Polynomial Maps of Modules

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And God said unto the animals: "Be fruitful and multiply."

But the snake answered: "How could I? I am an adder!"[1]

Argument

The article focuses on three different notions of polynomiality for maps of modules. In addition to the polynomial maps studied by Eilenberg and Mac Lane, and the strict polynomial maps ("lois polynomes") considered by Roby, we introduce *numerical maps* of modules and investigate their properties.

Even though our notion require the existence of binomial coefficients in the base ring, we argue that it constitutes the correct way to generalise Eilenberg and Mac Lane's original definition, of polynomial maps on abelian groups, to modules over more general rings. The main theorem propounds that our maps admit a description word by word corresponding to Roby's definition of strict polynomial maps.

Eilenberg and Mac Lane ([2]) studied *polynomial maps* of abelian groups. While their notion will remain valid for modules over any ring, it will clearly be deficient, for the simple and obvious reason that it does not take scalar multiplication into account.

Roby ([7]) was then led to consider *strict polynomial maps* of arbitrary modules. This concept is, as the name suggests, stronger, and it carries the advantage of making sense for an arbitrary commutative base ring.

In this note, we introduce *numerical maps* (Definition 5), which we believe furnish the proper way to extend Eilenberg and Mac Lane's weak notion of polynomiality to more general base rings, handling, as it does, scalar multiplication in a natural way. A key point is that the base ring is required to

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[1]In some retellings of this myth, it is further reported that God constructed a wooden table for the snakes to crawl upon, since even adders can multiply on a log table. God is not assumed to be familiar with tensor products.

possess binomial coefficients; such rings are commonly known as *binomial* or *numerical* (Ekedahl's terminology; see [3]). They were introduced by Hall ([6]).

We shall display the dependence of numerical maps on their deviations (Theorem 8), and also exhibit the universal numerical map of each degree (Theorem 4). A remarkable fact will ensue, namely, that numerical maps fit, like hand in glove, into the frame-work Roby erected for strict polynomial maps.

Recall that he defines a strict polynomial map of **B**-modules to be a natural transformation

$$\varphi\colon M \otimes_{\mathbf{B}} - \to N \otimes_{\mathbf{B}} -$$

between functors ${}_{\mathbf{B}}\mathfrak{CAlg} \to \mathfrak{Set}$, where ${}_{\mathbf{B}}\mathfrak{CAlg}$ denotes the category of commutative, unital algebras over the base ring **B**. An extremely broad and all-encompassing notion of polynomial maps as types of natural transformations (Definition 7) can be extracted as the essence of this.

Our final theorem (Theorem 10) will provide a beautiful and unexpected unification of the two notions:

Theorem. — *The map $\varphi\colon M \to N$ is numerical of degree n if and only if it can be extended to a natural transformation*

$$\varphi\colon M \otimes_{\mathbf{B}} - \to N \otimes_{\mathbf{B}} -$$

of functors ${}_{\mathbf{B}}\mathfrak{NAlg} \to \mathfrak{Set}$ of degree n, where ${}_{\mathbf{B}}\mathfrak{NAlg}$ denotes the category of numerical algebras over the ring **B**.

One point may deserve further elaboration. Is the existence of binomial coefficients a necessary evil? Would it be conceivable to extend the notion of polynomial map to modules over any ring? The question is justified, and especially so by the recent attempts by Gaudier & Hartl [5] to propose a notion of quadratic map valid for modules over arbitrary base rings.

To this we reply the following. Consider, for a moment, the original case of Eilenberg and Mac Lane: a map $\varphi\colon M \to N$ of **Z**-modules, that is, abelian groups. In this case, there can be no doubt concerning the proper definition of polynomial map, and, by virtue of the result just indicated, such a map (of degree n) is equivalent to a natural transformation

$$\varphi\colon M \otimes_{\mathbf{Z}} - \to N \otimes_{\mathbf{Z}} -$$

(of degree n). These functors are defined on the category ${}_{\mathbf{Z}}\mathfrak{NAlg}$, which is simply the category of numerical rings. Thus, even in the case of abelian groups, where no binomial coefficients *a priori* appear, numerical rings will nonetheless enter in a canonical fashion to render the definition akin to Roby's. This provides quite convincing evidence that our notion is the correct one.

In subsequent articles, we purport to employ numerical maps in order to define a corresponding notion of *numerical functor*.

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§0. Preliminaries

For the entirety of this article, **B** shall denote a fixed base ring of scalars, assumed commutative and unital. All modules, homomorphisms, and tensor products shall be taken over this **B**, unless otherwise stated. We let $\mathfrak{Mod} = {}_{\mathbf{B}}\mathfrak{Mod}$ denote the category of (unital) modules over this ring.

A *map* of modules shall always denote an *arbitrary* map — in general non-linear. On those rare occasions when a linear map is actually under consideration, we shall rather ostensibly proclaim it a "homomorphism".

