A direct proof that certain reduced products are countably saturated *

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Abstract

It is known that reduced products of the following form are ω_1-saturated: $\Pi_i \mathcal{A}_i/F$, where F is a countably incomplete, countably based filter and all $\mathcal{A}_i$ have the same countable signature. We give an elementary proof of this fact, which avoids reference to Feferman–Vaught's fundamental theorem for products, and quantifier elimination in atomless boolean algebras.

Jónsson and Olin [3] proved that reduced products of the form $\Pi_{i\in\omega}\mathcal{A}_i/F$ are ω_1-saturated, if F is the Frchet filter on ω, and all factors have the same countable signature. This is a rather remarkable result since there is no full 'Łos theorem' for these products. The result was generalised to countably incomplete, countably based filters by F. Galvin (see [3]). Pacholski and Ryll-Nardzewski [4] proved a still stronger theorem, where the condition that F is countably based was replaced by the requirement that the boolean algebra $\mathcal{P}(I)/F$ is atomless and basically connected. A complete characterisation of saturation for reduced products was finally obtained by Shelah [8]. In this paper we give a new proof for Galvin's generalisation. Our proof is more direct than those in [3] and [4]. In particular, it does not depend on Feferman–Vaught's fundamental theorem for generalised products [2] nor on elimination of quantifiers in atomless boolean algebras. Instead we use a property of the products, here called *the simple cover property*, apparently first noticed by Palyutin [7]. In another context, this property was crucial for providing denotational semantics of logic programs (Palmgren [6]). We also make an observation on subsaturated reduced products.

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1 The direct proof

We first recall some terminology and state a fundamental theorem. An *h-formula* is inductively generated from atomic formulas using only $\forall, \exists, \wedge$ as logical constants, and the construct $\exists x\, \varphi \wedge \forall x(\varphi \rightarrow \psi)$. For convenience we assume there is a symbol for falsity $\perp$ which is counted as atomic. By a *filter* we understand a proper filter, not containing the empty set. A filter is *countably incomplete* if it contains a countable subfamily with empty intersection. A *countably based* filter D on I is a filter that is generated by a countable family $\{B_n : n < \omega\}$ of subsets of I, i.e. $D = \{U \subseteq I : \exists n\, B_n \subseteq U\}$. The Frchet filter on ω, generated by the cofinite subsets, is countably incomplete.

Theorem 1 *(Palyutin) Let $\Pi_D \mathcal{A}_i = \Pi_{i \in I} \mathcal{A}_i / D$ be a reduced product of L-structures, where D is a filter on I. Let $\overline{a} = a_1, \ldots, a_n \in \Pi_{i \in \omega} \mathcal{A}_i$ be fixed, and let $\varphi(\overline{x})$ be an L-formula. Write $[\overline{a}] = [a_1], \ldots, [a_n]$ and $\overline{a}(i) = a_1(i), \ldots, a_n(i)$. If φ is an h-formula, then*

$$\{i : \mathcal{A}_i \models \varphi(\overline{a}(i))\} \in D \Longleftrightarrow \Pi_D \mathcal{A}_i \models \varphi([\overline{a}]).$$

Proof. The equivalence can be proven by a straightforward induction on the construction of h-formulas. QED

An L-structure $\mathcal{A}$ is said to have the *simple cover property* if for any h-formulas $\varphi(\overline{x}, \overline{y}), \psi_0(\overline{x}, \overline{y}), \ldots, \psi_n(\overline{x}, \overline{y})$ in L, $n \geq 0$,

$$\mathcal{A} \models \forall \overline{x}\, (\varphi \rightarrow \vee_{j=0}^{n} \psi_j) \rightarrow \vee_{j=0}^{n} \forall \overline{x}\, (\varphi \rightarrow \psi_j).$$

The converse implication is trivially true.

Lemma 2 *Let D be a countably incomplete, countably based filter, and let $\mathcal{A} = \Pi_D \mathcal{A}_i$ be a reduced product of L-structures. Then $\mathcal{A}$ has the simple cover property.*

Proof. We prove the contraposition of the property. Let $\{B_n : n < \omega\}$ be a base for D such that $B_n \supseteq B_{n+1}$ and $B_0 = I$. By incompleteness, $\cap_n B_n = \emptyset$.

