Predicativity problems in point-free topology

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1 Introduction

Predicativity has been an important issue in the foundations of mathematics and proof theory since the beginning of the last century. The notion received a particular technical and philosophically well-motivated meaning in the proof theory of ramified second order arithmetic, as analysed by Feferman and Schütte in the 1960s. On the other hand, in the foundations for constructive mathematics, which we shall be concerned with here, the notion has come to encompass a wide class of theories whose basic objects and logic have a clear inductive structure. Such theories have been called *generalised predicative* by Aczel. Modern type-theoretic frameworks make it possible to attempt a precise criterion for a predicative system, namely that the introduction rule of each type should be consistent with a recursion or elimination principle. We refer to Palmgren (1998), Dybjer (2000) and Dybjer and Setzer (2003) for further discussions.

Russell's ramified theory of types is the first example of a predicative theory intended for the formalisation of mathematics. It suffers from a well-known defect when it comes to the real numbers. If we consider a real number as a symmetric Dedekind cut, it is given by a pair of predicates (L, U), of a certain degree of the ramification, on the rational numbers. This has the consequence that the real numbers fall into a hierarchy

$$\mathbb{R}_1 \subseteq \mathbb{R}_2 \subseteq \mathbb{R}_3 \subseteq \cdots \subseteq \mathbb{R}_n \subseteq \cdots$$

of number systems, which fail to be order-complete. Instead of giving up the idea of predicativity, one may settle for weaker completeness properties. Weyl (1918) investigated the real numbers of the first level. Recent investigations of similar restrictions on set existence have been carried out in reverse mathematics (Simpson 1999). These pertain to the classical setting. The situation is quite different in the constructive setting.

Constructive set theory CZF (Aczel 1986) and type theory (Martin-Löf 1984) and Feferman's Explicit Mathematics are three main formal systems

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built on predicative principles. Of these the first two have been used extensively for the formalisation of constructive mathematics. Using the logical principle of dependent choice, which is a theorem of the type theory and is true in the standard model of CZF, one may construct real numbers as Cauchy sequences of rational numbers. These real numbers are then Cauchy-complete, but order-complete only in a restricted sense (cf. Bishop and Bridges 1985). Constructively, not every subset has a characteristic function, so even with dependent choice these completeness properties are not equivalent, unlike the classical setting. Topology in constructive mathematics following Bishop (1967) has essentially been restricted to metric spaces. More general topologies can be treated using non-classical or non-recursive axioms of Brouwer, but the results may then contradict classical or recursive mathematics, which is unsatisfactory for many reasons. The point-free topologies in the form of locale theory (Johnstone 1982) and formal topology (Sambin 1987) give a satisfactory treatment of general topology, which is also consistent with classical and effective interpretations. However, there are again many problems of predicativity. It is the purpose of this paper to survey some solutions and remaining problems in the point-free approach. We also present some new main results, Theorem 4.5 and 5.2. Some basic references in formal topology are Sambin (1987, 2003), Aczel (2002), Coquand (1995), Coquand *et al.* (2003), Negri and Soravia (1999) and Sigstam and Stoltenberg-Hansen (1997).

2 Sets in type theory

The established way of understanding a set in Bishop's constructive mathematics is as a type, or "preset", together with an equivalence relation $A = (\underline{A}, =_A)$. A function f from the set A to the set B is an operation from $\underline{A}$ to $\underline{B}$ respecting the equivalence relations. The resulting category of sets and functions, if built on a suitable type theory, can be seen to have many properties in common with toposes, such as existence of products, disjoint sums and quotients. We shall here assume that our type theory is Martin-Löf type theory with a cumulative sequence of universes (Martin-Löf 1972/1998)

$$\mathbb{U}_1 \subseteq \mathbb{U}_2 \subseteq \mathbb{U}_3 \subseteq \cdots \subseteq \mathbb{U}_n \subseteq \cdots$$

so that each type is contained some $\mathbb{U}_n$, where n is an index external to the theory. Moreover, for each n: $\mathbb{U}_n \in \mathbb{U}_{n+1}$ and $\mathbb{U}_n$ is closed under the type formers for dependent products (Π) and dependent sums (Σ), binary sums ($+$), identity types (Id) and well-founded trees (W). However, we do not assume the universe to be closed under full power sets. Instead, because of the hierarchical structure of universes, and hence the truth-values, there will be a ramification structure of subsets, which we now describe. A set A is an (m, n)-*set* if $\underline{A} \in \mathbb{U}_m$ and $=_A$ is a propositional function $\underline{A} \times \underline{A} \longrightarrow \mathbb{U}_n$. If

