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A categorical version of the Brouwer-Heyting-Kolmogorov interpretation

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In this paper we interpret (fragments of) intuitionistic logic in categories with weak closure properties, such as quasi left exact categories and locally cartesian closed categories (LCCC) with sums. We also interpret the full choice scheme in an LCCC. The interpretation can be seen as a categorical form of the usual Brouwer-Heyting-Kolmogorov (BHK) interpretation. The standard interpretation of geometric logic in a pretopos is obtained by applying the image functor to the BHK-interpretation.

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1. Introduction

This paper is partly expository, and aims to bridge the gap between two different interpretations of constructive logic: the original one due to Brouwer, Heyting and Kolmogorov, later crystallising as the Curry-Howard propositions-as-types interpretation (Martin-Löf 1984) on the one side, and the topos-theoretic standard semantics on the other side. First we explain the propositions-as-types interpretation in categories, and then its relation to standard semantics in exact completions (Carboni 1995), and how the standard semantics, for important fragments of intuitionistic logic, can be derived from this interpretation. Most of the results are well-known to the specialists in categorical logic, but the literature is maybe not so easily accessible to those outside the area. There are also some new results, e.g. the interpretation of fragments of logic in quasi left exact categories and weak hyperdoctrines. The idea behind this interpretation is the not so well-known Curry-Läuchli adjoint (Lawvere 1996).

The interpretation of a many-sorted first-order language L in a category $\mathbb{C}$ involves assignment of categorical entities to sorts (or types), constants, functions and relations. A sort S is naturally interpreted as an object $\mathcal{M}(S)$ in $\mathbb{C}$, a function $f : S_1 \times \cdots \times S_n \to T$

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is interpreted as a morphism $\mathcal{M}(f) : \mathcal{M}(S_1) \times \cdots \times \mathcal{M}(S_n) \to \mathcal{M}(T)$ in $\mathbb{C}$. A constant c of sort T can be considered as 0-ary function, and thus interpreted as a morphism $\mathcal{M}(c) : 1 \to \mathcal{M}(T)$. We shall assume that the category $\mathbb{C}$ has finite products. For the interpretation of relations there seems to be at least two natural choices.

In the standard interpretation of categorical logic (Makkai and Reyes 1977), a relation symbol R on $S_1 \times \cdots \times S_n$ is interpreted as a subobject of $\mathcal{M}(S_1) \times \cdots \times \mathcal{M}(S_n)$, represented by a monomorphism $\mathcal{M}(R) \rightarrowtail \mathcal{M}(S_1) \times \cdots \times \mathcal{M}(S_n)$. An interpretation of a formula φ with free variables of sorts $T_1, \ldots, T_m$ will be a subobject of $\mathcal{M}(T_1) \times \cdots \times \mathcal{M}(T_m)$. We may call this the *propositions-as-subobjects* interpretation. (Note that the definition of a subobject in a category requires quantification over all objects and morphisms. The foundational difficulties of such notions in relation to set theory have been recognised and addressed using Grothendieck universes or fibered category theory, see for instance (Borceux 1994).)

Another possibility is to interpret the relation as a morphism $f : \mathcal{M}(R) \to \mathcal{M}(S_1) \times \cdots \times \mathcal{M}(S_n)$. In this case there may, loosely speaking, be several "reasons" or "proofs" for a tuple to belong to the relation. The interpretation is then extended so that each formula φ with free variables of sorts $T_1, \ldots, T_m$ will be a morphism into $\mathcal{M}(T_1) \times \cdots \times \mathcal{M}(T_m)$. This is similar to the propositions-as-types interpretation by Curry and Howard (cf. (Martin-Löf 1984)), where a proposition is the type of its proofs, only that we replace types by categorical objects. This interpretation may be named the *propositions-as-objects* interpretation or the *categorical Brouwer-Heyting-Kolmogorov (BHK) interpretation.*

One important rationale for using internal logic in a category is to facilitate reasoning about the category. The categorical BHK-interpretation is indeed applicable to a wider class of categories than standard categorical interpretations. For instance to BHK-interpret the $\exists, \wedge$-fragment of intuitionistic logic, i.e. regular logic, it is enough to use a category with finite limits, or even a quasi left exact category (see Section 2 and 3). On the other hand, this internal logic is naturally different so that e.g. internal surjections correspond to external split epimorphisms. Nevertheless, it can be shown for pretoposes (or more generally *lextensive* regular categories (Carboni *et al.* 1993)) that the standard interpretation is obtained by applying the image functor to the BHK-interpretation (see Section 4). In a locally cartesian closed category (LCCC), with finite sums, the BHK-interpretation verifies not only first-order logic but also the full choice scheme (Awodey 1995); see Section 6 and 7. This is of course expected in view of Seely's interpretation (Seely 1984) of Martin-Löf type theory (Martin-Löf 1984). (However (Seely 1984) is slightly flawed by some subtle coherence problems relating to equality of types. These defects were corrected later by Curien (1993) by a change of the type theory, though some constructions of type theory have not been dealt with explicitly. Another solution due to M. Hofmann uses the Bénabou construction, which a priori assumes classical set theory as a meta-theory.) In the final section, we see that categories with finite limits, or even lextensive LCCCs, can be extended using the *exact completion* procedure (Carboni 1995; Carboni and Rosolini 2000). The standard interpretation in the completed category is then shown to be equivalent to the present BHK-interpretation of the original category. It seems that the BHK-interpretation has, among topos-theorists, been considered as the standard semantics of the completed category, so in this sense the result is not new. The verifications involved in the BHK-interpretation are fairly straightforward and follow the pattern of the standard interpretation (Makkai and Reyes 1977) quite closely. Hence we give few detailed proofs.

2. Quasi Left Exact Categories

The role of pullbacks in categorical interpretations of logic is to do substitution of terms into predicates and formulas. Let $\mathbb{C}$ be a category. A commutative square

$$\begin{tikzcd} D \arrow[r, "q"] \arrow[d, "p"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd} \tag{1}$$

is called a *weak pullback* if for all $p' : E \longrightarrow A$ and $q' : E \longrightarrow B$ such that $f \circ p' = g \circ q'$, there exists a map $r : E \longrightarrow D$ such that $p \circ r = p'$ and $q \circ r = q'$. It is an ordinary (strong) pullback square if, and only if, p and q are jointly monic. As weak pullbacks are unique up to an equivalence $\approx$ given by order reflection (see Section 3), they will suffice for the purpose of substitution. A *weak equalizer* is defined similarly to an equalizer, but dropping the uniqueness condition of the universal arrow. The following propositions are proved analogously to their "strong" counterparts.

