Constructive Mathematics*

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Introduction

In these notes we aim to give a short introduction to the foundations of constructive mathematics. We study mainly the constructive real number system, which already at a basic level provides illustrations to important issues of constructivism.

1. Nonconstructive Existence Proofs

Proposition 1 *There exist irrational numbers a and b such that a^b is rational.*

Proof. Consider the number $\sqrt{2}^{\sqrt{2}}$. If it is rational, then we are done, with $a = b = \sqrt{2}$. If not, let $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$. Then a^b is rational, since

$$a^b = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = (\sqrt{2})^2 = 2. \blacksquare$$

In this existence proof, it has not been decided whether $a = \sqrt{2}$ or $a = \sqrt{2}^{\sqrt{2}}$ yields the example. (In fact one can show that $\sqrt{2}^{\sqrt{2}}$ is irrational, but this proof requires a couple of pages, see S. Lang: Algebra.) Many classical existence proofs begin: "Assume there is no object such that ...", and then proceed to derive a contradiction. Here is a simple example.

Proposition 2 *Any infinite sequence $(a_k)_{k=0}^{\infty}$ of natural numbers has a smallest term, i.e. there exists n such that $a_k \geq a_n$ for all k.*

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Proof. Suppose there exists no smallest term, i.e. for every k, there exists $k' > k$ such that $a_k > a_{k'}$. Thus there is an infinite decreasing subsequence (a_{k_i}) of (a_k). Since this subsequence decreases by at least 1 with each term, we must have $a_{k_{a_0+1}} < 0$. A contradiction. ∎

This existence proof provides no information at all where to find a minimum. Information free existence theorems are not uncommon in analysis (and many more build on them): Ascoli's Lemma, Bolzano–Weierstrass Theorem and Cauchy–Peano's existence theorem for ordinary differential equations.

Theorem 3 (Bolzano–Weierstrass) *If $S \subseteq [a, b]$ is an infinite set of real numbers, then S has an accumulation point.* ∎

Closely related to this theorem is König's lemma: Every finitely branching, infinite tree has an infinite path.

Constructive mathematics builds on the idea of taking the notion of existence more seriously than in classical mathematics. To show that a certain object exists means to give a method of constructing it.

2. Constructive Interpretation of the Logical Connectives

The notion of construction must be explained. To construct a natural number means to actually give the number or to give a method of constructing it. To construct a infinite sequence of numbers $(a_n)_n$ means to provide a method or algorithm that given an index n, constructs a_n. More generally to construct a function $f : S \to T$ consists in giving a method or algorithm that to each element a of S produces an element $f(a)$ in T. By requiring an algorithm we exclude certain definition by cases that are common in classical mathematics.

$$f(n) = \begin{cases} 1 & \text{there are } n \text{ consecutive 7's in the decimal expansion of } \pi, \\ 0 & \text{otherwise} \end{cases}$$

This is not, apriori, a constructive function since no one has produced an algorithm to decide whether there are n 7's in the decimal expansion of π. Case distinctions are allowed in constructive functions only when we can effective decide what case applies.

The Brouwer–Heyting–Kolmogorov interpretation. We explain what it means to be a proof, or evidence, for a proposition A, by induction on the structure of A. We give it an slightly more formal version than is customary.

- $\perp$ has no proof.
- p proves $s = t$ iff $p = \mathtt{OK}$ and s and t calculates to the same thing.
- p is a proof of $A \wedge B$ iff p is a pair $\langle a, b \rangle$ such that a is a proof of A and b is a proof of B.
- p is a proof of $A \rightarrow B$ iff p is a function which to each proof a of A associates a proof $p(a)$ of B.
- p is a proof of $A \vee B$ iff p is a pair $\langle i, c \rangle$ such that $i = 0$ and c proves A *or* $i = 1$ and c proves B.
- p is a proof of $(\forall x \in S)A(x)$ iff p is a function which to each element $d \in S$, assigns a proof $p(d)$ of $A(d)$.
- p is a proof of $(\exists x \in S)A(x)$ iff p is a pair $\langle d, q \rangle$ such that $d \in S$ and q proves $A(d)$.

We say that A is valid under the Brouwer–Heyting–Kolmogorov (BHK) interpretation, if we can find a construction p such that p is a proof of A. The construction p is often called a *proof object*.