$m = n$, then A is called an *n-set*, and if $m = n + 1$, A is an *n-class.* A set S together with an injection $i : S \longrightarrow A$ is a *subset* of A. The subset (S, i) is *included* in another subset (T, j), symbolically $(S, i) \subseteq (T, j)$, if there is a function $f : S \longrightarrow T$ with $j \circ f = i$. If S is an (m, n)-set, then (S, i) is an (m, n)-subset of A. The derived notions *m-subset* and *m-subclass* are defined in the obvious way. Note that if A is an m-class, X is an m-set, and $f : X \longrightarrow A$ is a function, then the image of f in A, given by $\mathrm{Im}(f) = (A, =')$, where $x =' y$ iff $f(x) = f(y)$, and where the injection is f, is an m-subset. This is the crucial replacement property of classes relative to sets. The type of k-sets $S = (\underline{S}, =_S)$ belongs to $\mathbb{U}_{k+1}$ and is denoted Sets_k. Let A be an (m, n)-set. The set $\mathcal{P}_k(A)$ of k-subsets of A is given by the type

$$(\Sigma S \in \mathrm{Sets}_k)(\Sigma i : \underline{S} \longrightarrow \underline{A})[\ i : S \longrightarrow A \text{ is an injection}]$$

which is in $\mathbb{U}_{\max(k+1,m,n)}$, and the equivalence relation $(S, i, p) = (S', i', p')$ is given by the conjunction $((S, i) \subseteq (S', i')) \wedge ((S', i') \subseteq (S, i))$ which is a proposition in $\mathbb{U}_{\max(k,n)}$. The set $\mathcal{P}_k(A)$ is thus a $(\max(k+1, m, n), \max(k, n))$-set. In particular, if A is a k-set, or a k-class, $\mathcal{P}_k(A)$ is a k-class.

Following Aczel (2002) and Gambino (2002) we define a *partially ordered k-class* $(X, \leq)$ to consist of a k-class X and a k-subclass $\leq$ of $X \times X$, where $\leq$ is a partial order on X. Define $(X, \leq)$ to be *complete* if every k-subset of X has a supremum.

Example 2.1 *If A is a k-set, or a k-class, then $(\mathcal{P}_k(A), \subseteq)$ is a complete partially ordered k-class.*

A category $\mathbb{C}$ is *locally n-small,* if the morphisms between any two fixed objects form an n-set.

We remark that there is a predicative version of topos (Moerdijk and Palmgren 2002) which is an abstraction of the notion of sets in the mentioned type theory. In actual development of mathematics within type theory it seems rarely to be necessary to go beyond the first few levels of sets.

3 Formal topologies

The fundamentals of locale theory have been developed predicatively in CZF (Aczel 2002, Gambino 2002). We could use the notions of k-set and k-class introduced in the previous section to translate this development into type theory. However, we shall here stay closer to the usual presentations of formal topologies following Martin-Löf and Sambin, in that frames are not used explicitly. We use the term *formal space* almost interchangeably with *formal topology,* but prefer the former term when points have a significant role.

Definition 3.1 Let $(X, \leq)$ be a pre-ordered k-set, whose elements will be called *formal neighbourhoods.* Then for k-subsets U and V of X define their *formal intersection*, to be the k-subset

$$U \wedge V = \{x \in X : (\exists u \in U)(\exists v \in V)\, x \leq u \,\&\, x \leq v\}.$$

