**Consequences of cut elimination for coherent theories**
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What does the subformula property tell us about cut-free proofs in theories with "dynamic axioms" (Coste, Lombardi, Roy)?

References:
[CLR] M. Coste, H. Lombardi, M.-F. Roy: Dynamical method in algebra: Effective Nullstellensätze.
[ST] H. Schwichtenberg, A.S. Troelstra: Basic Proof Theory.

Consider Gentzen's sequent calculi GI and GC for intuitionistic and classical logic, respectively, as presented in [ST]. If $\mathcal{D}$ is a cut-free proof of $\Gamma \Rightarrow C$, then for every sequent $\Delta \Rightarrow D$ occuring in $\mathcal{D}$ we have [Prop. 4.2.1, ST]:

(1) The formulae of $\Delta$ occur positively in $\Gamma$ or negatively in $C$

(2) $D$ occurs positively in $C$ or negatively in $\Gamma$

(3) If $D$ occurs only positively in the formula $\wedge\Gamma \to C$, then $D$ is introduced by an R-rule.

(4) If $D$ occurs only negatively in $\wedge\Gamma \to C$, then $D$ is introduced by an L-rule.

**Ex.1.** Suppose that $\Gamma$ consists of universal Horn formulae, and that $C$ is atomic. Then if $\Delta \Rightarrow D$ sequent occuring in $\mathcal{D}$, $\Gamma$ is a universal Horn formula and $D$ is atomic. In the proof $\mathcal{D}$, no R-rules have been used, nor the rules L$\vee$ or L$\exists$.

**Ex.2** Suppose that $C$ is an equation, and that $\Gamma$ consists of dynamical formulas written as

$$\forall \overline{x}\,(A_1 \to (A_2 \to \cdots (A_n \to \exists \overline{y_1} B_1 \vee \cdots \vee \exists \overline{y_m} B_m) \cdots)),$$

where are $A_1, \ldots, A_n$ are atomic formulae, and $B_1, \ldots, B_m$ are conjunction of such. Then for sequents $\Delta \Rightarrow D$ occuring in $\mathcal{D}$, $\Delta$ consists of dynamical formulae and $D$ is an atomic formulae. In the intuitionistic proof no R-rules are used.

Is this a way to interpret the notion of dynamical proof of [CLR] in more conventional proof-theoretic terms?