TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 353, Number 9, Pages 3547–3562
S 0002-9947(01)02798-2
Article electronically published on April 18, 2001

MARKOV CHAINS IN RANDOM ENVIRONMENTS AND RANDOM ITERATED FUNCTION SYSTEMS

ÖRJAN STENFLO

ABSTRACT. We consider random iterated function systems giving rise to Markov chains in random (stationary) environments. Conditions ensuring unique ergodicity and a "pure type" characterization of the limiting "randomly invariant" probability measure are provided. We also give a dimension formula and an algorithm for simulating exact samples from the limiting probability measure.

1. Introduction

In this paper, we are going to consider random iteration of functions where the function to iterate is chosen *independently* in each iteration step at random from a *random* probability distribution selected according to a stationary and ergodic sequence. This procedure generates Markov chains in random (stationary) environments and generalizes the situation known as iterated function systems with probabilities (see e.g. Barnsley and Demko [1]) where the probability distribution, deciding which function to iterate in each step, is nonrandom.

We can interpret the random object constructed in two different ways depending on if we consider the whole process as random or not. That is, we can think of the random object as a random non-homogeneous Markov chain (as we will mainly do here) or (deterministically) as one Markov chain with random transition probabilities. The latter interpretation corresponds to iteration of functions, where the choice of function to iterate in each step, is determined by a stationary sequence of random variables which is a special case of recursive chains (Borovkov [3]).

Random iterations according to a stationary sequence has been considered e.g. by Elton [9] and Borovkov and Foss [4]. The special structure of our controlling stationary sequence, e.g. the two possible interpretations of the dynamics presented above, enables a more refined ergodic analysis which makes the theory of Markov chains in (stationary) random environments to more than a simple particular case of the stationary iteration model. A feature worth bringing to the readers attention is that the model of random iteration according to some stochastic sequence defined on the index space of some pre-described set of functions slightly differs from the

Received by the editors December 19, 1999 and, in revised form, October 2, 2000.
2000 *Mathematics Subject Classification.* Primary 28A80, 37H99, 60F05, 60J05, 60K37; Secondary 28A78, 60G57, 65C05.
Key words and phrases. Iterated Function Systems (IFS), Markov chains, pointwise dimension, random environments, exact sampling.
Supported by the The Royal Swedish Academy of Sciences.

above model in general. See Silvestrov and Stenflo [20] for ergodic results in the case of iteration according to a regenerative stochastic sequence.

The theory of Markov chains in random environments in the countable state space case was developed in papers by Cogburn [5],[6],[7] and Orey [15]. In the general case, Seppäläinen [18] and Kifer [11] proved large deviation theorems. In Kifer [12] also a central limit theorem and a law of iterated logarithms were proved. (For other convergence theorems, see also Lu and Mukherjea [14].)

The present paper is organized as follows. In Section 2 we will adapt the principle by Letac [13] studying reversed iterations to prove ergodic theorems for homogeneous Markov chains to our more general non-homogeneous situation. In the paper by Propp and Wilson [17] this method was used as a basis for their, in the theory of Markov Chain Monte Carlo (MCMC), already classical algorithm for exact simulation of random samples from the invariant probability measure of a homogeneous Markov chain. Below we show that this algorithm can be extended to an algorithm for exact sampling from the limiting probability distribution for Markov chains in random environments. Our basic distributional convergence theorem in Section 2 can be considered as a non-homogeneous generalization of a theorem in Stenflo [21]. This theorem is proved under contractivity assumptions. The paper by Kifer [11] is another main related reference here. In Section 3 we analyze the invariant probability regime generalizing a result in the homogeneous case by Dubins and Freedman [8]. This analysis is done under additional discreteness conditions posed on the family of functions. In Section 4 we change slightly the setup. We consider the unit interval as state space and give a dimension formula under further smoothness and separation conditions. A different feature from the previous sections is that no contractivity assumptions are made here. Section 4 is self-contained and may thus be read separately.

Let (X, d) be a complete separable metric space, and let $(\mathbb{R}, \mathcal{B})$ denote the set of real numbers with its Borel σ-field. Consider a measurable function $w : X \times \mathbb{R} \to X$. For each fixed $s \in \mathbb{R}$, we write $w_s(x) := w(x, s)$. We call the set $\{(X, d);\ w_s, s \in \mathbb{R}\}$ an iterated function system (IFS). Let $(\Omega, \mathcal{F}, P)$ be a probability space with an invertible P-preserving ergodic transformation $\theta : \Omega \to \Omega$. For each $\omega \in \Omega$ let P^ω be a probability measure on $\mathbb{R}$. Assume $P^\omega(A)$ is measurable in ω for each fixed $A \in \mathcal{B}$. Let $\{I_n\}$ be a sequence of independent, identically distributed (i.i.d.) random variables with values uniformly distributed in $(0, 1)$. Let for each $\omega \in \Omega$, $I_n^\omega := \inf\{y : P^\omega((-\infty, y]) \geq I_n\}$. Then for each $\omega \in \Omega$ we have that I_n^ω is distributed according to P^ω. Define for each fixed $x \in X$ and $\omega \in \Omega$,

$$Z_n^\omega(x) := w_{I_{n-1}^{\theta^{n-1}\omega}} \circ \cdots \circ w_{I_0^\omega}(x),\ n \geq 1, \qquad Z_0^\omega(x) = x,$$

and the reversed iterates

$$\hat{Z}_n^\omega(x) := w_{I_0^{\theta^{n-1}\omega}} \circ \cdots \circ w_{I_{n-1}^\omega}(x),\ n \geq 1, \qquad \hat{Z}_0^\omega(x) = x.$$

Note that the random variables $Z_n^\omega(x)$ and $\hat{Z}_n^\omega(x)$ are identically distributed for each fixed $x \in X$ and $\omega \in \Omega$.

We are going to give conditions implying that, for each fixed $\omega \in \Omega$, there exists a random variable $\hat{Z}^\omega$, such that

$$\hat{Z}_n^{\theta^{-n}\omega}(x) = w_{I_0^{\theta^{-1}\omega}} \circ \cdots \circ w_{I_{n-1}^{\theta^{-n}\omega}}(x) \overset{a.s.}{\to} \hat{Z}^\omega \quad \text{as } n \to \infty, \tag{1}$$

where the limit is independent of $x \in X$. If we then define μ^ω to be the probability distribution of $\hat{Z}^\omega$ and the "random transition kernels" are Feller continuous, then $\{\mu^\omega\}_{\omega\in\Omega}$ will satisfy an invariance equation (see (4) below).

The following principle for exact sampling can thus be formulated.

Principle (exact sampling): Let $\omega \in \Omega$ be fixed. Suppose there exist a random integer N with $N < \infty$ *a.s.* such that, $\hat{Z}_N^{\theta^{-N}\omega}(x)$ does not depend on $x \in X$. Then $\hat{Z}_N^{\theta^{-N}\omega}(x)$ is distributed according to μ^ω.

Remark 1. The case when $|\Omega| = 1$ is the coupling from the past (CFTP) algorithm by Propp and Wilson [17].

2. Convergence results

2.1. **Statements.** Let BL denote the class of bounded continuous functions, $f : X \to \mathbb{R}$ (with $\|f\|_\infty = \sup_{x\in X} |f(x)| < \infty$) that also satisfy the Lipschitz condition

$$\|f\|_L = \sup_{x\neq y} \frac{|f(x) - f(y)|}{d(x,y)} < \infty.$$

We set $\|f\|_{BL} = \|f\|_\infty + \|f\|_L$. For Borel probability measures ν_1 and ν_2 we define the metric

$$d_w(\nu_1, \nu_2) = \sup_{f\in BL} \{ | \int_X f d(\nu_1 - \nu_2)| : \|f\|_{BL} \leq 1\}.$$

It is well known see e.g. Shiryaev [19] that this metric metrizes the topology of weak convergence of probability measures (on separable metric spaces).

