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EXERCISES/ASSIGNMENTS 2
APPLIED LOGIC, Fall-10
2010-09-09

Predicate logic, its semantics and decidability problems

Assignment 2 for Applied Logic consists of the problems marked **A**. Number of points are indicated. Problems are to be solved individually and solutions are to be handed in by September 23. Please explain your solutions well and write (or type) neatly.

Some of the remaining exercises will be demonstrated during the second hour of the lecture September 15.

1. Some review exercises in predicate logic to do (if necessary):

   2.1: 1,3

   2.2: 3,4

   2.3: 2,3,4,7,13

   2.4: 2,6,11,12

   of the textbook (Huth and Ryan 2004).

2. Decide whether the following *instances* of Post's correspondence problem (PCP) are solvable. Provide a solution, or give a proof that no solution is possible!

   (a) $(11, 0),\ (10, 1)$ (**A** - 2pt)

   (b) $(000, 0),\ (0, 0000)$ (**A** - 2pt)

   (c) $(00, 10),\ (01, 0), (0, 110000)$ (**A** - 2pt)

3. Solve PCP for the sequence of pairs

   $$(001, 0),\ (01, 011),\ (01, 101),\ (10, 001).$$

   (Hint (?): It is not so easy if one gets started in the wrong way. One solution gives a binary string of total length 154. Use a computer program if it gets too tedious ...)

4. (Definability) Let $\mathcal{M}$ be a model for the language L and let $A = |\mathcal{M}|$ be its universe. A subset $S \subseteq A^n$ is *(first-order) definable* in $\mathcal{M}$ if there is an L-formula φ with free variables among $x_1, \ldots, x_n$ such that

$$S = \{(a_1, \ldots, a_n) \in A^n : \mathcal{M} \models_\ell \varphi \text{ and } \ell(x_1) = a_1, \ldots, \ell(x_n) = a_n\}$$

A relation $R \subseteq A^n$ is *definable in* $\mathcal{M}$ if the corresponding subset R is definable. A function $f : A^n \to A$ is *definable in* $\mathcal{M}$ if its graph

$$\operatorname{graph} f = \{(a_1, \ldots, a_n, b) \in A^{n+1} : f(a_1, \ldots, a_n) = b\}$$

is a definable subset in $\mathcal{M}$.

Show that the subsets, relations or functions in (a) – (h) below are definable in $\mathcal{N} = \langle \mathbb{N}; +, \cdot, 0, 1\rangle$ using as simple formulas as seems possible.

For instance the set of even numbers is defined by

$$\{m \in \mathbb{N} : \mathcal{N} \models_\ell (\exists x)\ x + x = y \text{ and } \ell(y) = m\}$$

This also shows that the predicate x *is even* is definable. The function $f(x) = x^2$ is defined by

$$\{(m, n) \in \mathbb{N}^2 : \mathcal{N} \models_\ell x \cdot x = y \text{ and } \ell(x) = m, \ell(y) = n\}.$$

(a) x is odd

(b) $y = x(x + 1)/2$

(c) $x \leq y$

(d) x divides y

(e) x is the sum of two prime numbers (**A** - 2pt)

(f) $z = \max(x, y)$ (**A** - 2pt)

(g) ** $y = x!$. [Look up and use Gödel's technique of the β-function and the Chinese remainder theorem.]

(h) $y = 2^{2^{2^{2^2}}}$

▷ Let L be a first-order language with finitely many symbols. A first order structure $\mathcal{M}$ for L is called *decidable,* if there is an algorithm which for every closed first order formula φ in the language L decides whether $\mathcal{M} \models \varphi$ holds or not. A well-known example of an *undecidable* structure is the structure of natural numbers $\mathcal{N} = \langle \mathbb{N}, +, \cdot, 0, 1\rangle$.

5. (Definability and decidability) Recall that in automata theory one studies languages as subsets of strings over a fixed alfabet. Let $\Sigma = \{a, b\}$ be an alfabet, and let Σ^* be the set of finite strings. Thus

$$\Sigma^* = \{\epsilon, a, b, aa, ab, ba, bb, aaa, aab, \ldots\}$$

Here ϵ is the empty string. Let & denote concatenation of strings, so $baba\&bba = bababba$. We may now regard $\langle \Sigma^*;\ \& \rangle$ as a first-order structure with concatenation as the only operation. Find elementary propositions (formulas) over $\langle \Sigma^*;\ \& \rangle$ that definies the following properties (note that $=$ may be used)

(a) x is a substring of y

(b) x is an empty string (you may not mention ϵ)

(c) x is a string of length 1 (you may not mention 0 or 1) (Hint: use (a) and (b). How many substrings can such a string have?)

(d) x is a string of length 4.

Consider now an extended structure $\langle \Sigma^*;\ \&, *, a, b, \epsilon \rangle$ where a, b, ϵ are constants (so they may be mentioned in elementary propositions) and moreover there is a "string duplicator" $*$ that satisfies the following

$$\begin{aligned} u * \epsilon &= \epsilon && \text{(erase)} \\ u * (a\&v) &= u * v && \text{(take a pause)} \\ u * (b\&v) &= (u * v)\&u && \text{(make a copy).} \end{aligned}$$

Thus $ab * bab = abab$ and $ab * aa = \epsilon$.

(e) Prove that the structure $\langle \Sigma;\ \&, *, a, b, \epsilon \rangle$ is undecidable, by showing that *if* it was decidable, then we could decide $\langle \mathbb{N}, +, \cdot, 0, 1 \rangle$ as well, contradicting a well-known theorem. (**A** - 5pt)

6. It is well-known by a result of Alfred Tarski (1951) that the first order structure $\mathcal{R} = \langle \mathbb{R}; +, \cdot, 0, 1 \rangle$ is decidable. By this follows that many problems in elementary (analytic) geometry are decidable. This can be demonstrated by simply formulating the problems in first order logic over the language $L = \{+, \cdot, 0, 1\}$. Formulate the following relations in first order language (a – f):

(a) $\vec{x} + \vec{y} = \vec{z}$

(b) $\vec{x} = A\vec{y}$ where A is a quadratic matrix

(c) the points $\vec{x}$, $\vec{y}$ and $\vec{z}$ are colinear in $\mathbb{R}^2$

(d) the point $\vec{x}$ lies between $\vec{y}$ and $\vec{z}$ on a line

(e) the two point triples $\vec{x}, \vec{y}, \vec{z}$ and $\vec{u}, \vec{v}, \vec{w}$ determine congruent triangles (**A** - 3pt)

(f) the two point triples $\vec{x}, \vec{y}, \vec{z}$ and $\vec{u}, \vec{v}, \vec{w}$ determine congruent angles.

(g) Use the above to formulate some theorems or problems of Euclidean geometry. (**A** - 2pt)

(h) Can you find some (famous) unsolved problem of geometry that can be formulated in first-order logic over L? Research the literature or the web.