Lemma 2.1. A category $\mathbb{C}$ with binary products, has all weak pullbacks if, and only if, it has all weak equalizers.

Proof. We prove the direction $\Rightarrow$. Suppose that $f, g : A \longrightarrow B$ are morphisms in $\mathbb{C}$. Then a weak equalizer of these is obtained by pulling back the diagonal $\langle 1, 1 \rangle : B \longrightarrow B \times B$ along $\langle f, g \rangle : A \longrightarrow B \times B$. □

Lemma 2.2. If the left and right square of the commutative diagram below both are weak pullbacks, then the outer rectangle is a weak pullback.

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \arrow[r] & \cdot \end{tikzcd}$$

□

A category is called a *quasi left exact category (qlex)* if it has all weak pullbacks and all finite products. A motivation for developing semantics from weak pullbacks is that a natural category arising from type theory (see Example 2.4 below) appears to lack strong pullbacks. From a categorical perspective a more natural notion is that of a *weak left exact category* in which products may be weak as well; see (Carboni and Vitale 1998).

Example 2.3 Let **Grpd** be the category having small groupoids as objects, equivalence classes of functors (between groupoids) as morphisms, and where two functors are identified if they are naturally isomorphic. This category has weak pullbacks and set-indexed products, but in general no pullbacks. We refer to (Palmgren 200?) for a proof.

Example 2.4 The category of types in Martin-Löf type theory is an example of a quasi left exact category. Its objects are types and the morphisms are functions on types with an equality determined point-wise by the identity type $\mathrm{Id}_A(a,b)$. More precisely $f : A \to B$ and $g : A \to B$ are equivalent if the following type is inhabited

$$(\Pi x \in A)\mathrm{Id}_B(f(x), g(x)).$$

The binary product is given by the usual type-theoretic product. The weak equalizer of $f : A \to B$ and $g : A \to B$ is then given by the construction

$$(\Sigma x \in A)\mathrm{Id}_B(f(x), g(x)).$$

It is not an equalizer unless uniqueness of identity proofs is assumed, i.e. that each identity type $\mathrm{Id}_D(u,v)$ contains at most one element up to Id-equality. However, we do not know whether this category has pullbacks (in some minimal axiomatization of type theory).

3. The Logic of Quasi Left Exact Categories

A map $r : R \to X$ in a category $\mathbb{C}$ may be regarded as a subobject of X. The intuitive interpretation is that x is in the subobject iff $r^{-1}(x)$ is inhabited, i.e. $x = r(x_0)$ for some $x_0 \in R$. If $s : S \to X$ is another map, we say that it is *included in* (R,r) if there is a map $f : S \to R$ with $r \circ f = s$. This inclusion relation, denoted $\lesssim$, defines a preorder on the slice category $\mathbb{C}/X$, where $1_X : X \to X$ is the terminal object. These maps are equivalent, $(R,r) \approx (S,s)$, if $(R,r) \lesssim (S,s)$ and $(S,s) \lesssim (R,r)$. The corresponding equivalence classes are called *pre-subobjects*. The induced partial order (the order reflection) is denoted $(\mathrm{Psub}_{\mathbb{C}}(X), \lesssim)$. Note that weak pullbacks are unique up to pre-subobject equivalence, so any map $f : A \to C$ in a qlex category $\mathbb{C}$, defines an order preserving function

$$f^{-1} : \mathrm{Psub}_{\mathbb{C}}(C) \to \mathrm{Psub}_{\mathbb{C}}(A)$$

by taking weak pullbacks along f. By Lemma 2.2, it follows that $g^{-1}f^{-1} = (fg)^{-1}$ for $g : X \to A$. Consider the weak pullback diagram

$$\begin{tikzcd} D \arrow[r, "q"] \arrow[d, "p"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

Let $r = fp = gq$. For $h : E \to C$ we have

$(E,h) \lesssim (A,f)$ and $(E,h) \lesssim (B,g)$ if, and only if, $(E,h) \lesssim (D,r)$.

When the weak pullback (D,r) of (A,f) and (B,g) is regarded as a pre-subobject of C, we denote it by $(A,f)\wedge(B,g)$. It follows, similarly as for ordinary subobjects, that inverse functions preserve infima, i.e. for any $k : X \to C$,

$$k^{-1}(A \wedge B) \approx k^{-1}(A) \wedge k^{-1}(B).$$

The interpretation $\mathcal{M}$ of the symbols of a many-sorted first-order language is the same as in a regular category (Makkai and Reyes 1977) except that a predicate symbol is interpreted as a pre-subobject rather than a subobject. In general, a formula φ with free variables (FV) among $\overline{x} = x_1, \ldots, x_n$ of sorts $S_1, \ldots, S_n$, respectively, will be interpreted as a pre-subobject $(\mathcal{M}_{\overline{x}}(\varphi), m)$ of the product $\mathcal{M}(S_1) \times \cdots \times \mathcal{M}(S_n)$. When the sorts can be inferred from the variables, we also denote the latter product by $\mathcal{M}(x_1, \ldots, x_n)$. A sequent $\varphi_1, \ldots, \varphi_n \overset{\overline{x}}{\Longrightarrow} \psi$ is *(BHK-)valid* in $\mathcal{M}$ iff

$$\mathcal{M}_{\overline{x}}(\varphi_1) \wedge \cdots \wedge \mathcal{M}_{\overline{x}}(\varphi_n) \lesssim \mathcal{M}_{\overline{x}}(\psi).$$

In particular, a sequent $\overset{\overline{x}}{\Longrightarrow} \psi$ is valid in $\mathcal{M}$ iff $1_{\mathcal{M}(\overline{x})} \lesssim \mathcal{M}_{\overline{x}}(\psi)$, i.e. the interpretation map $m : \mathcal{M}_{\overline{x}}(\psi) \to \mathcal{M}(\overline{x})$ has a section, i.e. there is some g such that $m \circ g = 1_{\mathcal{M}(\overline{x})}$.

