(i) For a 2-dimensional submanifold $M \subset \mathbb{R}^3$, the Gauss map $N : M \to S^2$ assigns to each $p \in M$ the unit normal vector to M at p. The shape operator S is the derivative of N which, since $T_pM = T_{N(p)}S^2$, can be regarded as a map $S_p : T_pM \to T_pM$ for any $p \in M$. The Gaussian curvature of M is given by $K(p) = \det S_p$.

Gauss's Theorema Egregium says that if in a local coordinates u, v on M, the induced metric is given by $g = Edu^2 + 2Fdudv + Gdv^2$ and we set $e =< \partial_u, \partial_u N >, f =< \partial_u, \partial_v N >, g =< \partial_v, \partial_v N >$ then

$$K = \frac{eg - f^2}{EG - F^2}.$$

Use this to show that K is the same as the only sectional curvature of M with the induced metric from $\mathbb{R}^3$.

(ii) Show that if M is simply connected and L is a complex line bundle on M such that $c_1(L) = 0$ then L is trivial.

(iii) A smooth covering is a smooth map $\pi : M \to N$ between two manifolds such that for each $p \in N$ there is a neighborhood $p \in V \subset N$ such that $\pi^{-1}(V)$ is a disjoint union of open sets $\{U_\alpha\}_\alpha$ each of which is mapped diffiomorphically to V by π. (In other words a fiber bundle with a discrete fiber.) If h is a Riemannian metric on N, show that there is a metric g on M which makes π into an isometry. Also show that (M, g) is complete if and only if (N, h) is.

(iv) Let $f : (M, g) \to (N, h)$ be a *Riemannian submersion* i.e. f as well as its derivative at each point are surjective and $||d_pf(u)|| = ||u||$ for any $p \in M$ and any $u \in T_pM$ which is orthogonal to $\ker d_pf$. (Such u are called horizontal vectors and elements of $\ker d_pf$ are called vertical. As an example, f can be a fiber bundle.) Let ∇, ∇' be the Levi-Civita connections for M and N respectively.

Show that for any vector field X defined in a neighborhood of $f(p)$ there is a unique horizontal lift X' defined in a neighborhood of p i.e. X' is horizontal and $d_qf(X'_q) = X_q$ for q in a neighborhood of p. Show that

$$\nabla'_{X'}Y' = (\nabla_X Y)' + \frac{1}{2}[X', Y']_v$$

where v denotes the vertical component. *Hint.* You can use the definition of the Levi-Civita connection.

(v) The n dimensional complex projective space $\mathbb{C}P^n$ is defined to be the set of all complex lines in $\mathbb{C}^{n+1}$. In other words

$$\mathbb{C}P^n = \{z = (z_0, \dots, z_n) \in \mathbb{C}^{n+1} \backslash \{0\}\}/(z \simeq \lambda z, \lambda \in \mathbb{C}\backslash\{0\}).$$

Note that in the above definition we can take z to have Euclidean norm one and therefore we have a projection $\pi : S^{2n-1} \to \mathbb{C}P^n$ whose fiber over each point of $\mathbb{C}P^n$ is a circle. It can be given a manifold structure with the atlas $\{(U_i, f_i)\}$ for $i = 0, \ldots, n$ given by $U_i = \{z | z_i \neq 0\}$ and $f_i : U_i \to \mathbb{C}^n$, $f(z) = (z_0/z_i, \ldots, \widehat{z_i/z_i}, z_n/z_i)$.

Consider the metric

$$\tilde{h}_z(X, Y) = \frac{Re < X, Y >}{||z||^2}$$

on $\mathbb{C}^{n+1} \backslash \{0\}$ where $< X, Y >= \sum_i \bar{X}_i Y_i$ is the standard Hermitian metric on $\mathbb{C}^{n+1}$ and Re denoted real part. Show that the action of S^1 on $\mathbb{C}^{n+1}$ given by $(e^{i\theta}, z) \to (e^{i\theta} z_0, \ldots e^{i\theta} z_n)$ perserves this metric and therefore there is a metric h on $\mathbb{C}P^n$ which makes π into a Riemannian submersion. This metric is called the Fubini-Study metric.