Regular universes and formal spaces

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Overview

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1. The connection: formal topologies — sites

Let $Q = (Q, \leq)$ be a preorder, i.e. a reflexively and transitively ordered set. (Q is trivially a category.)

For $X \subseteq Q$ define the *downwards closure*

$$X_{(\leq)} = \{u \in Q : (\exists x \in X)\, u \leq x\}$$

Formal intersection is defined by

$$X \wedge Y = X_{(\leq)} \cap Y_{(\leq)}.$$

Formal intersection with one element $a \in Q$:

$$X \cdot a = X \wedge \{a\}.$$

— *the restriction of X to a.*

A *formal topology over a preorder* Q

(Sigstam, Stoltenberg-Hansen 1990).

is $(Q, \leq, \lhd)$ where $\lhd \subseteq Q \times \mathcal{P}(Q)$ is a relation such that

(R) $a \in U \Longrightarrow a \lhd U$

(T) $a \lhd U$ and $U \lhd V \Longrightarrow a \lhd V$

(QL) $a \lhd U$ and $a \lhd V \Longrightarrow a \lhd U \wedge V$

(QG) $a \leq b \Longrightarrow a \lhd \{b\}$.

Grothendieck topologies on preorders

Note that $X = X \cdot a$ if, and only if

1) $x \leq a$ for all $x \in X$

2) $x \in X$, $y \leq x$ implies $y \in X$.

Such an X is called a *sieve on a.*

For a formal topology on Q we have

(G1) $a \lhd \{a\}_{(\leq)}$

(G2) $a \lhd U$ and $b \leq a \Longrightarrow b \lhd U \cdot b$

(G3) $a \lhd U$ and $b \lhd V \cdot b$, for each $b \in U \Longrightarrow a \lhd V$

(Note: The sets can be bounded above by an element)

A *Grothendieck topology* on Q is $(Q, \leq, \lhd)$ where

$$\lhd \subseteq \{(a, U) : a \in Q, U \text{ sieve on } a\}$$

satisfies the conditions G1–G3 on the previous page.

We note that for sieves U and V on a usual intersection will do:

$$a \lhd U \text{ and } a \lhd V \Longrightarrow a \lhd U \cap V.$$

Theorem 1 From any Grothendieck topology $(Q, \leq, \lhd)$ we obtain a formal topology $(Q, \leq, \lhd^*)$ over Q by defining

$$a \lhd^* U \iff a \lhd U \cdot a.$$

Remark. Grothendieck topologies (sites) are more generally defined on categories.

• Has the problem of predicative inductive generation been considered in this more general case? (The intersection of a set of sites is a site.)

2. Formal topologies without prescribed order

Cf. Martin-Löf 1986: *Formell (punktfri) topologi.*

Let S be a set. A relation $\lhd \subseteq S \times \mathcal{P}(S)$ is an *unlocalised covering relation* if

(R) $a \in U \Longrightarrow a \lhd U$

(T) $a \lhd U$ and $U \lhd V \Longrightarrow a \lhd V$

Points can be defined as usual.

For $U \subseteq S$ let $U^* = \{\alpha \in \mathrm{Pt}(S) : (\exists a \in U)\, a \in \alpha\}$.

Thm 2 $\tau = \{U^* : U \subseteq S\}$ is a topology on $\mathrm{Pt}(S)$.

Since:

1) $\cup_{i \in I} U_i^* = (\cup_{i \in I} U_i)^*$

2) $U^* \cap V^* = (U \downarrow \cap V \downarrow)^*$

where $U \downarrow = \{u \in S : (\exists x \in U)\, u \lhd \{x\}\}$.

The localisation axiom (L) seems necessary to obtain a reasonable category.

(L) $a \lhd U$ and $a \lhd V \Longrightarrow a \lhd U \downarrow \cap V \downarrow$

Extend a relation $R \subseteq S \times T$ as follows

$U\,R\,b$ means $(\forall u \in U)\ uRb$.

$a\,R\,V$ means $(\exists v \in V)\, aRv$,

The inverse image of V under R is

$$R^{-1}[V] =_{\text{def}} \{a \in S : a\,R\,V\}$$

Def. Let $(S, \lhd)$ and $(T, \lhd)$ be two formal topologies. A relation $R \subseteq S \times T$ is an *approximable mapping* if

(A1) $aRb,\ b \lhd V,\ \Rightarrow\ a \lhd R^{-1}[U]$,

(A2) $a \lhd U,\ U\,R\,b,\ \Rightarrow\ aRb$,

(A3) $S \lhd R^{-1}[T]$,

(A4) $a \lhd R^{-1}[U],\ a \lhd R^{-1}[V] \Rightarrow a \lhd R^{-1}[(U \downarrow) \cap (V \downarrow)]$,

Thm 3 The formal topologies (satisfying R,T,L) and the approximable mappings consistute a category **FTop** and there is a functor (the point-functor)

$$\mathrm{Pt} : \mathbf{FTop} \to \mathbf{Top}.$$

Remark. The localisation axiom (L) seems too strong for inductive generation of formal topologies.