For $FV(\varphi, \psi) \subseteq \overline{x}$, define

$$\mathcal{M}_{\overline{x}}(\varphi \wedge \psi) = \mathcal{M}_{\overline{x}}(\varphi) \wedge \mathcal{M}_{\overline{x}}(\psi).$$

Sequences of formulas are denoted by Γ, Δ etc. The following rules are immediately verified

L1. (Assumption) For $\overline{x} \supseteq FV(\varphi_1, \ldots, \varphi_n)$ and $i = 1, \ldots, n$:

$$\varphi_1, \ldots, \varphi_n \overset{\overline{x}}{\Longrightarrow} \varphi_i$$

L2. (Weakening) For any ψ with $FV(\psi) \subseteq \overline{x}$:

$$\frac{\Gamma \overset{\overline{x}}{\Longrightarrow} \varphi}{\Gamma, \psi \overset{\overline{x}}{\Longrightarrow} \varphi}$$

L3. (Conjunction rules)

$$(\wedge - I)\ \frac{\Gamma \overset{\overline{x}}{\Longrightarrow} \varphi_1 \qquad \Gamma \overset{\overline{x}}{\Longrightarrow} \varphi_2}{\Gamma \overset{\overline{x}}{\Longrightarrow} \varphi_1 \wedge \varphi_2} \qquad\qquad (\wedge - E)\ \frac{\Gamma \overset{\overline{x}}{\Longrightarrow} \varphi_1 \wedge \varphi_2}{\Gamma \overset{\overline{x}}{\Longrightarrow} \varphi_i}\ (i = 1 \text{ or } 2)$$

L4. (Cut rule) For $FV(\Gamma, \Delta, \varphi, \psi) \subseteq \overline{x}$:

$$\frac{\Gamma, \varphi, \Delta \overset{\overline{x}}{\Longrightarrow} \psi \quad \Gamma, \Delta \overset{\overline{x}}{\Longrightarrow} \varphi}{\Gamma, \Delta \overset{\overline{x}}{\Longrightarrow} \psi}$$

Since $\mathbb{C}$ has finite products the interpretation of terms is as usual (Makkai and Reyes 1977). If t is a term of sort T, with $FV(t) \subseteq \overline{x} = x_1, \ldots, x_n$, then its interpretation is a morphism $\mathcal{M}_{\overline{x}}(t) : \mathcal{M}(\overline{x}) \to \mathcal{M}(T)$, defined by induction on t.

(i) For a variable x_i: $\mathcal{M}_{x_1,\ldots,x_n}(x_i) = \pi_i : \mathcal{M}(x_1, \ldots, x_n) \to \mathcal{M}(x_i)$.

(ii) For a constant symbol c of sort T, $\mathcal{M}_{x_1,\ldots,x_n}(c) : \mathcal{M}(x_1, \ldots, x_n) \to \mathcal{M}(T)$ is the unique map factoring through $\mathcal{M}(c) : 1 \to \mathcal{M}(T)$.

(iii) For a function symbol $f : S_1 \times \cdots \times S_n \longrightarrow T$, and terms $t_1, \ldots, t_n$ of sorts $S_1, \ldots, S_n$, with free variables among $\overline{x}$

$$\mathcal{M}_{\overline{x}}(f(t_1, \ldots, t_n)) = \mathcal{M}(f) \circ \langle \mathcal{M}_{\overline{x}}(t_1), \ldots, \mathcal{M}_{\overline{x}}(t_n) \rangle.$$

Let $m : \mathcal{M}(R) \longrightarrow \mathcal{M}(S_1) \times \cdots \times \mathcal{M}(S_n)$ be an interpretation of the relation symbol R. If $t_1, \ldots, t_n$ are terms of sorts $S_1, \ldots, S_n$, respectively, with free variables among $\overline{x}$, the interpretation of $\mathcal{M}_{\overline{x}}(R(t_1, \ldots, t_n))$ is

$$\langle \mathcal{M}_{\overline{x}}(t_1), \ldots, \mathcal{M}_{\overline{x}}(t_n) \rangle^{-1}(\mathcal{M}(R), m)$$

The equality predicate symbol $=_S$ on a sort S is always interpreted by the diagonal

$$\mathcal{M}(=_S) = \langle 1, 1 \rangle : \mathcal{M}(S) \longrightarrow \mathcal{M}(S) \times \mathcal{M}(S).$$

If the maps $\mathcal{M}_{\overline{x}}(s), \mathcal{M}_{\overline{x}}(t) : \mathcal{M}(S_1) \times \cdots \times \mathcal{M}(S_n) \longrightarrow \mathcal{M}(S)$ are interpretations of terms s and t, then $\mathcal{M}_{\overline{x}}(s = t)$ is the weak equalizer of these maps, according to the proof of Lemma 2.1.

Lemma 3.1. Let $e : E \longrightarrow X$ be a weak equalizer of $f, g : X \longrightarrow Y$. Then $f = g$ if, and only if, $(X, 1_X) \lesssim (E, e)$. $\square$

Thus the sequent $\stackrel{\overline{x}}{\Longrightarrow} s = t$ is valid in $\mathcal{M}$ if, and only if, $\mathcal{M}_{\overline{x}}(s) = \mathcal{M}_{\overline{x}}(t)$.

One easily proves that for φ which are formed by conjunctions of atomic formulas:

$$\langle \mathcal{M}_{\overline{u}}(t_1), \ldots, \mathcal{M}_{\overline{u}}(t_n) \rangle^{-1}(\mathcal{M}_{\overline{x}}(\varphi)) \approx \mathcal{M}_{\overline{u}}(\varphi(t_1, \ldots, t_n / x_1, \ldots, x_n)). \qquad (2)$$

From this *substitution equivalence,* the following substitution rule is now easily verified:

L5. (Substitution) For a sequence of terms $\overline{t}$ whose sorts agree with those of the variable sequence $\overline{x}$, item-wise, and for which $FV(\overline{t}) \subseteq \overline{u}$

$$\frac{\Gamma \stackrel{\overline{x}}{\Longrightarrow} \varphi}{\Gamma(\overline{t}/\overline{x}) \stackrel{\overline{u}}{\Longrightarrow} \varphi(\overline{t}/\overline{x})}$$

To verify the following equality axioms it is sufficient to use weak equalizers.

L6. (Equality axioms)

$$\stackrel{x}{\Longrightarrow} x = x, \qquad x = y \stackrel{x,y}{\Longrightarrow} y = x, \qquad x = y, y = z \stackrel{x,y,z}{\Longrightarrow} x = z,$$

$$u_1 = v_1, \ldots, u_n = v_n \stackrel{\overline{u},\overline{v}}{\Longrightarrow} f(u_1, \ldots, u_n) = f(v_1, \ldots, v_n),$$

$$u_1 = v_1, \ldots, u_n = v_n, R(u_1, \ldots, u_n) \stackrel{\overline{u},\overline{v}}{\Longrightarrow} R(v_1, \ldots, v_n).$$

The existential quantifier is easy to BHK-interpret. For $\alpha : A \longrightarrow X$ and $f : X \longrightarrow Y$ define $\tilde{\exists}_f(A, \alpha) = f \circ \alpha$. Thus we have a well-defined function

$$\tilde{\exists}_f : \mathrm{Psub}(X) \longrightarrow \mathrm{Psub}(Y)$$

which satisfies the following properties. (Note that we do not take the image of $f \circ \alpha$ as in the standard categorical interpretation.)

