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Kompletterande material,
övningar och inlämningsuppgifter
Konstruktiv logik och λ-kalkyl
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1 On equivalent statements of adjoints

Lambek and Scott (1986) defines an *adjointness* between categories $\mathcal{A}$ and $\mathcal{B}$ to be a quadruple $(F, U, \eta, \varepsilon)$ where $F : \mathcal{A} \to \mathcal{B}$ and $U : \mathcal{B} \to \mathcal{A}$ are functors and $\eta : 1_{\mathcal{A}} \dot{\to} UF$ and $\varepsilon : FU \dot{\to} 1_{\mathcal{B}}$ are natural transformations that satisfy

$$(U\varepsilon) \circ (\eta U) = 1_U \tag{1}$$

and

$$(\varepsilon F) \circ (F\eta) = 1_F. \tag{2}$$

U is the *right adjoint* to F, and conversely F is the *left adjoint* to U.

The notation in (1) and (2) need some explication: $U\varepsilon$ is the natural transformation $UFU \dot{\to} U$ defined by

$$(U\varepsilon)_B = U(\varepsilon_B)$$

whereas ηU is the natural transformation $U \dot{\to} UFU$ given by

$$(\eta U)_B = \eta_{U(B)}.$$

1_U is the identity natural transformation $U \dot{\to} U$.

Exercise: check that these are indeed natural transformations.

(The composition $\circ$ of natural transformations is sometimes called *vertical composition* and denoted $\bullet$.) The equation (1) now reads that for all objects B of $\mathcal{B}$,

$$U(\varepsilon_B) \circ \eta_{U(B)} = 1_{U(B)}.$$

Similarly, (2) states that for all objects A of $\mathcal{A}$

$$\varepsilon_{FA} \circ F(\eta_A) = 1_{F(A)}.$$

This definition of adjointness involves two functors and two natural transformations. There are two very useful ways of saying that an adjointness exists given only one functor (see MacLane 1971). The first formulation occurs naturally with free structures over a set.

Theorem 1.1 *Let $U : \mathcal{B} \to \mathcal{A}$ be a functor. Suppose that for each A in $\mathcal{A}$ there exists an object $F_0(A)$ and an arrow $\eta_A : A \to UF_0(A)$ such that for: each object B of $\mathcal{B}$ and each arrow $f : A \to U(B)$ in $\mathcal{A}$ there exists a unique $\bar{f} : F_0(A) \to B$ with*

$$U(\bar{f}) \circ \eta_A = f. \tag{3}$$

Then U is part of an adjointness $(F, U, \eta, \varepsilon)$ between $\mathcal{A}$ and $\mathcal{B}$, where F_0 is the object part of the functor F and η is a natural transformation. The morphism part is defined as follows: for $g : A \to C$ the morphism $F(g)$ is the unique h such that

$$U(h) \circ \eta_A = \eta_C \circ g.$$

The natural transformation ε is given by: for each object B of $\mathcal{B}$ the component ε_B is the unique $h : FU(B) \to B$ such that

$$U(h) \circ \eta_{U(B)} = 1_{U(B)}.$$

Conversely, if $(F, U, \eta, \varepsilon)$ is an adjointness, then $\bar{f} = \varepsilon_B \circ F(f)$ is the unique solution to (3).

The next is a dual formulation, which seems most natural for cartesian closedness.

Theorem 1.2 *Let $F : \mathcal{A} \to \mathcal{B}$ be a functor. Suppose that for each B in $\mathcal{B}$ there exists an object $U_0(B)$ and an arrow $\varepsilon_B : FU_0(B) \to B$ such that: for each object A of $\mathcal{A}$ and each arrow $f : F(A) \to B$ in $\mathcal{B}$ there exists a unique $\bar{f} : A \to U_0(B)$ with*

$$\varepsilon_B \circ F(\bar{f}) = f. \tag{4}$$

Then F is part of an adjointness $(F, U, \eta, \varepsilon)$ between $\mathcal{A}$ and $\mathcal{B}$, where U_0 is the object part of the functor U and ε is a natural transformation. The morphism part is defined as follows: for $g : B \to D$ the morphism $U(g)$ is the unique h such that

$$\varepsilon_D \circ F(h) = g \circ \varepsilon_B.$$

The natural transformation η is given by: for each object A of $\mathcal{A}$ the component η_A is the unique $h : A \to UF(A)$ such that

$$\varepsilon_{F(A)} \circ F(h) = 1_{F(A)}.$$

Conversely, if $(F, U, \eta, \varepsilon)$ is an adjointness, then $\bar{f} = U(f) \circ \eta_A$ is the unique solution to (4).