3. How to generate formal topologies?

This problem has been investigated by Coquand, Sambin, Smith, Valentini (2000) through a careful analysis of the structure of possible derivation trees.

• Another more general (and brutal) method for this kind of inductive definitions will be presented here.

First a mild generalisation.

A family $\mathcal{G} = (g_i, G_i)_{i \in I}$ is an *axiom set* on S if I is a set, $g_i \in S$ and $G_i \subseteq S$ for each $i \in I$.

Define a binary relation

$$a \leq_{\mathcal{G}} b \Longleftrightarrow_{\text{def}} (\exists i \in I)\, a = g_i \wedge G_i = \{b\}.$$

Formal intersection $\wedge_{\mathcal{G}}$ is defined from this relation.

A $\mathcal{G}$*-extending cover relation* $\lhd$ satisfies

(R) $a \in U \Longrightarrow a \lhd U$

(T) $a \lhd U$ and $U \lhd V \Longrightarrow a \lhd V$

(GL) $a \lhd U$ and $a \lhd V \Longrightarrow a \lhd U \wedge_{\mathcal{G}} V$

(GE) $g_i \lhd G_i$ for all $i \in I$.

Note that

$$U \wedge_{\mathcal{G}} V \subseteq U \downarrow \cap V \downarrow .$$

Hence $(S, \lhd)$ is called a *formal topology* over $\mathcal{G}$

Theorem 4 The intersection of a set of cover relations over $\mathcal{G}$ is a cover relation over $\mathcal{G}$.

Remark. The theorem gives an impredicative way (†) (in ZFC, IZF, topos logic etc.) to define the least formal topology that extends $\mathcal{G}$.

Let C be the set (†) of cover relations $R \subseteq X \times \mathcal{P}(X)$. By Theorem 4, $\bigcap C$ is the minimal cover relation that contains $\mathcal{G}$.

• The set C is unacceptable from a *predicativistic* point of view. If one could restrict the inner power set $\mathcal{P}(X)$, then the relation $\lhd$ can be defined inductively, since the relation occurs strictly positively in the axioms R,T, GE and GL.

• From an *effective* point of view $\mathcal{P}(X)$ is also too large. The coverings should have some algorithmically enumerable base.

A type-theoretic definition of the power set of X is

$$\mathcal{P}(X) = X \to \mathrm{Set},$$

i.e. as the *type* of propositions over X, or equivalent as the type of families of sets indexed of X.

We want a limited power set $\mathcal{R}(X) = X \to U$ for each set X, such that $\lhd'$ can be defined.

Extend $\lhd' \in \mathcal{P}(X \times \mathcal{R}(X))$ to arbitrary subsets U

$$a \lhd U \iff (\exists U' \in \mathcal{R}(X))\ U' \subseteq U \& a \lhd' U'.$$

Whar closure properties are required of$\mathcal{R}(X)$?

The crucial axiom is that of *transitivity*

$$a \lhd U, U \lhd V \Longrightarrow a \lhd V.$$

LHS gives $a \lhd' U'$ for some $U' \subseteq U$ and for each $x \in U'$ there is $V_x \subseteq V$ with $x \lhd' V_x$. If now $V' = \cup_{x \in U'} V_x \in \mathcal{R}(X)$, the axiom is valid by monotonicity of $\lhd'$.

The type-theoretic axiom of choice was used above $V_{(\cdot)} : A \to \mathcal{R}(X)$ where

$$A = (\Sigma x : X)\, x \varepsilon U'.$$

Further: $y \varepsilon \cup_{x \in U'} V_x$ means more precisely $(\exists x : X)(\exists p : x \varepsilon U') y \varepsilon V_{\langle x,p \rangle}$

It suffices that U is closed under Σ!

Regular universes

Let $B : A \to Set$ be a family of sets. Then we assume that $U = U(A,B)$ och $T = T(A,B)$ is a family of sets such that

$$\frac{a : A}{\beta(a) : U}, \qquad T(\beta(a)) = B(a)$$

and

$$\frac{a : U \qquad b : (T(a))U}{\sigma(a,b) : U}$$

$$T(\sigma(a,b)) = \Sigma(T(a),(x)T(b(x))).$$

This is a universe closed under Σ and including A, B. The *canonical regular universe* including A, B.

- Note that $U(A,B)$ and $T(A,B,\cdot)$ is an inductive-recursive definition (Dybjer, JSL 1999) and a lean version of universe operators (P. *On universes in Type Theory* 1999.)
- One may consider $U(A,B)$ as the set of all sets that are hereditarily “enumerable” given A, B. This is the set-theoretic idea of a *regular cardinal.*

4. The formalisation

- carried out in Agda/Alfa using the graphical interface almost exclusively (Linux, 2 $\times$ Athlon MP 1800+, 3 GB)
- all relations are extensional. $\mathcal{R}(X)$ consists of extensional functions $X \to U(A,B)$
- The basic family A, B is chosen sufficiently large to accomodate the generators $\mathcal{G}$ and the relations $=_X$ and $\leq_{\mathcal{G}}$.