Let $[\overline{a}] \in \mathcal{A}$, so that $\mathcal{A} \models \neg \forall \overline{x}\, (\varphi(\overline{x}, [\overline{a}]) \rightarrow \psi_j(\overline{x}, [\overline{a}]))$ for all j. Whence there are $\overline{b_j}$ with

$$\mathcal{A} \models \varphi([\overline{b_j}], [\overline{a}]) \wedge \neg \psi_j([\overline{b_j}], [\overline{a}])$$

By Theorem 1, every base set B_n contains an index i such that

$$\mathcal{A}_i \models \neg \psi_j(\overline{b_j}(i), \overline{a}(i)). \tag{1}$$

Using this and $\cap_n B_n = \emptyset$ one finds a sequence $0 = k_0 < k_1 < k_2 < \cdots$ such that for every $j = 0, \ldots, n$ there is an $i \in B_{k_m} \setminus B_{k_{m+1}}$ satisfying (1). Clearly

$C_m = B_{k_m} \backslash B_{k_{m+1}}$ constitute a partition of I. Let $\overline{c}$ be the unique tuple of sequences that coincides with $\overline{b_j}$ on C_k for $k \equiv j \bmod (n+1)$. Thus $\varphi([\overline{c}], [\overline{a}])$ holds in $\mathcal{A}$. Suppose, to obtain a contradiction, that $\mathcal{A} \models \psi_j([\overline{c}], [\overline{a}])$. So $\mathcal{A}_i \models \psi_j(\overline{c}(i), \overline{a}(i))$ for all i in each sufficiently small base set, in particular, for all $i \in C_k$ for some $k \equiv j \bmod (n+1)$. This is a contradiction. QED

By the above lemma and Łos' theorem it is clear that a countably incomplete ultrafilter cannot be countably based. (There is of course a simple direct argument for this fact too.) The following striking lemma is due to Palyutin [7].

Lemma 3 *Let $\mathcal{A}$ be an L-structure with the simple cover property. Then for every L-formula $\varphi(\overline{x})$ there is a boolean combination $\psi(\overline{x})$ of h-formulas in L such that $\mathcal{A} \models \forall \overline{x} \, (\varphi(\overline{x}) \leftrightarrow \psi(\overline{x}))$.*

Proof. We prove the result for boolean combinations of h-formulas with an existential quantifer in front: $\varphi(\overline{x}) \equiv \exists y \, \varphi'(\overline{x}, y)$. The general result then follows by induction. Since we can move the disjunctions of φ' outmost and let $\exists y$ distribute over the disjuncts, it is sufficient to consider $\varphi(\overline{x})$ of the form

$$\exists y \, (\theta(\overline{x}, y) \wedge \neg\psi_1(\overline{x}, y) \wedge \cdots \wedge \neg\psi_n(\overline{x}, y)), \tag{2}$$

where θ, ψ_j are h-formulas. If $n = 0$, we are done. Suppose $n > 0$. By the simple cover property, (2) is equivalent to $\exists y \, (\theta \wedge \neg\psi_1) \wedge \cdots \wedge \exists y \, (\theta \wedge \neg\psi_n)$. It is clear that $\exists y \, (\theta \wedge \neg\psi_i)$ is equivalent to the following boolean combination of h-formulas,

$$\neg(\exists y \, \theta \wedge \forall y \, (\theta \rightarrow \psi_i)) \wedge \exists y \theta.$$

QED

Theorem 4 *If $\mathcal{A} = \Pi_D \mathcal{A}_i$ is a reduced product with a countable signature L, and D is a countably incomplete, countably based filter, then $\mathcal{A}$ is ω_1-saturated.*

Proof. In view of Lemma 3 it is sufficient to prove the theorem for boolean combinations of h-formulas. Assume therefore that $\Sigma(\overline{x}) = \{\sigma_n(\overline{x}) : n < \omega\}$ is a set of such combinations in the expanded language $L(C)$, where C is a countable set of constants. Write each $\sigma_n(\overline{x})$ on disjunctive form:

$$\sigma_n^1(\overline{x}) \vee \cdots \vee \sigma_n^{k_n}(\overline{x}),$$

so that σ_j^i is a conjunction of h-formulas or negations of h-formulas. By König's lemma $\Sigma(\overline{x})$ is finitely satisfiable in $\mathcal{A}$ if, and only if, there is a sequence $f : \omega \rightarrow \omega$, $1 \leq f(n) \leq k_n$, such that

$$\Sigma_f(\overline{x}) = \{\sigma_n^{f(n)}(\overline{x}) : n < \omega\}$$

is finitely satisfiable in $\mathcal{A}$. We can thereby further simplify the problem, and may assume that each σ_n is a conjunction of the kind described. Since we are looking for satisfaction of the whole set, we may as well assume that σ_n has the form $\varphi_n \wedge \neg\psi_n$, where φ_n and ψ_n are h-formulas. Assume that Σ is finitely satisfiable, i.e. that for each m there exists a tuple $[\overline{a_m}]$ of $\mathcal{A}$-elements such that

$$\mathcal{A} \models \varphi_0([\overline{a}_m]) \wedge \cdots \wedge \varphi_m([\overline{a}_m]) \wedge \neg\psi_0([\overline{a}_m]) \wedge \cdots \wedge \neg\psi_m([\overline{a}_m]). \tag{3}$$