Presumably, it was Eilenberg and Mac Lane who first studied non-additive maps of *abelian groups*, introducing in [2] (section 8) the so-called *deviations* of a map. Let $[n]$ denote the set $\{1, \dots, n\}$.

Definition 1. — Let $\varphi\colon M \to N$ be a map of modules. The nth **deviation** of φ is the map

$$\varphi(x_1 \diamond \cdots \diamond x_{n+1}) = \sum_{I \subseteq [n+1]} (-1)^{n+1-|I|} \varphi\left(\sum_{i \in I} x_i\right)$$

of $n+1$ variables. ◇

Let us, for clarity, point out that the diamond sign itself does not work as an operator; the entity $x \diamond y$ does not possess a life of its own, and cannot exist outside the scope of an argument of a map.

It is an immediate consequence of the definition that

$$\varphi(x_1 + \cdots + x_{n+1}) = \sum_{I \subseteq [n+1]} \varphi\left(\Diamond_{i \in I} x_i\right).$$

Loosely speaking, the nth deviation measures how much φ deviates from being polynomial of degree n. We have for example

$$\begin{aligned} \varphi(x \diamond y) &= \varphi(x+y) - \varphi(x) - \varphi(y) + \varphi(0), \\ \varphi(\diamond x) &= \varphi(x) - \varphi(0), \end{aligned}$$

and, of course,

$$\varphi(\diamond) = \varphi(0).$$

We abbreviate

$$\varphi\left(\Diamond_n x\right) = \varphi(\underbrace{x \diamond \cdots \diamond x}_{n}).$$

Definition 2. — The map $\varphi\colon M \to N$ is **polynomial** of degree n if its nth deviation vanishes:

$$\varphi(x_1 \diamond \cdots \diamond x_{n+1}) = 0$$

for any $x_1, \dots, x_{n+1} \in M$. ◇

Observe that, when we speak of maps of degree n, we always mean degree *n or less*.

Example 1. — A map is polynomial of degree 0 if and only if it is constant. A map is polynomial of degree 1 if and only if it is a translated group homomorphism. △

This definition of polynomiality is the classical one for abelian groups. It is valid, but inadequate, for arbitrary modules, because it does not regulate the behaviour of scalar multiplication.

We now present Roby's notion of strict polynomial map, as given in section 1.2 of [7]. Let $\mathfrak{CAlg} = {}_{\mathbf{B}}\mathfrak{CAlg}$ denote the category of commutative, unital algebras over the base ring $\mathbf{B}$.

Definition 3. — A natural transformation

$$\varphi\colon M \otimes - \to N \otimes -$$

between functors $\mathfrak{CAlg} \to \mathfrak{Set}$, will be called a **strict polynomial map**. ⋄

Recall that a *multi-set* is a set with repeated elements. When X is a multi-set, we shall denote by $|X|$ its *cardinality*, that is, the number of elements counted with multiplicity, and by $\#X$ the underlying set, called its *support*.

Theorem 1 ([7], Théorème I.1). — *Let $\varphi\colon M \to N$ be a strict polynomial map. For any $u_1, \dots, u_k \in M$ there exist unique elements $v_X \in N$ (only finitely many of which are non-zero) such that*

$$\varphi(u_1 \otimes x_1 + \cdots + u_k \otimes x_k) = \sum_{\#X \subseteq [k]} v_X \otimes x^X,$$

for all x_j in all (commutative, unital) algebras.

Definition 4. — If, in the theorem above, v_X is non-zero only when $|X| \leqslant n$, then φ is said to be of **degree** n. ⋄

Example 2. — A map is strict polynomial of degree 0 if and only if it is constant. A map is strict polynomial of degree 1 if and only if it is a translated module homomorphism. △

§1. Numerical Maps

The base ring $\mathbf{B}$ of scalars will now be assumed *binomial* or *numerical*, by which is simply meant a commutative ring with unity, equipped with binomial coefficients. These may be invoked in two different ways. One way is to postulate the existence of unary maps

$$\binom{-}{n}\colon \mathbf{B} \to \mathbf{B},$$

subject to certain axioms, which is the approach taken by Ekedahl ([3]). Alternatively, one may require $\mathbf{B}$ to be torsion-free and closed in $\mathbf{Q} \otimes_{\mathbf{Z}} \mathbf{B}$ under the operations

$$r \mapsto \frac{r(r-1)\cdots(r-n+1)}{n!}.$$

This was how Hall ([6]) originally introduced the concept. It is a non-trivial fact that these two definitions yield the same class of rings. A proof can be found in [8].