Remarks 4 In the case for $=$ it is required that we know how to decide when two elements are equal. This is possible for natural numbers, integers, rational numbers, and finitary data structures such as lists and trees.

In the $\vee$-case, i tells what disjunct is proven. Thus we must commit ourselves to which of the disjuncts has been proved.

Here are some examples of the BHK-interpretation. We use λ-notation for functions.

Examples 5 1. $p = \lambda x.x$ is a proof of the proposition $A \rightarrow A$. Clearly, if a is a proof of A, then $p(a) = (\lambda x.x)(a) = a$ is, trivially, a proof for A.

2. A proof of $A \wedge B \rightarrow B \wedge A$ is given by the construction $f(\langle a, b \rangle) = \langle b, a \rangle$.

3. Consider the proposition $\perp \rightarrow A$. A proof for this is any function, e.g. $f(x) = 42$: Suppose a is a proof of $\perp$. But according to the BHK-interpretation, $\perp$ has no proof. Hence we have arrived at a contradiction. Anything follows, in particular that 42 is a proof A.

Negation is defined by $\neg A \equiv A \rightarrow \perp$. Thus to prove $\neg A$ means to show that A leads to a contradiction. As usual we define $A \leftrightarrow B$ to be $(A \rightarrow B) \wedge (B \rightarrow A)$.

Example 6 The law of contraposition $(A \to B) \to (\neg B \to \neg A)$ is valid under the BHK-interpretation. Assume that f proves $A \to B$. We wish to find a construction for $(\neg B \to \neg A)$. Thus suppose that g proves $\neg B$ and a proves A. Then $f(a)$ proves B, and hence $g(f(a))$ proves $\bot$. Thus the construction $\lambda a.g(f(a))$ proves $\neg A$. Furthermore $\lambda g.\lambda a.g(f(a))$ proves evidently $\neg B \to \neg A$. Finally, the construction

$$\lambda f.\lambda g.\lambda a.g(f(a))$$

becomes the proof of the contraposition law.

Readers familiar with natural deduction will notice that the proof is really 'term decoration' of the following proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{[\neg B] \qquad \dfrac{[A \to B] \qquad\qquad [A]}{B}}{\bot}}{\neg A}}{\neg B \to \neg A}}{(A \to B) \to (\neg B \to \neg A)}$$

Notice that the *reductio ad absurdum* rule is not present, we have used only the 'intuitionistic' part of natural deduction. It turns out that the laws of intuitionistic logic are valid under this interpretation (see Exercises). The law of excluded middle

$$A \vee \neg A$$

has no obvious BHK-interpretation, since it seems that we need to find some uniform method for deciding whether A holds or not. In fact, we can give a so-called Brouwerian counterexample to it (see next section). The same is true for $\neg\neg A \to A$, which embodies the idea of indirect proof.

The Axiom of Choice The BHK-interpretation validates a certain form of choice axiom. Suppose that we have a proof p of

$$(\forall x \in S)\,(\exists y \in T)\,A(x, y).$$

Then for each $a \in S$, $p(a)$ proves $(\exists y \in T)\,A(a, y)$. But $p(a)$ is a pair $\langle c, q\rangle$ such that q proves $A(a, c)$. Hence $\pi_2(p(a))$ proves $A(a, \pi_1(p(a))$. Thus $f(x) = \pi_1(p(x))$ defines a choice function such that $\lambda x.\pi_2(p(a))$ proves $(\forall x \in S)\,A(x, f(x))$.

Remark 7 Some vagueness exists in the BHK-interpretation. In particular it is not decided what the constructions or proof objects p should be. There are two major ways of making it precise.

The historically first is Kleene's recursive realisability interpretation which interprets construction by a index for a partial recursive function (cf. [7]).

The followers of the Bishop school in constructive mathematics believe (rightly) that all theorems of constructive mathematics should be theorems of classical mathematics. A full committment to Kleene's realisability interpretation, would lead one to accept as an axiom that every function from $\mathbb{N}$ to $\mathbb{N}$ is recursive (which is false classically). Thus one should avoid such a strong interpretation. This does not mean that constructive mathematics cannot be formalised. Indeed all mathematics from Bishop's school can be formalised in Martin-Löf's constructive type theory [4, 5]. The constructions can be identified with elements of a type, and propositions can be identified with types. This is the Curry–Howard correspondence.

