Probability on Graphs

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17 April 2002

Preface

This is my licentiate thesis, summarizing my first two years as a PhD-student. In general terms, the thesis deals with problems concerning probability models on combinatorial structures. The first paper considers the random assignment problem, the second first-passage percolation. The third paper investigate subgraph relations of the Archimedean and Laves lattices, and has applications in classical percolation, first-passage percolation, and self-avoiding walks.

I was introduced to these fields by my supervisor, Sven Erick Alm, when I as an undergraduate was looking for a project for my master thesis. The title of the master thesis became *Lower and Upper Bounds for the Time Constant of First-Passage Percolation*, which is also the title of the second paper in the present thesis (co-authored with Sven Erick Alm). I am most grateful to him for introducing me to these fascinating subjects, and for his support and help.

Special thanks also to John C. Wierman, who is the co-author of the third paper in this thesis, *The Subgraph Partial Ordering of Archimedean and Laves Lattices*.

And finally, I would like to thank my colleagues at the department, friends, and family.

Introduction

The thesis consists of this introduction and 3 papers.

I. *Random Assignment with Integer Costs*, Robert Parviainen, submitted to *Combinatorics, Probability and Computing.*

II. *Lower and Upper Bounds for the Time Constant of First-Passage Percolation*, Sven Erick Alm and Robert Parviainen, to appear in *Combinatorics, Probability and Computing.*

III. *The Subgraph Partial Ordering of Archimedean and Laves Lattices*, Robert Parviainen and John C Wierman, U.U.D.M Report 2002:13, Department of Mathematics, Uppsala University.

Despite the lack of common nouns in the titles, there are several connections. Paper I and II treat problems concerning graphs with random edge weights, and in both papers we are interested in the minimum of sums of independent random variables, where the sums are defined by subsets of edge sets of the graphs. Although Paper III does not directly involve probability theory, the primary applications are in percolation theory and related subjects, in which the models can be stated as (randomly) weighted graphs.

Due to the graph theory connection, we find it convenient to start this introduction by presenting some graph theory terms that will be of use later on. Since the papers are written with a reader familiar with these terms in mind, much of this is omitted in the papers themself.

Graph theory

Make some points on a paper and draw some lines connecting pairs of points. This is a graph. Formally, a graph consists of a countable (finite or infinite) set V of *vertices*, and a set E of *edges* connecting pairs of vertices. We only consider undirected graphs, so each edge $e \in E$ is an unordered pair of vertices (u, v), $u, v \in V$. We denote the edge by $e = uv$. We will only consider simple graphs, disallowing loops, *i.e.* edges joining one vertex with itself, and multiple edges,*i.e.* and more than one edge between any pair of vertices.

If $e = uv$ is an edge, u and v are said to be *adjacent*. We also say that u and v are *incident* to e. Two edges sharing one vertex are also said to be adjacent. (Thus two neighboring objects of the same kind are adjacent, of different kinds are incident.) The degree of a vertex v is the number of adjacent vertices to v, that is, the number of edges incident to v.

A *walk* between v_0 and v_n is a sequence of vertices $\{v_0, v_1, \ldots, v_n\}$, such that v_k is adjacent to v_{k+1}, $k = 0, \ldots, n-1$. A graph is *connected* if there exist walks between all pairs of vertices.

A *self-avoiding walk* on a graph is a walk $v_0, v_2, \ldots, v_n$ such that no vertex is used more than once.

Two graphs are *isomorphic* if they can be drawn in the same way. Mathematically, $G = (V, E)$ and $G' = (V', E')$ are isomorphic if there exists a bijection $\phi : V \to V'$ such that $uv \in E$ if and only if $\phi(u)\phi(v) \in E'$.

A graph is *transitive* if all vertices play the same role, *i.e.*, for every pair of vertices, u and v, there is a graph isomorphism that maps u to v.

If the edge set of a graph $G = (V, E)$ is a subset of the edge set of another graph $G' = (V', E')$, we say that G is a subgraph of G'. Intuitively, G is obtained from G' by deleting some edges.

A graph G is *bipartite* if we can divide the vertex set into two parts V_1 and V_2 such that all edges have one vertex in V_1, and one in V_2. If V_1 has n_1 vertices, and V_2 has n_2 vertices, we say that G is a $n_1 \times n_2$-bipartite graph. If every vertex in V_1 is adjacent to every vertex in V_2, we say that G is a *complete* bipartite graph. A *matching* in a bipartite graph is a subset M of the edge set, such that all edges in M are *independent*, meaning that no two edges in M are adjacent. A matching is said to be a *complete matching* if M is such that every vertex is incident to exactly one edge in M.

A common theme in the thesis is that we deal with *weighted* graphs. We associate with each edge e a weight X_e. Furthermore, the weights are stochastic variables. For example, X_e may represent the cost of including e in a matching M, or if X_e are 0-1 variables, we may take the random subgraph with edge set $F = \{e : X_e = 1\}$.

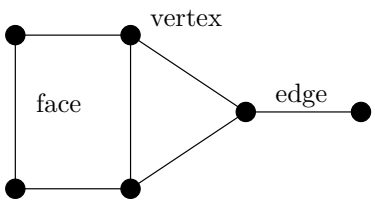

Figure 1: A simple planar graph, with 6 vertices, 7 edges, and 3 faces (one unbounded).

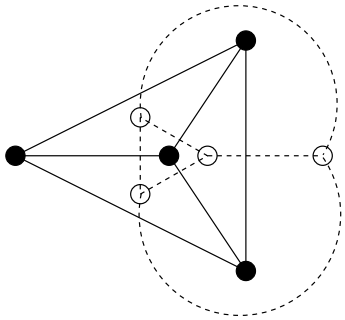

Figure 2: A graph and its dual.

A graph G is *planar* if it is possible to draw it in $\mathbb{R}^2$ so that no two edges intersect. A planar graph divides the plane into regions, separated by the edges, called *faces*. In the graph in Figure 1, we have three faces, one square, one triangular, and one unbounded face. A precise definition is the following: the *faces* of a planar graph are the connected regions in $\mathbb{R}^2$ that remain after we have removed the edges and vertices from (the planar embedding of G in) $\mathbb{R}^2$.

The *dual graph* $D(G)$ of a planar graph G is the graph in which every face of G is a vertex, and in which two vertices are adjacent if and only if the corresponding faces in G are adjacent, that is, share an edge. See Figure 2 for an example of a graph and its dual graph. Note that in this case, the two graphs are isomorphic.

Paper I. Random assignment.

Let us assume that we have n jobs that need to be done, and n workers available. If we know that worker i needs x_{ij} hours to finish job j, how shall we assign the jobs to the workers, such that the time until all jobs are finished is minimized?

If we impose the condition that each worker must do exactly one job, this problem is known as the *assignment problem*. It can be rephrased in the language of graph theory. Given a weighted complete $n \times n$-bipartite graph G,

what is the minimal cost among all complete matchings?

This is a combinatorial optimization problem, which can be solved by simply checking all complete matchings. However, there are $n!$ possible complete matchings, so for all but small n this is intractable. Fortunately, there exist faster algorithms for solving this problem. Instances with $n = 10000$ can be solved in about a minute on a standard personal computer, which is quite remarkable comparing to the naive method of checking all complete matchings.

Now it is time to introduce randomness. Assume that we do not know the costs exactly. We only know that they are independently distributed according to some probability distribution F. If we denote the (random) minimal cost by Z^*, we want to say something about the distribution of Z^*. This is the *random assignment problem*, and few answers are known. For instance, the expected value of Z^* is only known for very small values of n (when F is the exponential distribution, for n up to 7), and in the limit as n goes to infinity when F is continuous and positive. Part of the attractiveness of this problem are the conjectures. The best example is Parisi's conjecture, which says that, for exponential costs with mean 1, with $Z^*(n)$ denoting the minimal cost with n jobs,

$$EZ^*(n) = \sum_{k=1}^{n} \frac{1}{k^2}.$$

As mentioned above, we now know that this is true for n up to 7, and in the limit (which equals $\pi^2/6$).

In Paper I we consider the random assignment problem for some discrete distributions. First we let the workers rank every job, from 1 to n. Also, we imagine that there is a supervisor, ranking each combination of worker/job from 1 to n^2.

Formally, if the matrix of costs is denoted by $\mathbf{C}$, we consider the following cases.

- Each row in $\mathbf{C}$ is an independent random permutation of $\{1, 2, \ldots, n\}$, chosen uniformly from the set of all permutations.
- Each element in $\mathbf{C}$ is an independent random number, chosen uniformly from $\{1, 2, \ldots, n\}$.
- $\mathbf{C}$ is a random permutation of $\{1, 2, \ldots, n^2\}$ chosen uniformly.
- Each element in $\mathbf{C}$ is an independent random number, chosen uniformly from $\{1, 2, \ldots, n^2\}$.

Note that in the first case, costs in the same row are dependent, and in the third case, all costs are dependent.

By combining recent results by Aldous about the limit in the continuous case with coupling arguments, we find that the limiting expected costs for the third and fourth cases equal that for the continuous case.

For the first and second cases, we find bounds for the limiting expected costs, indicating that we get a slightly higher limit than in the continuous case.

We also study the four cases by Monte Carlo simulations, confirming the above conclusions. The simulations also give information about variances and rank statistics.

The interest in random assignment with integer cost distributions originated in computer science applications. Particularly, the case where the rows are random permutations is related to so called *hashing with linear probing*.

Paper II. First-passage percolation

In the bond percolation process, we produce a random subgraph of a given graph. More precisely, we let each edge remain in the subgraph with probability p, independently of all other edges. The most interesting question is for what values of p we get a large (infinite) connected component in the subgraph. For infinite transitive planar graphs, it is well known that there exists a critical value p_c, depending on the graph G, such that, for p below p_c, we almost surely get no infinite connected components, and for p above p_c, we get a unique infinite connected component with probability 1.

In first-passage percolation, the edge weights are independently distributed according to some distribution F, representing the time needed to walk along the edge. We are then interested in the shortest possible time needed to go from one vertex to another.

In Paper II, we study this problem on the square lattice. We consider the *time constant*, which is defined as the limit of the time needed to go from the origin to the vertex $(n, 0)$, normalized by the distance n. The limit can be shown to exist, but it is not known for any non-trivial distribution. We give improved lower and upper bounds for the time constant for the exponential and uniform distributions.

The lower bounds are found by improving a method of Janson, based on enumerating self-avoiding walks of finite length. The key idea is that by concatenating the finite self-avoiding walks, we overestimate the number of infinite walks. We introduce a transfer matrix, which makes it possible to substantially reduce the overestimation, resulting in improved lower bounds. The method is applicable for general distributions.

The upper bounds are found by doing exact calculations on very small subgraphs of the square lattice. Due to the high computational burden, it is important that the calculations are done in an efficient way.

For both the lower and upper bounds, computers were extensively used.

We also present a short simulation study, with the aim of estimating the time constants.

Paper III. Archimedean and Laves lattices.

The Archimedean lattices are infinite transitive planar graphs, where each face is a regular polygon. A regular polygon with n edges is a polygon where each edge is of length 1, and the inner angle between any two adjacent edges is $(n-2)\pi/n$ radians. Kepler proved in 1619 that there are exactly 11 Archimedean lattices. The Archimedean lattices include the square, triangular and hexagonal lattices. The dual graphs of the Archimedean lattices are called the Laves lattices. Since the square lattice is its own dual, and the triangular and hexagonal lattices are each others duals, the union of the Archimedean and Laves lattices consists of 19 graphs.

In Paper III we determine which Archimedean and Laves lattices are subgraphs of each other. For each of the $19 \cdot 18 = 342$ ordered pairs of graphs

(G, H), we either show how to get G from H by deleting edges, or prove that it is not possible.

The usefulness of this in percolation theory comes from the following easily proved facts. If H is a subgraph of G, the critical probabilities and the time constant are higher for H than for G.

Paper III is a first step towards finding the order of critical values (including critical probabilities and time constants) on the Archimedean lattices, which gives important clues on what properties of a graph these values depend on and how. This can be used for example to evaluate so called universal formulas, common in the physics literature on the subject, that try to predict the critical probabilities, based on simple properties of the graphs.

Random assignment with integer costs

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June 14, 2001

Abstract

The random assignment problem is to minimize the cost of an assignment in a $n \times n$ matrix of random costs. In this paper we study this problem for some integer valued cost distributions. We consider both uniform distributions on $1, 2, \ldots, m$, for $m = n$ or n^2, and random permutations of $1, 2, \ldots, n$ for each row, or of $1, 2, \ldots, n^2$ for the whole matrix. We find the limit of the expected cost for the n^2 cases, and prove bounds for the n cases. This is done by simple coupling arguments together with Aldous recent results for the continuous case. We also present a simulation study of these cases.

1 Introduction

In the assignment problem we are to choose n elements from a $n \times n$ matrix $\boldsymbol{C}$ of costs, one element from each row and each column, in such a way that the total cost is minimized. In other words, we are looking for a permutation π, that minimizes

$$Z = \sum_{i=1}^{n} C_{i\pi(i)}.$$

If we let the elements of $\boldsymbol{C}$ be random variables, we have the random assignment problem. Traditionally, the random costs have been independent, identically distributed, with the exponential or the uniform distribution.

When the costs are i.i.d. exponential (mean 1) there are strong conjectures for the more general case of k-assignment from a $m \times n$ cost matrix. Let $Z^*(k, m, n)$ denote the minimal cost. Mézard and Parisi [8, 9], conjectured that

$$\lim_{n\to\infty} E(Z^*(n, n, n)) = \pi^2/6.$$

This was proven by Aldous [1]. Parisi [11] has also conjectured that

$$E(Z^*(n, n, n)) = \sum_{i=1}^{n} \frac{1}{i^2},$$

which was improved by Coppersmith and Sorkin [3] to

$$E(Z^*(k,m,n)) = \sum_{i+j<k} \frac{1}{(m-i)(n-j)}.$$

The last conjecture was proven by Alm and Sorkin [2] for $k \leq 4, k = m = 5$, and $k = m = n = 6$. Linusson and Wästlund [7] extended this to $k \leq 6$, and $k = m = n = 7$.