Definition 5. — The map $\varphi\colon M \to N$ is **numerical** of degree (at most) n if it satisfies the following two equations:

$$\varphi(x_1 \diamond \cdots \diamond x_{n+1}) = 0, \qquad x_1, \dots, x_{n+1} \in M$$
$$\varphi(rx) = \sum_{k=0}^{n} \binom{r}{k} \varphi\left(\underset{k}{\Diamond} x\right), \qquad r \in \mathbf{B},\ x \in M.$$

$\diamond$

It is easy to prove that, over the integers, the second equation above is implied by the first, so that the concepts of polynomial and numerical map coincide.

It is clear that a strict polynomial map is also numerical of the same degree (provided of course the base ring be numerical). If the base ring is a $\mathbf{Q}$-algebra, the two concepts coincide, for then every algebra is numerical.

Example 3. — The map φ is of degree 0 if and only if it is constant, for when $n = 0$ the above equations read:

$$\varphi(x_1) - \varphi(0) = \varphi(\diamond x_1) = 0,$$
$$\varphi(rx) = \binom{r}{0}\varphi(\diamond) = \varphi(0).$$

$\triangle$

Example 4. — When $n = 1$, the equations read as follows:

$$\varphi(x_1 + x_2) - \varphi(x_1) - \varphi(x_2) + \varphi(0) = \varphi(x_1 \diamond x_2) = 0,$$
$$\varphi(rx) = \binom{r}{0}\varphi(\diamond) + \binom{r}{1}\varphi(\diamond x) = \varphi(0) + r(\varphi(x) - \varphi(0)).$$

The map

$$\psi(x) = \varphi(x) - \varphi(0)$$

is then a module homomorphism. Conversely, any translate of a module homomorphism is numerical of degree 1. $\triangle$

Example 5. — Let $\mathbf{B} = \mathbf{Z}$. It is a well-known fact, and not difficult to prove, that the numerical (polynomial) maps $\varphi\colon \mathbf{Z} \to \mathbf{Z}$ of degree n are precisely the ones given by numerical polynomials of degree n:

$$\varphi(x) = \sum_{k=0}^{n} c_k \binom{x}{k}.$$

$\triangle$

Example 6. — Let $\mathbf{B} = \mathbf{Z}$. The map

$$\xi\colon \mathbf{Z} \otimes - \to \mathbf{Z} \otimes -, \quad 1 \otimes x \mapsto 1 \otimes \binom{x}{2},$$

is numerical of degree 2, but not strict polynomial of any degree. The following tentative diagram, where $\beta\colon t \mapsto a$, indicates the impossibility of defining $\xi_{\mathbf{Z}[t]}$.

$$\begin{tikzcd}
\mathbf{Z} \otimes \mathbf{Z}[t] \arrow[r, "\xi_{\mathbf{Z}[t]}"] \arrow[d, "1\otimes\beta"'] & \mathbf{Z} \otimes \mathbf{Z}[t] \arrow[d, "1\otimes\beta"] \\
\mathbf{Z} \otimes \mathbf{Z} \arrow[r, "\xi_{\mathbf{Z}}"'] & \mathbf{Z} \otimes \mathbf{Z}
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \otimes t \arrow[r] \arrow[d] & ? \arrow[d] \\
1 \otimes a \arrow[r] & 1 \otimes \binom{a}{2}
\end{tikzcd}$$

Note that $\mathbf{Z}[t]$ is not a numerical ring; there is no such thing as $\binom{t}{2}$! △

Example 7. — Contrary to the situation for numerical maps, strict polynomial maps are not determined by the underlying maps. The most simple example is probably the following. Let $\mathbf{B} = \mathbf{Z}$, and define

$$\varphi_A\colon \mathbf{Z}/2 \otimes A \to \mathbf{Z}/2 \otimes A, \quad 1 \otimes x \mapsto 1 \otimes x(x-1).$$

This is a non-trivial strict polynomial map of degree 2, and its underlying map is zero!

We see here at play the well-known distinction between polynomials and polynomial maps, the former class being richer than the latter. The point is that the strict polynomial structure provides extra data, which make the zero map strict polynomial of degree 2 in a non-trivial way. △

§2. Polynomiality

Let us now conduct an investigation of *polynomiality* at its most general, and indicate how this perspective provides a unifying view.

Let $D \subseteq \mathfrak{Mod}$ be a *finitary algebraic category*, by which is simply meant an equational class in the sense of universal algebra. Since D is a subcategory of $\mathfrak{Mod}$, the objects of D are first of all $\mathbf{B}$-modules, possibly equipped with some extra structure.

For a set of variables V, let

$$\langle V \rangle_D$$

denote the free algebra on V in D. That the free algebra exists is a basic fact of universal algebra; see for example [1].

Definition 6. — Let M be a module, not necessarily in D. An element of

$$M \otimes \langle x_1, \dots, x_k \rangle_D$$

is called a **D-polynomial** over M in the variables $x_1, \dots, x_k$.

