Neighborhood spaces vs. formal spaces

Erik Palmgren

Notes November 22, 2002*

1 Neighborhood spaces

A *neighborhood space* is a pair consisting of a set X and set-indexed family $\{U_i\}_{i \in I}$ of subsets of X, called neighborhoods, satisfying the conditions

(N1) Every $x \in X$ is contained in some neighborhood

(N2) If U and U' are neighborhoods containing x, then there exists a neighborhood U'' such that

$$x \in U'' \subseteq U \cap U'.$$

A function $f : X \to Y$ is a *continuous* map from the neighborhood space $(X, \{U_i\}_{i \in I})$ to the neighborhood space $(Y, \{V_j\}_{j \in J})$, if whenever $f(x)$ belongs to a neighborhood V, then there exists a neighborhood U containing x with the property

$$f(U) \subseteq V.$$

It is straighforward to check that the neighborhood spaces and continuous maps forms a category, **Nbhd**.

Remark 1.1 Neighborhood spaces were considered in Bishop and Bridges (1985) as a constructive version of topological spaces. The axiom (N1) is not mentioned there, but is probably intended to be included (see page 76 op.cit.). Variants of these are called *concrete* topological spaces (cf. Sambin 1987).

*Addendum to the course notes in domain theory: E.Palmgren. *Topics in Domain Theory and Point-free Topology* U.U.D.M. LN2:2002. URL: www.math.uu.se/~palmgren/

2 Relation to formal spaces

From a neighborhood space $X = (X, \{U_i\}_{i \in I})$ we build a formal space with preorder $N(X) = (I, \leq, \lhd)$ taking the set of indexes I as formal opens and by introducing order and cover relations as follows

$i \leq j$ iff $U_i \subseteq U_j$,

$i \lhd S$ iff $U_i \subseteq \cup_{j \in S} U_j$.

We wish to extend N to a functor. We first adapt the definition of approximable mapping to formal spaces with preorder. We recall some notation: For a relation $R \subseteq S \times T$ and $U \subseteq S$, let $U \, R \, b$ mean $(\forall u \in U) \, uRb$. The *inverse image of* V *under the relation* R is, as usual,

$$R^{-1}[V] =_{\mathrm{def}} \{a \in S : (\exists b \in V) \, a \, R \, b\}$$

Notice that $R^{-1}[U] \subseteq R^{-1}[V]$ whenever $U \subseteq V$, and

$$R^{-1}[\cup_{i \in I} U_i] = \cup_{i \in I} R^{-1}[U_i].$$

Definition 2.1 Let $(S, \leq, \lhd)$ and $(T, \leq', \lhd')$ be formal spaces with preorder. A relation $R \subseteq S \times T$ is an *approximable mapping* from $(S, \leq, \lhd)$ to $(T, \leq', \lhd')$ if

(A1) $a \, R \, b$, $b \lhd' V$ implies $a \lhd R^{-1}[V]$,

(A2) $a \lhd U$, $U \, R \, b$, implies $a \, R \, b$,

(A3) $S \lhd R^{-1}[T]$,

(A4) $a \lhd R^{-1}[U]$, $a \lhd R^{-1}[V]$ implies $a \lhd R^{-1}[U_{\leq'} \cap V_{\leq'}]$,

Here $U_{\leq'} = \{x \in T : (\exists y \in U)(x \leq' y)\}$.

Remark 2.2 Note that an approximable mapping R satisfies

$$a \, R \, b, b \lhd \{b'\} \Longrightarrow a \, R \, b'.$$

The assumptions imply $a \lhd R^{-1}[\{b'\}]$ using (A1). Trivially, $U \, R \, b'$ for $U = R^{-1}[\{b'\}]$ so by (A2) we get $a \, R \, b'$. Note also that by $b \lhd' \{b\}$, (A1) and (A2)

$$a \, R \, b \Longleftrightarrow a \lhd R^{-1}[\{b\}]$$

Example 2.3 Let $f : X \to Y$ be a continuous function between neighborhood spaces $X = (X, \{U_i\}_{i \in I})$ and $Y = (Y, \{V_j\}_{j \in J})$. Define an approximable mapping $N(f) = R$ from $N(X)$ to $N(Y)$ by $R \subseteq I \times J$ and

$$iRj \iff f(U_i) \subseteq V_j.$$

It is straightfoward to check that (A1) — (A4) are satisfied.