We shall inductively construct sets $U_0 \supset U_1 \supset U_2 \supset \cdots$ forming a base for D, with the property that for all $k \geq 0$

(I) $(\forall i \in U_k)\mathcal{A}_i \models \varphi_0(\overline{a_k}(i)) \wedge \cdots \wedge \varphi_k(\overline{a_k}(i))$,

(II) $(\forall m \leq k-1)(\exists i \in U_{k-1} \setminus U_k)\, \mathcal{A}_i \models \neg\psi_m(\overline{a_{k-1}}(i))$.

Let $\{B_n : n < \omega\}$ be a base for D. By Theorem 1 and (3) let $U_0 = B_0 \cap \{i : \mathcal{A}_i \models \varphi_0(\overline{a_0}(i))\} \in D$. Assume that $U_0 \supset \cdots \supset U_p$ have been choosen so that (I) and (II) holds for $k = 0, \ldots, p$. We now choose U_{p+1}. Since (3) holds for $m = p$, none of the sets

$$V_k = \{i : \mathcal{A}_i \models \psi_k(\overline{a_p}(i))\}$$

$k = 0, \ldots, p$ include U_p; pick indices $i_k \in U_p \setminus V_k$. By incompleteness of D, there is a $B_j \subset U_p$, not including these indices. By the positive part of (3)

$$U' = \{i : \mathcal{A}_i \models \varphi_0(\overline{a}_{p+1}(i)) \wedge \cdots \wedge \varphi_{p+1}(\overline{a}_{p+1}(i))\} \in D.$$

Let $U_{p+1} = U' \cap B_j \cap B_{p+1}$. Hence the conditions (I) and (II) are fulfilled for all k. Also by construction $U_n \subseteq B_n$, so the sequence forms a base.

Define sequences $\overline{b}$ by letting

$$\overline{b}(i) = \overline{a_m}(i)$$

for $i \in U_0$, where m is the unique index satisfying $i \in U_m \setminus U_{m+1}$. For $i \notin U_0$ we may let $\overline{b}(i)$ be rather arbitrary, $\overline{a_0}(i)$, say. We prove that $\Sigma(\overline{x})$ is realised by $[\overline{b}]$ in $\mathcal{A}$. Let m be fixed. Suppose that $\mathcal{A} \models \psi_m([\overline{b}])$. Thus $U = \{i : \mathcal{A}_i \models \psi_m(\overline{b}(i))\} \in D$. The $U_n : s$ form a base, so $U_n \subseteq U$ for some $n \geq m$. By (II) there is $i \in U_n \setminus U_{n+1}$ with $\mathcal{A}_i \models \neg\psi_m(\overline{a_n}(i))$, contradicting the definition of $\overline{b}$. Hence $\mathcal{A} \models \neg\psi_m([\overline{b}])$. Now suppose $i \in U_m$, so there is some $k \geq m$, with $i \in U_k \setminus U_{k+1}$. Therefore by (I)

$$\mathcal{A}_i \models \varphi_0(\overline{a}_k(i)) \wedge \cdots \wedge \varphi_k(\overline{a}_k(i)).$$

In particular, $\mathcal{A}_i \models \varphi_m(b(i))$. This proves $\mathcal{A} \models \varphi_m([b])$. QED

For a given filter D on I, the mentioned theorem of Feferman and Vaught relates, roughly speaking, truth in a reduced product $\Pi_D \mathcal{A}_i$ to truth in the quotient boolean algebra $\mathcal{P}(I)/D$. We observe that $\mathcal{P}(I)/D$ is isomorphic to the reduced power $\mathcal{C} = \mathcal{B}^I/D$ of the two element boolean algebra $\mathcal{B}$. The proofs of Jónsson and Olin, and Pacholski and Ryll-Nardzewski [4, Theorem 1] rests on the assumption that this boolean algebra is atomless. We have instead used the simple cover property, which by Lemma 3 allowed us to bypass the Feferman–Vaught theorem. However, we note the following

Proposition 5 *Let D be a filter on I. Then the following are equivalent:*

(a) every reduced product $\Pi_D \mathcal{A}_i$ has the simple cover property,

(b) the boolean algebra $\mathcal{P}(I)/D$ is atomless.