A **linear form** over M in these variables is a polynomial of the form

$$\sum u_j \otimes x_j,$$

for some $u_j \in M$. ◇

Theorem 2: Ekedahl's Esoteric Polynomiality Principle. — *Let two modules M and N be given, and a family of maps*

$$\varphi_A : M \otimes A \to N \otimes A, \quad A \in D.$$

The following statements are equivalent:

A. *For every D-polynomial $p(x) = p(x_1, \dots, x_k)$ over M, there is a unique D-polynomial $q(x) = q(x_1, \dots, x_k)$ over N, such that for all $A \in D$ and all $a_j \in A$,*

$$\varphi_A(p(a)) = q(a).$$

B. *For every linear form $l(x)$ over M, there is a unique D-polynomial $q(x)$ over N, such that for all $A \in D$ and all $a_j \in A$,*

$$\varphi_A\,(l(a)) = q(a).$$

C. *The map*

$$\varphi : M \otimes - \to N \otimes -$$

is a natural transformation of functors $D \to \mathfrak{Set}$.

Proof. It is of course trivial that A implies B. Suppose statement B holds, and consider a homomorphism $\chi : A \to B$, along with finitely many elements $u_j \in M$. Define

$$l(x) = \sum u_j \otimes x_j,$$

and find the unique D-polynomial q satisfying B. Then, for any $a_j \in A$, there is a commutative diagram of the following form, proving that φ is natural:

$$\begin{tikzcd} M \otimes A \arrow[r, "\varphi_A"] \arrow[d, "1\otimes\chi"'] & N \otimes A \arrow[d, "1\otimes\chi"] \\ M \otimes B \arrow[r, "\varphi_B"'] & N \otimes B \end{tikzcd} \qquad \begin{tikzcd} \sum u_j \otimes a_j \arrow[r] \arrow[d] & q(a) \arrow[d] \\ \sum u_j \otimes \chi(a_j) \arrow[r] & q(\chi(a)) \end{tikzcd}$$

Thus, condition C holds.

Finally, suppose φ natural. We shall prove condition A. Given a D-polynomial

$$p(x) \in M \otimes \langle x_1, \dots, x_k \rangle_D\,,$$

define

$$q(x) = \varphi_{\langle x_1,\dots,x_k\rangle_D}(p(x)).$$

For any $A \in D$ and $a_j \in A$, define the homomorphism

$$\chi\colon \langle x_1,\dots,x_k\rangle_D \to A, \quad x_j \mapsto a_j.$$

Then since φ is natural, the following diagram commutes:

$$\begin{tikzcd}
M \otimes \langle x_1,\dots,x_k\rangle \arrow[r, "\varphi_{\langle x_1,\dots,x_k\rangle}"] \arrow[d, "1\otimes\chi"'] & N \otimes \langle x_1,\dots,x_k\rangle \arrow[d, "1\otimes\chi"] \\
M \otimes A \arrow[r, "\varphi_A"'] & N \otimes A
\end{tikzcd}
\qquad
\begin{tikzcd}
p(x) \arrow[r] \arrow[d] & q(x) \arrow[d] \\
p(a) \arrow[r, dotted] & q(a)
\end{tikzcd}$$

The uniqueness of q is evident, which proves A. □

Definition 7. — When the conditions of the theorem are fulfilled, we call φ a **D-polynomial map** from M to N. ⋄

According to part B of the theorem, φ_A maps

$$\sum u_j \otimes a_j \mapsto q(a)$$

for some (unique) D-polynomial q. In naïve language, the Polynomiality Principle amounts to the following. *If we want the coefficients a_j (in some algebra) of the module elements u_j to transform according to certain operations, the correct setting is the category of algebras using these same operations.*

Example 8. — A $\mathfrak{Mod}$*-polynomial map* $\varphi\colon M \to N$ is just a linear transformation $M \to N$. This is because, by B above, $\varphi_{\mathbf{B}}$ will map $\sum u_j \otimes r_j$ to $\sum v_j \otimes r_j$ for all $r_j \in \mathbf{B}$, and such a map is easily seen to be linear. Conversely, any module homomorphism induces a natural transformation $M \otimes - \to N \otimes -$. △

Example 9. — Let S be a $\mathbf{B}$-algebra; then ${}_S\mathfrak{Mod} \subseteq \mathfrak{Mod}$. An ${}_S\mathfrak{Mod}$*-polynomial map* $M \to N$ is a transformation

$$M \otimes A \to N \otimes A$$

which is natural in the S-module A. This is the same as a natural transformation

$$(M \otimes S) \otimes_S - \to (N \otimes S) \otimes_S -,$$

which is an ${}_S\mathfrak{Mod}$-polynomial map $M \otimes S \to N \otimes S$; or, as noted in the previous example, an S-linear map from $M \otimes S$ to $N \otimes S$. △

Example 10. — The $\mathfrak{CAlg}$*-polynomial maps* are precisely the strict polynomial ones.