3. Brouwerian Counterexamples

There are certain basic principles that no one believes will ever be proved constructively. These were called *principles of omniscience* by Bishop, and are usually stated in terms of infinite binary sequences. A binary sequence is a function $\alpha : \mathbb{N} \to \{0, 1\}$.

(LPO) For all binary sequences α: $\exists n\, \alpha_n = 1$ or $\forall n\, \alpha_n = 0$.

(WLPO) For all binary sequences α: $\forall n\, \alpha_n = 0$ or $\neg\forall n\, \alpha_n = 0$.

(LLPO) Let α be a binary sequence that contains at most one 1. Then $\forall n\, \alpha_{2n} = 0$ or $\forall n\, \alpha_{2n+1} = 0$.

(The acronyms stand for limited principle of omniscience, weak limited principle of omniscience and lesser limited principle of omniscience, respectively.) It can readily be shown that the following implications hold

$$(\text{LPO}) \Rightarrow (\text{WLPO}) \Rightarrow (\text{LLPO}).$$

Within a recursive interpretation they are all false. If one want avoid to speak about recursive functions, there is another way of giving evidence why one should not accept these principles. This consists in giving an example of an unsolved mathematical problem whose solution would follow readily if such a principle is valid. Such an example is a called a *Brouwerian counterexample*.

A Brouwerian counterexample to WLPO: Construct a binary sequence α as follows

$$\alpha_n = \begin{cases} 0 & \text{if } 2n+4 \text{ is a sum of two primes,} \\ 1 & \text{otherwise.} \end{cases}$$

Thus $\forall n\, \alpha_n = 0$ iff each even number ≥ 4 is the sum of two primes. If WLPO holds, we could immediate decide Goldbach's conjecture (unsolved as of yet[1]).

[1]Brouwerian counterexamples tend to age. Until a few years ago counterexamples involving Fermat's Last Theorem were quite popular.

A Brouwerian counterexample to LLPO: Let β be given by

$$\beta_n = \begin{cases} 1 & \text{if the } \textit{first} \text{ occurence of 100 consecutive 7's in the} \\ & \text{decimal expansion of } \pi \text{ start at the } n\text{th decimal,} \\ 0 & \text{otherwise.} \end{cases}$$

This sequence contains at most one 1. If LLPO was true, we would immediately know that such a row of 7's cannot start at even position, or cannot start at an odd position. (See the Exercises for a recursive counterexample.)

Consider a classical result P whose constructivity we doubt. If one can prove that P implies one of the principles LPO, WLPO or LLPO, we can stop looking for a constructive proof of P.

4. Constructive Real Numbers

The constructive definition of real number coincides with the classical definition. This complies to the general principle of Bishop's school, that all constructive mathematics should be meaningful also under the classical reading.

Definition 8 A *real number* $x = (r, c)$ consists of a fundamental sequence of rationals $r : \mathbb{N} \to \mathbb{Q}$ and a *convergence modulus* $c : \mathbb{N} \to \mathbb{N}$ such that for all k and all $m, n \geq c(k)$,

$$|r_m - r_n| < 2^{-k}.$$

We usually write x also for the fundamental sequence and use the notation $c^{(x)}$ for the convergence modulus.

The real numbers x and y are *equal*, written $x =_{\mathbb{R}} y$, if for each k there exists some ℓ so that for all $m \geq \ell$,

$$|x_m - y_m| < 2^{-k}.$$

Lemma 9 $=_{\mathbb{R}}$ *is an equivalence relation on* $\mathbb{R}$.

Proof. Exercise. ∎

Example 10 Classically we have by the principle of excluded middle

$$(\forall x, y \in \mathbb{R})\, [x = y \vee \neg\, x = y]. \tag{1}$$

We show that this statement implies WLPO. Let $\alpha : \mathbb{N} \to \{0, 1\}$. Define a real number $x^{(\alpha)}$ as follows

$$x_n^{(\alpha)} = \begin{cases} 2^{-n} & \text{if } \alpha_k = 0 \text{ for all } k \leq n \\ 2^{-k_0} & \text{if for some } k \leq n,\ \alpha_k = 1, \text{ and } k_0 \text{ is the least such.} \end{cases}$$

The fundamental sequence thus decreases to 0 as long as it does not encounter 1 in α; if it does it stays. A convergence modulus for $x^{(\alpha)}$ is $c(n) = n$. By the definition of equality it follows that

$$x^{(\alpha)} =_{\mathbb{R}} 0 \Leftrightarrow \forall n \; \alpha_n = 0.$$

Thus (1) entails WLPO.