Proof. (a) $\Rightarrow$ (b) By the above remark, we may consider the reduced power $\mathcal{C} = \mathcal{B}^I/D$ instead. Suppose that $[a]$ is an atom, so that

$$\mathcal{C} \models \forall x\,(0 \leq x \leq [a] \longrightarrow x = 0 \vee x = [a]).$$

Hence by the simple cover property, $[a] = 0$, which is a contradiction.

(b) $\Rightarrow$ (a) Palyutin [7, Lemma 1] proves (a) for the Frchet filter F, but uses, in fact, only that $\mathcal{P}(I)/F$ is atomless. QED

Note. A version of Lemma 2 appeared in Palmgren [5].

2 Subsaturation

One may ask whether not more reduced products have saturation properties if we limit the logical complexity of the formulas in the types. This is indeed the case. The phrase "$\mathcal{A}$ is λ-saturated for Φ-formulas" means that the formulas to be realised belong to Φ. We shall give an example of a reduced power that is ω_1-saturated for h-formulas but not for quantifier free formulas. For results on the quantifier hierarchy, see Shelah [9, Ch. VI.3].

Theorem 6 *Let D be a filter on I. The following are equivalent:*

(a) D is countably incomplete,

(b) if $\mathcal{A}_i$, $i \in I$, are structures of the same countable signature, then $\Pi_D \mathcal{A}_i$ is ω_1-saturated for h-formulas.

Proof. (a) $\Rightarrow$ (b) Similar to the proof for ultraproducts [1, Theorem 6.1.1], but use Theorem 1 instead of Łos' theorem.

(b) $\Rightarrow$ (a) We need only to note that Shelah's argument [8, Lemma 2.2] uses nothing but atomic formulas. Let $\mathcal{F} = (\mathcal{P}_{fin}(\omega), \subseteq)$ be a structure for the language with $\subseteq$ as its only symbol. Clearly $\Sigma(x) = \{[\lambda i.\{n\}] \subseteq x : n \in \omega\}$ is finitely satisfiable in the reduced power $\mathcal{H} = \mathcal{F}^I/D$. Thus for some $[a] \in \mathcal{H}$, $\mathcal{H} \models \Sigma([a])$. Hence $U_n = \{i : \mathcal{A}_i \models \{n\} \subseteq a(i)\}$ all belong to D, and they cannot have a nonempty intersection since then some $a(i)$ would have to be infinite. QED

A boolean algebra is *basically connected* if it contains no strictly decreasing countable sequence of elements whose infimum is 0. It is easy to see that any boolean algebra that is ω_1-saturated for quantifier free formulas, is basically connected. Suppose, namely, that $\{b_n : n < \omega\}$ is a strictly decreasing sequence. Then $\Sigma(x) = \{b_n \geq x \wedge b_n \neq x \wedge x \neq 0 : n < \omega\}$ is finitely satisfiable, and thus satisfiable. Hence the infimum of $\{b_n : n < \omega\}$ is greater than 0.

Example 1. Let U be a non-principal ultrafilter on ω. Define a filter D on $\omega \times \omega$ by

$$V \in D \Longleftrightarrow \forall n \, \{m : (n,m) \in V\} \in U.$$

We prove that the boolean algebra $\mathcal{B}^{\omega\times\omega}/D \cong \mathcal{P}(\omega \times \omega)/D$ is ω_1-saturated for h-formulas, but not for quantifier free formulas. Theorem 6 gives the first result, since D inherits countable incompleteness from the ultrafilter. It is easy to verify that the mapping $J/D \mapsto \langle \{m : (n,m) \in J\}/U : n < \omega\rangle$ defines an isomorphism

$$\mathcal{P}(\omega \times \omega)/D \cong (\mathcal{P}(\omega)/U)^\omega \cong \mathcal{B}^\omega.$$

Let $\{a_n : n < \omega\}$ be the atoms of the algebra, all distinct. Then $b_n = -(a_0 + \cdots + a_n)$ form a strictly decreasing sequence of elements, whose infimum is 0. By the above remark, the second result follows.

Question: Is there a constructive example?

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