Overspill and countable saturation implies LLPO

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This note is an addendum to [4]. Here we show that overspill and countable saturation implies the non-constructive principle LLPO.

Consider an infinite 0-1-sequence $(a_n)_{n<\omega}$. Recall that the *lesser limited principle omniscience* (LLPO) states that if the sequence $(a_n)_n$ contains at most one 1 it can be decided whether $a_{2n} = 0$ for all n, or whether $a_{2n+1} = 0$ for all n. Within constructive mathematics this principle is equivalent to the principle that for all real numbers x and y: $x \leq y \vee y \leq x$. For systematic account of the consequences of LLPO, see Ishihara [2].

We work within the framework of constructive nonstandard analysis as presented in [4]. A standard set is denoted ${}^\sigma S$ and its nonstandard version is denoted ${}^*S$. The *countable saturation principle* states that for any function $f : {}^\sigma\mathbb{N} \to {}^*S$ there is an internal extension $g : {}^*(\mathbb{N} \to S)$ such that $f(n) = g(n)$ for all $n \in {}^\sigma\mathbb{N}$. Suppose that $a : {}^\sigma\mathbb{N} \to {}^\sigma\{0,1\}$ is a sequence containing at most one 1. Let $\hat{a}$ be its internal extension $\hat{a} : {}^*(\mathbb{N} \to \{0,1\})$. Then for all $k \in {}^\sigma\mathbb{N}$:

$$(\forall m \leq k)\hat{a}_{2m} = 0 \vee (\forall m \leq k)\hat{a}_{2m+1} = 0.$$

Since the displayed statement is internal, we obtain by overspill an infinite k for which it holds. Thus $(\forall m \in {}^\sigma\mathbb{N})\, \hat{a}_{2m} = 0$ or $(\forall m \in {}^\sigma\mathbb{N})\, \hat{a}_{2m+1} = 0$. Consequently LLPO is valid in the sheaf model of constructive nonstandard analysis [3].

In [3] we showed, using only overspill, that LLPO holds for the more restricted set of functions, ${}^\sigma(\mathbb{N} \to \mathbb{N})$, a function space which does not supply enough choice functions.

References

[1] D.S. Bridges and F. Richman. *Varieties of Constructivism.* Cambridge University Press 1987.

[2] H. Ishihara. An omniscience principle, the König lemma and the Hahn-Banach Theorem. *Zeitschr. f. math. und Grundlagen d. Math.* 36(1990), 237 – 240.

[3] E. Palmgren. Developments in constructive nonstandard analysis. *Bulletin of Symbolic Logic* 4(1998), 233 – 272.

[4] E. Palmgren. Unifying constructive and nonstandard analysis. U.U.D.M. Report 1999:29.