Let **FPtop** be the following category of formal spaces with preorder and approximable mappings. For a formal space $(S, \leq, \lhd)$ we define an approximable mapping $I \subseteq S \times S$ (the identity) by

$$aIb \iff a \lhd \{b\}.$$

For approximable mappings $R_1 \subseteq S_1 \times S_2$ and $R_2 \subseteq S_2 \times S_3$ between formal spaces, define the composition

$$a(R_2 \circ R_1)b \iff a \lhd R_1^{-1}[R_2^{-1}[\{b\}]].$$

This is an approximable mapping $(R_2 \circ R_1) \subseteq S_1 \times S_3$. It is possible to check that this defines a category.

Lemma 2.4 $N : \mathbf{Nbhd} \to \mathbf{FPtop}$ *is a functor.*

We define a functor in the opposite direction. For a formal space $S = (S, \leq, \lhd)$ let $\mathrm{Qt}(S)$ be the points in S, i.e. $\alpha \subseteq S$ that are inhabited, filtering with respect to $\leq$: if $a, b \in \alpha$ then for some $c \in \alpha$ we have $c \leq a$ and $c \leq b$ and moreover satisfying the prime condition: $a \lhd U$, $a \in \alpha$ implies $U \cap \alpha$ non-void. For $a \in S$ let a^* consist of the points α such that $a \in \alpha$. Then $\mathrm{Qt}(S) = (\mathrm{Qt}(S), \{a^*\}_{a \in S})$ is a neighborhood space.

Lemma 2.5 *Let R be an approximable mapping between formal spaces S and T. Then $k = \mathrm{Qt}(R) : \mathrm{Qt}(S) \to \mathrm{Qt}(T)$ defined by*

$$k(\alpha) = \{b \in T : (\exists a \in \alpha)\, a\, R\, b\}$$

is a continuous function. In fact, Qt *defines a functor* **FPtop** $\to$ **Nbhd**.

Remark 2.6 Note that we assume that the points of a formal space consistute a set here, which is an impredicative assumption.

Let $X = (X, \{U_i\}_{i\in I})$ be a neighborhood space. Define $\eta_X : X \to \mathrm{Qt}(N(X))$ by letting

$$\eta_X(x) = \{i \in I : x \in U_i\}.$$

This is a point for each $x \in X$. Moreover it is continuous since

$$\eta_X(x) \in i^* \iff i \in \eta_X(x) \iff x \in U_i.$$

Next we see that the η_X forms a natural transformation

$$\eta : 1_{\mathbf{Nbhd}} \Longrightarrow \mathrm{Qt} \circ N.$$

Let $f : X \to Y$ be a continuous map between neighborhood spaces $X = (X, \{U_i\}_{i\in I})$ and $Y = (Y, \{V_j\}_{j\in J})$. We have to check

$$\mathrm{Qt}(N(f)) \circ \eta_X = \eta_Y \circ f.$$

A simple calculation shows

$$\mathrm{Qt}(N(f))(\eta_X(x)) = \{j \in J : (\exists i \in I)\, x \in U_i \,\&\, f(U_i) \subseteq V_j\}$$

Since f is continuous this is the same as $\eta_Y(f(x)) = \{j \in J : f(x) \in V_j\}$.

Theorem 2.7 *The functor N is left adjoint to* Qt *with η as unit of the adjunction.*

Proof. Suppose that $f : X \to \mathrm{Qt}(S)$ is continuous. Define an approximable mapping $R : NX \to S$ by

$$iRa \iff (\forall y \in U_i) a \in f(y).$$

Then an easy calculation shows that

$$\mathrm{Qt}(R)(\eta_X(x)) = \{a \in S : (\exists i \in I) x \in U_i \,\&\, iRa\}.$$

By the fact that f is continuous it follows that this the same as $f(x)$, so $f = \mathrm{Qt}(R) \circ \eta_X$. Now suppose that R' is some other approximable mapping solving this equation. Assume first iRa. Using the equation we get that for each $y \in U_i$ there exists some $k \in I$ with $y \in U_k$ and $kR'a$. This means

$$i \lhd (R')^{-1}[\{a\}].$$

Now since R' is approximable Remark 2.2 gives $iR'a$. To prove iRa from $iR'a$ is immediate using the equation. $\square$

References

E. Bishop and D.S. Bridges (1985). *Constructive Analysis,* Springer.

G. Sambin (1987). *Intuitionistic formal spaces* - a first communication, in: Mathematical Logic and its Applications, ed. D. Skordev, Plenum, New York 1987, pp. 187-204 (URL: www.math.unipd.it/˜sambin/)