$$

Finally, there exist $r_0 > 0$ and a Borel set $H_3 \subseteq H_2$ with positive μ-measure such that

$$r^{\frac{s+3\gamma}{\overline{\delta}-\varepsilon}} \leqslant \mu(B(x,r)) \leqslant r^{s+\frac{\gamma}{2}}$$

for every $x \in H_3$ and for all $r \in]0, r_0]$. Note that, for any $x \in \mathbb{R}^d$ and $r > 0$, either $H_3 \cap B(x,r) = \emptyset$ or $H_3 \cap B(x,r) \subseteq B(z,2r)$ for some $z \in H_3$. Since μ is a probability measure,

$$\mu(B(x,r)) \leqslant 1 = r^{-(s+\frac{\gamma}{2})} r^{s+\frac{\gamma}{2}} \leqslant \left(\frac{r_0}{2}\right)^{-(s+\frac{\gamma}{2})} r^{s+\frac{\gamma}{2}}$$

for all $r \geqslant \frac{r_0}{2}$. Therefore, there exists a constant $C = C(s,\gamma,r_0,\mu(H_3))$ such that

$$\mu_{H_3}(B(x,r)) \leqslant C r^{s+\frac{\gamma}{2}}$$

for every $x \in \mathbb{R}^d$ and $r > 0$.

We will now apply Corollary 4.3 to μ_{H_3} with $u = \frac{s+4\gamma}{\overline{\delta}-\varepsilon}$ and s replaced by $s + \frac{\gamma}{2}$. By Lebesgue's density theorem,

$$\lim_{r\to 0} \frac{\mu(H_3 \cap B(x,r))}{\mu(B(x,r))} = 1$$

for μ-almost every $x \in H_3$, hence μ_{H_3}-almost every x belongs to $\widehat{A}_k^u$ (see Equation (4.23)) for every k large enough (depending on x). Now observe that if $t < s_2(\underline{r})(\overline{\delta} - \varepsilon)\frac{s+\frac{\gamma}{2}}{s+4\gamma}$, then

$$t \frac{s+4\gamma}{(\overline{\delta}-\varepsilon)\left(s+\frac{\gamma}{2}\right)} < s_2(\underline{r}).$$

Recalling Equation (2.1), we have that

$$\sum_{k=1}^{\infty} \chi_{\widehat{A}_k^u}(x) r_k^{t\frac{s+4\gamma}{(\overline{\delta}-\varepsilon)(s+\frac{\gamma}{2})}} = \infty$$

for μ_{H_3}-almost every $x \in \mathbb{R}^d$. By Corollary 4.3, $f_{\mu_{H_3}}(\underline{r}) \geqslant t$. Lemma 3.5 now yields $f_\mu(\underline{r}) \geqslant t$. From this we deduce that $f_\mu(\underline{r}) \geqslant s_2(\underline{r})(\overline{\delta} - \varepsilon)\frac{s+\frac{\gamma}{2}}{s+4\gamma}$ and letting $\gamma \to 0$ and $\varepsilon \to 0$ completes the proof. □

7 | SHARPNESS OF THE BOUNDS IN THEOREM 2.5

In this section, we provide examples which demonstrate that the bounds in Theorem 2.5 are sharp. In particular, these examples show that there is no formula for $f_\mu(\underline{r})$ involving only the quantity $s_2(\underline{r})$ and the local dimensions of the measure. For simplicity, we do our constructions in $\mathbb{R}$ but similar examples can be constructed also in $\mathbb{R}^d$.

We start with an example showing that it is possible that $f_\mu(\underline{r}) = s_2(\underline{r})$ for every sequence $\underline{r}$ with $s_2(\underline{r}) < \overline{\dim}_{\mathrm{H}}\mu$, no matter how small the quantity $\overline{\delta}$ is. In particular, this example shows that for some measures, the trivial upper bound in Theorem 2.5 is the correct value for the almost sure dimension for all sequences of radii.