A *k-formal topology* $X = (X, \leq, \mathcal{A})$ arises when X is equipped with a *covering operator* $\mathcal{A} : \mathcal{P}_k(X) \longrightarrow \mathcal{P}_k(X)$ which is monotone, inflationary, idempotent and which satisfies

$$\mathcal{A}(U) \cap \mathcal{A}(V) = \mathcal{A}(U \wedge V).$$

We often write $a \lhd_{\mathcal{A}} U$ for $a \in \mathcal{A}(U)$, to be read *a is covered by U.* A subset U of X is *$\leq$-filtering,* if for any $x, y \in U$ there is some $z \in U$ with $z \leq x$ and $z \leq y$. A *k-point* α of the topology X is an inhabited k-subset of X which is $\leq$-filtering and such that $U \cap \alpha$ is inhabited whenever $U \in \mathcal{P}_k(X)$ is such that $\mathcal{A}(U) \cap \alpha$ is inhabited. For $k \geq m$, a k-formal topology X is said to be *m-set presented,* if there is a family $I(a)$ $(a \in X)$ of m-sets, and a family of m-subsets $C(a, i)$, $(a \in X, i \in I(a))$ such that

$$\mathcal{A}(U) = \{x \in X : (\exists i \in I(x))\, C(x, i) \subseteq U\}.$$

Let X be a k-formal topology. The k-class of *saturated* k-subsets is

$$\mathrm{Sat}_k(X) =_{\mathrm{def}} \{V \in \mathcal{P}_k(X) : \mathcal{A}(V) = V\}.$$

Noting that the supremum of a family of saturated subsets is

$$\mathcal{A}(\cup_{i \in I} V_i),$$

it is straightforward to prove

Theorem 3.2 *For each k-formal topology, $(\mathrm{Sat}_k(X), \subseteq)$ is a complete k-frame, i.e. complete lattice with k-set indexed infinite distributive law.*

Let $X = (X, \leq, \mathcal{A})$ be a k-formal topology. Define a $(k+1)$-formal topology $X^+ = (X, \leq, \mathcal{A}^+)$ by extending $\mathcal{A}$ to an operator $\mathcal{A}^+$ on $\mathcal{P}_{k+1}(X)$ as follows

$$\mathcal{A}^+(U) = \{x \in X : (\exists V \in \mathcal{P}_k(X))\, V \subseteq U \,\&\, x \in \mathcal{A}(V)\}.$$

The notation $\mathcal{A}^{+m}$ means that $\mathcal{A}$ has been extended m steps in this way. The collection of k-points of X is a k-class, which is denoted $\mathrm{Pt}_k(X)$. Then we have the following relations between a level and the next.

Proposition 3.3 *(i) $\mathcal{A}^+$ is a covering operator for $(X, \leq)$ which extends $\mathcal{A}$, in the sense that $\mathcal{A}^+(U) = \mathcal{A}(U)$, for all $U \in \mathcal{P}_k(X)$. Consequently, X^+ is a $(k+1)$-formal topology.*

(ii) $\mathrm{Pt}_k(X) = \mathrm{Pt}_{k+1}(X^+) \cap \mathcal{P}_k(X)$

Since $\mathcal{A}^+$ extends $\mathcal{A}$ it follows that X^+ is compact iff X is compact. (Recall that compactness means that if each neighbourhood is covered by some fixed set V of neighbourhoods, then there is a finite list of neighbourhoods in V that covers any neighbourhood.) We also have $x \wedge a \in \mathcal{A}^+(\emptyset)$ iff $x \wedge a \in \mathcal{A}(\emptyset)$, so $a \ll_{\mathcal{A}^+} b$ iff $a \ll_{\mathcal{A}} b$. From this follows that X^+ is regular iff X is regular. Recall also that a formal topology $(X, \leq, \mathcal{A})$ is regular if $z \lhd_{\mathcal{A}} \{x \in X : x \ll_{\mathcal{A}} z\}$ for each $z \in X$, and where $x \ll_{\mathcal{A}} z$ (x is *well covered by* z) means that any neighbourhood is covered by $\{z\} \cup \{a \in A : x \wedge a \lhd_{\mathcal{A}} \emptyset\}$. The latter is the neighbourhoods "apart from x", denoted x^*.

