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A criterion for uniqueness in G-measures and perfect sampling

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Abstract

Using coupling techniques, we prove uniqueness in G-measures under a weak regularity condition and give estimates of the associated rates of convergence. We also show how to generate a random variable distributed according to the unique G-measure on cylinder sets for any fixed level of precision.

1. *Introduction*

Let $\{X_n\}_{n\leqslant 0}$ be a stochastic process on $\{1,\ldots,N\}$. We may define random variables

$$G_k(y) \coloneqq P(X_n = y_n,\ 0 \geqslant n \geqslant -k+1 \mid X_m = y_m,\ m \leqslant -k),$$

where $y = y_0 y_{-1} y_{-2}\cdots$ and $k \geqslant 1$. Also, a.s.,

$$G_k(y) = \Pi_{i=0}^{k-1} g_i(\theta^i y),$$

where θ denotes the shift map, and

$$g_i(y) = P(X_{-i} = y_0 \mid X_{-i+m} = y_m,\ m \leqslant -1). \tag{1·1}$$

This presents the measure defining $\{X_n\}_{n\leqslant 0}$ as a G-measure. If the set of functions $G = \{g_i\}_{i\geqslant 0}$, given by (1·1) uniquely specify this measure, then we say that there is a unique G-measure.

The concept of G-measures originates from Brown and Dooley [**2**] and is a generalisation of the notion of g-measure introduced by Keane [**8**]. Keane's work was based on the consideration of a g-measure as a shift invariant measure on an infinite product space, corresponding to the case when $\{X_n\}$ is stationary. (Note that $g = g_i$ is independent of i if $\{X_n\}$ is stationary.)

One of the key questions asked in Keane's paper is whether continuity and positivity of g was a sufficient condition for uniqueness, but this was disproved by Bramson and Kalikow [**3**], and by Quas [**10**] for circle continuous g-functions.

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The question of sufficient conditions for uniqueness in g-measures has a long history within the theory of "chains with complete connections" in the case $\{X_n\}$ is assumed to be stationary, with Doeblin and Fortet [**5**] and Harris [**7**] containing some of the most important early results.

The notion of g-measure corresponds to the idea of an equilibrium state in statistical mechanics in the special case of a normalized potential. It is not known whether the well known conditions of Hölder-continuity and "summable variations" for uniqueness in equilibrium-states for general potentials, see Bowen [**1**] and Walters [**13**] respectively, can be relaxed to the corresponding best known conditions for non-normalized potentials.

In recent work, Dooley and Hamachi [**6**] showed that every non-singular ergodic dynamical system is orbit equivalent to a Markov odometer with a unique G-measure. Therefore it is of heightened interest to give the best possible conditions for uniqueness.

Keane proved uniqueness in g-measures for strictly positive differentiable g-functions, unaware of the already existing weaker conditions for uniqueness by Doeblin and Fortet [**5**] of summable variations and the even weaker condition by Harris [**7**]. In Brown and Dooley [**2**], sufficient conditions were given for uniqueness in G-measures, generalising those of Keane. As with Keane's conditions, it is clear that these conditions are not necessary.

In this paper, we shall generalise the coupling ideas of Harris [**7**] for proving uniqueness in g-measures to the case of G-measures, showing how the coupling method work in this more general case. In the next section, we give the definitions of our basic notions, and a precise statement of the results. The basic theorem (Theorem 1) is proved in Section 3. As a consequence of our method of proving Theorem 1, we are able to give a perfect sampling algorithm in Theorem 2.

2. *Preliminaries and statements of the results*

Let $\{N(j)\}_{j=0}^{\infty}$ be a sequence of positive integers, and let $\Sigma_n := \Pi_{j=n}^{\infty}\{1,2,\ldots,N(j)\}$ be a sequence of spaces. For each n, introduce a topology on Σ_n by the metric

$$\rho(x,y) := \begin{cases} 2^{-j}, & \text{if } x \text{ and } y \text{ differ for the first time in the} \\ & j\text{th digit} \\ 0, & \text{if } x = y. \end{cases} \tag{2·1}$$

The spaces (Σ_n, ρ) are compact metric spaces.

For $j \in \{1,2,\ldots,N(n)\}$ and $x = x_1x_2\cdots \in \Sigma_{n+1}$, let jx be the element in Σ_n defined by $jx = jx_1x_2\cdots$. Consider a family $\{g_n\}_{n=0}^{\infty}$, $g_n : \Sigma_n \to (0,\infty)$, of continuous functions, and suppose that the g_n's are normalised in the sense that

$$\sum_{j=1}^{N(n)} g_n(jx) = 1, \text{ for any } x \in \Sigma_{n+1}. \tag{2·2}$$

We call such a family $G := \{g_n\}$ a family of g-functions.