Example 7.1. Let $0 < s < u < 1$ and let $0 < \alpha, \beta < \frac{1}{2}$ be such that $s = \frac{\log 2}{-\log \alpha}$ and $u = \frac{\log 2}{-\log \beta}$. Set $C_0 := [0,1]$. Let $(N_k)_{k=1}^{\infty}$ be a rapidly growing sequence of integers and let $C := \bigcap_{n=0}^{\infty} C_n$ be the Cantor set, where C_n is obtained from C_{n-1} by removing the middle $(1-2\alpha)$-interval from each construction interval of C_{n-1} for $N_{2k} \leqslant n < N_{2k+1}$ and by removing the middle $(1-2\beta)$-interval from each construction interval of C_{n-1} for $N_{2k+1} \leqslant n < N_{2k+2}$. Let μ be the natural Borel probability measure on C giving equal weight to all construction intervals at same level. We then have that

$$r^u \lesssim \mu(B(x,r)) \lesssim r^s \tag{7.1}$$

for all $x \in C$ and $0 < r < 1$, where the notation $g(r) \lesssim h(r)$ means that there exists a constant c such that $g(r) \leqslant ch(r)$. If the sequence $(N_k)_{k=1}^{\infty}$ grows fast enough, then $\underline{\dim}_{\mathrm{loc}}\mu(x) = s$ and $\overline{\dim}_{\mathrm{loc}}\mu(x) = u$ for all $x \in C$.

Our aim is to show that $f_\mu(\underline{r}) = s_2(\underline{r})$ for any sequence $\underline{r}$ with $s_2(\underline{r}) < s$. To this end, let $x_0 \in C$ and $0 < r_0 < 1$. Let I_0 be the largest construction interval contained in $C \cap B(x_0, r_0)$. Let n_0 be the level of I_0. Then, $\mu(I_0) \approx \mu(B(x_0, r_0))$, $|I_0| \approx r_0$ and

$$I_t(\mu_{B(x_0,r_0)}) \lesssim I_t(\mu_{I_0}) \tag{7.2}$$

for all $t > 0$. Here, $g \approx h$ means that $g \lesssim h$ and $h \lesssim g$. Let $0 < t < s$ and fix $x \in I_0$. Let $N > n_0$ be large and for $j \in \{n_0, \dots, N-1\}$, set

$$D_j(x) := \{y \in C \mid C_j(y) = C_j(x) \text{ and } C_{j+1}(y) \cap C_{j+1}(x) = \emptyset\},$$

where $C_j(y)$ denotes the unique construction interval of C_j containing y. Note that $D_j(x)$ is a construction interval of C_{j+1} and $I_0 = C_N(x) \cup \bigcup_{j=n_0}^{N-1} D_j(x)$ with the union disjoint. Let γ_j denote the length of the intervals removed from C_{j-1} in the construction to obtain C_j. Since $\alpha < \beta$, we have that

$$\operatorname{dist}(x, D_j(x)) \geqslant \gamma_{j+1} \geqslant (1-2\beta)\alpha^{j-n_0}|I_0| \tag{7.3}$$

for $j \in \{n_0, \dots, N-1\}$. Finally, let ℓ_n denote the length of the construction intervals of C_n. By Lemma 3.2, Equations (7.3) and (7.1), we obtain

$$\begin{aligned}
\phi^t_{\mu|_{I_0}}(x) &= \int_{I_0} |x-y|^{-t}\, d\mu(y) = \int_{C_N(x)} |x-y|^{-t}\, d\mu(y) + \sum_{j=n_0}^{N-1} \int_{D_j(x)} |x-y|^{-t}\, d\mu(y) \\
&\lesssim |C_N(x)|^{s-t} + \sum_{j=n_0}^{N-1} \mu(D_j(x))\gamma_{j+1}^{-t} = \ell_N^{s-t} + \sum_{j=n_0}^{N-1} 2^{n_0-j-1}\mu(I_0)\gamma_{j+1}^{-t} \\
&\leqslant \ell_N^{s-t} + \mu(I_0)(1-2\beta)^{-t}|I_0|^{-t} \sum_{j=n_0}^{N-1} 2^{n_0-j-1}\alpha^{t(n_0-j)} \\
&\leqslant \ell_N^{s-t} + \frac{1}{2}\mu(I_0)(1-2\beta)^{-t}|I_0|^{-t} \sum_{j=0}^{\infty} \left(\frac{\alpha^{-t}}{2}\right)^j.
\end{aligned}$$

Since $t < s = \frac{\log 2}{-\log \alpha}$, it is true that $\alpha^{-t} < 2$. Hence, the series above converges. Integrating over I_0 and normalising, we obtain for some constant $c_1 = c(\alpha, \beta, t)$ that

$$I_t(\mu_{I_0}) \leqslant c_1 \left(\mu(I_0)^{-1} \ell_N^{s-t} + |I_0|^{-t} \right).$$

Letting $N \to \infty$ yields $I_t(\mu_{I_0}) \leqslant c_1 |I_0|^{-t}$. Recalling Equation (7.2), we obtain the estimate $I_t(\mu_{B(x_0,r_0)}) \lesssim |I_0|^{-t} \approx r_0^{-t}$. If $\underline{r}$ is a sequence of positive numbers such that $s_2(\underline{r}) < s$ and $0 < t < s_2(\underline{r})$, then there exists a constant c_2 such that $I_t(\mu_{B(x,r_k)}) \leqslant c_2 r_k^{-t}$ for all $x \in C$ and $k \in \mathbb{N}$. Hence, Theorem 2.8 implies that $f_\mu(\underline{r}) \geqslant t$. Letting $t \to s_2(\underline{r})$ through a countable sequence then yields the claim $f_\mu(\underline{r}) = s_2(\underline{r})$.