Confer Theorem 1. The equation

$$\varphi_A\left(\sum u_j \otimes a_j\right) = \sum v_X \otimes a^X.$$

shows that, intuitively, the coefficients of the elements u_j "transform as ordinary polynomials". △

Suppose now that $\mathbf{B}$ is numerical, and consider the category $\mathfrak{NAlg} = {}_{\mathbf{B}}\mathfrak{NAlg}$ of numerical algebras over $\mathbf{B}$. According to the definition, an $\mathfrak{NAlg}$-*polynomial map* is a natural transformation

$$\varphi_A \colon M \otimes A \to N \otimes A, \quad A \in \mathfrak{NAlg}.$$

The Polynomiality Principle guarantees that, for every linear form $\sum u_j \otimes x_j$ over M, there is a unique numerical polynomial $\sum v_X \otimes \binom{x}{X}$ over N such that for all numerical algebras A and all $a_j \in A$,

$$\varphi_A\left(\sum u_j \otimes a_j\right) = \sum v_X \otimes \binom{a}{X}. \tag{1}$$

Intuitively, the coefficients of the elements u_j "transform as numerical (binomial) polynomials".

While the right-hand side of equation (1) is finite, there is no reason to expect a uniform upper bound for its degree. Roby refers to this phenomenon as a "somme localement finie". In his definition of strict polynomial map, he does not include such an assumption on bounded degree, but he circumvents it by immediately restricting attention to homogeneous maps.

Definition 8. — We say that φ is of **degree** n if $v_X = 0$ whenever $|X| > n$ (independently of the linear form $\sum u_j \otimes x_j$). ◇

Example 11. — We present an example of infinite degree. Let

$$U = \langle u_0, u_1, u_2, \dots \rangle$$

be free on an infinite basis. The map

$$\varphi_A \colon U \otimes A \to U \otimes A, \quad \sum u_k \otimes a_k \mapsto \sum u_k \otimes \binom{a_k}{k}$$

is $\mathfrak{NAlg}$-polynomial, but not numerical of any finite degree n. △

The final theorem of this note will establish that a map is numerical of degree n if and only if it is $\mathfrak{NAlg}$-polynomial of degree n.

§3. The Universal Numerical Maps

There is an algebraic way of describing numerical maps, which turns out to be very fruitful. Recall that the free module on a set M is the set

$$\mathbf{B}[M] = \left\{ \sum a_j[x_j] \;\middle|\; a_j \in \mathbf{B},\ x_j \in M \right\}$$

of formal (finite) linear combinations of elements of M. If M itself is a module (or even abelian group), it carries a multiplication

$$[x][y] = [x+y],$$

which makes $\mathbf{B}[M]$ into a commutative, associative algebra with unity $[0]$.

Consider now the map

$$M \to \mathbf{B}[M], \quad x \mapsto [x],$$

and form its nth deviation

$$(x_1, \dots, x_{n+1}) \mapsto [x_1 \diamond \dots \diamond x_{n+1}].$$

Theorem 3. — *In the free algebra $\mathbf{B}[M]$, the following formula holds:*

$$[x_1 \diamond \cdots \diamond x_{n+1}] = ([x_1] - [0]) \cdots ([x_{n+1}] - [0]).$$

Proof. Simply calculate:

$$\begin{aligned}[x_1 \diamond \cdots \diamond x_{n+1}] &= \sum_{I \subseteq [n+1]} (-1)^{n+1-|I|} \left[\sum_{i \in I} x_i\right] \\ &= ([x_1] - [0]) \cdots ([x_{n+1}] - [0]).\end{aligned}$$

□

There is a filtration of $\mathbf{B}[M]$, given by the decreasing sequence of ideals

$$\begin{aligned} I_n &= (\, [x_1 \diamond \cdots \diamond x_{n+1}] \mid x_i \in M \,) \\ &\quad + \left([rx] - \sum_{k=0}^{n} \binom{r}{k} \left[\Diamond_k x \right] \,\middle|\, r \in \mathbf{B},\ x \in M \right), \quad n \geqslant -1. \end{aligned}$$

Definition 9. — The nth **augmentation algebra** is the quotient algebra

$$\mathbf{B}[M]_n = \mathbf{B}[M]/I_n.$$

◇

Theorem 4. — *The map*

$$\delta_n \colon M \to \mathbf{B}[M]_n, \quad x \mapsto [x],$$

is the universal *numerical map of degree n, in that every numerical map $\varphi\colon M \to N$ of degree n has a unique factorisation $\varphi = \hat{\varphi}\delta_n$ through it, as in the subsequent commutative diagram:*

$$\begin{tikzcd} M \arrow[r, "\delta_n"] \arrow[dr, "\varphi"'] & \mathbf{B}[M]_n \arrow[d, dotted, "\hat{\varphi}"] \\ & N \end{tikzcd}$$

Proof. Given a map $\varphi\colon M \to N$, extend it linearly to a homomorphism

$$\varphi\colon \mathbf{B}[M] \to N.$$

The theorem amounts to the trivial observation that φ is numerical of degree n if and only if it kills I_n. □

The augmentation quotients of a free module M are given by the next theorem.