• First define (extensional) $\lhd' \subseteq X \times \mathcal{R}(X)$ by

$$a \lhd' U \iff (\exists d : D)(a = c(d) \wedge C(d) = U)$$

where $D : Set$, $c : D \to X$ and $C : D \to \mathcal{R}(X)$ are defined inductive-recursively.

Inference rule for transitivity:

$$\frac{V : \mathcal{R}, p : D, q : \check{p} \to D, r : (\forall x : \hat{p}) c(qx) = x.1 \& C(qx) = V}{trans(V, p, q, r) : D}$$

where $\mathcal{R} = \mathcal{R}(X)$ and $\hat{p} = (\Sigma x \in X) x \varepsilon C(p)$. Further,

$$\begin{aligned} c(trans(V,p,q,r)) &= c(p) \\ C(trans(V,p,q,r)) &= V. \end{aligned}$$

Inference rule for localisation:

$$\frac{p, q : D \quad e : c(p) = c(q)}{loc(p, q, e) : D}$$

Further,

$$\begin{aligned} c(loc(p,q,e)) &= c(p) \\ C(loc(p,q,e)) &= C(p) \wedge_{\mathcal{G}} C(q). \end{aligned}$$

This gives rise to an induction principle for derivations, which can prove that

$$a \lhd U \iff_{def} (\exists U' \in \mathcal{R}(X))\, U' \subseteq U \& a \lhd' U$$

is the minimal cover relation $\subseteq X \times \mathcal{R}(X)$ containing $\mathcal{G}$.

Some experiences with Alfa/Agda

• When defining record types it would be useful to allow let-expression depending on previous record elements. This is needed for e.g. defining algebraic structures:

$op : Funct(A \times A, A)$ is an operation with extensionality proof. Give a name to the operation part, e.g "+".

• the constant suggestions to make circular proofs are often too tempting.

• the cursor jumps about too much in big proofs. Some compromise between moving to the next goal and not at all would be useful.

• Please implement check for inductive-recursive definitions! Also simultaneous recursive definitions fails sometime (cf "max" with three arguments reported earlier.)

5. Inductive generation of sites

Let $\mathbb{C}$ be a small category with extensional equality on objects. A *sieve U on an object c* is a set of morphism with codomain c such that for $f : d \to c$

$$g : e \to d \in U \Longrightarrow f \circ g \in U.$$

For every $h : c' \to c$ the restriction $U \cdot h = \{g : e \to c' \mid h \circ g \in U\}$ is a sieve c'.

For each object c there is a maximal sieve on c:

$$t_c = \{h \in \mathbb{C} : \mathrm{cod}\, h = c\} \subseteq \mathrm{Mor}(\mathbb{C}).$$

The sieves $S(c)$on c is a subclass of $\mathcal{P}(\hat{t_c})$.

The small sieves $S_R(c)$ on c is a subclass $\mathcal{R}(\hat{t_c})$.

Here $\hat{t_c}$ is the set corresponding to t_c.

Sats 5 For an axiom set $g_i \in \mathrm{Ob}(\mathbb{C})$, $G_i \in S(g_i)$ $(i \in I)$ where I is a set, there is a smallest site extending this set.

The proof is analogous to that of for topologies (but has not been formalised).

Let Geom be the inductive type closed under

• $atom(A)$: Geom for any set A,

• $P \wedge Q$: Geom whenever P : Geom and Q : Geom,

• $(\exists x : S)\, R(x)$: Geom for any set S and any "propositional function" $R : S \to$ Geom.

(For $S = \{0, 1\}$, we get a disjunction.)

A *site* on $\mathbb{C}$ is given by $\lhd \subseteq (\Sigma c \in \mathrm{Ob}(\mathbb{C}))\, S(c)$ such that

(R) $c \lhd t_c$,

(T) If $c \lhd U$, $V \in S(c)$, and $d \lhd V \cdot h$ for all morphism $h : d \to c$ in U, such that $c \lhd V$,

(L) $h : d \to c$ and $c \lhd U$ implies $d \lhd U \cdot h$.

Barr's theorem.

Barr's Theorem (a) If $A^i \to B^i$, then $A^g_F \to B^g_F$ for any F

(b) If A^g_F for all F, then A^i.

Remark The theorem has been proved in Agda/Alfa. It could in principle be used to import classical proofs from a resolution system to type theory.

There is the standard interpretation $(\cdot)^i : \mathrm{Geom} \to \mathrm{Prop}$

and a classical interpretation $(\cdot)^g_F : \mathrm{Geom} \to \mathrm{Prop}$ given by

$atom(P)^g_F = (P \to F) \to F$,

$(P \wedge Q)^g_F = P^g_F \wedge Q^g_F$,

$((\exists x : A)\, R(x))^g_F = ((\Pi x : A)\, R(x)^g_F \to F) \to F$.