Proposition 3.2. Let $\mathbb{C}$ be a qlex category. For relevant pre-subobjects A and B and maps f, g in $\mathbb{C}$:

(i) $A \lesssim B \Rightarrow \tilde{\exists}_f(A) \lesssim \tilde{\exists}_f(B)$.
(ii) $\tilde{\exists}_f(A) \lesssim B \Leftrightarrow A \lesssim f^{-1}(B)$.
(iii) $\tilde{\exists}_g \circ \tilde{\exists}_f \approx \tilde{\exists}_{g \circ f}$ and $\tilde{\exists}_1 = 1$.
(iv) $\tilde{\exists}_f(f^{-1}(B) \wedge A) \approx B \wedge \tilde{\exists}_f(A)$.
(v) $\tilde{\exists}_{f'}(h^{-1}(A)) \approx g^{-1}(\tilde{\exists}_f(A))$, if the following is a weak pullback

$$\begin{tikzcd} X' \arrow[r, "h"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

Proof. (i) and (iii) are immediate by the properties of composition. (ii) is direct by considering the pullback square associated with $f^{-1}(B)$. (iv) and (v) follow from Lemma 2.2 and the fact that weak pullbacks are unique up to the equivalence $\approx$. □

Let φ be a formula with free variables among $\overline{x}, y$, where y has sort S. Then the interpretation of the existential quantifier is given by

$$\mathcal{M}_{\overline{x}}(\exists y \in S.\varphi) =_{\mathrm{def}} \tilde{\exists}_p(\mathcal{M}_{\overline{x},y}(\varphi))$$

where p is the projection $\mathcal{M}(\overline{x}, y) \to \mathcal{M}(\overline{x})$. Using Proposition 3.2 it is now straightforward to verify the substitution equivalence (2) for formulas including existential quantifiers as well. The familiar rules for the existential quantifier are easy to validate.

L8. (Rules for existential quantification)

$$(\exists I)\ \frac{\Gamma \stackrel{\overline{x}}{\Longrightarrow} \varphi(b/y)}{\Gamma \stackrel{\overline{x}}{\Longrightarrow} \exists y \in S.\varphi} \quad (FV(b) \subseteq \overline{x}; b \text{ has sort } S)$$

$$(\exists E)\ \frac{\Gamma \stackrel{\overline{x}}{\Longrightarrow} \exists y \in S.\varphi \quad \Gamma, \varphi \stackrel{\overline{x},y}{\Longrightarrow} \psi}{\Gamma \stackrel{\overline{x}}{\Longrightarrow} \psi} \quad (y \notin \overline{x}, FV(\Gamma, \psi))$$

The rules and axioms of *regular logic* are L1 – L8. We summarize the above claims.

Theorem 3.3. Regular logic is valid under the BHK-interpretation in a quasi left exact category. □

The internal BHK-logic of a qlex category may be used to characterize categorical properties. Most notably, split epimorphisms correspond to surjective functions in this logic.

Proposition 3.4. Let $\mathbb{C}$ be a qlex category.

(i) $f : X \longrightarrow Y$ is a split epi iff this sequent is BHK-valid: $\overset{y}{\Longrightarrow} \exists x \in X. f(x) = y$.
(ii) $f : X \longrightarrow Y$ is mono iff this sequent is BHK-valid: $f(x) = f(y) \overset{x,y}{\Longrightarrow} x = y$.
(iii) Suppose $e : E \longrightarrow A$ satisfies $f \circ e = g \circ e$, where $f, g : A \longrightarrow B$. Then e is a weak equalizer iff this sequent is BHK-valid:

$$f(x) = g(x) \overset{x}{\Longrightarrow} \exists z \in E. e(z) = x. \Box$$

4. The Standard Interpretation as an Image of BHK

We show that the standard interpretation of regular logic in a regular category is the image of the BHK-interpretation. In the process we obtain an alternative proof for the basic properties of the standard existential quantifier (cf. Corollary 4.2). Thus one may take BHK as the fundamental interpretation and derive the standard interpretation from it.

Recall that a category is *regular* if

(i) it has all finite limits, i.e. is *left exact,*
(ii) every arrow can be factored into a regular epi (that is an epi which is a coequalizer) followed by a mono,
(iii) the property of being a regular epi is preserved when taking pullbacks along any morphism.

The regular-epi/mono factorisations are unique up to isomorphism, and if we let $\mathrm{Sub}_{\mathbb{C}}(X)$ denote the subobjects of X in $\mathbb{C}$, this factorisation induce a left adjoint $\mathrm{Im}_X : \mathrm{Psub}_{\mathbb{C}}(X) \longrightarrow \mathrm{Sub}_{\mathbb{C}}(X)$ to the inclusion $\mathrm{U}_X : \mathrm{Sub}_{\mathbb{C}}(X) \longrightarrow \mathrm{Psub}_{\mathbb{C}}(X)$. The following lemma is now an easy consequence of this adjunction.

Lemma 4.1. Let $\mathbb{C}$ be a regular category. Then the following holds

(i) $\mathrm{Im}_X \circ f^{-1} = f^{-1} \circ \mathrm{Im}_Y$ for $f : X \longrightarrow Y$,
(ii) $\mathrm{Im}_X(\alpha \wedge \beta) = \mathrm{Im}_X(\alpha) \wedge \mathrm{Im}_X(\beta)$,
(iii) $\mathrm{Im}_Y \circ \tilde{\exists}_f = \exists_f \circ \mathrm{Im}_X$, where $\exists_f = \mathrm{Im}_Y \circ \tilde{\exists}_f \circ U_X$. $\Box$

Define $\exists_f = \mathrm{Im}_Y \circ \tilde{\exists}_f \circ U_X$. Now the usual properties of the standard existential quantifier follows from the theorem above, the adjunction $\mathrm{Im}_X \dashv \mathrm{U}_X$, and Proposition 3.2.

Corollary 4.2. For relevant subobjects A and B and maps f, g in a regular category:

(i) $A \leq B \Rightarrow \exists_f(A) \leq \exists_f(B)$.
(ii) $\exists_f(A) \leq B \Leftrightarrow A \leq f^{-1}(B)$.
(iii) $\exists_g \circ \exists_f = \exists_{g \circ f}$ and $\exists_1 = 1$.
(iv) $\exists_f(f^{-1}(B) \wedge A) = B \wedge \exists_f(A)$
(v) $\exists_{f'}(h^{-1}(A)) = g^{-1}(\exists_f(A))$, if the following is a pullback

$$\begin{tikzcd} X' \arrow[r, "h"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd} \quad \Box$$

A *regular formula* is a formula built up from atoms using only $\exists$ and $\wedge$. Lemma 4.1 states that the image operator commutes with substitution, conjunction and existential quantification, so the following result is straightforwardly proved by induction.

