Partial algebras in type theory

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In this note we suggest one way of formalising Kleene's notation ($\downarrow$, $\uparrow$, $\simeq$) for partial operations. Let A, B be sets with equalities $=_A$ and $=_B$, respectively. A partial function f from A to B, denoted $f : A \to_p B$, is an extensional relation $R = R_f \subseteq A \times B$ satisfying

$$R(x,y), R(x,y') \Longrightarrow y =_B y'.$$

We want to formalise Kleene's notation in type theory for a given algebra of partial functions. First we need the notion of regular universe.

1 Regular universes in type theory

We define a type-theoretic counterpart to regular sets (Aczel 1986).

Let A : Set and $B : A \to$ Set be a family of sets. Then we assume $U = U(A,B)$ and $T = T(A,B)$ is a family of sets satisfying

$$\frac{a : A}{\varepsilon(a) : U}, \qquad T(\varepsilon(a)) = B(a)$$

and

$$\frac{a : U \qquad b : (T(a))U}{\sigma(a,b) : U}, \qquad T(\sigma(a,b)) = \Sigma(T(a),(x)T(b(x))).$$

It is thus a universe closed under Σ which includes the family A, B. Any such universe is called *regular universe enclosing* A, B (where ε is not necessarily a constructor). An ordinary type-theoretic universe (Martin-Löf 1984) may then be considered as a regular universe enclosing many different A, B. We call the minimal U, T defined above the *canonical regular universe* enclosing A, B (cf. also universe operators (Palmgren 1998)). This universe could also be considered as the set of "enumerable" sets relative to the family A, B.

Consider an arbitrary regular universe U, T enclosing A, B. We note that the set of propositions in U are closed under conjunction ($a \wedge b = \sigma(a, (x)b)$) and existential quantification over sets in U. Define the regular power set of X enclosing A, B as

$$\mathcal{R}(X) = \mathcal{R}_{A,B}(X) =_{\text{def}} X \to U(A, B)$$

2 Kleene's notation

Consider the algebra

$$f_i : A^{n_i} \to_p A \qquad (i : I)$$

of partial functions. Let U be a regular universe enclosing the relations R_{f_i} $(i : I)$ and the equality $=_A$ and with a code for A. Let A_p be the subsets $M : \mathcal{R}(A)$ such that for all $x \in M$ and all $y : A$

$$y \in M \Longleftrightarrow x =_A y.$$

Thus A_p consists of extensional subsets of A with at most one element; $\emptyset$ is the "Undefined". Then Kleene's notation is interpreted as follows for $a, b \in A_p$

- $a \uparrow$ (a *diverges*) means $a = \emptyset$
- $a \downarrow$ (a *converges*) means a is inhabited.
- $a \simeq b$ (a and b are *strongly equal*) means $a =_{A_p} b$.

For a partial $f : A^n \to_p A$ define a total $\bar{f} : A_p^n \to A_p$ by

$$\bar{f}(x_1, \ldots, x_n) = \{y : A \mid (\exists a_1 \in x_1) \cdots (\exists a_n \in x_n) R_f(a_1, \ldots, a_n, y)\}.$$

Thus for constants a we have that $\bar{a}$ is the singleton $\{a\}$. The universe is strong enough to allow the above comprehension for functions belonging to the algebra.

Thus for instance $\bar{g}(a) \downarrow$ means $a \downarrow$ and $a = \{x\}$, $R_g(x, y)$ for some unique x and y.

In this way partial functions within the algebra can be composed freely, thus saving us from unnecessary proofs of well-definedness (which occur in intermediate steps). This could possibly be useful in formalising category theory, where the composition operation is partial.

Note also that because of the restriction of $\mathcal{R}(X)$ its predicates may be considered as "generalised enumerable".