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Assignment 4
APPLIED LOGIC, Fall-10
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Assignment 4

Program extraction, the negative translation and Brouwerian counter examples

The solutions should preferably be handed in by October 12.

1. In this problem you are asked to modify your proof from Lab 1 (vii) to allow extraction of a decision algorithm. Use the predefined natural numbers "nat" and modify notation accordingly. Replace "Prop" by "Set". You need to use the stronger disjunctive operation written $\{A\}+\{B\}$ instead of $A \vee B$. Proof by induction are preferably done using the command `induction` x where x is the variable to do induction on.

   Finish proofs by "Defined" instead of "Qed" to ensure that the proof term is saved.

   Here is a suggestion how to start:

```
Section Extract_lab1.

Require Import Arith.

Lemma Peano3: (forall x y:nat, S x = S y -> x=y).
apply eq_add_S.
Defined.

Lemma ne_symm: (forall x y:nat, x <> y -> y <> x).
intros.
intro; apply H; symmetry; assumption.
Defined.

Lemma Peano4: (forall x:nat, ~ (S x = 0)).
intros.
apply ne_symm.
```

```
apply O_S.
Defined.

(* The following lemma may be useful *)

Lemma L1: (forall n:nat, ~(S n = n)).
induction n.
apply Peano4.
intro; apply IHn.
apply Peano3; assumption.
Defined.

(* To prove the actual theorem it is helpful
to introduce a new predicate that fixes one
of the arguments. *)

Definition Q (m:nat) := (forall n:nat, {n = m} + {~ (n = m)}).

(* Now  the problem to be solved
can be formulated as follows. Just replace
Q m by its definition. *)

Theorem EqDec: (forall m:nat, Q m).

(* Your proof here *)

Theorem Decide: (forall m n:nat, {n = m} + {~ (n = m)}).
intro.
apply EqDec.
Defined.

(* Now we can use the proof of this
theorem to decide whether two numbers
are equal or not. *)

Eval compute in (Decide 2 3).

(* The answer should be "= right ..." indicating that the
right disjunct holds, i.e. that the numbers are not equal *)

Eval compute in (Decide 4 4).
```

```
(* The answer should be "= left ..." indicating that
the numbers are equal. *)
```

(10 p)

2. Use the theory (and exercises) of Section 6 of the lecture notes *Constructive Logic and Type Theory* to show that: if $\neg(\exists xyz)[x^{17} + y^{17} = z^{17} \wedge x \neq 0 \wedge y \neq 0]$ is provable in classical logic from PA, then it is provable in intuitionistic logic from PA. (5 p)

3. *Brouwerian counter examples.* Prove Theorem A and B below showing the equivalence of two non-constructive principles in constructive logic. Use Coq or do the proof by hand. If you chose the latter method make sure you are not using any nonconstructive principle.

```
Section Brouwer_counter_ex.

Require Import Arith.

Definition Binary (f:nat -> nat) :=
(forall n:nat, f n = 0 \/ f n =1).

(* Limited principle of omniscience *)

Definition LPO : Prop :=
 (forall f : nat->nat,
  Binary f ->
  (exists n:nat, f n =1) \/ (forall n:nat, f n =0)).

(* Minimal term principle for sequences *)

Definition MTP : Prop :=
(forall g:nat -> nat,
  (exists n:nat,  (forall m:nat, g n <= g m))).

Theorem A: MTP ->LPO.

Theorem B: LPO -> MTP.
```

(Theorem A: 5 p, Theorem B (harder): 10 p)