Denote by $\mu_n^{\omega,x}$ the probability distribution of $Z_n^\omega(x)$.

We have the following theorem:

Theorem 1. *Suppose*

(A) *There exists a constant $c < 1$ such that*

$$Ed(w_{I_0^\omega}(x), w_{I_0^\omega}(y)) \leq cd(x,y),$$

for all $x, y \in X$ and all $\omega \in \Omega$.

(B) $\sup_{\omega\in\Omega} Ed(x_0, w_{I_0^\omega}(x_0)) < \infty$, *for some $x_0 \in X$.*

Let $K \subseteq X$ be a bounded set. Then there exists a positive constant γ_K and for all $\omega \in \Omega$, random variables $\hat{Z}^\omega$ such that

$$\sup_{x\in K} Ed(\hat{Z}_n^{\theta^{-n}\omega}(x), \hat{Z}^\omega) \leq \frac{\gamma_K}{1-c} c^n, \qquad n \geq 0. \tag{2}$$

For the random Markov chain, we obtain the following theorem.

Theorem 2. *Under the assumptions (A) and (B) above, for any bounded set $K \subseteq X$, there exists a positive constant γ_K, such that for each $\omega \in \Omega$ there exist a probability measure μ^ω such that*

$$\sup_{x\in K} d_w(\mu_n^{\theta^{-n}\omega,x}, \mu^\omega) \leq \frac{\gamma_K}{1-c} c^n, \qquad n \geq 0. \tag{3}$$

The family $\{\mu^\omega\}_{\omega\in\Omega}$ satisfies the invariance equation

$$\mu^{\theta\omega} = \int \mu^\omega \circ w_s^{-1} dP^\omega(s), \tag{4}$$

and are uniformly concentrated in the sense of a bounded first moment i.e. for any point $x_\star \in X$ there exists a finite constant B (depending on $x_\star$ but not on ω) such that

$$\int_X d(x_\star, x) d\mu^\omega(x) < B. \tag{5}$$

Furthermore, the family is unique satisfying (4) and (5).

Remark 2. If we consider the the whole process as one random sequence i.e. if we define $Z_n(x)$ to be $Z_n^\omega(x)$ with probability density $P(d\omega)$, and let

$$\mu(\cdot) := \int_\Omega \mu^\omega(\cdot) dP(\omega),$$

we have that the distribution of $Z_n(x)$ converges weakly to μ. The principle from the section above may be used for exact sampling from μ. That is, define $\hat{Z}_n(x)$ to be $\hat{Z}_n^{\theta^{-n}\omega}(x)$ with probability density $P(d\omega)$. Suppose there exist a random integer N with $N < \infty$ *a.s.* such that, $\hat{Z}_N(x)$ does not depend on $x \in X$. Then $\hat{Z}_N(x)$ is distributed according to μ.

Remark 3. An explicit expression and upper bound for γ_K is given by

$$\begin{aligned}\gamma_K &:= \sup_{x \in K} \sup_{\omega \in \Omega} Ed(x, w_{I_0^\omega}(x)) \\ &\leq \sup_{\omega \in \Omega} Ed(x_0, w_{I_0^\omega}(x_0)) + (c+1) \sup_{x \in K} d(x, x_0) < \infty.\end{aligned}$$

Remark 4. Note that the functions w_s, are not assumed to be continuous for any $s \in \mathbb{R}$. For an example when all maps are discontinuous but the theorem applies (in the case when $|\Omega| = 1$), see Stenflo [21].

Remark 5. If assumption (A) is relaxed to hold for P *a.a.* $\omega \in \Omega$, then the statements in Theorems 1 and 2 will be reduced to P *a.s.* statements.

2.2. **Proofs.**

Proof (Theorems 1 and 2). We start by observing that if there exists a family $\{\mu^\omega\}_{\omega \in \Omega}$ such that (for all $\omega \in \Omega$)

$$\sup_{x \in K} |Eg(Z_n^{\theta^{-n}\omega}(x)) - \int_X g d\mu^\omega| \to 0, \qquad \text{as } n \to \infty \tag{6}$$

for any bounded and continuous function $g : X \to \mathbb{R}$, and the family also satisfies (4) and (5), then this family must be unique satisfying these two equations.

In fact, suppose $\{\mu_\star^\omega\}$ is another family of probability measures satisfying (4) and (5). Then by a repeated use of (4), which equivalently can be formulated as (17) (see below), we see that for any for bounded and continuous function g,

$$\begin{aligned}|\int_X g d\mu_\star^\omega - \int_X g d\mu^\omega| &= |\int_X (Eg(Z_n^{\theta^{-n}\omega}(x)) - \int_X g d\mu^\omega) d\mu_\star^{\theta^{-n}\omega}(x)| \\ &\leq \int_X |Eg(Z_n^{\theta^{-n}\omega}(x)) - \int_X g d\mu^\omega| d\mu_\star^{\theta^{-n}\omega}(x)\end{aligned}$$

and, by writing $X = (X \backslash K) \cup K$ for a sufficiently large bounded set K and using (5) and (6) respectively on the two parts,we see that the the last sequence of integrals tends to 0 as $n \to \infty$. Thus for all bounded and continuous functions g we have that $\int_X g d\mu_\star^\omega = \int_X g d\mu^\omega$, for all $\omega \in \Omega$, and consequently, see e.g. Billingsley [2], μ^ω and $\mu_\star^\omega$ coincide.

We will prove that (6) holds true by proving the stronger statement that there exist a family of random variables $\hat{Z}^\omega$ such that for any sequence $\{x_n\}$ in K and any $\omega \in \Omega$, $\{\hat{Z}_n^{\theta^{-n}\omega}(x_n)\}$ is *a.s.* a Cauchy sequence which converges since X is complete. We then prove that the limit is independent of the sequence $\{x_n\}$.

For $N \leq n \leq m$ we have

$$d(\hat{Z}_n^{\theta^{-n}\omega}(x_n), \hat{Z}_m^{\theta^{-m}\omega}(x_m)) \leq \sum_{i=N}^{\infty} d(\hat{Z}_i^{\theta^{-i}\omega}(x_i), \hat{Z}_{i+1}^{\theta^{-(i+1)}\omega}(x_{i+1})). \tag{7}$$

Thus if we prove that

$$E\sum_{i=N}^{\infty} d(\hat{Z}_i^{\theta^{-i}\omega}(x_i), \hat{Z}_{i+1}^{\theta^{-(i+1)}\omega}(x_{i+1})) < \infty, \tag{8}$$

then

$$\sum_{i=N}^{\infty} d(\hat{Z}_i^{\theta^{-i}\omega}(x_i), \hat{Z}_{i+1}^{\theta^{-(i+1)}\omega}(x_{i+1})) < \infty \qquad a.s., \tag{9}$$

and from (7) and (9) we conclude that $\{\hat{Z}_n^{\theta^{-n}\omega}(x_n)\}$ *a.s.* forms a Cauchy sequence.