Definition 11 Let x and y be real numbers. Then $x <_{\mathbb{R}} y$ iff there exists k and n such that for all $m \geq n$,

$$x_m + 2^{-k} < y_m.$$

Lemma 12

(a) $<_{\mathbb{R}}$ *is irreflexive and transitive.*

(b) $x =_{\mathbb{R}} y$, $y <_{\mathbb{R}} z$, $z =_{\mathbb{R}} w$ *implies* $x <_{\mathbb{R}} w$. ∎

Example 13 The law of trichotomy is: for all real numbers x and y

$$x <_{\mathbb{R}} y \vee x =_{\mathbb{R}} y \vee y <_{\mathbb{R}} x.$$

This implies (1), since by Lemma 12, $\neg\, x =_{\mathbb{R}} y$ follows from $x <_{\mathbb{R}} y$.

The constructive failure of this law has consequences for the development of constructive analysis. But it is not as catastrophic as it may seem. The following is a good substitute.

Theorem 14 (Co-transitivity) *For all real numbers x, y and z:*

$$x <_{\mathbb{R}} z \rightarrow x <_{\mathbb{R}} y \vee y <_{\mathbb{R}} z.$$

Proof. Suppose $x <_{\mathbb{R}} z$. Then there are k, m such that for all $n \geq m$:

$$x_n + 2^{-k} < z_n. \tag{2}$$

Let $N \geq m, c^{(x)}(k+3), c^{(y)}(k+3), c^{(z)}(k+3)$. Thus the approximations x_N, y_N, z_N are correct to within $2^{-(k+3)}$. From (2) follows $x_N + 2^{-(k+1)} < z_N - 2^{-(k+1)}$. Thus either $x_N + 2^{-(k+1)} < y_N$ or $y_N < z_N - 2^{-(k+1)}$. The former implies that for all $n \geq N$, $x_n + 2^{-(k+2)} < y_n$, that is $x <_{\mathbb{R}} y$. The latter implies $y_n + 2^{-(k+2)} < z_n$, for each $n \geq N$, i.e. $y <_{\mathbb{R}} z$. ∎

For the development of constructive analysis, one needs a *positive* notion of inequality, a so-called apartness relation. Whereas one needs a *negative* notion corresponding to the classical $\leq$.

Definition 15

(a) Two real numbers x and y are said to be *apart*, $x \# y$, if $x <_{\mathbb{R}} y$ or $y <_{\mathbb{R}} x$.

(b) For real numbers x and y define: $x \leq_{\mathbb{R}} y \Leftrightarrow_{def} \neg y <_{\mathbb{R}} x$.

Lemma 16 *(a)* $\neg x \# y \rightarrow x =_{\mathbb{R}} y$.

(b) $\leq_{\mathbb{R}}$ *is a partial order.*

Proof. Exercise. ∎

In the sequel we often drop the subscript $\mathbb{R}$ on the relation symbols.

The arithmetical operations on real numbers are defined by letting the corresponding operations in $\mathbb{Q}$ act termwise on the fundamental sequences.

$$\begin{aligned}
(x_n)_n +_{\mathbb{R}} (y_n)_n &=_{def} (x_n + y_n)_n \\
(x_n)_n \cdot_{\mathbb{R}} (y_n)_n &=_{def} (x_n \cdot y_n)_n \\
-_{\mathbb{R}}(x_n)_n &=_{def} (-x_n)_n \\
|(x_n)_n|_{\mathbb{R}} &=_{def} (|x_n|)_n
\end{aligned}$$

One has, as well, to specify how the arithmetical operations act on convergence moduli. Thus $c^{(x+y)}(n) = \max(c^{(x)}(n+1), c^{(y)}(n+1))$ defines the modulus for $x+y$. We leave it to the reader to define the convergence modulus of xy, $-x$ and $|x|$. Furthermore its has to be shown that the real numbers are closed under these operations, and that the operations preserve equality between real numbers.