Theorem 5. — *In the polynomial algebra $\mathbf{B}[t_1, \dots, t_k]$, let J_n be the ideal generated by monomials of degree greater than n. Denote by $(e_i)_{i=1}^k$ the canonical basis of $\mathbf{B}^k$. Then*

$$\begin{aligned} \psi\colon \mathbf{B}[t_1, \dots, t_k]/J_n &\to \mathbf{B}[\mathbf{B}^k]_n \\ t^X &\mapsto \Big[\underset{i\in X}{\Diamond}\, e_i\Big] = ([e]-[0])^X \end{aligned}$$

is an isomorphism of algebras. In particular, $\mathbf{B}[\mathbf{B}^k]_n$ is a free module.

Proof. The map

$$\begin{aligned} \mathbf{B}[t_1, \dots, t_k] &\to \mathbf{B}[\mathbf{B}^k]_n \\ t^X &\mapsto \Big[\underset{i\in X}{\Diamond}\, e_i\Big] = ([e]-[0])^X \end{aligned}$$

is clearly a homomorphism of algebras, and since it annihilates J_n, it factors via $\mathbf{B}[t_1, \dots, t_k]/J_n$. This establishes the existence of ψ.

We now define the inverse of ψ. Each t_i is nilpotent in $\mathbf{B}[t_1, \dots, t_k]/J_n$, and so the powers

$$(1+t_i)^a = \sum_{j=0}^{\infty} \binom{a}{j} t_i^j = \sum_{j=0}^{n} \binom{a}{j} t_i^j$$

are defined for any $a \in \mathbf{B}$. Accordingly, for an element

$$x = a_1 e_1 + \dots + a_k e_k \in \mathbf{B}^k,$$

we define

$$\begin{aligned} \chi\colon \mathbf{B}[\mathbf{B}^k] &\to \mathbf{B}[t_1, \dots, t_k]/J_n \\ [a_1 e_1 + \dots + a_k e_k] &\mapsto (1+t_1)^{a_1} \cdots (1+t_k)^{a_k} + J_n. \end{aligned}$$

We write this more succinctly as

$$[x] \mapsto (1+t)^x + J_n.$$

The map χ is linear by definition, and also multiplicative, since

$$\chi([x][y]) = \chi([x+y]) = (1+t)^{x+y} = (1+t)^x(1+t)^y = \chi([x])\chi([y]).$$

It maps I_n into J_n, because, when $x_1, \ldots, x_{n+1} \in \mathbf{B}^k$,

$$\begin{aligned}
\chi([x_1 \diamond \cdots \diamond x_{n+1}]) &= \sum_{J \subseteq [n+1]} (-1)^{n+1-|J|} \chi\left(\left[\sum_{j \in J} x_j\right]\right) \\
&= \sum_{J \subseteq [n+1]} (-1)^{n+1-|J|} (1+t)^{\sum_{j \in J} x_j} \\
&= \prod_{j=1}^{n+1} \left((1+t)^{x_j} - 1\right) = 0.
\end{aligned}$$

Also, for $r \in \mathbf{B}$ and $x \in \mathbf{B}^k$,

$$\begin{aligned}
\chi\left([rx] - \sum_{m=0}^{n} \binom{r}{m} \left[\diamondsuit_m x\right]\right) &= \chi\left([rx] - \sum_{m=0}^{n} \binom{r}{m} \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} [jx]\right) \\
&= (1+t)^{rx} - \sum_{m=0}^{n} \binom{r}{m} \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} (1+t)^{jx} \\
&= (1+t)^{rx} - \sum_{m=0}^{n} \binom{r}{m} \left((1+t)^x - 1\right)^m \\
&= (p(t)+1)^r - \sum_{m=0}^{n} \binom{r}{m} p(t)^m,
\end{aligned}$$

where, in the last step, we have let $p(t) = (1+t)^x - 1$. Referring to the Binomial Theorem in [8], we have

$$(p(t)+1)^r = \sum_{m=0}^{\infty} \binom{r}{m} p(t)^m,$$

but since the terms of index $n+1$ and higher yield an $(n+1)$st degree polynomial, the above difference will belong to J_n. We therefore have an induced map

$$\chi\colon \mathbf{B}[\mathbf{B}^k]_n \to \mathbf{B}[t_1, \ldots, t_k]/J_n.$$

The inverse relationship of ψ and χ is easy to verify. □

Example 12. — The isomorphism

$$\psi\colon \mathbf{B}[t_1, t_2]/J_2 \to \mathbf{B}[\mathbf{B}^2]_2$$

is given by

$$\begin{aligned}
1 &\mapsto [\diamond] & t_1^2 &\mapsto [e_1 \diamond e_1] \\
t_1 &\mapsto [\diamond e_1] & t_1 t_2 &\mapsto [e_1 \diamond e_2] \\
t_2 &\mapsto [\diamond e_2] & t_2^2 &\mapsto [e_2 \diamond e_2].
\end{aligned}$$