Now by recursively using assumption (A) we obtain that

$$\begin{aligned}
&E\sum_{i=N}^{\infty} d(\hat{Z}_i^{\theta^{-i}\omega}(x_i), \hat{Z}_{i+1}^{\theta^{-(i+1)}\omega}(x_{i+1})) \\
&\quad= \sum_{i=N}^{\infty} Ed(\hat{Z}_i^{\theta^{-i}\omega}(x_i), \hat{Z}_{i+1}^{\theta^{-(i+1)}\omega}(x_{i+1})) \\
&\quad= \sum_{i=N}^{\infty} E(E(d(\hat{Z}_i^{\theta^{-i}\omega}(x_i), \hat{Z}_{i+1}^{\theta^{-(i+1)}\omega}(x_{i+1}))|w_{I_1^{\theta^{-2}\omega}}, \dots, w_{I_i^{\theta^{-(i+1)}\omega}})) \\
&\quad= \sum_{i=N}^{\infty} \big(E(E(d(w_{I_0^{\theta^{-1}\omega}}(w_{I_1^{\theta^{-2}\omega}} \circ \cdots \circ w_{I_{i-1}^{\theta^{-i}\omega}}(x_i)), \\
&\qquad w_{I_0^{\theta^{-1}\omega}}(w_{I_1^{\theta^{-2}\omega}} \circ \cdots \circ w_{I_i^{\theta^{-(i+1)}\omega}}(x_{i+1})))|w_{I_1^{\theta^{-2}\omega}}, \dots, w_{I_i^{\theta^{-(i+1)}\omega}}))\big) \\
&\quad\leq \sum_{i=N}^{\infty} cEd(w_{I_1^{\theta^{-2}\omega}} \circ \cdots \circ w_{I_{i-1}^{\theta^{-i}\omega}}(x_i), w_{I_1^{\theta^{-2}\omega}} \circ \cdots \circ w_{I_i^{\theta^{-(i+1)}\omega}}(x_{i+1})) \\
&\quad\leq \sum_{i=N}^{\infty} c^i Ed(x_i, w_{I_i^{\theta^{-(i+1)}\omega}}(x_{i+1})).
\end{aligned} \tag{10}$$

We consequently have that

$$E\sum_{i=N}^{\infty} d(\hat{Z}_i^{\theta^{-i}\omega}(x_i), \hat{Z}_{i+1}^{\theta^{-(i+1)}\omega}(x_{i+1})) \leq \frac{\sup_{x,y\in K}\sup_{\omega\in\Omega} Ed(x, w_{I_0^\omega}(y))}{1-c} c^N,$$

and since by using assumptions (A) and (B),

$$
\begin{aligned}
&\sup_{x,y\in K}\sup_{\omega\in\Omega} Ed(x, w_{I_0^\omega}(y)) \\
&\qquad \le \sup_{x\in K} d(x,x_0) + \sup_{\omega\in\Omega} Ed(x_0, w_{I_0^\omega}(x_0)) + \sup_{y\in K}\sup_{\omega\in\Omega} Ed(w_{I_0^\omega}(x_0), w_{I_0^\omega}(y)) \\
&\qquad \le \sup_{\omega\in\Omega} Ed(x_0, w_{I_0^\omega}(x_0)) + (c+1)\sup_{x\in K} d(x,x_0) < \infty, \qquad (11)
\end{aligned}
$$

we see that (8) holds. Thus $\{\hat{Z}_n^{\theta^{-n}\omega}(x_n)\}$ is *a.s.* a Cauchy sequence and converges since X is complete. Let us call the limit $\hat{Z}^\omega(\{x_n\})$.

It remains to prove that the limit is independent of the sequence $\{x_n\}$. Let us define $\hat{Z}^\omega := \hat{Z}^\omega(\{x_0\})$. By the Chebyshev inequality, and by a recursive use of assumption (A), we see that for any $\epsilon > 0$,

$$
\begin{aligned}
&Pr(d(\hat{Z}_n^{\theta^{-n}\omega}(x_n), \hat{Z}_n^{\theta^{-n}\omega}(x_0)) > \epsilon) \\
&\qquad\qquad \le \quad \frac{Ed(\hat{Z}_n^{\theta^{-n}\omega}(x_n), \hat{Z}_n^{\theta^{-n}\omega}(x_0))}{\epsilon} \\
&\qquad\qquad \le \quad \frac{1}{\epsilon} E(E(d(\hat{Z}_n^{\theta^{-n}\omega}(x_n), \hat{Z}_n^{\theta^{-n}\omega}(x_0)) | w_{I_1^{\theta^{-2}\omega}}, \ldots, w_{I_{n-1}^{\theta^{-n}\omega}})) \\
&\qquad\qquad \le \quad \frac{c}{\epsilon} Ed(w_{I_1^{\theta^{-2}\omega}} \circ \cdots \circ w_{I_{n-1}^{\theta^{-n}\omega}}(x_n), w_{I_1^{\theta^{-2}\omega}} \circ \cdots \circ w_{I_{n-1}^{\theta^{-n}\omega}}(x_0)) \\
&\qquad\qquad \le \quad \ldots \le \frac{c^n}{\epsilon} d(x_n, x_0).
\end{aligned}
$$

Thus

$$
\sum_{n=0}^{\infty} Pr(d(\hat{Z}_n^{\theta^{-n}\omega}(x_n), \hat{Z}_n^{\theta^{-n}\omega}(x_0)) > \epsilon) \le \sum_{n=0}^{\infty} \frac{c^n}{\epsilon} d(x_n, x_0) < \infty,
$$

and it follows (see e.g. Shiryaev [19]) that

$$
d(\hat{Z}_n^{\theta^{-n}\omega}(x_n), \hat{Z}_n^{\theta^{-n}\omega}(x_0)) \to 0 \quad a.s. \qquad (12)
$$

From (12), the triangle inequality, and the fact of almost sure convergence of $\hat{Z}_n^{\theta^{-n}\omega}(x_0)$ to $\hat{Z}^\omega$, it follows that $d(\hat{Z}_n^{\theta^{-n}\omega}(x_n), \hat{Z}^\omega) \overset{a.s.}{\to} 0$ as $n \to \infty$ establishing the *a.s.* independence of $\{x_n\}$. Thus (6) holds true.

For any $x \in X$, we have the following sequence of inequalities:

$$
\begin{aligned}
Ed(\hat{Z}_n^{\theta^{-n}\omega}(x), \hat{Z}^\omega) &= E \lim_{m\to\infty} d(\hat{Z}_n^{\theta^{-n}\omega}(x), \hat{Z}_m^{\theta^{-m}\omega}(x)) \\
&\le E \lim_{m\to\infty} \sum_{k=n}^{m-1} d(\hat{Z}_k^{\theta^{-k}\omega}(x), \hat{Z}_{k+1}^{\theta^{-(k+1)}\omega}(x)) \\
&= E \sum_{k=n}^{\infty} d(\hat{Z}_k^{\theta^{-k}\omega}(x), \hat{Z}_{k+1}^{\theta^{-(k+1)}\omega}(x)). \qquad (13)
\end{aligned}
$$