Let Γ_n denote the set of sequences $\gamma_0\gamma_1\cdots\gamma_{n-1}$ such that $\gamma_j \in \{1,2,\ldots,N(j)\}$, for any $0 \leqslant j \leqslant n-1$. For $\gamma = \gamma_0\cdots\gamma_{n-1} \in \Gamma_n$ and $x = x_0x_{-1}\cdots$ in Σ_0, let $\gamma(x) = \gamma_0\cdots\gamma_{n-1}x_{-n}x_{-(n+1)}\cdots$.

Definition 1. Let μ^n denote the measure $\sum_{\gamma\in\Gamma_n}\mu\circ\gamma$, and let $G_n(x)=\Pi_{i=0}^{n-1}g_i(\theta^i x)$, where θ denotes the shift map.

We say that a probability measure μ on Σ_0 is a G-**measure** if for any $n\geqslant 1$,

$$\frac{d\mu}{d\mu^n}(x) = G_n(x), \tag{2·3}$$

for μ-almost all $x\in\Sigma_0$.

In Brown and Dooley [**2**] it was shown that, provided the functions G_i are continuous, the existence of a unique G-measure is equivalent to the convergence (everywhere, or uniformly) of the sequence of functions

$$\lim_{n\to\infty}\sum_{\gamma\in\Gamma_n} f(\gamma(x))G_n(\gamma(x)) \tag{2·4}$$

for all $f\in C(\Sigma_0)$ and $x\in\Sigma_0$.

If (2·4) holds, then the limit is equal to $\int f d\mu$ for the unique G-measure. In Brown and Dooley [**2**] it was further shown that a unique G-measure is necessarily ergodic for the finite coordinate change action on Σ_0.

In the case when $N(j)$ is constant (say $=N$ for all j), we may identify Σ_n with Σ_0 via the shift map. If μ is shift invariant i.e. $\mu\circ\theta=\mu$ then all the functions g_i are identical under this identification, to a single function g, and we say that we have a g-measure. We are also interested in the case when there is a unique g-measure.

Notice that, in this setting, if there is a unique G-measure, there is necessarily a unique g-measure, since by definition if μ is a g-measure then μ is also a G-measure with $g_i=g$ for all i. However, the converse is not true. This is shown by the following.

Example 1. Let $N(j)=2$ for all j and define

$$g(x) = 1, \text{ if } x = 01* \quad\text{or } x = 10*$$

and $g(x)=0$ otherwise, where $*$ denotes an arbitrary ending of the infinite sequence x.

Let $x_0=010101\ldots.$ and $x_1=10101\ldots.$ It is not hard to see that the two Dirac measures δ_{x_0} and δ_{x_1} are both G-measures (associated with g), as is any convex combination of them. Thus we do not have uniqueness in G-measures. However, there is a unique g-measure (shift-invariant G-measure), *viz.* $(1/2)(\delta_{x_0}+\delta_{x_1})$.

In this paper we show that there is a unique G-measure provided that

$$\sum_{n=1}^{\infty}\prod_{m=1}^{n} cff_G(2^{-m}) = \infty, \tag{2·5}$$

where

$$cff_G(2^{-m}) = \inf_n \inf_{1\leqslant j_l\leqslant N(n+l),1\leqslant l\leqslant m-1}\sum_{i=1}^{N(n)}\inf_y g_n(ij_1\cdots j_{m-1}y).$$

The condition corresponding to (2·5) for uniqueness in g-measures was first considered by Comets et al. [**4**]. This condition is slightly stronger than the weakest known condition for uniqueness in g-measures of this type, see Stenflo [**12**], but has the advantage that it also gives the "uniform" convergence needed in our case.

THEOREM 1. *Let G be a family of g-functions satisfying condition* (2·5).

Let P denote the product Lebesgue measure on $((0,1)^{\mathbb{N}}, \mathcal{C})$, where $\mathcal{C}$ is the product Borel σ-field on $(0,1)^{\mathbb{N}}$.