In the next example, we construct a measure μ such that the quantity $\overline{\delta}$ can be made arbitrarily small, and for any $\gamma \in [\overline{\delta}, 1]$, there exists a sequence $\underline{r}$ of radii such that $f_\mu(\underline{r}) = s_2(\underline{r})\gamma$. In particular, this example shows that the lower bound in Theorem 2.5 can be attained.

Example 7.2. Let $0 < s < u < 1$. We will first construct a measure $\mu \in \mathcal{P}([0,1])$ such that $\underline{\dim}_{\mathrm{loc}}\mu(x) = s$ and $\overline{\dim}_{\mathrm{loc}}\mu(x) = u$ for every $x \in \operatorname{spt}\mu$. Fix some $\ell_0 \in]0,1[$ and define a sequence $(\ell_k)_{k=0}^\infty$ of positive numbers by setting $\ell_{k+1} := \ell_k^v$, where

$$v := \frac{u(1-s)}{s(1-u)}.$$

For every $k \in \mathbb{N} \setminus \{0\}$, let $L_k := \ell_k^{\frac{s}{u}} = \ell_{k-1}^{\frac{1-s}{1-u}}$. Note that $\ell_k < L_k < \ell_{k-1}$ for every $k \in \mathbb{N} \setminus \{0\}$ and $(l_k)_{k=0}^\infty$ tends to zero with super exponential speed. Set $C_0 := [0, \ell_0]$. Construct C_1 by partitioning C_0 into

$$N_1 := \left\lfloor \frac{\ell_0}{L_1} \right\rfloor$$

intervals of length L_1, and from each of these intervals, keep the leftmost segment of length ℓ_1 and discard the rest to obtain C_1. If C_k has been constructed and C_k is a disjoint union of $\prod_{j=1}^k N_j$ intervals of length ℓ_k, construct C_{k+1} by dividing each interval of C_k into $N_{k+1} := \lfloor \frac{\ell_k}{L_{k+1}} \rfloor$ intervals of length L_{k+1} and from each of these intervals keeping the leftmost segment of length ℓ_{k+1}. This way we obtain a decreasing sequence $(C_k)_{k=0}^\infty$ of compact sets with the properties that each C_k is a union of $\prod_{j=1}^k N_j$ intervals of length ℓ_k and these intervals are at least $(L_k - \ell_k)$-separated. Note that $L_k - \ell_k > \frac{L_k}{2}$ for all large $k \in \mathbb{N}$. Set $C := \bigcap_{k=0}^\infty C_k$. Let μ be the natural uniformly distributed Borel probability measure on C. Since $(l_k)_{k=0}^\infty$ tends to zero fast, $(N_k)_{k=1}^\infty$ tends to infinity fast. Therefore, the facts $N_k = \lfloor \frac{\ell_{k-1}}{L_k} \rfloor$ and

$$\frac{\ell_{k-1}}{L_k} \ell_k^s = \ell_{k-1}^{1+vs(1-\frac{1}{u})} = \ell_{k-1}^s$$

imply that $\mu(B(x,r)) \lesssim r^s$ for all $x \in \mathbb{R}$ and $r > 0$. Furthermore, for all $x \in C$, we have that

$$\mu(B(x, \tfrac{L_k}{2})) = \mu(B(x, \ell_k)) \approx \ell_k^s = L_k^u \tag{7.4}$$

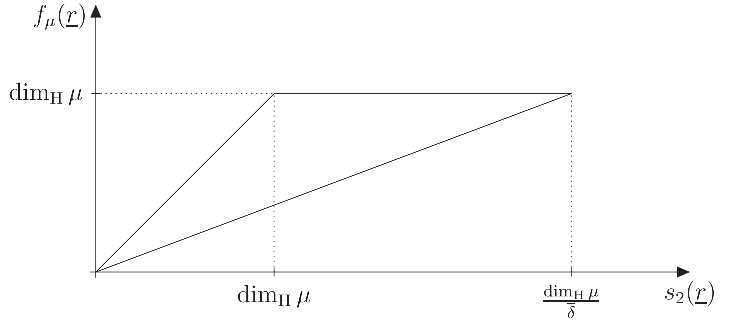

FIGURE 1 The almost sure dimension $f_\mu(\underline{r})$ depicted as a "function" of $s_2(\underline{r})$.

for all $k \in \mathbb{N}$, and $\mu(B(x,r)) \gtrsim r^u$ for all $0 < r < 1$. Thus, $\underline{\dim}_{\text{loc}}\mu(x) = s$ and $\overline{\dim}_{\text{loc}}\mu(x) = u$ for all $x \in C$ and, in particular, $\overline{\delta} = \frac{s}{u}$, where $\overline{\delta}$ is the quantity defined in Theorem 2.5.