Thus if we define

$$
\gamma_K := \sup_{x\in K}\sup_{\omega\in\Omega} Ed(x, w_{I_0^\omega}(x)),
$$

which by (11) is a finite constant, we obtain from (13) and (10) (with $x_i = x$), that

$$
\sup_{x\in K} Ed(\hat{Z}_n^{\theta^{-n}\omega}(x), \hat{Z}^\omega) \le \frac{c^n}{1-c}\gamma_K, \qquad n \ge 0,
$$

and Theorem 1 is proved. □

Proof (Theorem 2:(3)). Define $\mu^\omega(\cdot) = Pr(\hat{Z}^\omega \in \cdot)$. Since

$$\begin{aligned} d_w(\mu_n^{\theta^{-n}\omega,x}, \mu^\omega) &= \sup\{|\int_X f d(\mu_n^{\theta^{-n}\omega,x} - \mu^\omega)| : \|f\|_{BL} \le 1\} \\ &= \sup\{|E(f(\hat{Z}_n^{\theta^{-n}\omega}(x)) - f(\hat{Z}^\omega))| : \|f\|_{BL} \le 1\} \\ &\le \sup\{E|f(\hat{Z}_n^{\theta^{-n}\omega}(x)) - f(\hat{Z}^\omega)| : \|f\|_{BL} \le 1\} \\ &\le Ed(\hat{Z}_n^{\theta^{-n}\omega}(x), \hat{Z}^\omega), \end{aligned} \tag{14}$$

we see that Theorem 2:(3) is an immediate consequence of Theorem 1. □

Proof (Theorem 2:(4)). We shall prove that the probability measures $\{\mu^\omega\}$ satisfy the invariance equation (4). To do this, we prove that the random Markov chains $\{Z_n^\omega(x)\}$ has Feller continuous kernels which in our terminology means that $g : X \to \mathbb{R}$ being a bounded and continuous function implies that the mapping

$$x \mapsto Eg(w_{I_0^\omega}(x)) \tag{15}$$

is continuous (for each fixed $\omega \in \Omega$). It is well known that the limiting probability measure of an ergodic homogeneous Markov chain with the Feller property is invariant. An analogous statement holds also in our more general situation for Markov chains satisfying property (15). To be self-contained, we explain why before proving that this property is satisfied.

Since

$$Eg(Z_n^{\theta^{-n}(\theta\omega)}(x)) = \int_X Eg(w_{I_0^\omega}(y)) Pr(Z_{n-1}^{\theta^{-(n-1)}\omega}(x) \in dy), \tag{16}$$

the invariance equation

$$\int_X g d\mu^{\theta\omega} = \int_X Eg(w_{I_0^\omega}) d\mu^\omega \tag{17}$$

will follow by taking limits in (16) justified by using the continuity in (15) and the bounded convergence theorem.

To prove (15), let $\{y_n\}$ be a sequence in X with $\lim_{n\to\infty} y_n = y$. Since, for fixed $\epsilon > 0$, by the Chebyshev inequality, and from assumption (A),

$$Pr(d(w_{I_0^\omega}(y_n), w_{I_0^\omega}(y)) > \epsilon) \le \frac{Ed(w_{I_0^\omega}(y_n), w_{I_0^\omega}(y))}{\epsilon} \le \frac{cd(y_n, y)}{\epsilon} \to 0$$

as $n \to \infty$, we have proved that $w_{I_0^\omega}(y_n)$ converges in probability to $w_{I_0^\omega}(y)$ for any fixed $\omega \in \Omega$. Thus for any bounded and continuous function g

$$\lim_{n\to\infty} Eg(w_{I_0^\omega}(y_n)) = Eg(w_{I_0^\omega}(y)),$$

and (15) is established. Thus (17), which equivalently can be expressed as (4), holds true and Theorem 2:(4) is proved. □

Proof (Theorem 2:(5)). The validity of (5) is an immediate consequence of (2) in the case $n = 0$. □

This completes the proofs of Theorems 1 and 2. □

3. The invariant probability regime

In this section we will show, that in the case of a countable set of m to 1 maps, there is a generic structure of the family of probability measures obtained in Theorem 2. In the paper by Dubins and Freedman [8], the case corresponding to when Ω consists of one point was considered. We shall prove here, as suggested in Kifer [11], that their ideas can be extended also to give results in our setting.

A finite non-negative measure λ is called *continuous* if $\lambda(\{x\}) = 0$ for each $x \in X$ and *discrete* if $\sum_{x\in X} \lambda(\{x\}) = \lambda(X)$. (We interpret the latter sum as a sum over the (at most countable) set of $x \in X$ such that $\lambda(\{x\}) > 0$.)

We have the following theorem.

Theorem 3. *Suppose that*

(C) *For each map w_s and $x \in X$, $w_s^{-1}\{x\}$ is at most a countable set.*

(D) *The family of maps is at most countable.*

Then the family of measures $\{\mu^\omega\}$, obtained in Theorem 2 either contains only discrete or only continuous probability measures.

Proof. For each $\omega \in \Omega$, let μ^ω_{disk} (the discrete part of μ^ω) denote μ^ω restricted to the set of $x \in X$ with $\mu^\omega(\{x\}) > 0$, and μ^ω_{cont} (the continuous part of μ^ω) denote μ^ω restricted to the remaining part of X. For each $\omega \in \Omega$, we can thus write $\mu^\omega = \mu^\omega_{disk} + \mu^\omega_{cont}$.

Let us for non-negative measures λ on X define the operators $T^\omega\lambda := \int \lambda \circ w_s^{-1} dP^\omega(s)$. The invariance equation (4) may then be expressed as

$$\mu^{\theta\omega} = T^\omega \mu^\omega.$$

Since by definition, (4) and assumption (C), for any $x \in X$,

$$\mu^{\theta\omega}_{disk}(\{x\}) = \mu^{\theta\omega}(\{x\}) = T^\omega\mu^\omega(\{x\}) = T^\omega\mu^\omega_{disk}(\{x\}),$$

and since, following from assumption (D), $T\mu^\omega_{disk}$ is a discrete measure, we have that $\mu^{\theta\omega}_{disk} = T\mu^\omega_{disk}$. Thus the family of measures $\{\mu^\omega_{disk}\}$ and consequently also the family $\{\mu^\omega_{cont}\}$ satisfy the invariance equation (4). From the uniqueness of $\{\mu^\omega\}$ satisfying (4) and (5) (see Theorem 2 above), this implies that $\{\mu^\omega\}$ consist of probability measures of pure type, and the theorem is proved. $\square$

4. Dimensions

In this section, we are going to consider the local behavior of the limiting family of probability measures and give a dimension formula. To obtain this, we need to consider a particular setup.

Our setup is the following: Let $(\Omega, \mathcal{F}, P)$ be a probability space with an invertible P-preserving ergodic transformation $\theta : \Omega \to \Omega$. For each $\omega \in \Omega$, let $\mathcal{F}^\omega = \{w_1^\omega, ..., w^\omega_{m(\omega)}\}$, be a family of C^1 maps of $[0,1]$ into itself with continuous first order derivatives satisfying $0 < |(w_i^\omega)'(x)| < \infty$ (for all $x \in [0,1]$ and all i, $1 \leq i \leq m(\omega)$, where $m = m(\omega) \geq 2$ is an integer-valued random variable). Let $\{p_1^\omega, ..., p^\omega_{m(\omega)}\}$ be associated probabilities i.e. non-negative real numbers such that for each $\omega \in \Omega$, $\sum_{i=1}^{m(\omega)} p_i^\omega = 1$, and assume the functions p_j^ω and $(w_j^\omega)'(x)$ are measurable in ω. Define the function $i : [0,1) \times \Omega \to \{1, ..., m(\omega)\}$ as $i(x,\omega) = j$ if $\sum_{q=1}^{j-1} p_q^\omega \leq x < \sum_{q=1}^{j} p_q^\omega$, and for a sequence $\mathbf{i} = i_1 i_2 i_3 \ldots \in [0,1)^{\mathbb{N}}$ let

$$\hat{S}_n^{\omega}(\mathbf{i}) = w_{i(i_1,\theta^{-1}\omega)}^{\theta^{-1}\omega} \circ \cdots \circ w_{i(i_n,\theta^{-n}\omega)}^{\theta^{-n}\omega},\ n \geq 1, \qquad \hat{S}_0^{\omega}(\mathbf{i}) = id,$$

where id denotes the identity map.

We will make the following assumptions.