A predicativity problem similar to Russell's is now: for a given k-formal topology X, does the notion of point stabilize at some level? We call a k-formal topology T_1 if each of its points α is (absolutely) maximal, in the sense that $\beta = \alpha$ whenever $\beta \supseteq \alpha$ is a $(k+m)$-point in X^{+m}. By Proposition 3.3.(ii), the collection of points stabilise at level k for such a topology. Another question is whether the points of a k-formal topology form a k-set, and not only a k-class. This fails for one of the simplest examples: when X is the Sierpinski topology; see e.g. (Palmgren 2002a). The phenomenon is explained by the fact that there exist partial points (i.e. positively non-maximal) with respect to inclusion. They may then be used to represent the class of proposition as points. On the other hand, in case X is a 1-set-presented T_1-topology, it is shown in the same paper that its points X are isomorphic to a set in $\mathbb{U}_1$. It had already been established that the points of a locally compact regular topology form a set (Curi 2002), and later that the same is true for regular topologies (Aczel 2002). The conditions on these topologies pertain to the covers only, and are thus more in the spirit of point-free topology than the T_1-condition.

A crucial method for constructing new formal topologies is the inductive generation of covers or operators; see Johnstone (1982) and Coquand *et al.* (2002). Here is a method similar to (Aczel 2002).

For any pre-ordered k-set $(X, \leq)$ and a pair of functions $g : I \longrightarrow X$ and $G : I \longrightarrow \mathcal{P}_k(X)$, where I is a k-set (the pair (g, G) is called the *generator*) define an operator $\mathcal{A} : \mathcal{P}_k(X) \longrightarrow \mathcal{P}_k(X)$ as follows. For each k-subset $U \subseteq X$, let $\mathcal{A}(U)$ be the smallest subset $V \subseteq X$ including U and such that

- $b \in V$, $a \leq b$ implies $a \in V$
- $a_1, \ldots, a_n \in X$, $n \geq 0$, $G_i \wedge \{a_1\} \wedge \cdots \wedge \{a_n\} \subseteq V$ implies $\{g_i\} \wedge \{a_1\} \wedge \cdots \wedge \{a_n\} \subseteq V$

By assuming that $\mathbb{U}_k$ contains well-founded tree-types (W-types) one may prove that this set V is a k-subset of X. Then we have the following theorem which entails an induction principle for covers.

Theorem 3.4 *The structure $(X, \leq, \mathcal{A})$ defined as above is a k-formal topology whose covering operator satisfies*

$$g_i \in \mathcal{A}(G_i) \qquad (i \in I) \tag{1}$$

and if $\mathcal{B}$ is any other covering operator on $\mathcal{P}_n(X)$, $n \geq k$, satisfying this condition, then $\mathcal{A}^{+(n-k)}(U) \subseteq \mathcal{B}(U)$ for all $U \in \mathcal{P}_n(X)$.

4 Morphisms of formal topology

A continuous function f between topological spaces induces a relation $U \subseteq f^{-1}[V]$ among the open sets of the respective spaces. The notion of morphism between formal topologies is an abstraction of this relation (Sambin 1987). Since morphisms are generally represented by relations, rather than functions, we now encounter further predicativity problems in the category of formal topologies. For a relation $R \subseteq X \times Y$ the *inverse image of $V \subseteq Y$ under the relation R* is, as usual, $R^{-1}[V] =_{\text{def}} \{a \in X : (\exists b \in V)\, a\, R\, b\}$. We write $R^{-1}b$ for $R^{-1}[\{b\}]$.

Definition 4.1 Let $X = (X, \leq_X, \mathcal{A}_X)$ and $Y = (Y, \leq_Y, \mathcal{A}_Y)$ be k-formal topologies. A relation $R \in \mathcal{P}_k(X \times Y)$ is an *approximable mapping,* or *formal morphism* (of level k) from X to Y if

(A1) $R^{-1}[\mathcal{A}_Y(U)] \subseteq \mathcal{A}_X(R^{-1}[U])$,

(A2) $\mathcal{A}_X(R^{-1}b) \subseteq R^{-1}b$,

(A3) $X = \mathcal{A}_X(R^{-1}[Y])$,

(A4) $R^{-1}b \cap R^{-1}c \subseteq \mathcal{A}_X(R^{-1}[b \wedge c])$.