Then, for any $x \in \Sigma_0$, we can construct a sequence of random variables $\{\hat{Z}_n(x)\}$, $\hat{Z}_n(x): (0,1)^{\mathbb{N}} \to \Sigma_0$ with $P(\hat{Z}_n(x) = \gamma(x)) = G_n(\gamma(x))$, such that $\hat{Z}_n(x) \to \hat{Z}$, P a.s., where $\hat{Z}: (0,1)^{\mathbb{N}} \to \Sigma_0$, is a random variable (independent of x). We have

$$E \sup_{x \in \Sigma_0} \rho(\hat{Z}_n(x), \hat{Z}) \leqslant E 2^{-Y_n},$$

where the metric ρ is defined in (2·1) *above, and $\{Y_n\}$ is a Markov chain with state space $\mathbb{N}$ starting at $Y_0 = 1$, with $P(Y_{n+1} = k+1 \mid Y_n = k) = cf f_G(2^{-k})$, and $P(Y_{n+1} = 1 \mid Y_n = k) = 1 - cf f_G(2^{-k})$, for any $k \geqslant 1$.*

An explicit definition of $\hat{Z}_n(x)$ $(= \hat{Z}_n(x, \omega))$ is given in the proof below. The random variables $\hat{Z}_n(x, \omega)$ only depend on the first n coordinates of $\omega \in (0,1)^{\mathbb{N}}$.

Remark 1. Note that $\{Y_n\}_{n=0}^{\infty}$ is a non-ergodic Markov chain under condition (2·5), see e.g. Prabhu [**9**, p. 80, example 18], so $E2^{-Y_n} \to 0$.

Define $\mu(\cdot) = P(\hat{Z} \in \cdot)$. As a consequence of Theorem 1 and the well known fact that almost sure convergence implies convergence in distribution, see e.g. Shiryaev [**11**], we obtain.

COROLLARY 1. *Let G be a family of g-functions satisfying condition* (2·5). *Then there exists a unique G-measure, μ, i.e.*

$$\lim_{n \to \infty} \sum_{\gamma \in \Gamma_n} f(\gamma(x)) G_n(\gamma(x)) = \int f d\mu \tag{2·6}$$

for all $f \in C(\Sigma_0)$ and $x \in \Sigma_0$.

Even though the definition of μ is implicit, we can correctly simulate μ-distributed random variables up to any specified degree of accuracy. The following theorem generalizes results from Comets et al. [**4**].

THEOREM 2 (Perfect sampling). *Let G be a family of g-functions satisfying condition* (2·5).

For $s \in (0,1)$, and integers $m \geqslant 1$, let $f_s(m) = m+1$, if $s < cf f_G(2^{-m})$, $f_s(m) = 1$ if $s \geqslant cf f_G(2^{-m})$.

Let $N_\star$ be an arbitrary fixed integer.

Algorithm: *generate independent, uniformly distributed random variables on the unit interval, $U_1, U_2, \ldots, U_T$, where the stopping time T is the first integer such that $f_{U_1} \circ \cdots \circ f_{U_T}(1) > N_\star$.*

Let μ be the unique G-measure. Then the first $N_\star$ (common) coordinates of $\hat{Z}_{N_\star}(x)$ (defined in the proof below) is a random variable taking value $(i_0, \ldots, i_{N_\star - 1})$ with probability $\mu([i_0, \ldots, i_{N_\star - 1}])$, for an arbitrary cylinder set $[i_0, \ldots, i_{N_\star - 1}] = \{x \in \Sigma_0 \colon x_j = i_j, 0 \leqslant j \leqslant N_\star - 1\}$, of length $N_\star$ in Σ_0.

3. *Proofs*

Fix an integer $n \geqslant 0$, and $\omega \in (0,1)^{\mathbb{N}}$. We first define the function $\hat{Z}_n : \Sigma_0 \times (0,1)^{\mathbb{N}} \to \Sigma_0$.

Intuitively, for $y = y_0 y_{-1} y_{-2} \cdots \in \Sigma_0$, $\hat{Z}_n(y,\omega)$ will correspond to the "history available" at time 0 of a realization ω of a stochastic sequence with conditional distributions prescribed by the family of g-functions G, if the "history available" at time $-n$ is fixed to be $y_{-n} y_{-n+1} \cdots$.

To make this intuition precise, let for $\omega = \omega_0\omega_1\omega_2 \cdots \in (0,1)^{\mathbb{N}}$, and $y \in \Sigma_0$,

$$\hat{Z}_n(y,\omega) := X_0^n(\theta^n y, \omega), \tag{3·1}$$

where $\{X_j^n(x,\omega)\}_{j=-n}^{0}$ is a sequence of functions $X_{-j}^n : \Sigma_n \times (0,1)^{\mathbb{N}} \to \Sigma_j$ defined recursively in the following way.