△

§4. Properties of Numerical Maps

Let us elaborate somewhat on the behaviour of numerical maps, and investigate their elementary properties. To begin with, we note that the binomial coefficients themselves, considered as maps $\mathbf{B} \to \mathbf{B}$, are numerical.

Theorem 6. — *The binomial coefficient $x \mapsto \binom{x}{n}$ is numerical of degree n.*

Proof. $\binom{x}{n}$ is given by a polynomial in the enveloping $\mathbf{Q}$-algebra. □

Next, we prove an alternative characterisation of numericality.

Lemma 1. — *For r in a numerical ring and natural numbers $m \leqslant n$, the following formula holds:*

$$\sum_{k=m}^{n} (-1)^k \binom{r}{k}\binom{k}{m} = (-1)^n \binom{r}{m}\binom{r-m-1}{n-m}.$$

Proof. Induction on n. □

Theorem 7. — *Let the map $\varphi\colon M \to N$ be polynomial of degree n. It is numerical (of degree n) if and only if it satisfies the equation*

$$\varphi(rx) = \sum_{m=0}^{n} (-1)^{n-m} \binom{r}{m}\binom{r-m-1}{n-m}\varphi(mx), \quad r \in \mathbf{B},\ x \in M.$$

Proof. This follows from the lemma:

$$\begin{aligned}
\sum_{k=0}^{n} \binom{r}{k} \varphi\left(\underset{k}{\Diamond}\, x\right) &= \sum_{k=0}^{n} \binom{r}{k} \sum_{m=0}^{k} (-1)^{k-m} \binom{k}{m} \varphi(mx) \\
&= \sum_{m=0}^{n} (-1)^{-m} \left(\sum_{k=m}^{n} (-1)^k \binom{r}{k}\binom{k}{m} \right) \varphi(mx) \\
&= \sum_{m=0}^{n} (-1)^{n-m} \binom{r}{m}\binom{r-m-1}{n-m} \varphi(mx).
\end{aligned}$$

□

Finally, not only do the nth deviations of an nth degree map vanish, but its lower order deviations are also quite pleasant.

Theorem 8. — *The map $\varphi\colon M \to N$ is numerical of degree n if and only if the following equation holds:*

$$\varphi(a_1x_1 \diamond \cdots \diamond a_kx_k) = \sum_{\substack{\#X=[k] \\ |X| \leqslant n}} \binom{a}{X} \varphi\left(\underset{X}{\Diamond}\, x\right), \quad a_i \in \mathbf{B},\ x_i \in M.$$

Proof. If the equation is satisfied, it follows that

$$\varphi(x_1 \diamond \cdots \diamond x_{n+1}) = \sum_{\substack{\#X=[n+1]\\|X|\leqslant n}} \binom{1}{X} \varphi\left(\Diamond_X x\right) = 0$$

and

$$\varphi(a_1 x_1) = \sum_{\substack{\#X=[1]\\|X|\leqslant n}} \binom{a}{X} \varphi\left(\Diamond_X x\right) = \sum_{k=0}^{n} \binom{a_1}{k} \varphi\left(\Diamond_k x_1\right).$$

Conversely, if φ is of degree n, a calculation in the augmentation algebra $\mathbf{B}[M]_n$ yields

$$\begin{aligned}
[a_1 x_1 \diamond \cdots \diamond a_k x_k] &= ([a_1 x_1] - [0]) \cdots ([a_k x_k] - [0]) \\
&= \sum_{q_1=1}^{\infty} \binom{a_1}{q_1} \left[\Diamond_{q_1} x_1\right] \cdots \sum_{q_k=1}^{\infty} \binom{a_k}{q_k} \left[\Diamond_{q_k} x_k\right] \\
&= \sum_{q_1=1}^{\infty} \cdots \sum_{q_k=1}^{\infty} \binom{a_1}{q_1} \cdots \binom{a_k}{q_k} \left[\Diamond_{q_1} x_1 \diamond \cdots \diamond \Diamond_{q_k} x_k\right].
\end{aligned}$$

The theorem follows after application of φ. □

This proof is pure magic! It is absolutely vital that the calculation be carried out in the augmentation algebra, as there would have been no way to perform the above trick had the map φ been applied directly.