1.1 Discrete variants

We will study four discrete variants of the random assignment problem.

Case I Each row in $\boldsymbol{C}$ is an independent random permutation of $\{1, 2, \ldots, n\}$, chosen uniformly from the set of all permutations.

Case II Each element in $\boldsymbol{C}$ is an independent random number, chosen uniformly from $\{1, 2, \ldots, n\}$.

Case III $\boldsymbol{C}$ is a random permutation of $\{1, 2, \ldots, n^2\}$ chosen uniformly.

Case IV Each element in $\boldsymbol{C}$ is an independent random number, chosen uniformly from $\{1, 2, \ldots, n^2\}$.

In the first two cases we normalize by n, and in cases III and IV by n^2, thus considering the problem of minimizing

$$Z = \frac{1}{n}\sum_i C_{i\pi(i)} \text{ or } Z = \frac{1}{n^2}\sum_i C_{i\pi(i)}.$$

The (random) minimal costs will be denoted by $Z_i^*(n)$, for the four discrete cases, and by $Z_c^*(n)$ in the case of continuous costs.

In [1], Aldous proves the following theorems, valid for any non-negative continuous distribution, such that the density of the independent costs have value 1 at 0. Let π denote the permutation giving an optimal assignment.

Theorem 1.1.

$$\lim_{n\to\infty} EZ_c^*(n) = \frac{\pi^2}{6}.$$

Theorem 1.2. *$nC_{i\pi(i)}$ converges in distribution. The limit distribution has density*

$$h(x) = \frac{e^{-x}(e^{-x} - 1 + x)}{(1 - e^{-x})^2}, \quad 0 \leq x < \infty.$$

Theorem 1.3.

$$\lim_{n\to\infty} P(C_{i\pi(i)} \textit{ is the } k\textit{th smallest element of the } i\textit{th row in } \mathbf{C}) = 2^{-k}.$$

Remark. In a simulation study in [10], Olin noted that, even for as small dimensions as $n = 50$, the row rank distribution is surprisingly close to the above.

2 Coupling arguments

In this section we will prove the following theorem.

Theorem 2.1. *Let* $EZ_i^* = \lim_{n\to\infty} EZ_i^*(n)$. *Then*

$$\frac{\pi^2}{6} \le EZ_1^* \le 2, \quad \frac{\pi^2}{6} + \frac{12}{24} \le EZ_2^* \le \frac{\pi^2}{6} + \frac{13}{24},$$
$$EZ_3^* = \frac{\pi^2}{6}, \quad EZ_4^* = \frac{\pi^2}{6}.$$

The idea is to compare a discrete case of the problem with the case of (continuous) uniform costs. We want to generate matrices for both cases simultaneously, such that an optimal assignment for one matrix is close to optimal for the other.

When we say that π is optimal for the matrix $\boldsymbol{C}$, we mean that π is a permutation giving an optimal assignment for the random assignment problem, with cost matrix $\boldsymbol{C}$.

2.1 Case I

Let $\boldsymbol{U}$ be a $n \times n$ matrix of i.i.d. $U(0,n)$ (uniform on $(0,n)$) random variables. It will be convenient to denote the rows of $\boldsymbol{U}$ by $\boldsymbol{U}^{(i)}$. We want to use $\boldsymbol{U}$ to get an independent random permutation for each row. To achieve this, we can use the row ranks of the matrix $\boldsymbol{U}$. If we let

$$P_i(j) = \text{rank } U_j^{(i)},$$

each P_i will be an independent random permutation, chosen uniformly from the set of all permutations. By Theorem 1.3 we have,

$$\lim_{n\to\infty} P(\text{rank } U_j^{(i)} = k) = 2^{-k}.$$

This gives, if π is the optimal assignment for $\boldsymbol{U}$,

$$\lim_{n\to\infty} EZ_1^*(n) \le \lim_{n\to\infty} E\left(\frac{1}{n}\sum_{i=1}^{n} P_i(\pi(i))\right) = \lim_{n\to\infty} E(\text{rank } U_j^{(i)}) = 2.$$

For a lower bound, assume that, for $1 \le i \le n$, P_i is a random permutation of $\{1, 2, \ldots, n\}$, and that $\boldsymbol{V}$ is a $n \times n$ matrix with i.i.d. $U(0,n)$ random variables as elements. We will now use the permutations P_i to rearrange the rows of $\boldsymbol{V}$. This will give us another matrix, $\boldsymbol{U}$, also with i.i.d. $U(0,n)$ elements, such that U_{ij} is close to $P_i(j)$. To be precise, let

$$U_j^{(i)} = V_{(P_i(j))}^{(i)} = \text{ the } P_i(j)\text{th smallest element in row } i \text{ of } \boldsymbol{V},$$

and note that $E(k - V_{(k)}^{(i)}) = k - nk/(n+1) = k/(n+1)$, since $V_{(k)}^{(i)}/n$ is $Beta(k, n+1-k)$. We therefore have, for all permutations π,

$$E(P_i(\pi(i)) - U_{\pi(i)}^{(i)}) > 0.$$

Now assume that π is an optimal assignment for the discrete problem. The cost can then be bounded below by the cost of the problem with cost matrix $\boldsymbol{U}$:

$$EZ_1^*(n) = E\left(\frac{1}{n}\sum_{i=1}^{n} P_i(\pi(i))\right) = E\left(\frac{1}{n}\sum_{i=1}^{n}\left(U_{\pi(i)}^{(i)} + P_i(\pi(i)) - U_{\pi(i)}^{(i)}\right)\right) >$$

$$> E\left(\frac{1}{n}\sum_{i=1}^{n} U_{\pi(i)}^{(i)}\right) \geq EZ_c^*(n) \to \frac{\pi^2}{6}.$$

2.2 Case II

Let U_{ij} be i.i.d. $U(0, n)$. To get i.i.d. random variables from the discrete uniform distribution on $\{1, 2, \ldots, n\}$, we can simply take the integer part of U_{ij} and add 1. Let

$$Y_{ij} = [U_{ij}] + 1,$$

where $[x]$ denotes the integer part of x. Then Y_{ij} are i.i.d. with the desired distribution, and the differences $Y_{ij} - U_{ij}$ are uniform on $(0, 1)$. Assume that π is an optimal assignment for $\boldsymbol{Y}$. Then we still have

$$Y_{i\pi(i)} - U_{i\pi(i)} \in U(0,1) \text{ and } E(Y_{i\pi(i)} - U_{i\pi(i)}) = 1/2,$$

and for the lower bound of $EZ_2^*(n)$,

$$EZ_2^*(n) = E\left(\frac{1}{n}\sum_{i=1}^{n} Y_{i\pi(i)}\right) = E\left(\frac{1}{n}\sum_{i=1}^{n} U_{i\pi(i)}\right) + \frac{1}{2} \geq EZ_c^*(n) + \frac{1}{2}.$$

Now for the other direction. Assume that π is the optimal assignment for $\boldsymbol{U}$. Svante Janson [5] has calculated the expectation of the fractional part of one element in the optimal assignment, $\{U_{i\pi(i)}\} = U_{i\pi(i)} - [U_{i\pi(i)}]$, with respect to the limit distribution, given by Theorem 1.2.

$$\lim_{n\to\infty} E(U_{i\pi(i)} - [U_{i\pi(i)}]) = \int_0^\infty \{x\}h(x)dx = \frac{1}{2} - \frac{1}{24} + \sum_{k=1}^{\infty} \frac{\pi^2}{\sinh^2(2\pi^2 k)} = \frac{11}{24} + c,$$

where $c \approx 2.83 \cdot 10^{-16}$. Let Z_2^π be the cost of $\boldsymbol{Y}$ given by the assignment π.

$$\lim_{n\to\infty} E(Z_c^* - Z_2^\pi) = \lim_{n\to\infty} E\left(\frac{1}{n}\sum_{i=1}^{n} U_{i\pi(i)} - Y_{i\pi(i)}\right)$$

$$= \lim_{n\to\infty} E(U_{1\pi(1)} - [U_{1\pi(1)}]) - 1 = -\frac{13}{24} + c.$$

Since $Z_2^* \leq Z_2^\pi$, we get the upper bound

$$EZ_2^* \leq \frac{\pi^2}{6} + \frac{13}{24}.$$

2.3 Case III

This is similar to the first case, but for ease of notation we consider a vector of n^2 elements instead of a $n \times n$ matrix.

Given a random permutation P of $\{1, 2, \ldots, n^2\}$, and a vector $\boldsymbol{V}$ of n^2 i.i.d. $U(0, n^2)$ random variables, let U_i be the $P(i)$th smallest element of $\boldsymbol{V}$, that is, $U_i = V_{(P(i))}$.

Conversely, given random variables U_i, $1 \le i \le n^2$, i.i.d. $U(0, n^2)$, define the random permutation by $P(k) = \text{rank } U_k$.

This gives our desired relations between $\boldsymbol{U}$ and P. By noting that, since $V_{(k)}/n^2$ is $Beta(k, n^2 + 1 - k)$ distributed,

$$E(k - V_{(k)}) = \frac{k}{n^2 + 1},$$

we also have for i in the optimal assignment for either case

$$\frac{1}{n^2 + 1} \le E(P(i) - U_i) \le \frac{n^2}{n^2 + 1}.$$

Now, if π is optimal for P,

$$EZ_3^*(n) = E\left(\frac{1}{n^2}\sum_{i \in \pi} P(i) - U_i + U_i\right) \ge EZ_c^*(n) + \frac{1}{n(n^2 + 1)},$$

and if π is optimal for $\boldsymbol{U}$,

$$EZ_c^*(n) = E\left(\frac{1}{n^2}\sum_{i \in \pi} U_i - P(i) + P(i)\right) \ge EZ_3^*(n) - \frac{n}{n^2 + 1}.$$

And by letting n tend to infinity, we get the limit

$$EZ_3^* = \lim_{n \to \infty} EZ_3^*(n) = \frac{\pi^2}{6}.$$

2.4 Case IV

As in the second case, given the i.i.d. uniform $(0, n^2)$ variables U_{ij}, define X_{ij} and Y_{ij} by

$$X_{ij} = [U_{ij}], \quad Y_{ij} = X_{ij} + 1.$$

If π is optimal for $\boldsymbol{Y}$,

$$Z_4^*(n) = \frac{1}{n^2}(Y_{1\pi(1)} + \cdots + Y_{n\pi(n)}) \ge \frac{1}{n^2}(U_{1\pi(1)} + \cdots + U_{n\pi(n)}) \ge Z_c^*(n).$$

If π is optimal for $\boldsymbol{U}$,

$$Z_c^*(n) = \frac{1}{n^2}(U_{1\pi(1)} + \cdots + U_{n\pi(n)}) \ge \frac{1}{n^2}(X_{1\pi(1)} + \cdots + X_{n\pi(n)}) \ge Z_4^*(n) - \frac{1}{n}.$$

Combining this, we get by letting n tend to infinity

$$EZ_4^* = \lim_{n \to \infty} EZ_4^*(n) = \frac{\pi^2}{6}.$$

3 Simulation

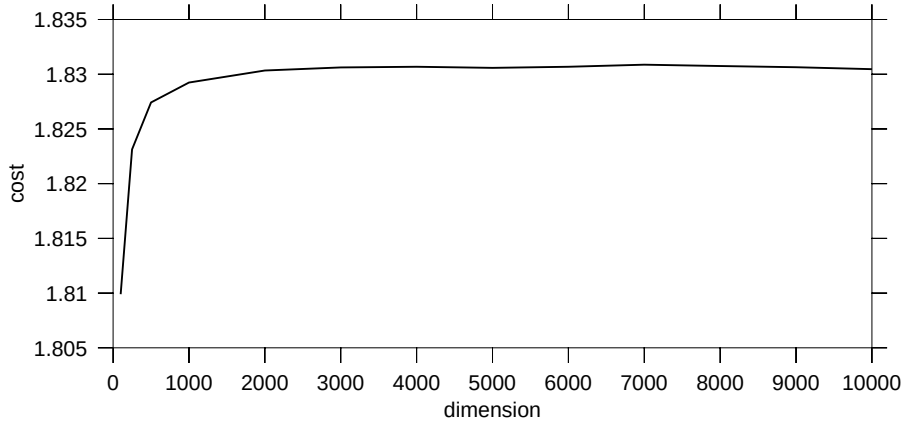

Figure 1: Simulation results, case I

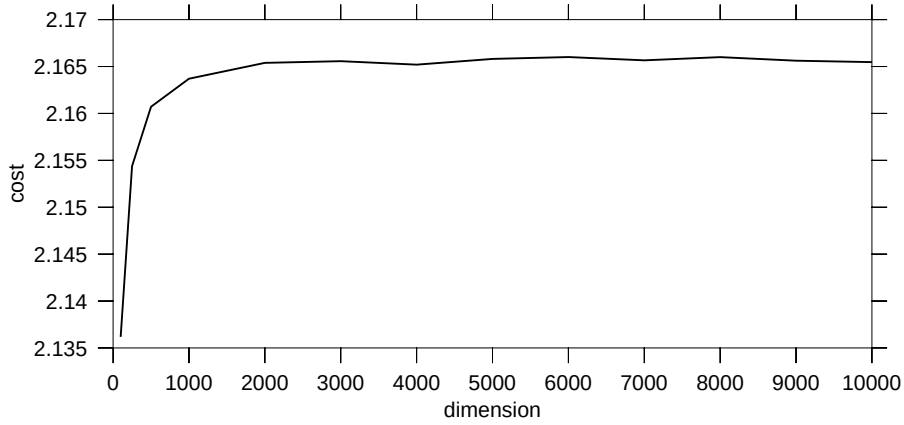

Figure 2: Simulation results, case II

The primary purpose of the simulation study is of course to estimate the expected minimal cost. Besides that, we look at the variance of the expected minimal cost, as well as the row rank distribution.