Let $X_{-n}^n(x,\omega) = x$, for any $x \in \Sigma_n$. Suppose that for some k_0, $X_{-(k_0+1)}^n(x,\omega)$ has already been defined. We then proceed to define $X_{-k_0}^n(x,\omega)$ as follows. Let $M = M(\omega)$ be the largest integer such that $X_{-(k_0+1)}^n(x,\omega)$ belongs to the cylinder set $\{i_1 i_2 \cdots i_M y : y \in \Sigma_{k_0+1+M}\}$, for some $i_j \in \{1,\ldots,N(k_0+j)\}$, $j = 1,\ldots,M$, and any $x \in \Sigma_n$.

For $1 \leqslant j \leqslant N(k_0)$, let

$$A_0(j) := \{s \in (0,1) : \sum_{i=1}^{j-1} \inf_y g_{k_0}(iy) \leqslant s < \sum_{i=1}^{j} \inf_y g_{k_0}(iy)\},$$

$$\begin{aligned} A_1(ji_1) := \{s \in (0,1) : &\sum_{i=1}^{N(k_0)} \inf_y g_{k_0}(iy) + \sum_{i=1}^{j-1}(\inf_y g_{k_0}(ii_1 y) - \inf_y g_{k_0}(iy)) \\ \leqslant s < &\sum_{i=1}^{N(k_0)} \inf_y g_{k_0}(iy) + \sum_{i=1}^{j}(\inf_y g_{k_0}(ii_1 y) - \inf_y g_{k_0}(iy))\}, \end{aligned}$$

and for $m \geqslant 2$,

$$\begin{aligned} A_m(ji_1\cdots i_m) := \{s \in (0,1) : &\sum_{i=1}^{N(k_0)} \inf_y g_{k_0}(ii_1 \cdots\cdot i_{m-1} y) \\ &+ \sum_{i=1}^{j-1}(\inf_y g_{k_0}(ii_1 \cdots\cdot i_m y) - \inf_y g_{k_0}(ii_1 \cdots i_{m-1} y)) \\ \leqslant s < &\sum_{i=1}^{N(k_0)} \inf_y g_{k_0}(ii_1 \cdots\cdot i_{m-1} y) \\ &+ \sum_{i=1}^{j}(\inf_y g_{k_0}(ii_1 \cdots i_m y) - \inf_y g_{k_0}(ii_1 \cdots i_{m-1} y))\}. \end{aligned}$$

Define $X_{-k_0}^n(x,\omega) = jX_{-(k_0+1)}^n(x,\omega)$, if $\omega_{k_0} \in \bigcup_{k=0}^{M} A_k(ji_1\cdots i_k)$, or

$$\begin{aligned} \omega_{k_0} \in \{s \in (0,1) : &\sum_{i=1}^{N(k_0)} \inf_y g_{k_0}(ii_1 \cdots\cdot i_M y) \\ &+ \sum_{i=1}^{j-1}(g_{k_0}(iX_{-(k_0+1)}^n(x,\omega)) - \inf_y g_{k_0}(ii_1 \cdots i_M y)) \\ \leqslant s < &\sum_{i=1}^{N(k_0)} \inf_y g_{k_0}(ii_1 \cdots\cdot i_M y) \\ &+ \sum_{i=1}^{j}(g_{k_0}(iX_{-(k_0+1)}^n(x,\omega)) - \inf_y g_{k_0}(ii_1 \cdots i_M y))\}. \end{aligned}$$

Let P be the product Lebesgue measure on $(0,1)^{\mathbb{N}}$. By construction

$$P(X^n_{-k_0}(x,\omega) = j\gamma_{k_0+1}\cdots\gamma_{n-1}x \mid X^n_{-(k_0+1)}(x,\omega)=\gamma_{k_0+1}\cdots\gamma_{n-1}x) = g_{k_0}(j\gamma_{k_0+1}\cdots\gamma_{n-1}x).$$

Thus $X^n_{-k}(x,\omega)$ can be viewed as random variables, for each fixed x, with

$$P(X^n_{-k}(x,\omega) = \gamma_k\cdots\gamma_{n-1}x) = \prod_{i=k}^{n-1} g_i(\gamma_i\cdots\gamma_{n-1}x).$$

(In the formula above we have calculated the probability on the left as a product of a finite collection of conditional probabilities.)