Theorem 4.3. Let $\mathbb{C}$ be a regular category. Suppose that $\mathcal{M}$ is an interpretation where all relation symbols are interpreted as subobjects. Denote by $\widetilde{\mathcal{M}}$ the BHK-interpretation function and let $\overline{\mathcal{M}}$ be the standard interpretation function. Then for regular formulas φ with $FV(\varphi) \subseteq \overline{x}$:

$$\overline{\mathcal{M}}_{\overline{x}}(\varphi) = \mathrm{Im}(\widetilde{\mathcal{M}}_{\overline{x}}(\varphi)). \square$$

5. Geometric Logic

We consider the BHK-interpretation of disjunction and extend the results of the previous section.

Recall that a category has finite sums if, and only if, it has binary sums and an initial object 0. Denote by 0_A the unique map from 0 to A. Suppose that $\alpha : A \longrightarrow X$ and $\beta : B \longrightarrow X$. Let $[\alpha, \beta] : A + B \longrightarrow X$ be the universal map given by the binary sum. For pre-subobjects $(A, \alpha), (B, \beta), (C, \gamma)$ of X we have

(S1) $(A, \alpha) \lesssim (A + B, [\alpha, \beta])$ and $(B, \beta) \lesssim (A + B, [\alpha, \beta])$,
(S2) If $(A, \alpha) \lesssim (C, \gamma)$ and $(B, \beta) \lesssim (C, \gamma)$, then $(A + B, [\alpha, \beta]) \lesssim (C, \gamma)$.

Sums are *weakly stable* if for any map $f : Y \longrightarrow X$ we have as pre-subobjects of Y:

$$\begin{array}{rcl} f^{-1}(0) & \approx & 0 \\ f^{-1}(A+B) & \approx & f^{-1}(A) + f^{-1}(B). \end{array}$$

Letting q_C denote the projection $f^{-1}(C) \longrightarrow Y$ of the weak pullback of $\gamma : C \longrightarrow X$ along f, we can express this condition more fully as

$$\begin{array}{rcl} (f^{-1}(0), q_0) & \approx & (0, 0_Y) \\ (f^{-1}(A+B), q_{A+B}) & \approx & (f^{-1}(A) + f^{-1}(B), [q_A, q_B]). \end{array}$$

The stability condition ensures that substitution works also for formulas involving absurdity and disjunction.

Let $\mathbb{C}$ be a qlex category with finite sums that are weakly stable. Define $\mathcal{M}_{\overline{x}}(\bot)$ to be the unique map $0 \longrightarrow \mathcal{M}(\overline{x})$. Since this is the least element in $\mathrm{Psub}(\mathcal{M}(\overline{x}))$, the absurdity rule is directly verified.

L9. (Absurdity) For any φ with $FV(\varphi) \subseteq \overline{x}$:

$$\frac{\Gamma \xRightarrow{\overline{x}} \bot}{\Gamma \xRightarrow{\overline{x}} \varphi}.$$

Let φ and ψ be formulas with $FV(\varphi, \psi) \subseteq \overline{x}$. Suppose that their interpretations are $\alpha : \mathcal{M}_{\overline{x}}(\varphi) \longrightarrow \mathcal{M}(\overline{x})$ and $\beta : \mathcal{M}_{\overline{x}}(\psi) \longrightarrow \mathcal{M}(\overline{x})$. Then the interpretation of the disjunction is

$$[\alpha, \beta] : \mathcal{M}_{\overline{x}}(\varphi \vee \psi) =_{\mathrm{def}} \mathcal{M}_{\overline{x}}(\varphi) + \mathcal{M}_{\overline{x}}(\psi) \longrightarrow \mathcal{M}(\overline{x}).$$

The following rules are now easily verified using (S1) and (S2).

L10. (Disjunction rules) For $FV(\varphi_1, \varphi_2) \subseteq \overline{x}$:

$$(\vee I)\ \frac{\Gamma \overset{\overline{x}}{\Longrightarrow} \varphi_i}{\Gamma \overset{\overline{x}}{\Longrightarrow} \varphi_1 \vee \varphi_2}\ (i = 1 \text{ or } 2) \qquad (\vee E)\ \frac{\Gamma \overset{\overline{x}}{\Longrightarrow} \varphi_1 \vee \varphi_2 \quad \Gamma, \varphi_1 \overset{\overline{x}}{\Longrightarrow} \psi \quad \Gamma, \varphi_2 \overset{\overline{x}}{\Longrightarrow} \psi}{\Gamma \overset{\overline{x}}{\Longrightarrow} \psi}$$

Geometric logic is defined by the rules L1 – L10. We summarize the verification as

Theorem 5.1. Geometric logic is valid under the BHK-interpretation in a qlex category with finite sums that are weakly stable. □

There is a standard interpretation of this logic in a pretopos, which is a factor of the abstract BHK-interpretation (see Theorem 5.4). The standard interpretation works also for more general categories: *lextensive regular* categories. A category $\mathbb{C}$ with finite sums is *extensive* if the canonical functor $+ : \mathbb{C}/A \times \mathbb{C}/B \longrightarrow \mathbb{C}/(A + B)$ is an equivalence. In the presence of finite limits this extensiveness can be characterised as follows.

Lemma 5.2 (Carboni, Lack and Walters (1993)). Let $\mathbb{C}$ be a left exact category. Then $\mathbb{C}$ is extensive if, and only if, every commutative diagram of the form

$$\begin{tikzcd} A_1 \arrow[r] \arrow[d] & A \arrow[d] & A_2 \arrow[l] \arrow[d] \\ X_1 \arrow[r, "i_1"] & X_1 + X_2 & X_2 \arrow[l, "i_2"'] \end{tikzcd} \tag{3}$$

constitutes a pair of pullbacks in $\mathbb{C}$ precisely when the top row is a co-product diagram in $\mathbb{C}$. □

A category is called *lextensive* if it is left exact and extensive. From Lemma 5.2 follows (Carboni *et al.* 1993, Prop. 2.8) that in such a category, any arrow into 0 is invertible, i.e. *initials are strict;* thus all finite sums are stable under pullbacks. Any pretopos is lextensive and regular.