(E) For any $\omega \in \Omega$,

$$w_i^{\omega}((0,1)) \cap w_j^{\omega}((0,1)) = \emptyset, \qquad 1 \leq i,j \leq m(\omega),\ i \neq j.$$

(F) There exists a constant c_0 such that,

$$p_{max} := \sup_{\omega\in\Omega} \max_{1\leq i\leq m(\omega)} p_i^{\omega} < c_0 < 1$$

and

$$\int_{\Omega} \sum_{i=1}^{m(\omega)} p_i^{\omega} \log p_i^{\omega} dP(\omega) > -\infty.$$

(G) $\int_{\Omega} \log \min_{1\leq i\leq m(\omega)} \inf_{x\in[0,1]} |(w_i^{\omega})'(x)| dP(\omega) > -\infty$
and $\int_{\Omega} \sum_{i=1}^{m(\omega)} p_i^{\omega} \log^+ \sup_{x\in[0,1]} |(w_i^{\omega})'(x)| dP(\omega) < \infty$.

Denote by ν, by the Kolmogorov extension theorem, the unique probability measure on $\Sigma := [0,1]^{\mathbb{N}}$ generated by the finite dimensional Lebesgue measures. For a bounded set K, let $\operatorname{diam}(K)$ denotes its diameter (in the usual Euclidean metric).

We have the following lemma.

Lemma 1. *Assumptions (E) and (F) imply that for all $\omega \in \Omega$,*

$$\limsup_{n\to\infty} c_0^{-n} \operatorname{diam}(\hat{S}_n^{\omega}(\mathbf{i})([0,1])) \leq 1,\ \textit{for } \nu \textit{ a.a. } \mathbf{i}.$$

(The proof of Lemma 1 is given later in this section.)

Using this lemma, we can, for ν-almost all sequences $\mathbf{i} = i_1 i_2 i_3 \ldots \in \Sigma$ and all $\omega \in \Omega$, define

$$\hat{Z}^{\omega}(\mathbf{i}) := \lim_{n\to\infty} w_{i(i_1,\theta^{-1}\omega)}^{\theta^{-1}\omega} \circ \cdots \circ w_{i(i_n,\theta^{-n}\omega)}^{\theta^{-n}\omega}(x),$$

where $x \in [0,1]$ can be chosen arbitrary since the limit does not depend on x.

Define, for all $\omega \in \Omega$, $\mu^{\omega}(\cdot) := \nu(\mathbf{i} : \hat{Z}^{\omega}(\mathbf{i}) \in \cdot)$. Note that

$$\mu^{\omega}(w_{i_1}^{\theta^{-1}\omega} \circ w_{i_2}^{\theta^{-2}\omega} \circ \cdots \circ w_{i_n}^{\theta^{-n}\omega}([0,1])) = p_{i_1}^{\theta^{-1}\omega} p_{i_2}^{\theta^{-2}\omega} \cdots p_{i_n}^{\theta^{-n}\omega},$$

for any possible index sequence $i_1, ..., i_n$, and any n.

Let $B(x,r)$ denote a ball centered in x of radius r (in Euclidean metric which we will denote here by d).

We have the following theorem.

Theorem 4. *Under assumptions (E)-(G) above, for P a.a. $\omega \in \Omega$, we have that μ^{ω} is exact dimensional, and the pointwise dimension is given by*

$$dim(\mu) := \frac{\int_{\Omega} \sum_{i=1}^{m(\omega)} p_i^{\omega} \log p_i^{\omega} dP(\omega)}{\int_{\Omega} \sum_{i=1}^{m(\omega)} p_i^{\omega} \int_{[0,1]} \log |(w_i^{\omega})'(x)| d\mu^{\omega}(x) dP(\omega)}$$

i.e. $\lim_{r\to 0} \frac{\log \mu^{\omega}(B(x,r))}{\log r}$ *exists, does not depend on x or ω and is equal to the above expression for μ^{ω} a.a. $x \in [0,1]$ for P a.a. $\omega \in \Omega$.*

Remark 6. Theorem 4 implies that the Hausdorff and entropy (or Rényi) dimensions exist and coincide with $dim(\mu)$. For a proof of this and further correspondences between different notions of dimension, see Young [23].

Remark 7. A related result in the case $|\Omega| = 1$ can be found in Strichartz [22]. Note that we make no contractivity assumptions here except the implicit contraction condition provided by assumption (E).

Remark 8. Assumption (E) is the key for convenient local characterizations of μ^ω. Even in the case of two affine maps and $|\Omega| = 1$ a relaxation of (E) to allow overlaps leads to very hard local estimation problems. For an overview of an interesting particular case of this corresponding to Bernoulli convolutions, see Peres et al. [16].

Remark 9. Theorem 4 implies results about the frequency of digits in random base expansions. For more on this topic and related results, see Kifer [10].

Proof (Theorem 4). Consider a fixed $\omega \in \Omega$ and an $\mathbf{i} \in \Sigma$ for which $\hat{Z}^\omega(\mathbf{i})$ is well defined. For such $\mathbf{i}$'s define the functions,

(18)
$$d_n^\omega(\mathbf{i}) = d(\hat{Z}^\omega(\mathbf{i}), \partial\hat{S}_n^\omega(\mathbf{i})([0,1])) := \min\{d(\hat{Z}^\omega(\mathbf{i}), \hat{S}_n^\omega(\mathbf{i})(0)), d(\hat{Z}^\omega(\mathbf{i}), \hat{S}_n^\omega(\mathbf{i})(1))\},$$

where $\partial\hat{S}_n^\omega(\mathbf{i})([0,1])$ denotes the boundary of the set $\hat{S}_n^\omega(\mathbf{i})([0,1])$.

Letting $\phi : \Sigma \to \Sigma$ denote the shift operator, i.e. $\phi(i_1 i_2 \dots) = i_2 i_3 \dots$, we see from the definitions that for any n,

$$\hat{Z}^\omega(\mathbf{i}) = \hat{S}_n^\omega(\mathbf{i})(\hat{Z}^{\theta^{-n}\omega}(\phi^n(\mathbf{i}))).$$

Consequently, by the mean value theorem and the monotonicity assumptions,

$$\inf_{x\in[0,1]} |(\hat{S}_n^\omega(\mathbf{i}))'(x)| d_0^{\theta^{-n}\omega}(\phi^n(\mathbf{i})) \le d_n^\omega(\mathbf{i}) \le \sup_{x\in[0,1]} |(\hat{S}_n^\omega(\mathbf{i}))'(x)|. \tag{19}$$

Since d_n^ω is (uniformly) non-increasing in n and tends to 0 by Lemma 1, we have that for each $0 < r < d_0^\omega(\mathbf{i})$, there exists an integer $n(r)(\omega, \mathbf{i})$ such that

$$d^\omega_{n(r)(\omega,\mathbf{i})+1}(\mathbf{i}) \le r < d^\omega_{n(r)(\omega,\mathbf{i})}(\mathbf{i}),$$

and thus $B(\hat{Z}^\omega(\mathbf{i}), r) \subseteq \hat{S}^\omega_{n(r)}(\mathbf{i})([0,1])$. (For notational convenience we will sometimes drop the $(\omega, \mathbf{i})$ in what follows.)