The collection of k-level morphisms $X \longrightarrow Y$ is a k-class, which we denote $\mathrm{Hom}_k(X, Y)$. A formal morphism $R : X \longrightarrow Y$ gives rise to a k-frame morphism $\mathrm{Sat}_k(Y) \longrightarrow \mathrm{Sat}_k(X)$

$$f_R(U) = \mathcal{A}(R^{-1}(U)).$$

The preservation of suprema is shown by applying (A1) and noting that R^{-1} commutes with unions. The corresponding point-function $\mathrm{Pt}_k(X) \longrightarrow \mathrm{Pt}_k(Y)$ is given by

$$\mathrm{Pt}_k(R)(\alpha) = \{y \in Y : (\exists x \in \alpha)\, x\, R\, y\}.$$

Let $\mathbf{FTop}_k$ be the category of k-formal topologies and formal morphisms of level k. For a formal topology X we define the identity $I : X \longrightarrow X$ by

$$aIb \Longleftrightarrow a \in \mathcal{A}_X\{b\}.$$

For morphisms $R_1 : X_1 \longrightarrow X_2$ and $R_2 : X_2 \longrightarrow X_3$ between formal topologies, define the composition

$$a(R_2 \circ R_1)b \Longleftrightarrow a \in \mathcal{A}_{X_1}(R_1^{-1}[R_2^{-1}b]).$$

This is a morphism $(R_2 \circ R_1) : X_1 \longrightarrow X_3$.

Example 4.2 For any k-formal topology X the morphisms in $\mathrm{Hom}_k(1, X)$ stand in bijective correspondence with $\mathrm{Pt}_k(X)$, via

$$R \mapsto \{a \in X : \star\, R\, a\}.$$

Since these points need not form a k-set (for e.g. the $X =$ Sierpinski topology) the same is true for the morphisms of level k. The object 1 above is the terminal k-formal topology given by a singleton set $\{\star\}$ and the identity operator.

This example shows that $\mathbf{FTop}_k$ is not locally k-small. The analogy of points and morphisms leads to the consideration of the inclusion order between morphisms. The morphism $R \in \mathrm{Hom}_k(X, Y)$ is *maximal,* if for any $R' \in \mathrm{Hom}_{k+m}(X^{+m}, Y^{+m})$ with $R' \supseteq R$ (as graphs) we have $R' = R$. We have as in Proposition 3.3

$$\mathrm{Hom}_k(X, Y) = \mathrm{Hom}_{k+1}(X^+, Y^+) \cap \mathcal{P}_k(X \times Y).$$

The Hom-sets stabilise at k when all k-morphisms are maximal.

The following is a generalisation of the result by Sambin that the points of a regular topology are maximal.

Theorem 4.3 *Let X be a k-formal topology and Y be a regular k-formal topology. Then all the morphisms in $\mathrm{Hom}_k(X, Y)$ are maximal.*

Proof. Suppose $R \in \mathrm{Hom}_k(X, Y)$ and $R' \in \mathrm{Hom}_{k+m}(X^{+m}, Y^{+m})$, $m \geq 0$, with $R' \supseteq R$. We show that $R' \subseteq R$ follows from the implication

$$a\, R'\, c,\ c \ll b \Longrightarrow a\, R\, b. \tag{2}$$

Since Y is regular, we have $b \lhd_Y \{c \in Y : c \ll b\}$, and hence using (2)

$$\begin{aligned} \mathcal{A}_{X^{+m}}(R'^{-1}b) &\subseteq \mathcal{A}_{X^{+m}}(R'^{-1}\{c \in Y : c \ll b\}) \\ &\subseteq \mathcal{A}_{X^{+m}}(\{a' \in X : a'\, R\, b\}) = \mathcal{A}_{X^{+m}}(R^{-1}b) \end{aligned}$$

Now if $a\, R'\, b$, then $a \in \mathcal{A}_{X^{+m}}(R'^{-1}b)$, and hence by the inclusions above, $a\, R\, b$.