To solve the realizations, we used an algorithm by Jonker and Volgenant [6]. In a recent survey [4], it came out as one of the fastest available algorithms for problems like ours. Source code written by Jonker is available on the Internet[1], and a C++ version was used for these simulations. The algorithm has time complexity $O(n^3)$. Beside the dimension, the time also depends on the size of

[1] http://www.magiclogic.com/assignment.html

the matrix elements, which makes the simulations of cases III and IV more time consuming.

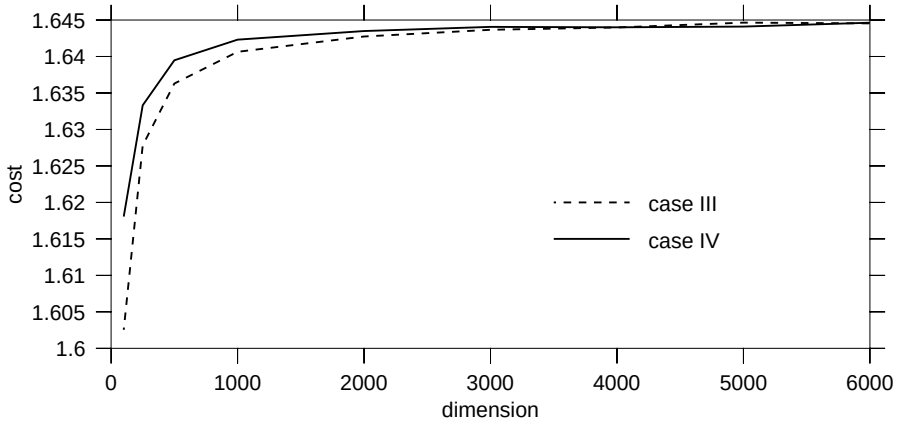

Figure 3: Simulation results, cases III and IV

As an indication of how fast the implemented algorithm really is, we note that in the permutation cases, the generation of the matrices takes about the same time as solving the assignment problem. In the independent cases the proportion of the time, spent generating the matrices, is about 0.25–0.4, depending on the dimension. An instance of dimension 1000 is solved in less than a second for all cases. For cases I and II it takes about 75–95 seconds to solve the problem with dimension 10000, and 30 seconds to generate the matrix. Almost 400 MB of RAM is needed for this dimension. The high dimension cases was run on a computer with two 1000 MHz Pentium III processors and 2 GB of RAM.

3.1 Results

3.1.1 Mean

The results are summarized in Tables 1–4 and Figures 1–3. Note that n in the tables is the number of realizations.

For case I and case II we simulated problems with dimensions up to 10000. The number of realizations varies between 40000 and 4000. We see that the estimated means stabilize quite fast. The difference between dimensions 2000 and 10000 is of order 10^{-4}, the same order as the standard error.

The n^2 cases III and IV behaves as expected. The mean increases nicely towards $\pi^2/6$, with case IV slightly ahead. Since these cases are more time consuming, and the limit is known to be $\pi^2/6$, we was content with simulations up to dimension 6000.

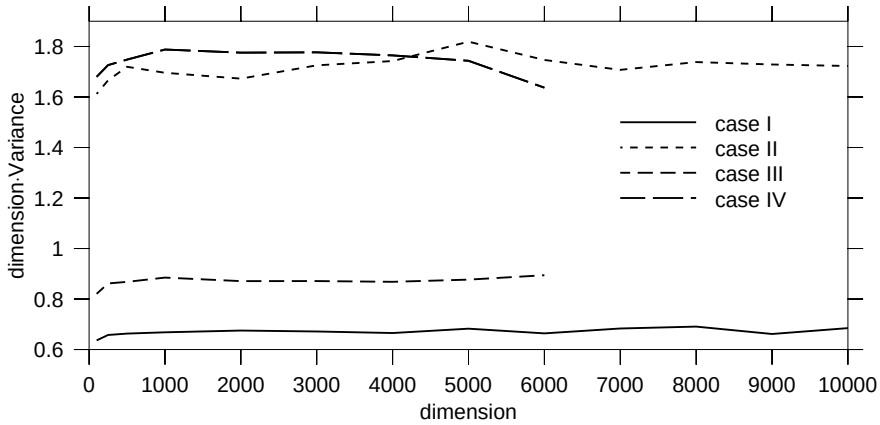

Figure 4: Estimated variance

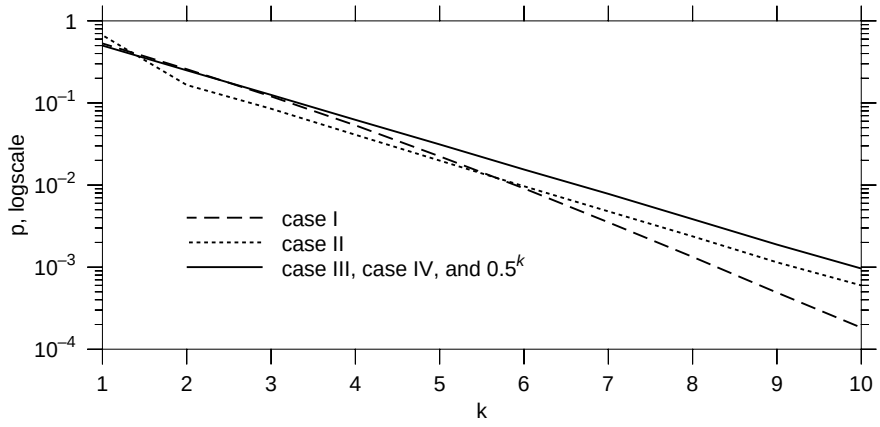

Figure 5: Estimated rank distribution, log-scale

Table 1: Simulation results, case I.

dimension	n	mean	std. dev.	s.e. mean
100	40000	1.80990	0.0797616	0.000398808
250	40000	1.82313	0.0512889	0.000256445
500	20000	1.82742	0.0364174	0.000257510
1000	20000	1.82924	0.0258506	0.000182791
2000	10000	1.83034	0.0183741	0.000183741
3000	10000	1.83062	0.0149616	0.000149616
4000	5000	1.83068	0.0128971	0.000182393
5000	5000	1.83058	0.0116841	0.000165238
6000	4000	1.83068	0.0105195	0.000166328
7000	4000	1.83087	0.0098804	0.000156223
8000	4000	1.83075	0.0092924	0.000146927
9000	4000	1.83064	0.0085737	0.000135563
10000	4000	1.83046	0.0082750	0.000130840

Table 2: Simulation results, case II.

dimension	n	mean	std. dev.	s.e. mean
100	40000	2.13618	0.1269710	0.000634857
250	40000	2.15438	0.0816278	0.000408139
500	20000	2.16071	0.0586444	0.000414678
1000	20000	2.16370	0.0411801	0.000291188
2000	10000	2.16539	0.0289196	0.000289196
3000	10000	2.16557	0.0239816	0.000239816
4000	5000	2.16520	0.0208703	0.000295150
5000	5000	2.16581	0.0190736	0.000269742
6000	4000	2.16601	0.0170615	0.000269765
7000	4000	2.16566	0.0156181	0.000246944
8000	4000	2.16600	0.0147405	0.000233067
9000	4000	2.16562	0.0138590	0.000219130
10000	4000	2.16547	0.0131255	0.000207532

Table 3: Simulation results, case III.

dimension	n	mean	std. dev.	s.e. mean
100	40000	1.60254	0.0905816	0.000452908
250	40000	1.62781	0.0587116	0.000293558
500	20000	1.63629	0.0416650	0.000294616
1000	20000	1.64064	0.0297464	0.000210339
2000	10000	1.64274	0.0208671	0.000208671
3000	10000	1.64366	0.0170409	0.000170409
4000	2000	1.64398	0.0147322	0.000329422
5000	2000	1.64464	0.0132429	0.000296119
6000	2000	1.64454	0.0122086	0.000272994

Table 4: Simulation results, case IV.

dimension	n	mean	std. dev.	s.e. mean
100	40000	1.61806	0.1296140	0.000648068
250	40000	1.63331	0.0830974	0.000415487
500	20000	1.63948	0.0591263	0.000418086
1000	20000	1.64231	0.0422843	0.000298995
2000	10000	1.64349	0.0297958	0.000297958
3000	10000	1.64406	0.0243364	0.000243364
4000	2000	1.64400	0.0210041	0.000469665
5000	2000	1.64410	0.0186728	0.000417537
6000	2000	1.64463	0.0165169	0.000369328

3.1.2 Variance

Alm and Sorkin [2] conjectures that the variance in the exponential case is $2/n + O(\log n/n^2)$. It is natural to suspect the same behavior in all our four cases. Figure 4 shows n times the estimated variance plotted against n. It is interesting to note is that the variance in the permutation cases is about half of that in the independent cases.

3.1.3 The rank distribution

In the continuous cases, the limiting rank distribution is geometric, with parameter 1/2. For comparison, we generated 1000 matrices of dimension 2000 for each discrete case, and determined the rank of every element in the optimal assignment given by the program. (Optimal assignments are not necessarily unique.) In the case of ties, we gave the element the lowest rank.

As suspected, cases III and IV seems to have the same limiting distribution as in the continuous case. Also in case II a geometric distribution, but with extra weight on 1, fit the data very well. For case I the picture looks a bit different. When plotted on a logarithmic scale, (Figure 5) we no longer get a straight line, but a slightly concave curve. (In this scale, a polynomial in k of degree 2 fit the data well.)

Acknowledgement

I would like to thank Svante Janson for suggesting both the problem and the coupling approach.

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Lower and Upper Bounds for the Time Constant of First-Passage Percolation

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We present improved lower and upper bounds for the time constant of first-passage percolation on the square lattice. For the case of lower bounds, a new method, using the idea of a transition matrix, has been used. Numerical results for the exponential and uniform distributions are presented. A simulation study is included, which results in new estimates and improved upper confidence limits of the time constants.

1. Introduction

1.1. Percolation

The percolation process was introduced as a mathematical model for the spread of a fluid through a random medium by Broadbent and Hammersley [2]. The term fluid has a broad interpretation and can for instance mean a liquid, an epidemic or a particle. The medium is represented by a connected graph, with a (possible finite) countable set of *vertices (sites)* and *edges (bonds)* joining the vertices.

Broadbent and Hammersley considered bond and site percolation, where each edge or vertex is open or closed for the fluid, with given probability.

In *first–passage percolation*, introduced by Hammersley and Welsh [3], each edge is open, and associated with a random variable, representing the time for the fluid to pass the bond.

1.2. First-passage percolation

We will study the time constant for first-passage percolation on the graph given by the square lattice. The vertices are the points $(x, y) \in \mathbb{Z}^2$. The edges are the lines of length 1 joining adjacent points, and with each edge e we associate a random variable X_e. We assume the variables X_e to be non-negative, independent and identically distributed with finite mean. We will be interested in the first-passage time to the line $x = n$, starting from the origin. (We will, with a slight abuse of notation, denote the line $\{(x, y) \in \mathbb{Z}^2 : x = n\}$ by $x = n$.)

A walk on the lattice is an alternating sequence $v_0, e_1, v_1, e_2, \ldots, e_n, v_n$ of vertices and edges. The walk is self-avoiding if all vertices are distinct. $(x(\gamma), y(\gamma))$ will be used to denote v_n, the endpoint for some walk γ, and $|\gamma|$ its length. We will need some notation for sets of walks. Γ is the set of all self-avoiding walks starting from the origin. There are some subsets of Γ that will be of use; Γ_n is the subset consisting of all self-avoiding walks of length n, $\Gamma(n, m)$ consists of all self-avoiding walks that end in (n, m), or more generally $\Gamma(R)$ for walks that end in some non-empty subset $R \subset \mathbb{Z}^2$. $F(n) = |\Gamma_n|$ will denote the number of self-avoiding walks of length n.

For a given walk γ we define the passage time $S_\gamma = \sum_{e \in \gamma} X_e$. We also define the first-passage times $T(R) = \inf_{\gamma \in \Gamma(R)} S_\gamma$, the first-passage time to the set R, starting at the origin, and $T_G = \inf_{\gamma \in G} S_\gamma$, the first-passage time over the subset $G \subset \Gamma$. For convenience, we write $T(m, n)$ instead of $T((m, n))$ for the first-passage time to the point (m, n).

We will on occasion use the term *infected* for sets of edges and vertices that may be reached from the origin in a given time, that is, the edge e (or vertex v) is infected at time t if there exists a path γ containing e (or v) with $v_0 = (0, 0)$, and $S_\gamma \leq t$.

1.3. The time constant

The time constant of first-passage percolation is defined as the limit of the normalized first-passage time from the origin to the line $x = n$. However, one can show (see the book by Smythe and Wierman [5]) that the same limit also arises if we only consider cylinder restricted walks. This allows us to use subadditivity, which will be important for the upper bounds.

Let C_{0n} be the subset of Γ consisting of the walks with end-vertex on the line $x = n$, and with all other vertices inside the cylinder $0 \leq x < n$.

Now, define s_{0n} as the first-passage time from the origin to the line $x = n$ over walks in C_{0n}. The time constant τ is defined as

$$\tau = \lim_{n \to \infty} \frac{s_{0n}}{n}.$$

The function Es_{0n} is subadditive (for a proof see [5]), which implies, for $n \geq 1$,

$$\tau = \inf_{k \geq 1} \frac{Es_{0k}}{k} \leq \frac{Es_{0n}}{n},$$

a key fact for our upper bounds.