Proposition 17 $(\mathbb{R}; +, \cdot, <)$ *is a constructive field, i.e. a commutative ring where every element that is apart from* 0 *has an inverse:*

$$x \# 0 \rightarrow \exists y \, xy = 1. \blacksquare$$

Some rules for manipulating inequalities are found in the exercises.

5. Continuous Functions

In constructive analysis the notion of *uniformly* continuous function is basic. The reason is that, constructively, there is no proof that continuous function on a compact interval is uniformly continuous. This is one of the points where Bishop constructivism and Brouwer's intuitionism differ. Brouwer brought extra assumptions into constructivism to allow such a proof to go through (see Heyting [3]). These assumptions are however classically false. (Nevertheless they can be shown consistent using non-classical models.)

Definition 18 A function $f : [a, b] \to \mathbb{R}$ is *uniformly continuous* if there exists a modulus of continuity $\mu : \mathbb{N} \to \mathbb{N}$ such that for all k and all $x, y \in [a, b]$:

$$|x - y| < 2^{-\mu(k)} \to |f(x) - f(y)| < 2^{-k}.$$

The classical intermediate value theorem tells us that we can find a root to the equation

$$f(x) = 0$$

for any continuous function $f : [a, b] \to \mathbb{R}$ such that $f(a) \leq 0 \leq f(b)$. The familiar proof using successive bisection might seem constructive, but on closer scrutiny it requires us to decide whether $f(x_i) \leq 0$ or $f(x_i) \geq 0$ at the bisection point x_i. In fact the intermediate value theorem (IVT) is not constructively valid in its classical form.

Proposition 19 *IVT implies* $(\forall u \in \mathbb{R})\, [u \leq 0 \vee u \geq 0]$.

Proof. Let u be a real number. Define a continuous function by

$$f(x) = \min(x - 1, 0) + \max(0, x - 2) + u.$$

This a piecewise linear function. Suppose that x_0 satisfies $f(x_0) = 0$. Clearly $x_0 > 1$ or $x_0 < 2$, by co-transitivity. If $x_0 > 1$,

$$f(x_0) = x_0 - 1 + \max(0, x_0 - 2) + u = 0,$$

and hence $u = 1 - x_0 - \max(0, x_0 - 2) \leq 0$. If $x_0 < 2$,

$$f(x_0) = \min(x_0 - 1, 0) + u = 0,$$

and hence $u = -\min(x_0 - 1, 0) \geq 0$. ∎

An approximate version of IVT can be proven constructively.

Theorem 20 *Let* $f : [a, b] \to \mathbb{R}$ *be uniformly continuous with* $f(a) \leq 0 \leq f(b)$. *Then for each* $\varepsilon > 0$ *there exists* $x \in [a, b]$ *with* $|f(x)| < \varepsilon$.

Proof. Let $\varepsilon > 0$. Since f is uniformly continuous we can find $\delta > 0$, such that for $|x - y| < \delta$,

$$|f(x) - f(y)| < \varepsilon/4.$$

Let N be a positive integer such that $(b - a)/N < \delta$, and let $t_j = a + j(b - a)/N$. Find rational numbers q_i such that $|f(t_i) - q_i| < \varepsilon/4$ and $q_1 \leq 0 \leq q_N$. By finitely many comparisons of rational numbers we find some i, such that $q_i \leq 0 \leq q_{i+1}$. We have

$$\begin{aligned} |q_{i+1} - q_i| &\leq |q_{i+1} - f(t_{i+1})| + |f(t_{i+1}) - f(t_i)| + |f(t_i) - q_i| \\ &\leq \varepsilon/4 + \varepsilon/4 + \varepsilon/4 = 3\varepsilon/4. \end{aligned}$$

Hence $|f(t_i)| < \varepsilon$. ∎

However the exact IVT can be salvaged if one can find enough bisection points for the function.