2 Recommended exercises, week 38

From C. McLarty, *Elementary Categories, Elementary Toposes*:

- Exercises 1.1, 1.3, 2.2, 2.3.

From J. Lambek and P.J. Scott, *Introduction to Higher Order Categorical Logic*:

- Exercises 0.3.1, 0.3.4, I.1.1, I.1.4, I.3.2, I.8.1, I.8.4.

3 Recommended exercises, week 41

From J. Lambek and P.J. Scott, *Introduction to Higher Order Categorical Logic*:

- Exercises I.9.1–2, I.12.1–2, I.15.1–3.

4 Hand-in problems, set 1 (inlämningsuppgifter 1)

The problems should be solved individually and are due by October 21. The solutions should be clearly written and well explained.

1.1 Exercise 0.3.1 in Lambek and Scott.

1.2 Prove the following isomorphisms in an arbitrary bicartesian closed category by showing that an object of one the member is part of diagram for the other member.

   (a) $A \times (B + C) \cong A \times B + A \times C$

   (b) $A^{B \times C} \cong (A^B)^C$

1.3 Determine all weak natural numbers objects in the category of sets.

1.4 Exercise I.12.1 in Lambek and Scott.

1.5 Translate the following propositions into dependent types and show that they are inhabited. (Hint: the procedure described in *Konstruktiv logik* is helpful here.)

   (a) $(\forall x \in S)\, A \wedge B(x) \rightarrow (A \wedge (\forall x \in S)\, B(x))$

   (b) $A \vee (\exists x \in S)\, B(x) \rightarrow (\exists x \in S)\, (A \vee B(x))$

   (c) $(\forall x \in N)\, 0 + x = x$ (Use the definition of $+$ and $=$ given by Martin-Löf, *An intuitionistic theory of types,* pp. 154 – 156)

5 Hand-in problems, set 2 (inlämningsuppgifter 2)

2.1 The category $\mathbb{G}$ of labelled directed graphs: The obejcts are tuples (A, V, s, t) where A and V are sets (possibly empty) and $s, t : A \to V$ are functions. A morphism $(A, V, s, t) \to (A', V', s', t')$ is a pair (f, g) such that $f : A \to A'$, $g : V \to V'$ and $s'f = gs$ and $t'f = gt$. Composition is given by $(f', g') \circ (f, g) = (f'f, g'g)$. Prove the following properties of the category

(a) it has a terminal object,

(b) it has binary products,

(c) it has a natural numbers object N,

(d) it is isomorphic to the functor category $\mathbf{Sets}^{\bullet \rightrightarrows \bullet}$,

(e)** it is cartesian closed. Calculate N^N explicitly.

2.2 Consider a category $\mathbb{C}$ with an endofunctor T. Suppose that $\alpha : TA \to A$ is an initial T-algebra and that $\omega : \Omega \to T\Omega$ is a terminal T-algebra. Prove that there exists a unique $f : A \to \Omega$ such that

$$\omega \circ f \circ \alpha = T(f).$$

2.3 Consider the functor $F = T + (-) : \mathbf{Sets} \to \mathbf{Sets}$ where T is the terminal object. Determine the terminal T-coalgebra. (Hint: the object could be $\mathbb{N} \cup \{\infty\}$.)

2.4 Let R be a fixed set. Investigate whether the double exponential

$$FX = R^{R^X} = (R \Leftarrow (R \Leftarrow X))$$

extends to morphisms and forms a functor $\mathbf{Sets} \to \mathbf{Sets}$.

6 Hand-in problems, set 3 (inlämningsuppgifter 3)

Problems 3.1 - 3.2 are to be solved using the functional programming language Haskell. Problems 3.3 - 3.4 refer to the introductory paper E. Palmgren, *Constructive Mathematics,* Uppsala 1997.

3.1 Implement interval arithmetic with rational bounds. The following operations and tests should be defined

$$+, -, \cdot, /, <, \subseteq$$

(se e.g. Palmgren, E., Topics in Domain Theory and Point-free Topology, Uppsala 2002.)

3.2 Define data types for binary signed digit expansions and an operation truncating such an expansion to an interval. Define also an algorithm for converting a rational number to a signed digit.

3.3 Give a constructive proof that for all real numbers x and y

$$x + y > 0 \Longrightarrow x > 0 \text{ or } y > 0.$$

3.4 Show that constructively the following two statements are equivalent

(1) LLPO

(2) For all real numbers x: $x \geq 0$ or $x \leq 0$.