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Applied Mathematics
Fall 2011

Dynamical systems problems

1. Find the general solution and sketch the phase portrait for each of the following systems. Characterize the systems as to type (node etc.) and stability.

(a)

$$\begin{cases} x' = -3x + 4y \\ y' = -2x + 3y \end{cases}$$

(b)

$$\begin{cases} x' = 7x + 6y \\ y' = 2x + 6y \end{cases}$$

(c)

$$\begin{cases} x' = -x + y \\ y' = -x - y \end{cases}$$

2. Determine the values of $b \in \mathbb{R}$ for which the system

$$\mathbf{x}' = \begin{pmatrix} 3 & b \\ 1 & 1 \end{pmatrix} \mathbf{x}$$

undergoes a bifurcation.

3. Determine the critical points of the system

$$\begin{cases} x' &= x - y \\ y' &= x^2 + y^2 - 2 \end{cases}$$

and investigate their nature and stability properties.

4. Consider the system

$$\begin{cases} x' &= 4x + 4y - x(x^2 + y^2), \\ y' &= -4x + 4y - y(x^2 + y^2). \end{cases}$$

(a) Show that there is a closed orbit in the region $1 \le r \le 3$, where $r^2 = x^2 + y^2$.
(b) Find the general solution. (Hint: Use polar coordinates.)

Answers or hints:

1. (a) The matrix has eigenvalues 1 and -1 with eigenvectors $\begin{pmatrix}1\\1\end{pmatrix}$ respectively $\begin{pmatrix}2\\1\end{pmatrix}$.
So $(0,0)$ is an (unstable) saddle point.
(b) The matrix has eigenvalues 10 and 3 with eigenvectors $\begin{pmatrix}2\\1\end{pmatrix}$ respectively $\begin{pmatrix}3\\-2\end{pmatrix}$.
So $(0,0)$ is an unstable node.
(c) The matrix has eigenvalues $-1+i$ and $-1-i$ with eigenvectors $\begin{pmatrix}1\\i\end{pmatrix}$ respectively $\begin{pmatrix}1\\-i\end{pmatrix}$.
So $(0,0)$ is an asymptotically stable spiral point.

2. The eigenvalues are $2\pm\sqrt{1+b}$. Thus, $(0,0)$ is an unstable spiral point if $b<-1$. If $-1<b<3$ both eigenvalues are positive, so in this case $(0,0)$ is an unstable node. If $b>3$ one eigenvalue is positive and one is negative, so in this case $(0,0)$ is saddle point. Hence, the system bifurcates at $b=-1$ and $b=3$.

3. The critical points are at $(1,1)$ and $(-1,-1)$. At $(1,1)$ the linearized system has coefficient matrix:

$$A=\begin{pmatrix}1 & -1\\ 2 & 2\end{pmatrix}.$$

The eigenvalues are $\frac{3\pm\sqrt{7}i}{2}$. Thus, $(1,1)$ is an unstable spiral point.
At $(-1,-1)$ the linearized system has coefficient matrix:

$$A=\begin{pmatrix}1 & -1\\ -2 & -2\end{pmatrix}.$$

The eigenvalues are $\frac{-1\pm\sqrt{17}}{2}$. Thus, $(1,1)$ is an (unstable) saddle point.

4. (a) Show: $r'>0$ when $r=1$ and $r'<0$ when $r=3$. Then use the Poincaré-Bendixson theorem.
(b) In polar coordinates the system becomes:

$$\begin{cases} r' &= 4r-r^3,\\ \theta' &= -4.\end{cases}$$

This gives

$$r(t)=\frac{2}{\sqrt{1+\frac{4-r_0^2}{r_0^2}e^{-8t}}},$$
$$\theta(t)=-4t+\theta_0.$$