Recall the definition of the metric ρ in (2·1). Define the random variables $D^n_j(\omega) \coloneqq \sup_{x,y}\rho(X^n_j(x,\omega), X^n_j(y,\omega))$. Then by construction, from (3·1),

$$\sup_{x,y\in\Sigma_0} \rho(\hat{Z}_n(x,\omega),\hat{Z}_n(y,\omega)) \leqslant D^n_0(\omega) \tag{3·2}$$

and

$$P(D^n_{-k_0} = 2^{-(M+2)} \mid D^n_{-(k_0+1)} = 2^{-(M+1)})$$
$$\geqslant \inf_n \inf_{1\leqslant j_l\leqslant N(n+l), 1\leqslant l\leqslant M} \sum_{i=1}^{N(n)} \inf_y g_n(ij_1\cdots j_M y) = cff_G(2^{-(M+1)}),$$

for any $0\leqslant k_0\leqslant n-1$, $n\geqslant 1$.

Define

$$\hat{Y}_n(\omega) = f_{\omega_0}\circ\cdots\circ f_{\omega_{n-1}}(1),\ n\geqslant 1 \qquad \hat{Y}_0(\omega) = 1,$$

where for $s\in(0,1)$, and integers $m\geqslant 1$, $f_s(m) = m+1$, if $s < cff_G(2^{-m})$, $f_s(m) = 1$ if $s\geqslant cff_G(2^{-m})$. Then $\hat{Y}_n(\omega)$ is nondecreasing in n, and since $D^n_{-(n-k)}(\omega)\leqslant 2^{-\hat{Y}_k(\omega)}$, for any $0\leqslant k\leqslant n$, we obtain in the particular case (when $k=n$) using (3·2) that

$$\sup_{x,y\in\Sigma_0} \rho(\hat{Z}_n(x,\omega),\hat{Z}_n(y,\omega)) \leqslant 2^{-\hat{Y}_n(\omega)} \tag{3·3}$$

for any $\omega\in(0,1)^{\mathbb{N}}$. In particular this means that if $\hat{Y}_T(\omega) > N_\star$, for some $T = T(\omega)$, then the first $N_\star$ digits of $\hat{Z}_n(x,\omega)$ do not depend on x for any $n\geqslant T(\omega)$. Thus the proof of Theorem 2 will be completed if we prove that $\hat{Y}_n\to\infty$ as $n\to\infty$ a.s.

Let $\{Y_n\}_{n=0}^{\infty}$, be a stochastic sequence with $Y_n : (0,1)^{\mathbb{N}}\to\mathbb{N}$, $n\geqslant 0$, defined inductively in the following way: let $Y_0(\omega) = 1$, for all $\omega\in(0,1)^{\mathbb{N}}$. Suppose $Y_n(\omega) = m$. Let $Y_{n+1}(\omega) = m+1$ if $\omega_{n+1} < cff_G(2^{-m})$ and $Y_{n+1}(\omega) = 1$ otherwise.

It follows that $\{Y_n\}$ is a homogeneous Markov chain with $Y_0 = 1$,

$$P(Y_{n+1} = m+1 \mid Y_n = m) = cff_G(2^{-m})$$

and

$$P(Y_{n+1} = 1 \mid Y_n = m) = 1 - cff_G(2^{-m}),\ m\geqslant 1.$$

Note that Y_n and $\hat{Y}_n$ are identically distributed for any fixed n. Therefore, by (3·3),

$$E\sup_{x,y\in\Sigma_0}\rho(\hat{Z}_n(x),\hat{Z}_n(y)) \leqslant E2^{-Y_n},\ n\geqslant 0. \tag{3·4}$$

Since Y_n is a non-ergodic Markov chain by assumption, see e.g. Prabhu [**9**, p. 80, example 18], and $\hat{Y}_n$ is monotone, it follows that $\hat{Y}_n\to\infty$ a.s. as $n\to\infty$, and thus using (3·3), we obtain

$$\sup_{x,y\in\Sigma_0} \rho(\hat{Z}_n(x),\hat{Z}_n(y)) \to 0, \ a.s. \tag{3·5}$$

as $n \to \infty$.

Note that if for some $M \leqslant n-k_0$, $\omega_{k_0+j} \in \bigcup_{k=0}^{M-j} A_k(i_{j+1}\cdots i_{j+k})$ for all $j=0,\ldots,$ $M-1$, then $X_{-k_0}^n \in \{i_1 i_2 \cdots i_M y : \ y \in \Sigma_{k_0+M}\}$ for all $n \geqslant M$.

From this property (in the case $k_0=0$) in combination with (3·5) it follows that there exists a Σ_0-valued random variable $\hat{Z}$, such that $\hat{Z}_n(x,\omega)$ converges almost surely to $\hat{Z}(\omega)$, uniformly in x. From (3·4) it follows that

$$E \sup_{x\in\Sigma_0} \rho(\hat{Z}_n(x),\hat{Z}) \leqslant E2^{-Y_n}.$$

This completes the proofs of the theorems.

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