The time constant measures the speed with which the fluid spreads, and is unknown for all non-trivial distributions. Previous lower and upper bounds are given by Janson [4] and Smythe and Wierman [5].

2. Lower bounds

In [4], Janson derives a method for calculating lower bounds for the time constant. The method is based on counting finite self-avoiding walks. The basic idea is to consider too many infinite walks, which is done by joining short self-avoiding walks at their endpoints.

Time constants may be defined in general directions as well, and Janson's method

treats all directions simultaneously. Define the set

$$N^* = \{(a,b) \in \mathbb{R}^2 | \lim_{(m,n)\to\infty} P(T(m,n) \leq am + bn) = 0\}.$$

The (horizontal) time constant τ may then be defined by

$$\tau = \sup_{a\in\mathbb{R}^+} \{(a,0) \in N^*\}.$$

In the following theorem from [4], we use the generating function

$$F_n(s,t) = \sum_{\gamma\in\Gamma_n} s^{x(\gamma)} t^{y(\gamma)},$$

and the Laplace transform for X_e,

$$\psi(\nu) = E(e^{-\nu X_e}),$$

to achieve criteria for $(a,0)$ to belong to N^*, and thus lower bounds for the time constant τ.

Theorem 2.1 (Janson). *If* $F_n\left(e^{a\nu}, e^{b\nu}\right)^{\frac{1}{n}} < \frac{1}{\psi(\nu)}$ *for some* $\nu > 0$ *and* $n \geq 1$, *then* $(a,b) \in N^*$.

For a proof, see [4]. Since $F_n(e^{a\nu}, 1)$ can be computed for a given n, Theorem 2.1 allows us to compute lower bounds for τ, by finding a and ν such that the assumptions are satisfied.

In a note in [4], Janson points out that Theorem 2.1 may be improved by noting that when joining two walks, for the resulting walk to be self-avoiding there are at most three possible directions for the first step of the second walk.

However, as we shall see, it is possible to join short self-avoiding walks in a more clever may, to reduce the number of non-self-avoiding walks in the limit, thereby improving the lower bounds.

2.1. An improved method

In [1], Alm uses a version of a method introduced by Wakefield [6] to find upper bounds for the connective constant of self-avoiding walks. A modified version of this method can be used to improve the bounds for the time constant as well.

Again, we count the number of self-avoiding walks of a given length n, but this time we also remember the m first and last steps, as well as the m-th last x-coordinate. This gives us a $F(m) \times F(m)$ matrix $\mathbf{B}$, see below. Each element in the matrix is the sum of two polynomials, one in s and one in s^{-1}, both of degree at most $n - m$. The largest eigenvalue of a matrix $\mathbf{A}$ will be denoted by $\lambda_1(\mathbf{A})$. A row vector $(1, 1, \ldots, 1)$ of suitable length will be denoted by $\mathbf{1}$. We will use the norm $\|\mathbf{A}\| = \sum_i \sum_j a_{ij} = \mathbf{1A1'}$.

Let m be fixed, let the walks of Γ_m be denoted $\gamma_1, \gamma_2, \ldots, \gamma_{F(m)}$, and let $a_{ij}^{(n)}(k)$ be the number of self-avoiding walks of length n that starts with γ_i and ends with a translation of γ_j, $|\gamma_i| = |\gamma_j| = m$, and has m-th last x-coordinate k. Define the matrix $\mathbf{A}$ by

$$\mathbf{A}^{(n)}(k) = \left(a_{ij}^{(n)}(k)\right), 1 \leq i, j \leq F(m),$$

and the matrix $\mathbf{B}$ by

$$\mathbf{B}^{(n)}(s) = \sum_{k=-n+m}^{n-m} \mathbf{A}^{(n)}(k)s^k = \left(b_{ij}^{(n)}(s)\right), 1 \leq i, j \leq F(m).$$

Now, every self-avoiding walk of length $2n-m$ that starts with γ_i and ends with a translation of γ_j, having m-th last x-coordinate k may be constructed by joining two self-avoiding walks of length n, the first starting with γ_i, and ending with a translation of γ_l, with m-th last x-coordinate r, the second starting with γ_l, and ending with a translation of γ_j, with m-th last x-coordinate $k-r$. Their composition $\gamma_1 \circ \gamma_2$ will then have m-th last x-coordinate k. Therefore

$$a_{ij}^{(2n-m)}(k) \leq \sum_{r=-n+m}^{n-m} \sum_{l=1}^{F(m)} a_{il}^{(n)}(r)a_{lj}^{(n)}(k-r) = \sum_{r=-n+m}^{n-m} \left(\mathbf{A}^{(n)}(r)\mathbf{A}^{(n)}(k-r)\right)_{ij},$$

and, for $\mathbf{B}$,

$$\begin{aligned}
b_{ij}^{(2n-m)}(s) &= \sum_{k=-2n+2m}^{2n-2m} a_{ij}^{(2n-m)}(k)s^k \\
&\leq \sum_{k=-2n+2m}^{2n-2m} s^k \sum_{r=-n+m}^{n-m} \sum_{l=1}^{F(m)} a_{il}^{(n)}(r)a_{lj}^{(n)}(k-r) \\
&= \sum_{k=-2n+2m}^{2n-2m} \sum_{r=-n+m}^{n-m} \sum_{l=1}^{F(m)} a_{il}^{(n)}(r)s^r a_{lj}^{(n)}(k-r)s^{k-r} \\
&= \sum_{l=1}^{F(m)} \sum_{r=-n+m}^{n-m} a_{il}^{(n)}(r)s^r \left(\sum_{k=-2n+2m}^{2n-2m} a_{lj}^{(n)}(k-r)s^{k-r} \right) \\
&= \sum_{l=1}^{F(m)} b_{il}^{(n)}(s)b_{lj}^{(n)}(s) = \left(\mathbf{B}^{(n)}(s)\mathbf{B}^{(n)}(s)\right)_{ij}.
\end{aligned}$$

In the same way we get

$$b_{ij}^{(k(n-m)+m)}(s) \leq \left(\mathbf{B}^{(n)}(s)\right)_{ij}^k. \tag{1}$$

Let the $F(m) \times 1$ column vector $\mathbf{R}^{(m)}$ be defined by the relation

$$F_{2n-m}(s,1) = \sum_{\gamma \in \Gamma_{2n-m}} s^{x(\gamma)} = \mathbf{1}\mathbf{B}^{(2n-m)}(s)\mathbf{R}^{(m)}.$$

$\mathbf{R}^{(m)}$ is just a correction vector, depending only on m, with elements from the set $\{s^{-m}, s^{-(m-1)}, \ldots, s^{m-1}, s^m\}$, due to the fact that we are using the m-th last x-coordinate. By (1), we then get, for all $k \geq 1$,

$$F_{k(n-m)+m}(s,1) = \mathbf{1}\mathbf{B}^{(k(n-m)+m)}(s)\mathbf{R}^{(m)} \leq \mathbf{1}\left(\mathbf{B}^{(n)}(s)\right)^k \mathbf{R}^{(m)}. \tag{2}$$

For $s > 1$ (we will use $s = e^{a\nu}$ in Theorem 2.2), we have $s^{-m} \le \mathbf{R}_i \le s^m$, for all i, and

$$\mathbf{1}\left(\mathbf{B}^{(n)}(s)\right)^k \mathbf{R}^{(m)} \le \mathbf{1}\left(\mathbf{B}^{(n)}(s)\right)^k \mathbf{1}' s^m = \| \left(\mathbf{B}^{(n)}(s)\right)^k \| s^m$$
$$\mathbf{1}\left(\mathbf{B}^{(n)}(s)\right)^k \mathbf{R}^{(m)} \ge \mathbf{1}\left(\mathbf{B}^{(n)}(s)\right)^k \mathbf{1}' s^{-m} = \| \left(\mathbf{B}^{(n)}(s)\right)^k \| s^{-m}.$$

Now let $k \to \infty$. Since

$$\lim_{k\to\infty} \left(s^{-m}\right)^{\frac{1}{k(n-m)+m}} = \lim_{k\to\infty} \left(s^{m}\right)^{\frac{1}{k(n-m)+m}} = 1$$

we get, by the Power method,

$$\lim_{k\to\infty} \left(1\left(\mathbf{B}^{(n)}(s)\right)^k \mathbf{R}^{(m)}\right)^{\frac{1}{k(n-m)+m}} = \lim_{k\to\infty} \left(\| \left(\mathbf{B}^{(n)}(s)\right)^k \|\right)^{\frac{1}{k(n-m)+m}}$$
$$= \left(\lambda_1\left(\mathbf{B}^{(n)}(s)\right)\right)^{\frac{1}{n-m}}.$$

And finally, by (2),

$$\left(F_{k(n-m)+m}(s,1)\right)^{\frac{1}{k(n-m)+m}} \le \left(\lambda_1\left(\mathbf{B}^{(n)}(s)\right)\right)^{\frac{1}{n-m}}.$$

With the help of Theorem 2.1 we have thus proved

Theorem 2.2. *If* $\left(\lambda_1\left(\mathbf{B}^{(n)}(e^{a\nu})\right)\right)^{\frac{1}{n-m}} < \frac{1}{\psi(\nu)}$ *for some* $\nu > 0$, *then* $(a, 0) \in N^*$.

We thus have a criterion for lower bounds. If the largest eigenvalue of the matrix $\mathbf{B}$ at the point $(e^{a\nu}, 1)$, to the power $(n-m)^{-1}$, is less than $\frac{1}{\psi(\nu)}$ then a is a lower bound for the time constant.

2.1.1. Reducing B The matrix $\mathbf{B}$ is actually unnecessary large. We can use a reduced $K(m) \times K(m)$ matrix $\tilde{\mathbf{B}} = (\tilde{b}_{ij})$, where $K(m)$ is the number of equivalence classes of walks of length m. We consider two walks equivalent if one can be mapped on the other by reflection in the x-axis. Every walk except those two that only uses horizontal steps (to $(m, 0)$ and $(-m, 0)$) has exactly one equivalent walk, so that $K(m) = \frac{F(m)}{2} + 1$. Denote the walks in the equivalence class $\tilde{\gamma}_1, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_{K(m)}$. Let $\tilde{\gamma}_1$ and $\tilde{\gamma}_{K(m)}$ be the walks of length m that goes straight along the x-axis to $(m, 0)$ and $(-m, 0)$. We define $\tilde{b}_{ij} = b_{ij}$ for $j = 1$ and $j = K(m)$, and $\tilde{b}_{ij} = b_{ij} + b_{ij'}$, where γ_j and $\gamma_{j'}$ are equivalent, otherwise. The following theorem shows that we can use $\tilde{\mathbf{B}}$ instead of $\mathbf{B}$.

Theorem 2.3. $\lambda_1(\tilde{\mathbf{B}}) = \lambda_1(\mathbf{B})$

Proof. Let $\tilde{\lambda}_1 = \lambda_1(\tilde{\mathbf{B}})$, with corresponding right eigenvector $\tilde{\mathbf{h}}$. Define $\mathbf{h}_{F(m)\times 1}$ by

$$h_j = \tilde{h}_s \text{ if } \gamma_j \text{ is equal to or equivalent to } \tilde{\gamma}_s.$$

If γ_i is equivalent to $\tilde{\gamma}_r$, then

$$\lambda_1 h_i = \sum_{j=1}^{F(m)} b_{ij} h_j = b_{i1}\tilde{h}_1 + \sum_{s=2}^{K(m)-1} (b_{is} + b_{is'})\tilde{h}_s + b_{iK(m)}\tilde{h}_{K(m)} = \sum_{s=1}^{K(m)} \tilde{b}_{rs}\tilde{h}_s = \tilde{\lambda}_1 \tilde{h}_r,$$

so $\tilde{\lambda}_1$ is an eigenvalue for $\mathbf{B}$. It remains to show that this is the largest eigenvalue, which is easily done by the Power method. In the recursion $\mathbf{v}^{(n)} = \frac{\mathbf{B}\mathbf{v}^{(n-1)}}{c_n}$, choose $\mathbf{v}(0) = \mathbf{h}$, and we get $c_n = \lambda_1(\tilde{\mathbf{B}})$ for all $n \geq 1$. Therefore $\lambda_1(\mathbf{B}) = \lim_{n\to\infty} c_n = \lambda_1(\tilde{\mathbf{B}})$. □

Table 1 Lower bounds for the time constant, exponential distribution

m / **n**	0	1	2	3	4	5	6	7
2	0.286787							
3	0.289423	0.298253						
4	0.292680	0.299266	0.299631					
5	0.293828	0.299473	0.299789					
6	0.295207	0.299685	0.299968	0.300186				
7	0.295900	0.299780	0.300025	0.300201				
8	0.296518	0.299860	0.300077	0.300223	0.300245			
9	0.296934	0.299913	0.300106	0.300233	0.300252			
10	0.297292	0.299955	0.300130	0.300242	0.300258	0.300272		
11	0.297561	0.299988	0.300147	0.300247	0.300261	0.300274		
12	0.297794	0.300015	0.300161	0.300251	0.300264	0.300275	0.300279	
13	0.297984	0.300037	0.300172	0.300254	0.300266	0.300276	0.300279	
14	0.298150	0.300056	0.300181	0.300257	0.300267	0.300277	0.300280	0.300282
15	0.298291	0.300072	0.300189	0.300259	0.300268	0.300277	0.300280	0.300282
16	0.298416	0.300086	0.300196	0.300261	0.300269	0.300277	0.300281	0.300282
17	0.298524	0.300098	0.300202	0.300263	0.300270	0.300278	0.300281	0.300282
18	0.298621	0.300109	0.300205	0.300265	0.300271	0.300278	0.300281	0.300282
19	0.298708	0.300119	0.300207	0.300266	0.300272	0.300279	0.300281	0.300282
20	0.298786	0.300127	0.300214	0.300267	0.300273	0.300279	0.300281	0.300282
21	0.298851	0.300135	0.300217	0.300268	0.300274	0.300280	0.300281	0.300282
22	0.298921	0.300142	0.300222	0.300268	0.300275	0.300280	0.300281	0.300282

2.2. Numerical results

The results are summarized in Tables 1 and 2. In Table 1, the entries for $m = 0$ correspond to the improved version of Theorem 2.1. The previous lower bounds, given in [4], were 0.29842 for the exponential distribution, by the improved version of Theorem 2.1 with $n = 16$, and 0.24294 for the uniform, by a result not included here, which only uses the Laplace transform $\psi(\nu)$. We improve these bounds already with $n = 4, m = 1$. The best lower bounds obtained here are 0.300282 and 0.243666, respectively.