Proof of (2): From $c \ll b$, i.e. $Y \lhd \{b\} \cup c^*$, and (A3), it follows that for any a

$$a \lhd_X R^{-1}[\{b\} \cup c^*] \wedge a.$$

The latter is covered by $R^{-1}b$: since if $x\,R\,d$, $d \in c^*$ and $x \leq a$, we have also $x\,R'\,d$ by the assumption that $R \subseteq R'$. If $a\,R'\,c$, then by localisation $x\,R'\,d \wedge c$. Hence x is covered by $\emptyset$, and therefore trivially covered by $R^{-1}b$. Thus $a\,R\,b$. $\square$

Corollary 4.4 *For the formal topology $\mathcal{R}$ of real numbers, the 1-class $\mathrm{Hom}_1(\mathcal{R},\mathcal{R})$ is actually a 1-set.*

Proof. Each $F \in \mathrm{Hom}_1(\mathcal{R},\mathcal{R})$ gives a continuous function $f = \mathrm{Pt}_1(F) : \mathrm{Pt}_1(\mathcal{R}) \longrightarrow \mathrm{Pt}_1(\mathcal{R})$ in the sense of Bishop (Palmgren 2003). It is straightforward to prove that $A_f \supseteq F$, where A_f is the morphism given by

$$(a,b)\, A_f\, (c,d) \iff f(a,b) \subseteq (c,d).$$

Since morphisms are all maximal, by the theorem above, we get $A_f = F$. Each morphism is thereby represented by a function. $\square$

Assuming that universes $\mathbb{U}_k$ are closed under formation of the next regular universe — a constructive version of the Hartogs function (Palmgren 2002b) — we can show a stronger result.

Theorem 4.5 *Suppose that $\mathbb{U}_1$ is closed under the formation of regular universes. For 1-set presented 1-formal topologies X and Y, where Y is a regular topology, the 1-class $\mathrm{Hom}_1(X,Y)$ is a 1-set.*

Proof. Suppose that X and Y are 1-set presented by $C(x,i)$ ($i \in I(x)$) and $D(y,j)$ ($j \in J(y)$) respectively. Then $R \in \mathrm{Hom}_1(X,Y)$ iff for all $x \in X$, $y \in Y$

(A1') $x\,R\,y \Rightarrow (\forall j \in I(y))(\exists i \in I(x))\,(\forall u \in C(x,i))\,(\exists v \in D(y,j))\,u\,R\,v$,

(A2') $(\forall i \in I(x))[(\forall u \in C(x,i))\,u\,R\,y \Rightarrow x\,R\,y]$,

(A3') $(\exists i \in I(x))(\forall u \in C(x,i))(\exists v \in Y)\,u\,R\,v$,

(A4') $(\forall b,c \in Y)[x\,R\,b, x\,R\,c \Rightarrow (\exists i \in I(x))(\forall u \in C(x,i))(\exists y \leq b,c)\,u\,R\,y]$.

Since R is maximal, according to Theorem 4.3, we may try to reconstruct R inductively from below. All the morphisms R are contained in $\mathcal{P}_1(X \times Y)$. We shall show that they are actually contained in a more restricted regular power set $\mathcal{R}(X \times Y)$, which is a 1-set. This is built from a certain regular universe $\mathcal{U}$. Using the W-type construction we can find $W \in \mathcal{U}$ which has for generating rules

$$0 \in W \qquad \frac{\alpha \in W}{S(\alpha) \in W} \qquad \frac{\alpha \in W \quad \beta \in W}{(\alpha,\beta) \in W}$$

and

$$\frac{x \in X \quad i \in I(x) \quad f : C(x,i) \longrightarrow W}{\sup_{x,i}(f) \in W}. \tag{3}$$

Then construct a family $R_\alpha \in \mathcal{R}(X \times Y)$ ($\alpha \in W$) so that $R_\alpha \subseteq R$, which satisfies the conditions obtained by modifying (A1') – (A4') as follows. Replace the successive R's by R_α and $R_{S(\alpha)}$ in (A1'), by $R_{f(u)}$ and $R_{\sup_{x,i}(f)}$ in (A2'), where $f : C(x,i) \longrightarrow W$, moreover by R_0 in (A3') and by R_α, R_β and $R_{(\alpha,\beta)}$ in (A4'). These may be constructed by recursion on W, and using the type-theoretic choice principles. (R is thus used as "template" for the sequence.) Let

$$R' = \bigcup_{\alpha \in W} R_\alpha \in \mathcal{R}(X \times Y).$$

Then $R' \subseteq R$. By the construction of the sequence R_α is straightforward to see that R' satisfies the original (A1') – (A4'). For (A2') use the type-theoretic choice principle and (3). Thus R' is a morphism $X \longrightarrow Y$ and by maximality $R' = R$. Since R was arbitrary, the result is proved.