Example 13. — Cubic maps φ are characterised by the following formulæ:

$$\begin{aligned}
\varphi(\diamond a_1 x_1) &= \binom{a_1}{1} \varphi(\diamond x_1) + \binom{a_1}{2} \varphi(x_1 \diamond x_1) + \binom{a_1}{3} \varphi(x_1 \diamond x_1 \diamond x_1) \\
\varphi(a_1 x_1 \diamond a_2 x_2) &= \binom{a_1}{1} \binom{a_2}{1} \varphi(x_1 \diamond x_2) \\
&\quad + \binom{a_1}{2} \binom{a_2}{1} \varphi(x_1 \diamond x_1 \diamond x_2) + \binom{a_1}{1} \binom{a_2}{2} \varphi(x_1 \diamond x_2 \diamond x_2) \\
\varphi(a_1 x_1 \diamond a_2 x_2 \diamond a_3 x_3) &= \binom{a_1}{1} \binom{a_2}{1} \binom{a_3}{1} \varphi(x_1 \diamond x_2 \diamond x_3).
\end{aligned}$$

△

There results the following very explicit description of numerical maps.

Theorem 9. — *The map $\varphi \colon M \to N$ is numerical of degree n if and only if for any $u_1, \ldots, u_k \in M$ there exist unique elements $v_X \in N$, X varying over multi-sets with $\#X \subseteq [k]$ and $|X| \leqslant n$, such that*

$$\varphi(a_1 u_1 + \cdots + a_k u_k) = \sum_{X} \binom{a}{X} v_X,$$

for any $a_1, \ldots, a_k \in \mathbf{B}$.

Proof. Assume φ is numerical of degree n. By the preceding theorem, we have

$$\begin{aligned}\varphi(a_1u_1+\cdots+a_ku_k) &= \sum_{I\subseteq[k]} \varphi\left(\mathop{\Diamond}_{i\in I} a_iu_i\right)\\ &= \sum_{I\subseteq[k]} \sum_{\substack{\#X=I\\ |X|\leqslant n}} \binom{a}{X}\varphi\left(\mathop{\Diamond}_{i\in X} u_i\right)\\ &= \sum_{\substack{\#X\subseteq[k]\\ |X|\leqslant n}} \binom{a}{X}\varphi\left(\mathop{\Diamond}_{i\in X} u_i\right),\end{aligned}$$

which establishes the existence of the elements v_X.

We now prove uniqueness. When S is a multi-set, we shall let $\deg_S z$ denote the *degree* or *multiplicity* of z in S. Let thus $Q\subseteq[k]$ be a multi-set, with $q_i=\deg_Q i$, and let

$$S=\{X \mid \#X\subseteq[k] \ \wedge\ \forall i: \deg_S i\leqslant q_i\}.$$

Then

$$\begin{aligned}\varphi(q_1u_1+\cdots+q_ku_k) &= \sum_X \binom{q}{X} v_X = \sum_{X\in S}\binom{q}{X}v_X\\ &= v_Q + \sum_{X\in S\setminus\{Q\}} \binom{q}{X}v_X.\end{aligned}$$

We see that v_Q is determined by all v_X such that X precedes Q in the lexicographical ordering on the set of all multi-sets on $[k]$, which can be identified with $\mathbf{N}^k$. By induction, each v_X is uniquely determined.

Conversely, assume φ is of the form specified in the theorem. It then readily follows that

$$v_X=\varphi\left(\mathop{\Diamond}_{i\in X} u_i\right)$$

for all X. In particular, the nth deviations of φ will vanish, and also

$$\varphi(au)=\sum_{m=0}^{n}\binom{a}{m}v_m=\sum_{m=0}^{n}\binom{a}{m}\varphi\left(\mathop{\Diamond}_{m} u\right).$$

□

§5. NUMERICALITY VERSUS $\mathfrak{NAlg}$-POLYNOMIALITY

And so, finally, we shall tie things together in our main theorem, and show that the definition we have given of numerical map coincides with $\mathfrak{NAlg}$-polynomiality.

THEOREM 10. — *The map $\varphi\colon M\to N$ is numerical of degree n if and only if it may be extended to a (unique) $\mathfrak{NAlg}$-polynomial map of degree n.*

Proof. If

$$\varphi_A : M \otimes - \to N \otimes -$$

is an $\mathfrak{NAlg}$-polynomial map of bounded degree n, it is clear from the Polynomiality Principle that

$$\varphi_{\mathbf{B}} : M \to N$$

has the property of Theorem 9.

Conversely, let a numerical map $\varphi \colon M \to N$ be given. Given elements $u_1, \ldots, u_k \in M$, fix the elements v_X from Theorem 9. We may then extend φ in the obvious way to a natural transformation:

$$\varphi_A : M \otimes A \to N \otimes A, \quad \sum u_j \otimes a_j \mapsto \sum_X v_X \otimes \binom{a}{X},$$

for any $A \in \mathfrak{NAlg}$. □

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