It follows that, if $x = \hat{Z}^\omega$,

$$\begin{aligned}
\frac{\log \mu^\omega(B(x,r))}{\log r} &\ge \frac{\log \mu^\omega(\hat{S}^\omega_{n(r)}([0,1]))}{\log r} \\
&\ge \frac{\log \mu^\omega(\hat{S}^\omega_{n(r)}([0,1]))}{\log d^\omega_{n(r)+1}} \\
&\ge \frac{\log \mu^\omega(\hat{S}^\omega_{n(r)}([0,1]))}{n(r)} \frac{n(r)}{\log d^\omega_{n(r)+1}}. \qquad (20)
\end{aligned}$$

Define $R_n = R_n(\omega, \mathbf{i})$ by $R_n(\omega, \mathbf{i}) := \operatorname{diam}(\hat{S}_n^\omega(\mathbf{i})([0,1]))$. By definition $\hat{S}_n^\omega([0,1]) \subseteq B(\hat{Z}^\omega, R_n)$. It follows that, if $x = \hat{Z}^\omega$

$$\begin{aligned}\frac{\log \mu^\omega(B(x,R_n))}{\log R_n} &\le \frac{\log \mu^\omega(\hat{S}_n^\omega([0,1]))}{\log R_n}\\ &\le \frac{\log \mu^\omega(\hat{S}_n^\omega([0,1]))}{n}\frac{n}{\log R_n}.\end{aligned}\tag{21}$$

We will see that the proof of Theorem 4 will follow from (20) and (21) and the following three lemmas:

Lemma 2 (The entropy)**.** *Assumptions* (*E*)-(*F*) *imply that*

$$\lim_{n\to\infty} \frac{\log \mu^\omega(\hat{S}_n^\omega([0,1]))}{n} = \int_\Omega \sum_{i=1}^{m(\omega)} p_i^\omega \log p_i^\omega \, dP(\omega), \qquad P \times \nu \textit{ a.s.}$$

Lemma 3 (The Lyapunov exponent)**.** *Assumptions* (*E*)-(*G*) *imply that*

$$\begin{aligned}\lim_{n\to\infty} \frac{\log \sup_{x\in[0,1]} |(\hat{S}_n^\omega)'(x)|}{n} &= \lim_{n\to\infty} \frac{\log \inf_{x\in[0,1]} |(\hat{S}_n^\omega)'(x)|}{n}\\ &= \int_\Omega \sum_{i=1}^{m(\omega)} p_i^\omega \int_{[0,1]} \log|(w_i^\omega)'(x)| d\mu^\omega(x) dP(\omega), \qquad P\times\nu \textit{ a.s.}\end{aligned}$$

Lemma 4. *Assumptions* (*E*)-(*G*) *imply that*

$$\lim_{n\to\infty} \frac{\log d_0^{\theta^{-n}\omega}(\phi^n(\mathbf{i}))}{n} = 0 \qquad \textit{for } P\times\nu \textit{ a.a. } (\omega,\mathbf{i}).$$

In fact, from (20), (19), Lemma 2, Lemma 3 and Lemma 4 we see that

$$\begin{aligned}&\liminf_{r\to 0} \frac{\log \mu^\omega(B(x,r))}{\log r}\\ &\quad\ge \liminf_{r\to 0} \frac{\log \mu^\omega(\hat{S}_{n(r)}^\omega(\mathbf{i})([0,1]))}{n(r)} \frac{n(r)}{\log d_{n(r)+1}^\omega(\mathbf{i})}\\ &\quad\ge \liminf_{r\to 0} \Bigg[\frac{\log \mu^\omega(\hat{S}_{n(r)}^\omega(\mathbf{i})([0,1]))}{n(r)}\\ &\quad\times \frac{n(r)}{\log \inf_{x\in[0,1]} |(\hat{S}_{n(r)+1}^\omega(\mathbf{i}))'(x)| + \log d_0^{\theta^{-(n(r)+1)}\omega}(\phi^{n(r)+1}(\mathbf{i}))}\Bigg]\\ &\quad= \dim(\mu), \qquad P\times\nu \textit{ a.s.}\end{aligned}\tag{22}$$

(Note, as a consequence of Lemma 4, that $n(r) \to \infty$ as $r \to 0$, $P \times \nu$ *a.s.*)

From the mean value theorem and monotonicity of the maps, we see that

$$\inf_{x\in[0,1]} |(\hat{S}_n^\omega)'(x)| \le R_n \le \sup_{x\in[0,1]} |(\hat{S}_n^\omega)'(x)|.$$

By Lemma 3 this implies that

$$\lim_{n\to\infty}\frac{\log \operatorname{diam}(\hat{S}_n^\omega([0,1]))}{n} = \lim_{n\to\infty}\frac{\log R_n}{n}$$
$$= \int_\Omega \sum_{i=1}^{m(\omega)} p_i^\omega \int_{[0,1]} \log|(w_i^\omega)'(x)| d\mu^\omega(x) dP(\omega) < 0, \tag{23}$$
$$P\times\nu \ a.s.,$$

where the last inequality is a consequence of Lemma 1. Since $\{R_n\}$ is non-increasing in n and tends to 0 *a.s.*, we obtain that for any (small) r there is *a.s.* an n such that $R_{n+1}\le r\le R_n$.

Thus

$$\frac{\log\mu^\omega(B(x,r))}{\log r}\le\frac{\log\mu^\omega(B(x,R_{n+1}))}{\log R_n},$$

and this combined with (21), (23) and Lemma 2 implies that

$$\limsup_{r\to 0}\frac{\log\mu^\omega(B(x,r))}{\log r}\le \dim(\mu), \qquad P\times\nu \ a.s. \tag{24}$$

From (22) and (24) we therefore see that to complete the proof it remains to prove the lemmata.

Proof (Lemma 1). Since (trivially) the number of disjoint "iterated intervals" of diameter$>x$ can be at most $1/x$, we have using assumptions (E) and (F) that

$$\nu(\bigcup_{n=1}^\infty\{\mathbf{i}:\operatorname{diam}(\hat{S}_n^\omega(\mathbf{i})([0,1]))>c_0^n\})\le\sum_{n=1}^\infty c_0^{-n}p_{max}^n<\infty.$$

By the Borel-Cantelli lemma this proves Lemma 1. □

Proof (Lemma 2). Since by assumption (E),

$$\mu^\omega(\hat{S}_n^\omega(\mathbf{i})([0,1])) = p_{i(i_1,\theta^{-1}\omega)}^{\theta^{-1}\omega}p_{i(i_2,\theta^{-2}\omega)}^{\theta^{-2}\omega}\cdots p_{i(i_n,\theta^{-n}\omega)}^{\theta^{-n}\omega}, \tag{25}$$

we obtain from assumption (F) and Birkhoff's ergodic theorem that

$$\lim_{n\to\infty}\frac{\log\mu^\omega(\hat{S}_n^\omega(\mathbf{i})([0,1]))}{n} = \lim_{n\to\infty}\frac{\sum_{j=1}^n\log p_{i(i_j,\theta^{-j}\omega)}^{\theta^{-j}\omega}}{n}$$
$$= \int_\Omega\sum_{i=1}^{m(\omega)}p_i^\omega\log p_i^\omega dP(\omega) \qquad \text{for } P\times\nu \ a.a.\ (\omega,\mathbf{i}).$$