The limitation in n is the time available. When going from n to $n+1$ the time needed increases roughly by a factor 3. In m, it is the amount of available internal computer memory that is the limiting factor, increasing by a factor 9 for each step in m. The time needed also increases in m, due to the fact that we must find the largest eigenvalue for a matrix that is roughly three times as wide. For $m = 7$ (the largest m we used), we need around 180 MB of RAM, and the computations for the largest n took a couple of days on a standard microcomputer. However, as can be seen in the Tables 1 and 2, there is little gained by doing larger calculations, especially by increasing n.

Table 2 Lower bounds for the time constant, uniform distribution

m / **n**	1	2	3	4	5	6	7
3	0.242941						
4	0.243325	0.243479					
5	0.243399	0.243518					
6	0.243468	0.243572	0.243643				
7	0.243500	0.243589	0.243647				
8	0.243526	0.243604	0.243653	0.243658			
9	0.243543	0.243613	0.243655	0.243660			
10	0.243557	0.243620	0.243657	0.243661	0.243664		
11	0.243568	0.243625	0.243658	0.243662	0.243665		
12	0.243577	0.243629	0.243659	0.243662	0.243665	0.243666	
13	0.243584	0.243633	0.243659	0.243663	0.243665	0.243666	
14	0.243591	0.243635	0.243659	0.243663	0.243665	0.243666	0.243666
15	0.243596	0.243638	0.243659	0.243663	0.243665	0.243666	0.243666
16	0.243601	0.243640	0.243661	0.243663	0.243665	0.243666	0.243666
17	0.243605	0.243641	0.243661	0.243663	0.243665	0.243666	0.243666
18	0.243608	0.243643	0.243661	0.243664	0.243666	0.243666	0.243666
19	0.243614	0.243644	0.243662	0.243664	0.243666	0.243666	0.243666
20	0.243614	0.243646	0.243662	0.243664	0.243666	0.243666	0.243666
21	0.243619	0.243647	0.243662	0.243664	0.243666	0.243666	0.243666
22	0.243619	0.243648	0.243662	0.243664	0.243666	0.243666	0.243666

3. Upper bounds

3.1. The method

In principle, it is easy to find upper bounds. We only have to calculate the expected first-passage time for some small subset of walks. We formalize this in the following proposition. Let C_{0n}, as before, be the subset of Γ consisting of walks with their endpoints on the line $x = n$, and with all other vertices inside the cylinder $0 \le x < n$.

Proposition 3.1. $\tau \le \frac{E(T_G(x=n))}{n}$, *where* $T_G(x = n)$ *is the first-passage time from the origin to the line* $x = n$ *over walks in* $G \subset C_{0n}$.

Proof. First, if $G_1 \subset G_2$, then

$$T_{G_1} = \inf_{\gamma \in G_1} S_\gamma \ge \inf_{\gamma \in G_2} S_\gamma = T_{G_2},$$

so that,

$$\tau \le \frac{E(s_{0n})}{n} \le \frac{E(T_G(x = n))}{n}.$$

□

Therefore, if we can compute $t = \frac{E(T_G(x=n))}{n}$ for some set G, t will be an upper bound for the time constant τ.

3.2. The exponential distribution

The nice properties of the exponential distribution makes it easy to compute upper bounds. We will consider rectangular subsets, with the origin on the left side. If the rectangle has M lines in the y-direction, the origin is placed at (the integer part of) $M/2$.

We start with only the origin infected. The expected first-passage time is then rewritten using the law of total probability, conditioning on the first step, giving an expression in terms of expected first-passage times with two infected vertices. All uninfected edges, adjacent to some infected vertex, still have, by the lack of memory property, the exponential mean 1 distribution, and we can continue in this way, successively conditioning on the next step, until we get explicit expressions for the expected first-passage times. Let $T_{k_1 \ldots k_l}$ denote the first-passage time with $l+1$ infected vertices, with k_1 as the first edge used, k_2 as the second, and so on. In the same way, let $N_{k_1 \ldots k_l}$ be the number of uninfected edges, adjacent to some infected vertex. Finally, let $X_{[1]}$ denote the minimum of $\{X_1, \ldots, X_N\}$. Then,

$$\begin{aligned}
E(T) &= \sum_{k_1=1}^{N} E(T|X_{k_1} = X_{[1]})P(X_{k_1} = X_{[1]}) \\
&= \sum_{k_1=1}^{N} E(X_{k_1} + T_{k_1}|X_{k_1} = X_{[1]})P(X_{k_1} = X_{[1]}) \\
&= P(X_1 = X_{[1]}) \left[\sum_{k_1=1}^{N} E(X_{k_1}|X_{k_1} = X_{[1]}) + E(T_{k_1}|X_{k_1} = X_{[1]}) \right] \\
&= \frac{1}{N} + \frac{1}{N} \sum_{k_1=1}^{N} E(T_{k_1}), \\
E(T_{k_1}) &= \frac{1}{N_{k_1}} + \frac{1}{N_{k_1}} \sum_{k_2=1}^{N_{k_1}} E(T_{k_1 k_2}),
\end{aligned}$$

and so on, until all vertices but those on the right borderline are infected. Then the expected first-passage time is $1/M$. By back substitution, we then find the wanted expected first-passage time. Note that the expected first-passage time only depends on the set of infected vertices. The order of infection, and the presence or not of edges between two infected vertices does not matter. This can be used to significantly reduce the number of cases by equating all configurations with the same set of infected vertices.

A PASCAL program was written with the purpose to automatically find these equations. Since the rational numbers involved soon have very large numerators and denominators, they where rounded to double precision reals, but *always upward*, to assure us that we really get an upper bound. The results are summarized in Table 3. The best upper bound found here is 0.503425. Smythe and Wierman [5] calculated the expected first-passage time to the line $x = 1$, which gives the upper bound 0.59726.

Table 3 Upper bounds, exponential distribution.

To line M	x=2	x=3	x=4	x=5	x=6	x=7
2	0.722222	0.709684	0.703017	0.698968	0.696262	0.694328
3	0.606463	0.597846	0.594009	0.591976	0.590729	0.589879
4	0.582480	0.567714	0.559863	0.555199	0.552170	0.550056
5	0.562535	0.544247	0.534522	0.528867	0.525309	0.522908
6	0.558398	0.537586	0.525706	0.518352	0.513503	0.510129
7	0.554433	0.531414	0.517790	0.509168	0.503425	
8	0.553678	0.529793	0.515191	0.505626		
9	0.552931	0.528208	0.512692			
10	0.552801	0.527828				

3.3. The uniform distribution

Without the lack of memory property, things are now a bit more complicated. We must keep track of the order of infection, and the time of each infection. However, the same approach still works, but in smaller scale.

Assume that l edges have been infected, in the order $k_1, k_2, \ldots, k_l$, with associated edge variables $X_{k_i} = x_{k_i}$. Also assume that there are n uninfected edges incident to at least one infected vertex. Only edges whose sole incident infected vertex is the last infected will be $U(0,1)$ distributed. The distributions of the other edges will still be uniform, but on $(0, z_i)$, where $z_i = 1 - \sum_{j \in S_i} x_{k_j}$, and S_i is a subset of $\{1, \ldots, l\}$. For a given configuration, the subsets S_i can be found by inspection, by considering the order of infection.

Let $f_{k_1 k_2 \ldots k_l}$ denote the joint density function for the n uninfected edges. The joint conditional density, given that the next edge infected is k_{l+1}, is denoted by $f_{k_1 \ldots k_l | k_1 \ldots k_{l+1}}$. We can express this in terms of the unconditional densities as $\frac{f_{k_1 \ldots k_l}}{f_{k_1 \ldots k_{l+1}}}$.

Let $T_{k_1 k_2 \ldots k_l}$ denote the first-passage time if we start with these edges infected, and $H_{k_1 k_2 \ldots k_l}$ the expected time until the next vertex gets infected. So, $H_{k_1 k_2 \ldots k_l}$ is the expectation of the minimum of the m edges that are incident to *exactly* one infected vertex,

$$H_{k_1 \ldots k_l} = \mathrm{E}(\min\{\mathrm{X}_1, \mathrm{X}_2, \ldots, \mathrm{X}_\mathrm{m}\}),$$

where $X_i \in \mathrm{U}(0, \mathrm{z_i})$.

The expected first-passage time may be decomposed as

$$E(T_{k_1 \ldots k_l}) = H_{k_1 \ldots k_l} + \sum_{k_{l+1}} \int_0^a E(T_{k_1 \ldots k_{l+1}}) \frac{f_{k_1 \ldots k_l}}{f_{k_1 \ldots k_{l+1}}} dx_{k_{l+1}},$$

where the sum is over the m edges that are incident to exactly one infected vertex, and $a = \min\{z_1, \ldots, z_m\}$. An important fact is that every integrand will be a polynomial in the variable of integration. Otherwise the number of integrals necessary to compute, even for very small areas, would make the calculations intractable. Still, we have so far only been able to perform the calculations for the rectangle bounded by the lines $x = 0, x = 2, y = 2, y = -1$. This gives the upper bound 0.403141. The best upper bound in Smythe and Wierman [5] is 0.425 in this case.

It is possible to implement the rules for these equations in a computer program, which generates as output the equations as a Maple file, which is read and processed (in Maple). The calculations have been cross-checked by simulations, and by other methods for smaller areas.

4. Simulation

A simulation study of the cylinder restricted process s_{0n} has been performed. The purpose of this study is to estimate the time constant, and to obtain upper confidence limits for the time constant. We also use the simulation program to cross-check the computations for the upper bounds. The simulation program generates a number of independent realizations of the process, and outputs means of first-passage times to each line from $x = 1$ up to some predetermined line $x = n$, as well as the y-coordinate for the first hit on the line $x = n$ for each simulation. To avoid unnecessary and time consuming programming the walks are restricted in y-direction as well. This is not a problem since it is easy to choose the restriction such that the probability that a walk will be restricted in the y-direction is virtually zero, which can in part be confirmed by the data of hitting points on the line $x = n$.

4.1. Upper confidence limits

The first-passage times to the line $x = n$ are used to compute upper confidence limits for the time constant. Since

$$\tau \leq E\left(\frac{s_{0n}}{n}\right),$$

an upper confidence limit for $E\left(\frac{s_{on}}{n}\right)$ will also be an upper confidence limit for τ, with a higher level of confidence. The confidence limits are constructed in the usual way, using the Gaussian 95% quantile.

4.1.1. Numerical results For the exponential distribution, 5000 realizations of the process $s_{0,500}$ were generated. The estimated expected first-passage time to the line $x = 500$ was 204.786, with standard deviation 2.99526, giving a 95% upper confidence limit of 0.409711.

Since the simulations for the uniform distribution are less time consuming than for the exponential distribution, a sample of 1000 realizations of the process $s_{0,750}$ could be generated. The estimated expected first-passage time to the line $x = 750$ was 236.629, with standard deviation 2.23546, giving a 95% upper confidence limit of 0.315660.

4.2. Estimates

The same data used for the confidence limits was also used to estimate the time constant. From the first-passage times we try to extrapolate towards infinity to get estimates. Here we use a simple method. We plot the (by n) normalized first-passage times against $\frac{1}{\sqrt{n}}$, and fit a regression line. The intercept will then estimate the time constant. The choice of $\frac{1}{\sqrt{n}}$ as the predictor is based only on data exploration, and not on theoretical ideas.