Lemma 5.3. Let $\mathbb{C}$ be a lextensive and regular category. Then for pre-subobjects $\alpha : A \longrightarrow X$ and $\beta : B \longrightarrow X$,

$$\mathrm{Im}([\alpha, \beta]) = \mathrm{Im}(\alpha) \vee \mathrm{Im}(\beta). \tag{4}$$

Also, $\mathrm{Im}(0 \longrightarrow X) = (0 \longrightarrow X)$.

Proof. Let $\alpha : A \longrightarrow X$ and $\beta : B \longrightarrow X$ be arbitrary morphisms in $\mathbb{C}$. Consider regular epi-mono factorisations of these maps $e_1 : A \longrightarrow \mathrm{Im}(\alpha)$, $m_1 : \mathrm{Im}(\alpha) \longrightarrow X$ and $e_2 : B \longrightarrow \mathrm{Im}(\beta)$, $m_2 : \mathrm{Im}(\beta) \longrightarrow X$. Using the fact that coequalizer diagrams are closed under the coproduct operation, it follows that $e_1 + e_2 : A + B \longrightarrow \mathrm{Im}(\alpha) + \mathrm{Im}(\beta)$ is a regular epi. Then take the regular epi-mono factorisation of $[m_1, m_2]$, which is $e : \mathrm{Im}(\alpha) + \mathrm{Im}(\beta) \longrightarrow \mathrm{Im}([m_1, m_2])$, $m : \mathrm{Im}([m_1, m_2]) \longrightarrow X$. The right hand side of (4) is the subobject m. But $e \circ (e_1 + e_2)$ and m is an regular epi-mono factorisation of $[\alpha, \beta] : A + B \longrightarrow X$. By the uniqueness of such factorisations up isomorphism, the equality (4) follows.

To prove $\mathrm{Im}(0 \longrightarrow X) = (0 \longrightarrow X)$ it suffices to observe that maps from a strict initial object are mono. □

A formula is *geometric* if it is built up from atomic formulas using only $\bot, \wedge, \vee, \exists$. Induction on formulas yields:

Theorem 5.4. Theorem 4.3 extends to geometric formulas in a lextensive, regular category. $\square$

Remark 5.5 Notice the following consequence of this theorem. If a sequent $\varphi \overset{\overline{x}}{\Longrightarrow} \psi$, where φ and ψ are geometric, is BHK-valid in a lextensive regular category, then it is also valid in the standard interpretation. The converse need not hold, in view of Proposition 3.4.(i).

6. First-Order Logic

Now suppose that $\mathbb{C}$ is qlex, has finite sums and that for every $f : X \to Y$ there is an order preserving $\forall_f : \mathrm{Psub}(X) \to \mathrm{Psub}(Y)$ such that

$$f^{-1}(A) \lesssim B \Leftrightarrow A \lesssim \forall_f(B). \tag{5}$$

Call such a category a *weak hyperdoctrine*. Using the above adjunction (5) one easily proves

Proposition 6.1. The finite sums of a weak hyperdoctrine are weakly stable. $\square$

The following is a consequence of the adjunction above and the Beck-Chevalley condition for $\tilde{\exists}$.

Proposition 6.2. In a weak hyperdoctrine,

$$\forall_{f'}(h^{-1}(A)) \approx g^{-1}(\forall_f(A)),$$

if the following is a weak pullback

$$\begin{tikzcd} X' \arrow[r, "h"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd} \quad \square$$

For $FV(\varphi) \subseteq \overline{x}, y$ define

$$\mathcal{M}_{\overline{x}}(\forall y \in S.\varphi) =_{\mathrm{def}} \forall_p(\mathcal{M}_{\overline{x},y}(\varphi))$$

where $p : \mathcal{M}(\overline{x}, y) \to \mathcal{M}(\overline{x})$ is the obvious projection. For $(B,\beta),(C,\gamma) \in \mathrm{Psub}(X)$ define

$$H((B,\beta),(C,\gamma)) = \forall_\beta(B \wedge C, p),$$

where p is the projection $B \wedge C \to B$. Then it follows that for $(A,\alpha) \in \mathrm{Psub}(X)$,

$$A \wedge B \lesssim C \Longleftrightarrow A \lesssim H(B,C).$$

For $FV(\varphi,\psi) \subseteq \overline{x}$ define

$$\mathcal{M}_{\overline{x}}(\varphi \to \psi) = H(\mathcal{M}_{\overline{x}}(\varphi), \mathcal{M}_{\overline{x}}(\psi)).$$

The substitution equalities for $\forall$ and $\longrightarrow$ are consequences of Proposition 6.2. The verification of rules L11 – L12 below follows the usual pattern.

L11. (Rules for universal quantification)

$$(\forall I)\ \frac{\Gamma \overset{\overline{x},y}{\Longrightarrow} \varphi}{\Gamma \overset{\overline{x}}{\Longrightarrow} \forall y \in S.\varphi} \quad (y \notin FV(\Gamma))$$

$$(\forall E)\ \frac{\Gamma \overset{\overline{x}}{\Longrightarrow} \forall y \in S.\varphi}{\Gamma \overset{\overline{x}}{\Longrightarrow} \varphi(b/y)} \quad (FV(b) \subseteq \overline{x}; b \text{ has sort } S)$$

L12. (Implication rules)

$$(\rightarrow I)\ \frac{\Gamma, \varphi \overset{\overline{x}}{\Longrightarrow} \psi}{\Gamma \overset{\overline{x}}{\Longrightarrow} \varphi \rightarrow \psi} \qquad (\rightarrow E)\ \frac{\Gamma \overset{\overline{x}}{\Longrightarrow} \varphi \rightarrow \psi \quad \Gamma \overset{\overline{x}}{\Longrightarrow} \varphi}{\Gamma \overset{\overline{x}}{\Longrightarrow} \psi}$$

The rules L1 – L12 determine *first-order intuitionistic logic.*

Theorem 6.3. The rules of first-order intuitionistic logic are BHK-valid in a weak hyperdoctrine. $\Box$

Example 6.4 The category of types introduced in Example 2.4 is a weak hyperdoctrine.

Remark 6.5 The sums in type theories are not necessarily disjoint. (Smith 1988) showed that in the absence of universes, the Peano axiom $\neg\mathrm{Id}_N(0, S(x))$ is not provable in Martin-Löf type theory. The same argument shows that disjointness of sums $\neg\mathrm{Id}_{A+B}(\mathrm{inl}(x), \mathrm{inr}(y))$ is not provable either in this weak context. However, all that is needed to prove these negative statements is the assumption that there exists a family of types $L(z)$, where $z \in \{0, 1\}$, such that $L(0)$ is empty and $L(1)$ is inhabited.