This theorem shows that the subcategory of set-presented regular topologies is locally small. The theorem is an improvement of the result of Curi (2003) which shows that $\mathrm{Hom}_1(X, Y)$ is a set when X is locally compact and Y is regular. See also Sigstam (1995) for a recursive version of Hyland's construction of exponential locales.

5 Some categorical constructions

Important basic constructions of classical topological spaces (cf. Armstrong 1983) may be expressed in category-theoretic language: products, disjoint unions (coproducts), quotient spaces and identification spaces (coequalisers and pushouts). We may then ask for the same categorical constructions in the category of formal topologies. This leads to predicativity problems again. For classical constructions in locales, see Borceux (1994).

The only known way of constructing the product of two formal topologies is to assume that they are inductively defined (Coquand *et al.* 2003). Equalizers can be constructed using a similar method, which we sketch below. Coproducts do not need this assumption. However quotients or coequalizers seem to present serious problems. The standard locale-theoretic solution is to consider the equalizer in frames. Simply translating this into formal topologies, means that we must consider the class of saturated sets as new basic neighbourhoods. This uses up one universe, and seems to make it impossible to prove the universal property predicatively. Some subtler construction, or restriction on topologies or morphisms, need to be applied. An equivalent problem is to construct pushouts. Here we have indeed a nontrivial special case which may easily be converted into a predicative proof, using e.g. Palmgren (2003). For a locale L and $a \in L$ consider the locale $L_a = \{x \in L : x \geq a\}$, and the locale morphism $L_a \longrightarrow L$ given by the frame morphism $x \mapsto x \vee a$. This is the embedding of the closed subspace L_a determined by the complement of a into L.

Proposition 5.1 *(gluing for closed subspaces) Let L be a locale and $a, b \in L$ with $a \wedge b = 0$. Then the following is a pushout*

$$\begin{tikzcd} L_{(a \vee b)} \arrow[r] \arrow[d] & L_b \arrow[d] \\ L_a \arrow[r] & L \end{tikzcd}$$

This pushout construction may, for instance, be used to join, continuously, two functions defined on adjacent intervals. Using equalizers we may construct zero sets or graphs of functions as subspaces.

Theorem 5.2 *In* $\mathbf{FTop}_1$ *the equalizer of $F_1, F_2 : X \longrightarrow Y$ exists when X is 1-set presented.*

Proof. Let $C(x,i)$ $(i \in I(x))$ be a 1-set presentation of X. Then $\lhd_X$ is the minimal cover relation so that $x \lhd_X C(x,i)$ for all $x \in X$ and $i \in I(x)$. Let $E = (X, \leq_X, \mathcal{E})$ where is $\mathcal{E}$ is the operator generated by the pairs (g, G) given by

(G1) $(x, C(x,i))$ for $x \in X$, $i \in I(x)$,

(G2) $(a, F_2^{-1}b)$ for $b \in B$, $a \in F_1^{-1}b$,

(G3) $(a, F_1^{-1}b)$ for $b \in B$, $a \in F_2^{-1}b$.

Then $\mathcal{E}(F_1^{-1}b) = \mathcal{E}(F_2^{-1}b)$. The equalizing morphism $F : E \longrightarrow X$ is given by

$$a\,F\,b \Longleftrightarrow a \lhd_E \{b\}.$$

To verify condition (A1) one proves that $b\,K\,V \Leftrightarrow \forall a\,(a\,F\,b \longrightarrow a \lhd_E F^{-1}V)$. defines cover relation on X. Then by minimality: $b \lhd_X V$ implies $b\,K\,V$ which verifies (A1). The conditions (A2) – (A4) are straightforward to check, as well proving that $F_1 \circ F = F_2 \circ F$. If $G : C \longrightarrow X$ is any other morphism making $F_1 \circ G = F_2 \circ G$, then $H : C \longrightarrow E$ defined by $H = G$ is the unique morphism so that $F \circ H = G$. To verify (A1) for H one need to show that

$$b\,L\,V \Leftrightarrow_{\text{def}} \forall a\,(a\,H\,b \longrightarrow a \lhd_C H^{-1}V)$$

defines a cover relation.

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