Theorem 21 *Let $f : [a, b] \to \mathbb{R}$ be continuous with $f(a) \leq 0 \leq f(b)$ and suppose that in each nonvoid subinterval $[c, d] \subseteq [a, b]$ there is some t such that $f(t) \# 0$. Then for some $x \in [a, b]$: $f(x) = 0$.* ∎

Exercises

1. Give BHK-interpretations of the following propositions:

   (a) $A \wedge B \to A$ and $A \wedge B \to B$

   (b) $A \to (B \to A \wedge B)$

   (c) $A \to A \vee B$ and $B \to A \vee B$

   (d) $(A \to C) \to ((B \to C) \to (A \vee B \to C))$

   (e) $A \to (B \to A)$

   (f) $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$ [Hint: the S-combinator]

   (g) $\perp \to A$

   (h) $A(t) \to (\exists x \in D)A(x)$

   (i) $(\forall x \in D)(A(x) \to B) \to ((\exists x \in D)A(x) \to B))$, where x does not occur free in B

   (j) $(\forall x \in D)A(x) \to A(t)$

   (k) $(\forall x \in D)(B \to A(x)) \to (B \to (\forall x \in D)A(x))$, where x does not occur free in B.

   These are the logical *axioms* of first order intuitionistic logic. The full logic requires the modus ponens rule and the rule of generalisation: from A, deduce $(\forall x \in D)A$.

2. Give BHK-interpretations of the following propositions:

   (a) $\neg\neg\neg A \to \neg A$

   (b) $\neg(A \vee B) \leftrightarrow \neg A \wedge \neg B$.

   (c) $\neg(\exists x \in D)A(x) \leftrightarrow (\forall x \in D)\neg A(x)$

3. Does any of the classical true formulae below have BHK-intepretations? Discuss.

   (a) $\neg(\forall x \in D)A(x) \to (\exists x \in D)\neg A(x)$.

(b) $\neg(A \wedge B) \to \neg A \vee \neg B$.

4. Prove Lemma 9.

5. Prove, constructively, that for real numbers x, y, z and t:

   (a) $x \leq z$ and $y \leq t$ implies $x + y \leq z + t$,
   (b) $x \leq z$ and $y < t$ implies $x + y < z + t$,
   (c) $x \leq z$ and $y \geq 0$ implies $xy \leq zy$,
   (d) $x < z$ and $y > 0$ implies $xy < zy$,
   (e) $x < y$ implies $-y < -x$,
   (f) $|x + y| \leq |x| + |y|$.

6. Give constructive proofs of the following. Let x, y be real numbers.

   (a) $x + y > 0 \to x > 0$ or $y > 0$
   (b) $xy < 0 \to x < 0$ or $y < 0$

7. (a) Prove, constructively, that for all real numbers x, y: $xy \# 0$ implies $x \# 0$ and $y \# 0$.
   (b) The following is classically true: for all real numbers x and y,

   $$xy = 0 \to x = 0 \vee y = 0.$$

   Give a Brouwerian counterexample to this implication.

8. Prove the implications: (LPO) $\Rightarrow$ (WLPO) $\Rightarrow$ (LLPO).

9. Show that constructively the following two statements are equivalent

   (a) LLPO
   (b) For all real numbers x: $x \geq 0$ or $x \leq 0$.

10. Give a Brouwerian counterexample to Proposition 2.

11. Prove constructively: if $(a_n)_n$ and $(b_n)_n$ are two infinite sequences of natural numbers then there exists $m < n$ such that both $a_m \leq a_n$ and $b_m \leq b_n$.

12. Classically we know that any system of linear equations with fewer equations than unknowns admits a nontrivial solution. A Brouwerian counterexample can be given to this.

    (a) Show that

    $$(\forall a, b \in \mathbb{R})\,(\exists x, y \in \mathbb{R})[(x \# 0 \vee y \# 0) \wedge ax + by = 0] \qquad (3)$$

    implies the statement in Ex. 7.(b).

(b) Formulate and prove an approximate version of (3). Generalise.

13. Give a recursive counterexample to LLPO by proving the following (classically). There is total recursive function $a(i, j)$, written a^i_j, taking values 0 or 1 such that

  (i) for each i there is at most one j with $a^i_j = 1$,

  (ii) there is no total recursive function $f : \mathbb{N} \to \{0, 1\}$ such that for all i and n:
$$a_{2n+f(i)} = 0.$$

  This says that there is no uniform recursive way of deciding whether even or odd positions are 0. [*Hint.* Use recursively inseparable sets, or see F. Richman: Polynomials and Linear Transformations, *Linear Algebra and its Applications* 131(1990), p. 135.]

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