7. The Axiom of Choice and LCCC

The axiom of choice, which in its set-theoretic and topos-theoretic formulation has such strong consequences, is in Martin-Löf type theory a triviality. This is due to the BHK-interpretation of the existential quantifier as a (strong) Σ-type. Since (Seely 1984) it has been known that type theories with strong Σ-types can be interpreted in a locally cartesian closed category. The combination of these facts did not, as far as the author knows, make an immediate impression in categorical logic. It appeared in print only in 1995 as an abstract (Awodey 1995).

Recall that a locally cartesian closed category (LCCC) is a category $\mathbb{C}$ with finite limits, in which the pullback functor

$$f^* : \mathbb{C}/Y \longrightarrow \mathbb{C}/X$$

has a right adjoint Π_f, for any $f : X \longrightarrow Y$ in $\mathbb{C}$. The left adjoint Σ_f always exists and is just a composition with f. The "order reflection" of Σ_f and Π_f is $\tilde{\exists}_f$ respectively $\tilde{\forall}_f$ (cf. Proposition 3.2).

The logic obtained by a BHK-interpretation into a LCCC turns out to be quite powerful. The full choice scheme is valid.

Proposition 7.1. Any LCCC, with finite sums, is a weak hyperdoctrine.

Proof. The only verification which is not direct is the existence of the right adjoint $\tilde{\forall}_f$. For this use that Π_f is the right adjoint of f^*. $\square$

Corollary 7.2. The rules of first-order intuitionistic logic are BHK-valid in an LCCC with finite sums. $\square$

The idea of the following proof is that, under the BHK-interpretation, a proof of a quantifier combination $\forall\exists$ already contains the choice function. See (Martin-Löf 1984) for a type-theoretic proof.

Theorem 7.3. In an LCCC the following general choice scheme is BHK-valid:

$$(\forall x \in X)\,(\exists y \in Y)\,\varphi(u,x,y) \overset{u}{\Longrightarrow} (\exists f \in Y^X)\,(\forall x \in X)\,\varphi(u,x,f(x)).$$

Proof. Let $r : R \to U \times X \times Y$ be the interpretation $\mathcal{M}_{uxy}(\varphi) \to U \times X \times Y$. Let $q = \langle \pi_1, \pi_2 \rangle : U \times X \times Y \to U \times X$ and $p = \pi_1 : U \times X \to U$ be projections. Let $ev : Y^X \times X \to Y$ be the evaluation operator. The interpretation $m : \mathcal{M}_{ufx}(\varphi(f(x)/y)) \to U \times Y^X \times X$ is the pullback of r along $\varepsilon = \langle \pi_1, \pi_3, ev \circ \langle \pi_2, \pi_3 \rangle \rangle$. Let $\alpha = \langle \pi_1, \pi_2 \rangle : U \times Y^X \times X \to U \times Y^X$. Thus $m_1 : \mathcal{M}_u((\forall x \in X)\,(\exists y \in Y)\,\varphi(u,x,y)) \to U$ is $\Pi_p \Sigma_q(r) \to U$ and $\mathcal{M}_u((\exists f \in Y^X)\,(\forall x \in X)\,\varphi(u,x,f(x))) \to U$ is $\Sigma_{p_1}\Pi_\alpha(m) \to U$, the composition of $p_2 : \Pi_\alpha(m) \to U \times Y^X$ and the first projection $p_1 : U \times Y^X \to U$.

We sketch the rest of the proof. Extract the choice function by composing the evaluation $ev' : p^*(\Pi_p \Sigma_q(r)) \to \Sigma_q(r)$ with $\pi_3 \circ r : \Sigma_q(r) \to Y$. Using that $p^*(\Pi_p \Sigma_q(r))$ is the pullback of m_1 along p one obtains $c : X \times \Pi_p \Sigma_q(r) \to Y$ by composing $\pi_3 \circ r \circ ev'$ with the universal map into the pullback. Form the transpose $\hat{c} : \Sigma_q(r) \to Y^X$ and then $\langle m_1, \hat{c} \rangle : \Pi_p \Sigma_q(r) \to U \times Y^X$. Now it is fairly easy to construct $\psi : X \times \Pi_p(\Sigma_q(r)) \to \mathcal{M}_{ufx}(\varphi(f(x)/y)$ such that $m \circ \psi = \langle \langle m_1, \hat{c} \rangle \circ \pi_2, \pi_1 \rangle$. Then since $X \times \Pi_p(\Sigma_q(r))$ is isomorphic to $\alpha^*(\Pi_p(\Sigma_q(r)))$ it follows by the adjunction (α^*, Π_α) that there is a map $\theta : \Pi_p(\Sigma_q(r)) \to \Pi_\alpha(m)$ such that $p_2 \circ \theta = \langle m_1, \hat{c} \rangle$. By composition with p_1 we obtain a map $\Pi_p \Sigma_q(r) \to \Sigma_{p_1} \Pi_\alpha(m)$ over U as desired. $\square$

Example 7.4 The category of types with arbitrary equivalence relations (Moerdijk and Palmgren 2000) in Martin-Löf type theory is an LCCC with finite sums (and a pretopos as well).

Remark 7.5 Awodey (1995) proves a version of Theorem 7.3. (See also Remark 2.6 in (Moerdijk and Palmgren 2000).)

Remark 7.6 Notice that the image functor does not commute with implication and universal quantification, since this, together with Theorem 7.3, would imply that AC holds in any topos.

Remark 7.7 Consider the category of sets in ZF set theory. This is a topos and in particular an LCCC. Does this mean that Theorem 7.3 gives a relative consistency proof of choice? Unfortunately not. The principle of excluded middle

$$\overset{\overline{x}}{\Longrightarrow} \varphi \vee \neg\varphi$$

is BHK-valid for all φ if, and only if, ZF proves AC. The direction $\Leftarrow$ is clear. As for $\Rightarrow$, consider a surjective function $f : A \longrightarrow X$. Let R be a unary predicate symbol whose interpretation is this function. Suppose that $\stackrel{x}{\Longrightarrow} R(x) \vee \neg R(x)$ is BHK-valid. Since $f^{-1}(x)$ is non-empty for each x, it must be the case that all fibers of $\mathcal{M}_x(\neg R(x)) \longrightarrow \mathcal{M}(x)$ are empty. Hence $\stackrel{x}{\Longrightarrow} R(x)$ is BHK-valid, but this means that f has a section. Since f was arbitrary, AC holds. This argument holds more generally in any topos (Awodey 1995).

