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Assignment 5
APPLIED LOGIC, Fall-10
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Assignment 5

Term algebras, term rewriting and termination

The problems should be solved individually and are due by November 2.

1. Decide whether the following pairs of terms are unifiable and if so determine their mgus.

   (a) $h(x, y)$ and $h(u, u)$

   (b) $f(h(z, y), y)$ and $f(h(x, g(u)), g(x))$

   (c) $f(h(z, y), y)$ and $f(z, h(z, y))$ (5 p)

2. The Dutch National Flag game. Red, blue and white marbles are placed in a row in no particular order. Adjacent marbles may exchanged according to the following rules

$$\begin{aligned} WR &\to RW \\ BR &\to RB \\ BW &\to WB \end{aligned}$$

   This can be regarded as an example of an abstract rewriting system: let $\Sigma = \{B, R, W\}$ and $A = \Sigma^*$ and write $u \to v$ if and only if $u = u_1 r u_2$ and $v = u_1 s u_2$ where $r \to s$ is one of the three rules above. Thus for example

$$BRW\underline{WR} \to \underline{BR}WRW \to R\underline{BW}RW \to RWBRW.$$

   Note that the rewriting possibilities may overlap

$$\underline{BW}R \to WBR \qquad B\underline{WR} \to BRW.$$

   (a) What are the normal forms of the ARS $(A, \to)$?

   (b) Show that $(A, \to)$ is weakly confluent.

(c) Prove that $(A, \rightarrow)$ is strongly normalising. First define the following wellorder on the marbles $R < W < B$. For two rows of n marbles $a_1 \cdots a_n$ and $b_1 \cdots b_n$ define

$$a_1 \cdots a_n < b_1 \cdots b_n$$

iff $a_1 \cdots a_n$ comes before $b_1 \cdots b_n$ in the lexicographic order given by $R < W < B$. Clearly $<$ is wellorder. Now show that

$$u \rightarrow v \Longrightarrow u > v.$$

(Why does this prove strong normalisation?)

(d) Conclude that $(A, \rightarrow)$ has unique normal forms.

(e) On the basis of (a) – (d) devise an efficient decision procedure for when two strings u and v have a common normal form. (10 p)

3. Prove that the substring relation over $\{0, 1\}^*$ is not a well-quasi-order. (3 p)

4. Let $(A, \leq)$ and $(B, \leq')$ be two quasi-orders, and let $f : A \rightarrow B$ be function such that for all $x, y \in A$:

$$f(x) \leq' f(y) \Longrightarrow x \leq y.$$

Show that if $(B, \leq')$ is a well-quasi-order then so is $(A, \leq)$. (2 p)

5. Consider the following term rewriting system for simplifying boolean expressions (see Klop 1992, pp. 31-32)

$$\begin{array}{rcl} \neg\neg x & \rightarrow & x \\ \neg(x \vee y) & \rightarrow & \neg x \wedge \neg y \\ \neg(x \wedge y) & \rightarrow & \neg x \vee \neg y \\ x \wedge (y \vee z) & \rightarrow & (x \wedge y) \vee (x \wedge z) \\ (y \vee z) \wedge x & \rightarrow & (y \wedge x) \vee (z \wedge x) \end{array}$$

(a) Prove that it is strongly normalising by completing the proof begun in Klop.

(b) Compute the normal form of the following expression

$$\neg(x \vee (y \vee \wedge(z \vee (u \wedge v)))).$$

(c) Implement the term rewriting system above in Coq using the `rewrite` tactics.

(d) Can you describe all the normal forms in some general way?

(10 p)