□

Proof (Lemma 3). Since by the chain rule,

$$\begin{aligned}(\hat{S}_n^\omega(\mathbf{i}))'(x) &= (w_{i(i_1,\theta^{-1}\omega)}^{\theta^{-1}\omega})'(w_{i(i_2,\theta^{-2}\omega)}^{\theta^{-2}\omega}\circ\cdots\circ w_{i(i_n,\theta^{-n}\omega)}^{\theta^{-n}\omega}(x))\\ &\cdot(w_{i(i_2,\theta^{-2}\omega)}^{\theta^{-2}\omega})'(w_{i(i_3,\theta^{-3}\omega)}^{\theta^{-3}\omega}\circ\cdots\circ w_{i(i_n,\theta^{-n}\omega)}^{\theta^{-n}\omega}(x))\\ &\cdots(w_{i(i_n,\theta^{-n}\omega)}^{\theta^{-n}\omega})'(x),\end{aligned}$$

it follows that for any fixed m,

$$
\begin{aligned}
\log \sup_{x\in[0,1]} |(\hat{S}^{\omega}_n(\mathbf{i}))'(x)| &\le \sum_{k=1}^{n} \log \sup_{x\in \hat{S}^{\theta^{-k}\omega}_{n-k}(\phi^k\mathbf{i})([0,1])} |(w^{\theta^{-k}\omega}_{i(i_k,\theta^{-k}\omega)})'(x)| \\
&\le \sum_{k=1}^{n-m} \log \sup_{x\in \hat{S}^{\theta^{-k}\omega}_{m}(\phi^k\mathbf{i})([0,1])} |(w^{\theta^{-k}\omega}_{i(i_k,\theta^{-k}\omega)})'(x)| \\
&\quad + \sum_{k=n-m+1}^{n} \log \sup_{x\in[0,1]} |(w^{\theta^{-k}\omega}_{i(i_k,\theta^{-k}\omega)})'(x)| \qquad (26)
\end{aligned}
$$

where in the last inequality we used the fact that the sets $\hat{S}^{\omega}_n(\mathbf{i})([0,1])$ are nested and non-increasing in n.

We obtain similarly, that

$$
\begin{aligned}
\log \inf_{x\in[0,1]} |(\hat{S}^{\omega}_n(\mathbf{i}))'(x)| &\ge \sum_{k=1}^{n} \log \inf_{x\in \hat{S}^{\theta^{-k}\omega}_{n-k}(\phi^k\mathbf{i})([0,1])} |(w^{\theta^{k}\omega}_{i(i_k,\theta^{-k}\omega)})'(x)| \\
&\ge \sum_{k=1}^{n-m} \log \inf_{x\in \hat{S}^{\theta^{-k}\omega}_{m}(\phi^k\mathbf{i})([0,1])} |(w^{\theta^{-k}\omega}_{i(i_k,\theta^{-k}\omega)})'(x)| \\
&\quad + \sum_{k=n-m+1}^{n} \log \inf_{x\in[0,1]} |(w^{\theta^{-k}\omega}_{i(i_k,\theta^{-k}\omega)})'(x)|. \qquad (27)
\end{aligned}
$$

It follows from (26), assumption (G) and Birkhoff's ergodic theorem, that

(28)

$$
\begin{aligned}
&\limsup_{n\to\infty} \frac{\log \sup_{x\in[0,1]} |(\hat{S}^{\omega}_n(\mathbf{i}))'(x)|}{n} \\
&\quad\le \limsup_{n\to\infty} \frac{\sum_{k=1}^{n-m} \log \sup_{x\in \hat{S}^{\theta^{-k}\omega}_{m}(\phi^k\mathbf{i})([0,1])} |(w^{\theta^{-k}\omega}_{i(i_k,\theta^{-k}\omega)})'(x)|}{n-m} \frac{n-m}{n} \\
&\qquad + \limsup_{n\to\infty} \frac{\sum_{k=n-m+1}^{n} \log \sup_{x\in[0,1]} |(w^{\theta^{-k}\omega}_{i(i_k,\theta^{-k}\omega)})'(x)|}{n} \\
&\quad= \int_{\Omega}\int_{\Sigma} \sum_{i=1}^{m(\omega)} p^{\omega}_i \log \sup_{x\in \hat{S}^{\omega}_m(\mathbf{i})([0,1])} |(w^{\omega}_i)'(x)| d\nu(\mathbf{i}) dP(\omega) \text{ for } P\times\nu \text{ } a.a. \ (\omega,\mathbf{i}).
\end{aligned}
$$

Similarly from (27), assumption (G) and Birkhoff's ergodic theorem it follows that

$$
\begin{aligned}
&\liminf_{n\to\infty} \frac{\log \inf_{x\in[0,1]} |(\hat{S}^{\omega}_n)'(x)|}{n} \\
&\qquad \ge \int_{\Omega}\int_{\Sigma} \sum_{i=1}^{m(\omega)} p^{\omega}_i \log \inf_{x\in \hat{S}^{\omega}_m(\mathbf{i})([0,1])} |(w^{\omega}_i)'(x)| d\nu(\mathbf{i}) dP(\omega), \qquad P\times\nu \ a.s. \qquad (29)
\end{aligned}
$$

From (28) and the monotone convergence theorem (using Lemma 1 and the fact that all maps have continuous derivatives) it follows that

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{\log \sup_{x\in[0,1]} |(\hat{S}_n^\omega(\mathbf{i}))'(x)|}{n} \\
&\qquad \leq \quad \lim_{n\to\infty} \int_\Omega \int_\Sigma \sum_{i=1}^{m(\omega)} p_i^\omega \log \sup_{x\in \hat{S}_n^\omega(\mathbf{i})([0,1])} |(w_i^\omega)'(x)| d\nu(\mathbf{i}) dP(\omega) \\
&\qquad = \quad \int_\Omega \int_\Sigma \sum_{i=1}^{m(\omega)} p_i^\omega \log |(w_i^\omega)'(\hat{Z}^\omega(\mathbf{i}))| d\nu(\mathbf{i}) dP(\omega) \qquad \text{for } P\times\nu \ a.a. \ (\omega, \mathbf{i})
\end{aligned}$$

and similarly from (29),

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{\log \inf_{x\in[0,1]} |(\hat{S}_n^\omega)'(x)|}{n} \\
&\qquad \geq \quad \lim_{n\to\infty} \int_\Omega \int_\Sigma \sum_{i=1}^{m(\omega)} p_i^\omega \log \inf_{x\in \hat{S}_n^\omega(\mathbf{i})([0,1])} |(w_i^\omega)'(x)| d\nu(\mathbf{i}) dP(\omega) \\
&\qquad = \quad \int_\Omega \int_\Sigma \sum_{i=1}^{m(\omega)} p_i^\omega \log |(w_i^\omega)'(\hat{Z}^\omega(\mathbf{i}))| d\nu(\mathbf{i}) dP(\omega), \qquad P\times\nu \ a.s.
\end{aligned}$$

Since by a change of variables

$$\begin{aligned}
&\int_\Omega \int_\Sigma \sum_{i=1}^{m(\omega)} p_i^\omega \log |(w_i^\omega)'(\hat{Z}^\omega(\mathbf{i}))| d\nu(\mathbf{i}) dP(\omega) \\
&\qquad\qquad = \quad \int_\Omega \sum_{i=1}^{m(\omega)} \int_{[0,1]} p_i^\omega \log |(w_i^\omega)'(x)| d\mu^\omega(x) dP(\omega),
\end{aligned}$$

we see that the proof of Lemma 3 is completed. □

Proof (Lemma 4). We need to show that

$$\lim_{n\to\infty} \frac{\log d_0^{\theta^{-n}\omega}(\phi^n(\mathbf{i}))}{n} = 0 \qquad \text{for } P\times\nu \ a.a. \ (\omega, \mathbf{i}).$$