8. Exact Completions

In the practice of constructive mathematics (Bishop and Bridges 1985), and as well as in computer checked formalisations in Martin-Löf type theories, there is a well-established notion of set. A set is here a type, or preset, X together with an equivalence relation R. There is a categorical version of this construction, the *exact completion* $\mathbb{B}_{\mathrm{ex}}$ of a (weak) left exact category $\mathbb{B}$ (see (Carboni and Vitale 1998)). Here $\mathbb{B}$ plays the role of types and $\mathbb{B}_{\mathrm{ex}}$ that of constructive sets. For qlex categories $\mathbb{B}$ we may describe this construction entirely in the internal language. Let X be an object in $\mathbb{B}$. Suppose that $r : \mathcal{M}(R) \longrightarrow X \times X$ is a BHK-interpretation of R (this corresponds to pseudo-relations of (Carboni and Vitale 1998)). Then $r : \mathcal{M}(R) \longrightarrow X \times X$ is a (pseudo) equivalence relation in the sense of (Carboni and Vitale 1998) if, and only if, the usual conditions on equivalence relation holds of R under the BHK-interpretation. These conditions are formulated as the obvious sequents. Such equivalence relations (X, r) are the objects of $\mathbb{B}_{\mathrm{ex}}$. Suppose that $s : \mathcal{M}(S) \longrightarrow Y \times Y$ is another equivalence relation. Then $f : X \longrightarrow Y$ in $\mathbb{B}$ is a morphism of $\mathbb{B}_{\mathrm{ex}}$ if, and only if, the sequent

$$R(x, y) \stackrel{xy}{\Longrightarrow} S(f(x), f(y)) \tag{6}$$

is BHK-valid in $\mathbb{C}$. Furthermore, two such morphisms $f, g : X \longrightarrow Y$ are considered equivalent if, and only if, the sequent

$$\stackrel{x}{\Longrightarrow} S(f(x), g(x))$$

is BHK-valid, i.e. in case they are extensionally equal. This is of course the familiar method for defining extensional maps in constructive mathematics.

Denote the equivalence class containing f by $[f]$. The regular-epi/mono factorisation of a $\mathbb{C}_{\mathrm{ex}}$-morphism $[f] : (X, r) \longrightarrow (Y, s)$ is $[1_X] : (X, r) \longrightarrow (X, p)$ and $[f] : (X, p) \longrightarrow (Y, s)$ where p is the arrow

$$\mathcal{M}_{xy}(S(f(x), f(y))) \rightarrow X \times X$$

obtained from s by substitution (pullback). Carboni (1995) shows that for left exact $\mathbb{B}$ there is a canonical embedding y of $\mathbb{B}$ as the subcategory of projective objects of $\mathbb{B}_{\mathrm{ex}}$. It takes the object A to the diagonal $\Delta_A : A \longrightarrow A \times A$, and a morphism $f : A \longrightarrow B$ to $[f] : (A, \Delta_A) \longrightarrow (B, \Delta_B)$. The condition (6) will be automatically satisfied. The assignment of monos $[f]$ in $\mathbb{B}_{\mathrm{ex}}$ to morphisms f induces an order isomorphism (Carboni 1995, p. 128) between the different kinds of sub-objects

$$k_X : \mathrm{Psub}_{\mathbb{B}}(X) \longrightarrow \mathrm{Sub}_{\mathbb{B}_{\mathrm{ex}}}(y(X)).$$

Theorem 8.1 (Carboni (1995)). Let $\mathbb{B}$ be a lex category. Then the exact completion $y : \mathbb{B} \longrightarrow \mathbb{B}_{\mathrm{ex}}$ embeds $\mathbb{B}$ into an exact category. Moreover,

(i) if $\mathbb{B}$ is lextensive, then $\mathbb{B}_{\mathrm{ex}}$ is a pretopos,
(ii) if $\mathbb{B}$ is LCC, then so is $\mathbb{B}_{\mathrm{ex}}$,
(iii) if N is a natural number object in $\mathbb{B}$, then so is $y(N)$ in $\mathbb{B}_{\mathrm{ex}}$. $\square$

Suppose that the basic category $\mathbb{B}$ is a lextensive LCCC. Then $\mathbb{B}_{\mathrm{ex}}$ is a Π-pretopos (Moerdijk and Palmgren 2000,2002). In particular it is a Heyting pretopos, i.e. a pretopos where the inverse image $f^{-1} : \mathrm{Sub}(Y) \longrightarrow \mathrm{Sub}(X)$ has a right adjoint $\forall_f$, for each $f : X \longrightarrow Y$. The map

$$k_X : \mathrm{Psub}_{\mathbb{B}}(X) \longrightarrow \mathrm{Sub}_{\mathbb{B}_{\mathrm{ex}}}(y(X))$$

is then a Heyting algebra isomorphism, since it is an order isomorphism, and both kinds of subobjects form Heyting algebras. To show that also quantifiers are preserved we need a lemma which is easily proved.

Lemma 8.2. Let $\mathbb{B}$ be a lex category. Then for $(A, \alpha) \in \mathrm{Psub}_{\mathbb{B}}(X)$ and $f : X \longrightarrow Y$ in $\mathbb{B}$:

$$k_X(f^{-1}(A, \alpha)) = f^{-1}(k_Y(A, \alpha)). \square$$

From this lemma follows now readily using the uniqueness of $\exists_f$ and $\forall_f$ as left and right adjoints to f^{-1}:

Theorem 8.3. If $\mathbb{B}$ is a lextensive LCCC, then $k_X : \mathrm{Psub}_{\mathbb{B}}(X) \longrightarrow \mathrm{Sub}_{\mathbb{B}_{\mathrm{ex}}}(y(X))$ is an isomorphism of Heyting algebras which preserves quantifiers in the sense that for $f : X \longrightarrow Y$ and $(B, \beta) \in \mathrm{Psub}_{\mathbb{B}}(Y)$

(i) $k_Y(\tilde{\exists}_f(B, \beta)) = \exists_f(k_X(B, \beta))$,
(ii) $k_Y(\tilde{\forall}_f(B, \beta)) = \forall_f(k_X(B, \beta))$.

If $\mathbb{B}$ is lex (lextensive), then k_X is an upper semilattice (distributive lattice) isomorphism such that (i) holds, for arbitrary f and (B, β). $\square$

Now $y : \mathbb{B} \longrightarrow \mathbb{B}_{\mathrm{ex}}$ is a left exact functor, so products and monomorphisms are preserved. The BHK-interpretation in $\mathbb{B}$ can therefore be seen as the standard semantics in $\mathbb{B}_{\mathrm{ex}}$. This suggests that several of the results presented here could have been proved in standard semantics, *under the assumption that $\mathbb{B}$ is extensive.* For instance the verification of Theorem 7.3 can be done in this manner using the fact that $y(X)$ is projective and that y preserves exponentials (Carboni and Rosolini 2000).

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