The removal of some data points corresponding to the lines closest to the origin results in slightly higher estimates than using the whole data set. Further removal and fitting

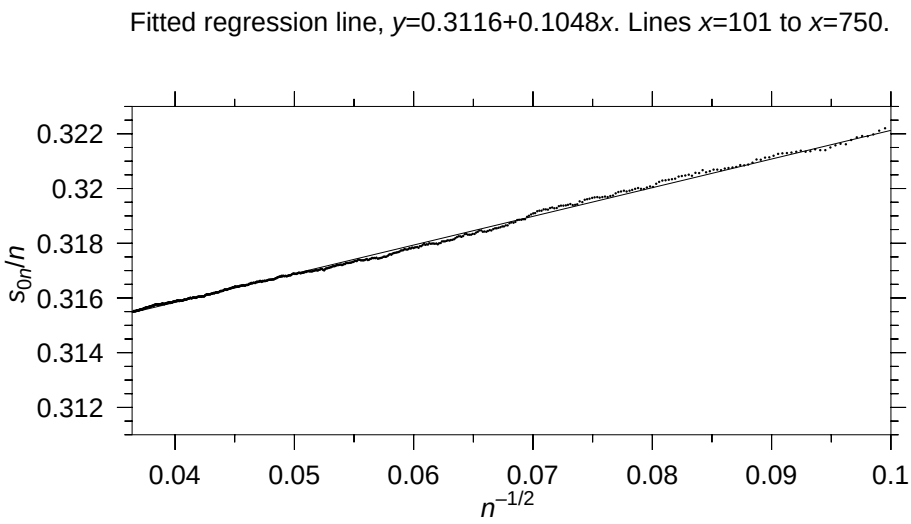

Figure 1 Estimating τ, uniform distribution

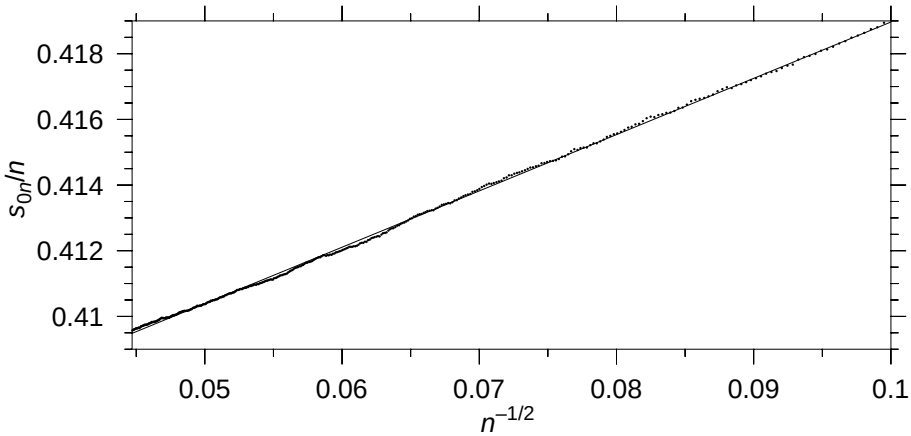

Figure 2 Estimating τ, exponential distribution

polynomials with higher degree had in most cases only minor or no influence on the estimates. Examples are found in Figures 1 and 2. The chosen estimates are 0.402 for the exponential distribution, and 0.312 for the uniform distribution.

5. Algorithms

5.1. Lower bounds

The general algorithm implemented consists of three steps.

1 *Generate an enumeration of the $K(m)$ self-avoiding walks of length m.* This is done

Table 4 Summary

	Exponential	Uniform
Upper bound	0.503425	0.403141
Upper confidence limit	0.409711	0.315660
Estimate	0.402	0.312
Lower bound	0.300282	0.243666

recursively, by starting with the walk that goes straight to $(m, 0)$, and successively altering more and more of the end of the walk, enumerating all walks as we pass them, up to equivalence.

2 *Generate the $K(m) \times K(m)$ matrix $\tilde{\mathbf{B}}$.* For each enumerated walk γ_i, we cycle through all walks that start with γ_i. If a walk ends with a translation of γ_j with m-th last x-coordinate z we increment the element (i, j, z) in the three dimensional array representing $\tilde{\mathbf{B}}$.

3 *Look for a good lower bound.* We start with values of a and ν that we know fulfills the criteria. Then a good ν is found and held fixed. a is then increased to the best possible value. This is repeated a few times (typically two or three times).

The program was written in C++. The time-consuming parts are step 2 (for large n) and step 3 (for large m). Good start values of a and ν are critical for fast performance, but this is easily accomplished by extrapolation from smaller n or m. Also, ν is quite stable for varying n and m.

5.2. Simulations

We start with a list (implemented as a doubly linked list) of the vertices that can be infected, that is, they are adjacent to some already infected vertex. In the first step the list contains the points $(1, 0), (0, -1)$ and $(1, 0)$. The list is sorted with respect to the times of the forthcoming infection. The first vertex in the list (the next vertex to be infected) is then removed from the list, and the list is expanded with eventual new vertices that are adjacent to the vertex being removed, but not yet infected. Of course only vertices inside the cylinder may be in the list. We then continue removing vertices until the infection has reached the line $x = n$, and we are done.

5.3. Upper bounds, exponential distribution

The calculations outlined in Section 3.2 were implemented in a recursive PASCAL program, where all calculation were rounded upward to double precision. All calculated expectations were saved. As the number of possible configurations in some of the cases was very large ($M = 7$, $x = 6$ gives 2^{42} configurations), memory was used dynamically. The computing time was several days for the larger cases, with close to 1 GB of memory used.

5.4. Upper bounds, uniform distribution

We see each configuration as a node in a rooted tree, with the case of only the origin infected as the root, and where each (tree) node's children are the configurations with one more edge infected than the parent. The algorithm makes a postorder traversal

of this tree, where in each step the equation for the expectation for this configuration is determined. The processing of a given configuration is rather involved, due to the somewhat complex rules for the set of edges that should be considered, and the difficulties to implement these in an efficient way.

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The Subgraph Partial Ordering of Archimedean and Laves Lattices

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April 12, 2002

Abstract

The subgraph relation defines a partial order on graphs. In this paper, we determine this partial order completely for the Archimedean and Laves lattices.

1 Introduction

A *regular tiling* is a tiling of the plane which consists entirely of regular polygons. An *Archimedean lattice* is the graph of vertices and edges of a regular tiling which is vertex-transitive, i.e., for every pair of vertices, u and v, there is a graph isomorphism that maps u to v. There are exactly 11 Archimedean lattices. A proof that these are the only vertex-transitive regular tilings is given in Grünbaum and Shephard [7, Ch. 2]. Archimedean lattices were studied and named by Kepler [13], due to connections with the Archimedean solids.

A notation for Archimedean lattices, which can also serve as a prescription for constructing them, is given in Grünbaum and Shephard. Around any vertex (since all are equivalent, by vertex-transitivity), starting with the smallest polygon touching the vertex, list the number of edges of the successive polygons around the vertex. For convenience, an exponent is used to indicate that a number of successive polygons have the same size.

Since the Archimedean lattices are planar graphs, each has a planar dual graph. The square lattice, (4^4), is self-dual, and the triangular, (3^6), and hexagonal, (6^3), lattices are a dual pair of graphs. The other 8 Archimedean lattices have dual graphs that are not Archimedean. We will denote the dual of an Archimedean lattice G by $D(G)$. The duals of the Archimedean lattices have

applications in crystallography, where they are called Laves lattices [16, 17]. There are a total of 19 different Archimedean and Laves lattices, which are illustrated in Figure 1.

Let $A \subseteq B$ denote that A is isomorphic to a subgraph of B. The relation $\subseteq$ is reflexive, anti-symmetric, and transitive, and thus is a partial order on the set of Archimedean and Laves lattices. In this paper, we determine this *subgraph partial order* completely, showing all cases in which lattices are subgraphs of others, and showing which pairs of lattices are incomparable. While at first glance this task may seem trivial, creativity is required to find some unusual inclusions, and subtleties and intricate reasoning (although with elementary methods) are required to prove incomparability in many cases.

This paper is organized as follows: Section 2 provides an introduction to our motivating applications in the theories of classical percolation, first-passage percolation, and self-avoiding walks. A summary of the results are presented in Section 3, in a Hasse diagram showing the subgraph partial ordering and in a Table which also indicates the method of proof. All subgraph inclusion results are demonstrated in Figures in Section 4. In Section 5, several techniques for checking for non-inclusion are described. Non-inclusion proofs that are special cases, not handled by the general techniques, are provided in Section 6.

2 Motivating Applications

This study is motivated by applications to models from probability, combinatorics, and mathematical physics – bond and site percolation, first-passage percolation, and self-avoiding walks. We briefly describe each of these models in the following subsections, the comment on similarities relevant to the subgraph order problem.

2.1 Bond and Site Percolation Models

The two classical percolation models were introduced as a models for the spread of fluid through a random medium. The medium is represented by an infinite connected locally finite graph. In the bond percolation model, each edge of the graph is *open* to the flow of fluid with probability p, $0 \leq p \leq 1$. In the site percolation model, each vertex is open with probability p, and fluid is permitted to flow through the subgraph induced by the set of open vertices. The key concept is the *critical probability*, or *percolation threshold*, denoted p_c, such that for $p < p_c$ there are almost surely no infinite connected components of open edges or vertices, and for $p > p_c$ there exists an infinite connected component with probability one. Considerable scientific interest focuses on percolation as a simple mathematical model for a phase transition, which is represented by the critical probability. See [21, 23] for descriptions of applications of percolation in engineering and physics. See Grimmett [6] for the most complete discussion of the mathematical theory.

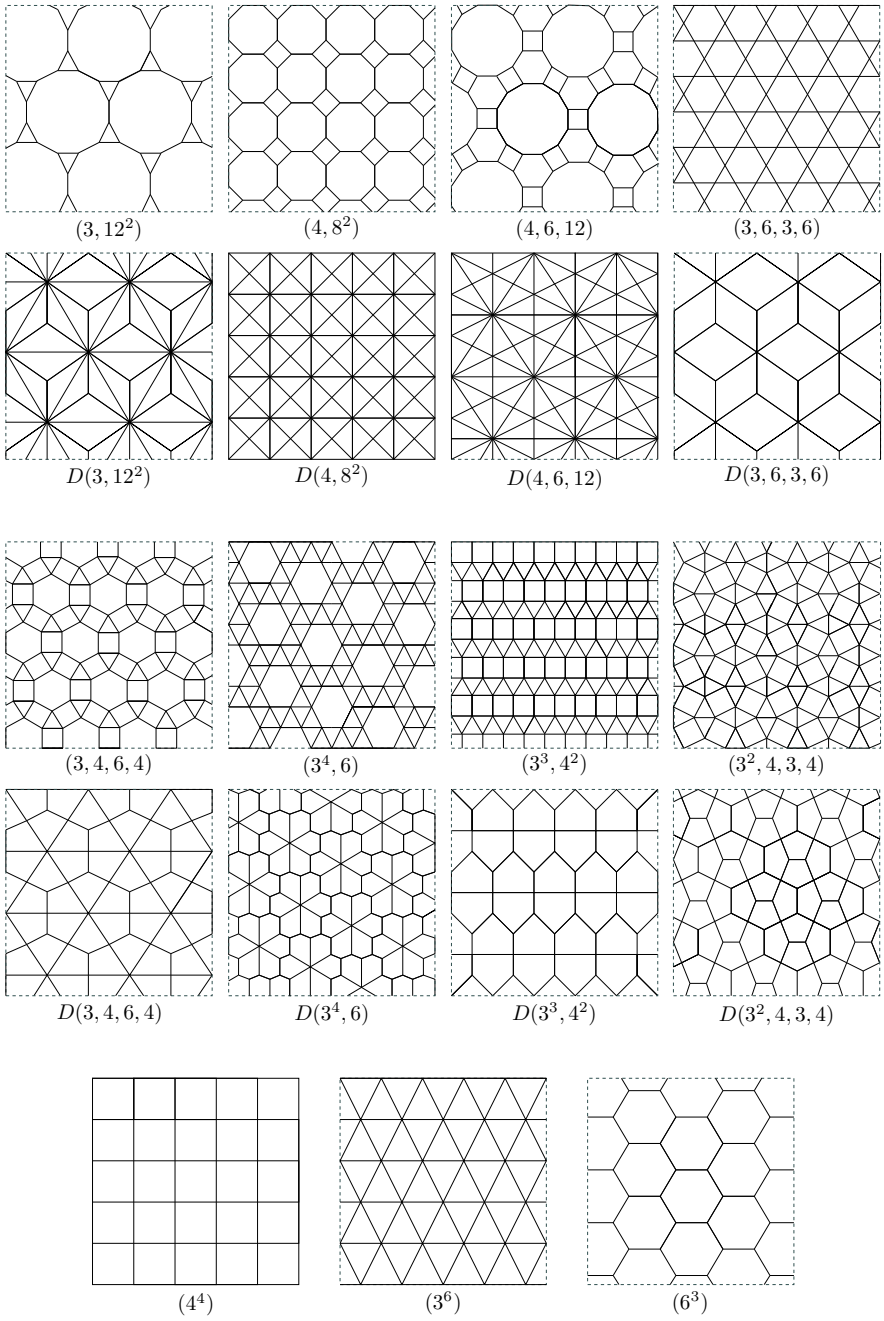

Figure 1: The 11 Archimedean lattices and their dual Laves lattices.

2.2 First-passage Percolation

In *first-passage percolation*, introduced by Hammersley and Welsh [9], each edge of a graph is open, and associated with a random variable representing the time for the fluid to pass through the edge. Often the graph is taken to be the square lattice $\mathbb{Z}^2$ or some hypercubic lattice $\mathbb{Z}^d$, where it is natural to measure the speed of the spread of the fluid by a *first-passage time*, e.g., the time needed for the fluid to pass from the origin to a point $(n, 0)$ or to a line $x = n$. The *time constant* is defined as the limit of the first-passage time, normalized by the graph distance (number of edges in the shortest path), as the graph distance tends to infinity. The time constant may be interpreted as the reciprocal of the velocity of spread.

2.3 Self-avoiding Walks

A *self-avoiding walk* is a path of adjacent vertices such that no vertex occurs more than once. For a regular graph, the *connective constant* is the limit of the n-th root of the number of self-avoiding walks with n edges (starting at a fixed vertex). Self-avoiding walks have been used as a lattice model for the excluded volume problem in the theory of polymers. The first mathematically rigorous analysis of the subject was by Hammersley and Morton [8] in 1954. Upper bounds for the connective constant were provided by Alm [2]. Hughes [10, Ch. 7] gives a nice review of the field, while a more substantial treatment is given by Madras and Slade [18].

2.4 Similarities

In the study of percolation models, first-passage percolation, and self-avoiding walks, considerable interest focuses on quantities which depend on the structure of an underlying graph in an extremely complicated fashion, so much so that there are few (if any) exact values known, and only very crude bounds in the many unsolved cases. Rigorous lower and upper bounds can be found for these critical values for several lattice graphs, usually by methods that require extensive computer calculations.

In classical percolation theory, the exact bond model critical probabilities or site model critical probabilities are known for only a few graphs [14, 15, 25, 26], thus making it important to determine rigorous bounds for unsolved graphs [4, 27, 28, 29, 30, 31] . Many simulation studies have estimated critical probabilities of various graphs, in particular the Archimedean lattices [24].