In the following, we will demonstrate that by choosing a suitable sequence $\underline{r}$, the almost sure dimension $f_\mu(\underline{r})$ can take any value in the interval $[s_2(\underline{r})\frac{s}{u}, s_2(\underline{r})]$, that is, the possible pairs $(s_2(\underline{r}), f_\mu(\underline{r}))$ fill the entire triangle depicted in Figure 1. Let $\gamma \in [\frac{s}{u}, 1]$ and let $s_0 \in\,]0, \frac{s}{\gamma}]$. We will show that there exists a sequence $\underline{r}$ such that $s_2(\underline{r}) = s_0$ and $f_\mu(\underline{r}) = s_0\gamma$. Varying γ and s_0 through their allowed ranges fills the triangle in Figure 1.

Define a sequence $(M_j)_{j=1}^{\infty}$ of integers by setting

$$M_j := \left\lfloor \left(\frac{1}{\ell_j^{\gamma}}\right)^{s_0} \right\rfloor$$

for every $j \in \mathbb{N} \setminus \{0\}$. Set $M_0 := 0$. Consider the sequence $\underline{r}$, where $r_k := \frac{\ell_j^{\gamma}}{2}$ for all $k = M_{j-1} + 1, \dots, M_j$. Then, $s_2(\underline{r}) = s_0$, since

$$\sum_{n=1}^{\infty} r_n^t = \sum_{j=1}^{\infty} \sum_{k=M_{j-1}+1}^{M_j} r_k^t \approx \sum_{j=1}^{\infty} \ell_j^{\gamma(t-s_0)},$$

and this sum converges when $t > s_0$. Observe that since $\ell_j^{\gamma} \leqslant \ell_j^{s/u} = L_j$ and the construction intervals of C_j are at least $\frac{L_j}{2}$-separated, we have for any $x \in C$ and $M_{j-1} < k \leqslant M_j$ that

$$C \cap B(x, \tfrac{\ell_j}{2}) \subset C \cap B(x, r_k) \subset C \cap B(x, \ell_j) = C \cap B(x, (2r_k)^{\frac{1}{\gamma}}). \tag{7.5}$$

Thus, by denoting $\rho_k := (2r_k)^{\frac{1}{\gamma}}$, we have that

$$f_\mu(\underline{r}) \leqslant s_2(\underline{\rho}) = s_2(\underline{r})\gamma = s_0\gamma. \tag{7.6}$$

Next, we will show that $s_0\gamma \leqslant f_\mu(\underline{r})$. To this end, let $0 < t < s_0\gamma$. Recall that $s_0\gamma \leqslant s$ and that, for some constant $D > 0$, $\mu(B(x,r)) \leqslant Dr^s$ for all $x \in C$ and $r > 0$. In Equation (7.5), we saw that, for all $x \in C$, the ball $B(x, (2r_k)^{\frac{1}{\gamma}})$ contains exactly one construction interval I of C_j and

$\mu(B(x,r_k)) \geqslant \frac{1}{2}\mu(I)$. Therefore,

$$I_t(\mu_{B(x,r_k)}) \lesssim I_t\left(\mu_{B(x,(2r_k)^{\frac{1}{\gamma}})}\right).$$

Since $\mu(B(x,(2r_k)^{\frac{1}{\gamma}})) \approx l_j^s = ((2r_k)^{\frac{1}{\gamma}})^s$, Lemma 3.2 yields that

$$I_t\left(\mu_{B(x,(2r_k)^{\frac{1}{\gamma}})}\right) \lesssim r_k^{-\frac{t}{\gamma}}.$$

Note that

$$\sum_{k=1}^{\infty} r_k^{\frac{t}{\gamma}} = \infty$$

since $t < s_0\gamma$. Thus, by Theorem 2.8, we have that

$$f_\mu(\underline{r}) \geqslant t.$$

Letting $t \nearrow s_0\gamma$ through a countable sequence yields the desired lower bound.

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