Define for each fixed $\omega\in\Omega$, $c^\omega := \min_{1\leq i\leq m(\omega)} \inf_{x\in[0,1]} |(w_i^\omega)'(x)|$. Let

$$A_n^\omega = \{\mathbf{i} : d_0^{\theta^{-n}\omega}(\phi^n(\mathbf{i})) < c^{\theta^{-(n+1)}\omega} c^{\theta^{-(n+2)}\omega} \cdots c^{\theta^{-(n+\lfloor\sqrt{n}\rfloor)}\omega}\},$$

or equivalently expressed

$$A_n^\omega = \Big\{\mathbf{i} : \Big| \frac{\log d_0^{\theta^{-n}\omega}(\phi^n(\mathbf{i}))}{n} \Big| > \Big| \frac{\sum_{k=n+1}^{n+\lfloor\sqrt{n}\rfloor} \log c^{\theta^{-k}\omega}}{n} \Big| \Big\}. \tag{30}$$

(Recall the definition of d_n^ω in equation (18).) Since $\mathbf{i} = i_1 i_2 \ldots$ belonging to A_n^ω forces $S_{\lfloor\sqrt{n}\rfloor}^{\theta^{-n}\omega}(\phi^n(\mathbf{i}))$ to assume at most two values (corresponding to the two intervals closest to the endpoints of $[0,1]$), we obtain from (25) that

$$\nu(A_n^\omega) \leq 2 p_{max}^{\lfloor\sqrt{n}\rfloor}.$$

From assumption (F) it follows that $\sum_{n=1}^{\infty}\nu(A_n^{\omega})<\infty$, which by the Borel-Cantelli lemma implies that $\nu(A_n^{\omega}\ i.o.)=0$. Thus

$$\limsup_{n\to\infty}\left|\frac{\log d_0^{\theta^{-n}\omega}(\phi^n(\mathbf{i}))}{n}\right|\le\limsup_{n\to\infty}\left|\frac{\sum_{k=n+1}^{n+\lfloor\sqrt{n}\rfloor}\log c^{\theta^{-k}\omega}}{n}\right|, \tag{31}$$

for ν *a.a.* $\mathbf{i}$.

Now since by using assumption (G) and Birkhoff's ergodic theorem,

$$\begin{aligned}\lim_{n\to\infty}\frac{\sum_{k=n+1}^{n+\lfloor\sqrt{n}\rfloor}\log c^{\theta^{-k}\omega}}{n}&=\lim_{n\to\infty}\frac{\sum_{k=1}^{n+\lfloor\sqrt{n}\rfloor}\log c^{\theta^{-k}\omega}}{n+\lfloor\sqrt{n}\rfloor}\frac{(n+\lfloor\sqrt{n}\rfloor)}{n}\\ &\quad-\lim_{n\to\infty}\frac{\sum_{k=1}^{n}\log c^{\theta^{-k}\omega}}{n}=0,\qquad P\ a.s.,\end{aligned} \tag{32}$$

we see from (31) and (32) that

$$\lim_{n\to\infty}\left|\frac{\log d_0^{\theta^{-n}\omega}(\phi^n(\mathbf{i}))}{n}\right|=0\qquad\text{for } P\times\nu\ a.a.\ (\omega,\mathbf{i}),$$

which completes the proof of Lemma 4. □

Theorem 4 is proved. □

Acknowledgements

I am grateful to Yuri Kifer for the inspiration I found in his work, for useful discussions, and for his hand in making this research possible. Thanks also to Anders Johansson, Anders Öberg and Yuval Peres for valuable discussions related to Section 4. This paper was written during a postdoctoral visit at the Institute of Mathematics at the Hebrew University of Jerusalem. I am grateful to the Royal Swedish Academy of Sciences for financial support.

References

[1] Barnsley, M. F. and Demko, S. (1985) Iterated function systems and the global construction of fractals, *Proc. Roy. Soc. London Ser. A*, **399**, 243–275. MR **87e:**58051

[2] Billingsley, P. (1968) *Convergence of probability measures*, Wiley, New York. MR **38:**1718

[3] Borovkov, A. A. (1998) *Ergodicity and stability of stochastic processes*, Wiley, Chichester. MR **2000a:**60001

[4] Borovkov, A. A. and Foss, S. G. (1994) Two ergodicity criteria for stochastically recursive sequences, *Acta Appl. Math.*, **34**, 125–134. MR **95g:**60083

[5] Cogburn, R. (1984) The ergodic theory of Markov chains in random environments, Z. Wahrsch. Verw. Gebiete, **66**, 109–128. MR **85k:**60100

[6] Cogburn, R. (1990) On direct convergence and periodicity for transition probabilities of Markov chains in random environments, *Ann. Probab.*, **18**, 642–654. MR **92e:**60131

[7] Cogburn, R. (1991) On the central limit theorem for Markov chains in random environments *Ann. Probab.*, **19**, 587–604. MR **92h:**60029

[8] Dubins, L. E. and Freedman, D. A. (1966) Invariant probabilities for certain Markov processes, *Ann. Math. Statist.*, **37**, 837–848. MR **33:**1884

[9] Elton, J. H. (1990) A multiplicative ergodic theorem for Lipschitz maps, *Stochastic Process. Appl.*, **34**, 39–47. MR **91a:**47010

[10] Kifer, Y. (1995) Fractals via random iterated function systems and random geometric constructions. *Fractal geometry and stochastics (Finsterbergen, 1994)*, 145–164, Progr. Probab., **37**, Birkhäuser, Basel. MR **97j:**28013

[11] Kifer, Y. (1996) Perron-Frobenius theorem, large deviations, and random perturbations in random environments, *Math. Z.*, **222**, 677–698. MR **97f:**60131

[12] Kifer, Y. (1998) Limit theorems for random transformations and processes in random environments, *Trans. Amer. Math. Soc.*, **350**, 1481–1518. MR **98i:**60021

[13] Letac, G. (1986) A contraction principle for certain Markov chains and its applications, *Contemp. Math.*, **50**, 263–273. MR **88a:**60121

[14] Lu, G. and Mukherjea, A. (1997) Invariant measures and Markov chains with random transition probabilities, *Probab. Math. Statist.*, **17**, 115–138. MR **98d:**60126

[15] Orey, S. (1991) Markov chains with stochastically stationary transition probabilities, *Ann. Probab.*, **19**, 907–928. MR **92i:**60065

[16] Peres, Y., Schlag, W. and Solomyak, B. (2000) Sixty years of Bernoulli convolutions, *Fractal geometry and stochastics II (Greifswald, 1998)*, 39–65, Progr. Probab., **46**, Birkhäuser, Basel. CMP 2001:02

[17] Propp, J. G. and Wilson, D. B. (1996) Exact sampling with coupled Markov chains and applications to statistical mechanics, *Random Structures Algorithms*, **9**, 223–252. MR **99k:**60176

[18] Seppäläinen, T. (1994) Large deviations for Markov chains with random transitions, *Ann. Probab.*, **22**, 713–748. MR **95j:**60044

[19] Shiryaev, A. N. (1996) *Probability, Second Edition*, Springer-Verlag, New York. MR **97c:**60003

[20] Silvestrov, D. S. and Stenflo, Ö. (1998) Ergodic theorems for iterated function systems controlled by regenerative sequences, *J. Theoret. Probab.*, **11**, 589–608. MR **99j:**60100

[21] Stenflo, Ö. (2001) Ergodic theorems for Markov chains represented by iterated function systems, *Bull. Polish Acad. Sci. Math.*, **49**, 27–43.

[22] Strichartz, R. S. (1993) Self-similar measures and their Fourier transforms. III., *Indiana Univ. Math. J.*, **42**, 367–411. MR **94j:**42025

[23] Young, L.-S. (1982) Dimension, entropy and Lyapunov exponents. *Ergodic Theory Dynam. Systems*, **2**, 109–124. MR **84k:**58087

Department of Mathematics, Umeå University, SE-90187 Umeå, Sweden
E-mail address: `stenflo@math.umu.se`