In first-passage percolation, other than its counterpart for infinite trees, the exact value of the time constant is not known for any non-trivial distribution on any non-trivial lattice. Determining rigorous bounds has been very challenging, with some progress by Janson [11] and Alm and Parviainen [3].

The connective constant is not known for any non-trivial lattice, although Nienhuis [19] has, by non-rigorous methods, derived the value $\sqrt{2+\sqrt{2}}$ for the hexagonal lattice. (Jensen and Guttmann [12] uses this to conjecture that the

connective constant for the $(3, 12^2)$ lattice is 1.71104.) See Alm and Parviainen [3], Conway and Guttmann [5], and Pönitz and Tittmann [20] for bounds on the connective constant.

However, for critical probabilities, time constants, and connective constants, the values for two graphs are ordered if one is a subgraph of the other. If H is a subgraph of G, the critical probabilities and the time constant are higher for H than for G, and the connective constant is lower for H than for G. Thus, knowledge of the subgraph order will allow the use of exact values or bounds for some graphs to provide bounds for other graphs. Only a few subgraph relationships, involving the triangular, hexagonal, and square lattices, have been observed and used in these theories in the past. To our knowledge, there has been no systematic study to determine the complete set of of subgraph relationships for any class of lattices.

The Archimedean and Laves lattices include the most common examples of 2-dimensional graphs studied in the three applications above. Thus, they are an appealing starting point when trying to obtain a deeper understanding of dependence of the critical parameters on the properties of the underlying graph.

3 Results

The results of this investigation can be best reported in the form of the Hasse diagram of the subgraph ordering, shown in Figure 4.

The Hasse diagram is accompanied by Table 1, which also provides a summary of the proof. Each entry indicates whether the lattice named at the left margin includes the lattice named at the top margin as a subgraph. A "+" or "T" indicates that inclusion holds, while any other symbol indicates that it does not. Entries of "T" indicate that the inclusion holds by transitivity, in which case the inclusions that imply it may be found by consulting the Hasse diagram or the "+" entries in Table 1. Other symbols indicate the lemma or method in Section 5 which is used to prove that inclusion does not hold.

Each subgraph inclusion result implies an inequality for the quantity of interest in each of the motivating applications. For critical probabilities in classical percolation theory, the inclusion results established in this paper all provide strict inequalities for critical probabilities, by a result of Aizenman and Grimmett [1]. For each of these motivating applications, the results of this paper may be combined with results from other techniques to make progress in determining the ordering of the quantities of interest.

4 Inclusion Proofs

Figures 2 – 4 demonstrate the 35 subgraph inclusion relationships that are denoted by + entries in Table 1. These are the covering relationships in the Hasse diagram for the subgraph ordering. Since the graphs are periodic, in each

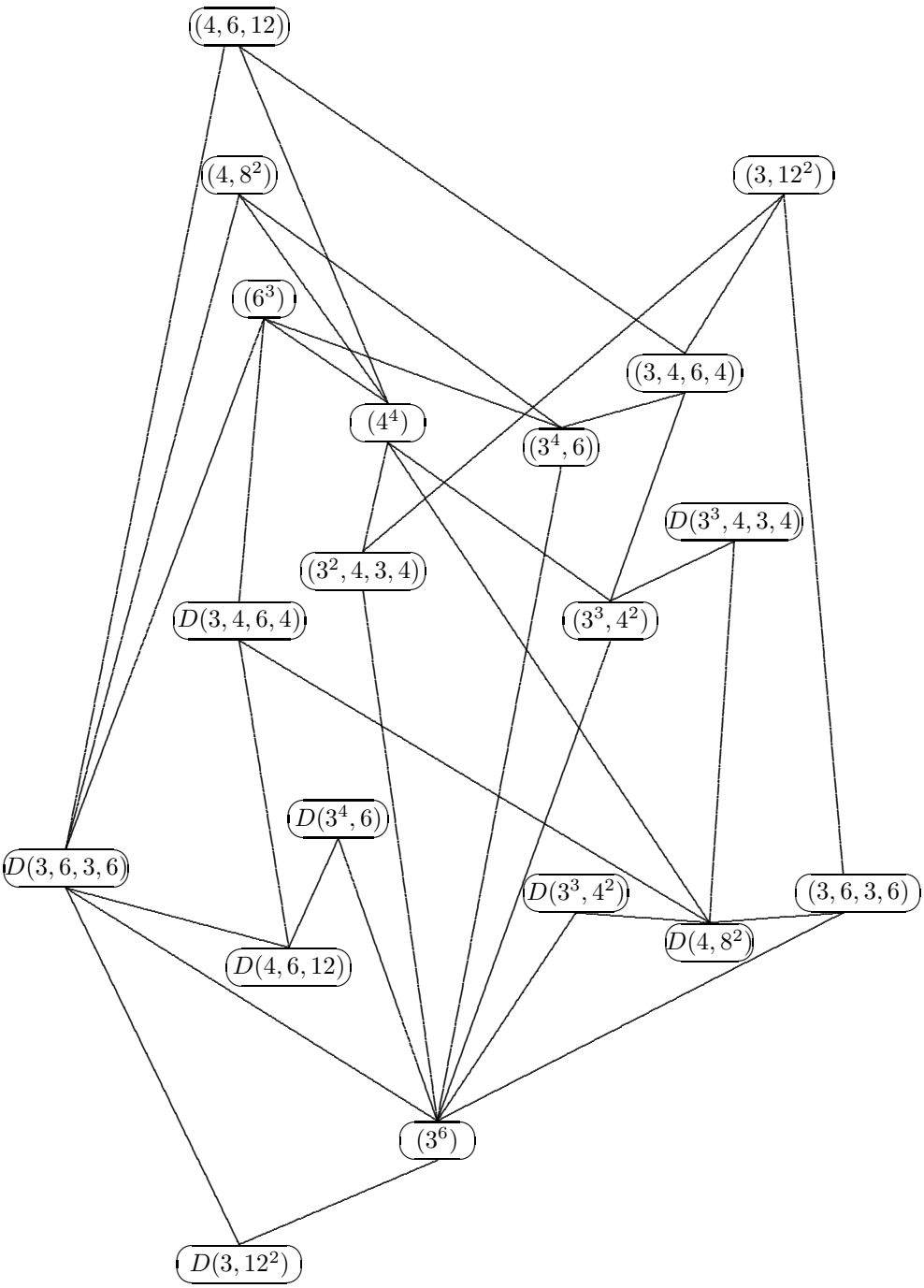

Figure 2: The Hasse diagram of the subgraph order of the Archimedean and Laves lattices. Edges of the diagram indicate covering relationships, in which the lattice higher in the diagram is a subgraph of the lattice lower in the diagram. Additional subgraph relationships, valid by transitivity, are implied, but not shown.

	$(3,12^2)$	$(4,6,12)$	$(4,8^2)$	(6^3)	$D(3^2,4,3,4)$	$D(3^3,4^2)$	$D(3^4,6)$	$(3,6,3,6)$	$(3,4,6,4)$	(4^4)	$D(3,4,6,4)$	$D(3,6,3,6)$	$(3^4,6)$	$(3^3,4^2)$	$(3^2,4,3,4)$	(3^6)	$D(4,8^2)$	$D(4,6,12)$	$D(3,12^2)$
$(3,12^2)$		K	K	K	K	K	K	K	K	K	K	K	K	K	K	K	K	K	K
$(4,6,12)$	K		K	K	K	K	K	K	K	K	K	K	K	K	K	K	K	K	K
$(4,8^2)$	K	K		K	K	K	K	K	K	K	K	K	K	K	K	K	K	K	K
(6^3)	K	K	K		K	K	K	K	K	K	K	K	K	K	K	K	K	K	K
$D(3^2,4,3,4)$	ϕ	ϕ	ϕ	C		L	L	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	Δ	Δ	ϕ	Δ
$D(3^3,4^2)$	ϕ	ϕ	ϕ	C	L		L	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	Δ	Δ	ϕ	Δ
$D(3^4,6)$	ϕ	ϕ	ϕ	C	L	L		ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	Δ
$(3,6,3,6)$	+	C	C	C	Φ	Φ	Φ		C	Φ	Φ	Φ	Δ	Φ	Φ	Δ	Δ	Φ	Δ
$(3,4,6,4)$	+	+	S	S	Φ	Φ	Φ	I		Φ	Φ	Φ	Δ	Δ	Φ	Δ	Δ	Φ	Δ
(4^4)	ϕ	+	+	+	χ	χ	χ	χ	χ		L	L	χ	χ	χ	χ	χ	χ	Δ
$D(3,4,6,4)$	ϕ	S	S	+	χ	χ	χ	χ	χ	L		L	χ	χ	χ	χ	χ	χ	Δ
$D(3,6,3,6)$	ϕ	+	+	+	χ	χ	χ	χ	χ	L	L		χ	χ	χ	χ	χ	χ	Δ
$(3^4,6)$	T	T	+	+	Φ	Φ	Φ	S	+	Φ	Φ	Φ		Φ	Φ	Δ	Φ	Φ	Δ
$(3^3,4^2)$	T	T	T	T	+	S	Δ	I	+	+	Δ	Δ	I		I	Δ	Φ	Φ	Δ
$(3^2,4,3,4)$	+	T	T	T	S	S	Δ	S	S	+	Δ	Δ	A	A		Δ	Φ	Φ	Δ
(3^6)	T	T	T	T	T	+	+	+	T	T	S	+	+	+	+		L	L	Δ
$D(4,8^2)$	T	T	T	T	+	+	S	+	+	+	S	S	S	V	V	L		L	Δ
$D(4,6,12)$	+	T	T	T	S	S	S	V	V	V	+	+	V	V	V	L	L		S
$D(3,12^2)$	T	T	T	T	T	T	T	T	T	T	+	T	T	T	T	+	S	S	

+	inclusion
T	transitivity
K	3-connectivity
ϕ	minimum polygon size
Φ	maximum polygon size
C	combining polygons
Δ	maximum degree
V	variation in degree
L	Laves lattice
I	incident polygons
A	adjacent polygons
χ	chromatic number
S	special case

Table 1: All inclusions and non-inclusions among the Archimedean and Laves lattices. Each entry indicates if the lattice listed at the top is included in the lattice listed at the left, and if not, indicates the reasoning that proves non-inclusion. The key at the right provides an interpretation for each symbol used in the table. A "+" indicates an inclusion which is demonstrated in the figures in Section 4. A "T" indicates an inclusion which is valid by transitivity. An "S" indicates that non-inclusion is proved in a special argument given in Section 6. All other symbols refer to a lemma or method, named by the symbol, in Section 5 for proving non-inclusion.

case sufficiently large induced subgraphs of the graphs are shown to demonstrate that the inclusion relationship can be extended throughout the infinite graphs.

Transitivity implies the remaining 37 inclusions, denoted by T entries in Table 1. In each case, a sequence of covering relationships may be found in the Hasse diagram to demonstrate the inclusion.

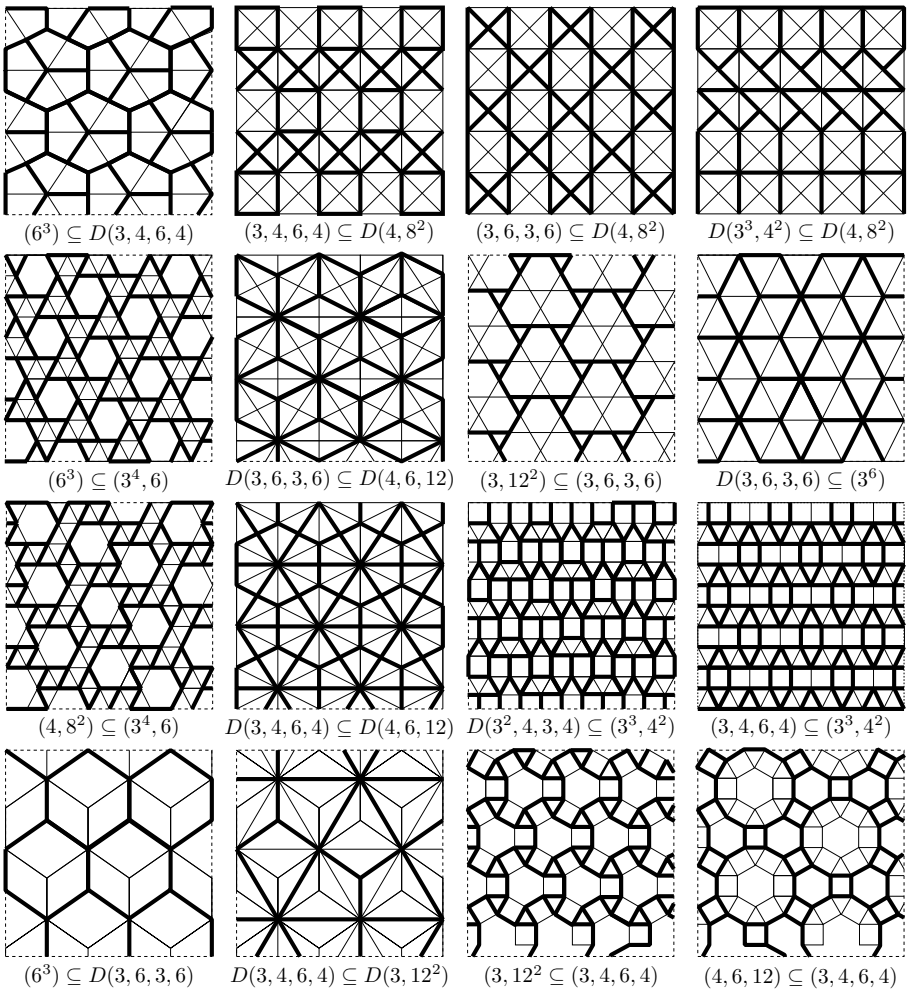

Figure 3: 16 Inclusions. Each drawing shows that one Archimedean or Laves lattice is a subgraph of another. The edges of the subgraph are indicated by thicker lines.

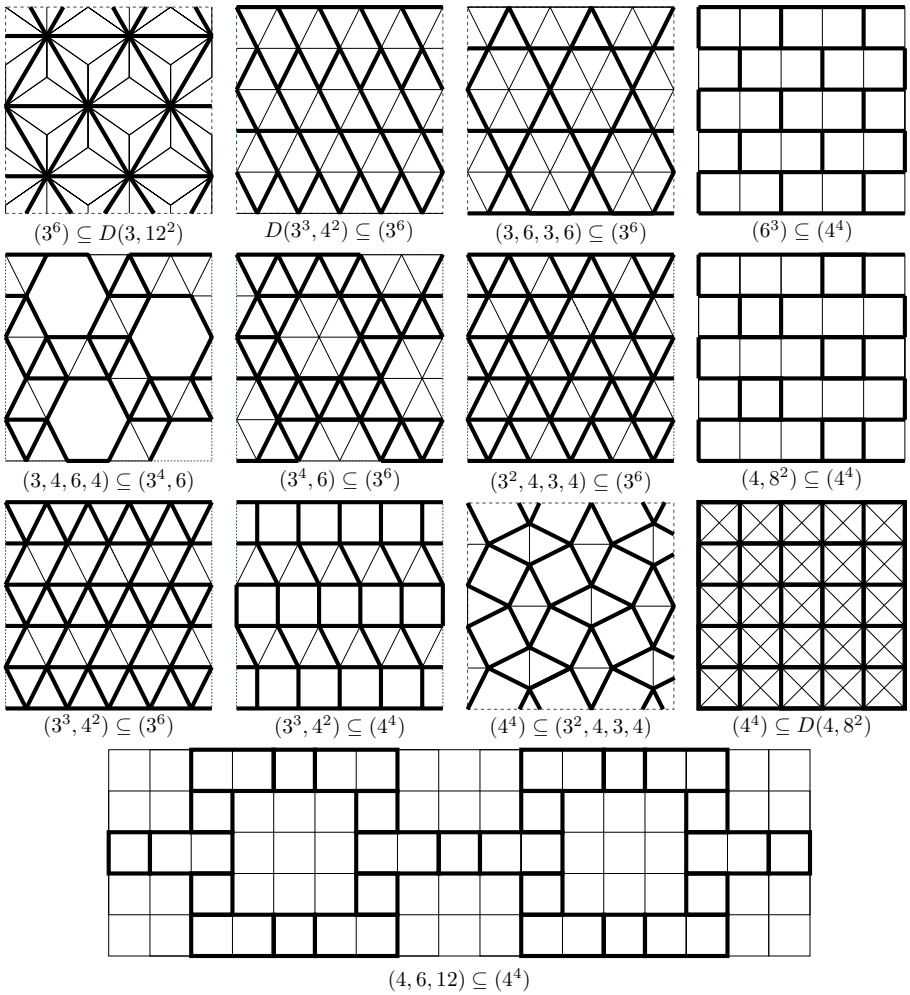

Figure 4: 13 Inclusions. Each drawing shows that one Archimedean or Laves lattice is a subgraph of another. The edges of the subgraph are indicated by thicker lines.

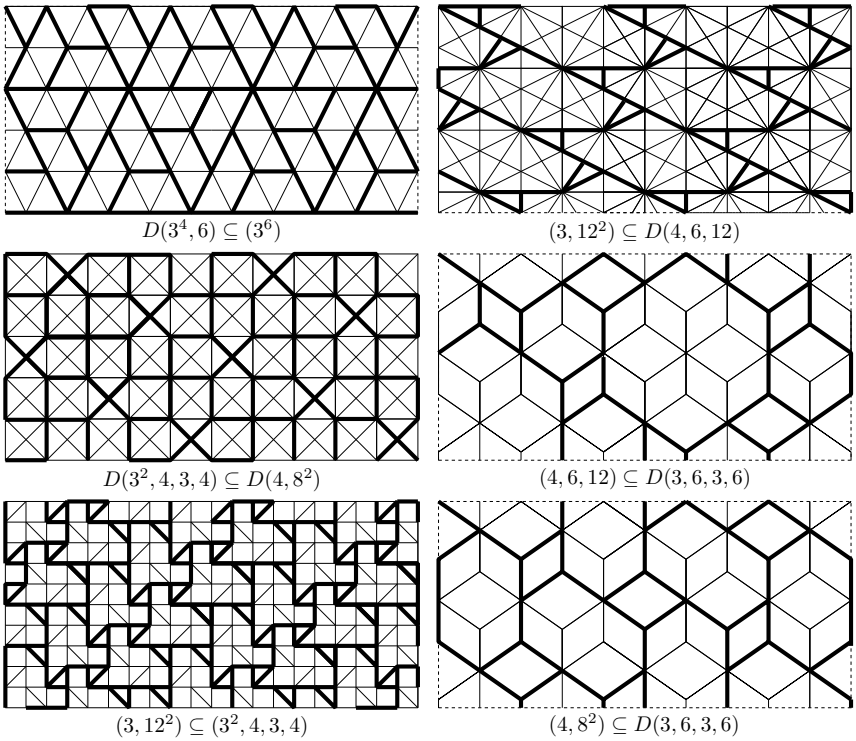

Figure 5: 6 Inclusions. Each drawing shows that one Archimedean or Laves lattice is a subgraph of another. The edges of the subgraph are indicated by thicker lines.

5 Non-inclusion Proofs

This section establishes several conditions under which one lattice cannot be a subgraph of another. While these allow us to prove the majority of the non-inclusion results for the Archimedean and Laves lattices, there are still a number of special cases which require individualized reasoning, which are discussed in Section 6.

Let $\mathcal{A}$ denote the set of Archimedean lattices and $\mathcal{L}$ denote the set of Laves lattices. Note that $\mathcal{A} \cap \mathcal{L} = \{(4^4), (3^6), (6^3)\}$.

We use the symbol $\subseteq$ to denote the inclusion relationship, letting $G \subseteq H$ denote that G is isomorphic to a subgraph of H. In many cases we will write as if H is a subgraph of G, rather than a separate graph on a different set of vertices. To create subgraphs of a given graph, we will delete vertices and edges. When we refer to deleting a vertex, we mean that the vertex and all edges incident to it are deleted from the graph.

For any graph G, let $V(G)$ and $E(G)$ denote the sets of vertices and edges of G, respectively. Let $d_G(v)$ denote the degree of vertex v in graph G. A vertex

with degree n will be called an n-vertex.

For a graph G, denote the maximum degree by $\Delta(G) = \max_{v \in G} d_G(v)$ and the minimum degree by $\delta(G) = \min_{v \in G} d_G(v)$.

Since $G \in \mathcal{A} \cup \mathcal{L}$ is a 3-connected planar graph, it has a unique dual graph G^* by Whitney's 2-Isomorphism Theorem [32]. Consequently, any plane representation of G determines a set of faces of G, denoted $F(G)$, and each vertex $v \in V(G^*)$ corresponds to a unique face $f \in F(G)$.

We will refer to a face as a polygon, to the number of sides of a polygon as the *size* of the polygon, and to a polygon of size k as a k-gon. For a graph G, we denote the maximum polygon size by $\Phi(G)$ and the minimum polygon size by $\phi(G)$. Similarly, we will refer to cycle of length k as a k-cycle. The term polygon is only used for cycles with empty interior.

Similarly to the notation for Archimedean lattices, we will say a vertex is of type $(a_1, a_2, \ldots)$ if successive faces around the vertex are of size a_1, a_2, etc. A polygon is said to be adjacent to each of its vertices. An edge of a polygon is said to be adjacent to the polygon.

5.1 Degree

Lemma Δ: If $\Delta(H) > \Delta(G)$, then $H \not\subseteq G$.

Proof: If $v \in H \subseteq G$, then $d_H(v) \leq d_G(v)$. Thus, $H \subseteq G$ implies that $\Delta(H) \leq \Delta(G)$.

This criterion is easy to check quickly, and is used in many cases to show that an Archimedean lattice cannot contain a Laves lattice as a subgraph.

Approach V: For several pairs of lattices, there is a large disparity between the maximum degrees, so that lowering the large degree in one lattice creates faces of a larger size than exist in the other lattice. We give an individual argument for each case in the following:

Examples: $(4^4) \not\subseteq D(4,6,12)$. To obtain (4^4) by deletions from $D(4,6,12)$, each 12-vertex in $D(4,6,12)$ must either be deleted or have its degree lowered to 4. Deleting the vertex creates an n-gon with $n \geq 12$, which does not exist in (4^4), so the degree must be lowered by deleting 8 incident edges. However, since each edge-deletion increases an incident polygon by at least one size, only 4 edges can be deleted to create the 4 4-gons needed to create (4^4).

$(3,6,3,6) \not\subseteq D(4,6,12)$. To create 2 3-gons and 2 6-gons around a 12-vertex in $D(4,6,12)$, at most 6 incident edges can be deleted. However, exactly 8 must be deleted to reduce the degree to 4.

$(3^4,6) \not\subseteq D(4,6,12)$. To create 4 3-gons and one 6-gon around a 12-vertex in $D(4,6,12)$, only 3 incident edges can be deleted, but exactly 7 must be to reduce the degree to 5.

$(3,4,6,4) \not\subseteq D(4,6,12)$. At a 12-vertex in $D(4,6,12)$, at most 5, but exactly 8 incident edges must be deleted.

$(3^3, 4^2) \not\subseteq D(4,6,12)$ and $(3^2, 4, 3, 4) \not\subseteq D(4,6,12)$. At a 12-vertex in $D(4,6,12)$, at most 2, but exactly 7 incident edges must be deleted.

$D(4,8^2) \not\subseteq D(4,6,12)$. $D(4,8^2)$ has 8-vertices surrounded by 3-gons. Deletion of any edge from a 12-vertex in $D(4,6,12)$ would create a larger polygon, but 4 edges must be deleted to reduce the degree to 8.

$(3^3, 4^2) \not\subseteq D(4,8^2)$ and $(3^2, 4, 3, 4) \not\subseteq D(4,8^2)$. $D(4,8^2)$ has 8-vertices surrounded by 3-gons. 3 incident edges must be deleted, but only 2 edges can be without creating more or larger polygons than 3 3-gons and 2 4-gons.

5.2 Chromatic Number

A proper vertex coloring of G is a coloring of the vertices of G in which no two adjacent vertices have the same color. The vertex-chromatic number of G, denoted $\chi(G)$, is the minimum number of colors used in any proper coloring.

Lemma χ: If $\chi(H) > \chi(G)$, then $H \not\subseteq G$.

Proof: If C is a proper coloring of G, and $H \subset G$, then the restriction of C to H is a proper coloring of H. Thus, $H \subset G$ implies $\chi(H) \leq \chi(G)$.

Shrock and Tsai [22] give the vertex-chromatic numbers of all Archimedean and Laves lattices. Among these, the only lattice with vertex-chromatic number 4 is $D(3,12^2)$. The lattices (4^4), (6^3), $(4,6,12)$, $(4,8^2)$, D$(3,4,6,4)$, and D$(3,6,3,6)$ have vertex-chromatic number 2, and the remaining lattices have vertex-chromatic number 3. Letting L_k denote any lattice with vertex-chromatic number k, we have the following non-inclusions:

$$D(3,12^2) \not\subseteq L_3 \not\subseteq L_2.$$

Remark: Similar reasoning is valid for the edge-chromatic number. However, for all $G \in \mathcal{A} \cup \mathcal{L}$, the edge-chromatic number is equal to the maximum degree, so no additional information is gained from edge-chromatic number.

5.3 Edge-Connectivity

A disconnecting set of edges is a set $F \subseteq E(G)$ such that the subgraph of G with edge set $E(G) - F$ has more than one component. A graph is k-edge connected if every disconnecting set has at least k edges. The edge-connectivity of G, denoted $K(G)$, is the minimum size of a disconnecting set, or, equivalently, the maximum k such that G is k-edge-connected.

Each $G \in \mathcal{A} \cup \mathcal{L}$ is 3-edge connected. In particular, each graph $G \in \mathcal{A} \cup \mathcal{L}$ has minimum vertex degree greater than or equal to three. If a vertex or edge is deleted, the remaining graph may have vertices of degree 2 or less. If so, we may continue by repeatedly deleting all vertices of degree two or less (and edges incident to these vertices) until the resulting graph has minimum degree 3 or larger. We call this process 3-deletion, and denote the graph obtained by 3-deletion of an edge e from G by $(G-e)_{(3)}$.

Lemma K: Let $G, H \in \mathcal{A} \cup \mathcal{L}$. If $(G-e)_{(3)} = \emptyset$ for every $e \in G$, then $H \not\subseteq G$.

Proof: Suppose $H \subset G$ is 3-edge connected. If an edge $e \in G - H$ were deleted, then $H \subset (G-e)_{(3)}$, since all vertices $v \in H$ have $d_H(v) \geq 3$. Thus, if $(G-e)_{(3)} = \emptyset$ for every $e \in G$, then $G \not\supseteq H \in \mathcal{A} \cup \mathcal{L}$.

Since the hexagonal, $(3, 12^2)$, $(4, 8^2)$, $(4, 6, 12)$ lattices are regular with degree three, by Lemma K none can contain any of the other Archimedean or Laves lattices.

5.4 Polygon Size

Polygon sizes in the Archimedean and Laves lattices satisfy a monotonicity property: Let H be constructed from $G \in \mathcal{A} \cup \mathcal{L}$ by deleting a set of vertices and edges. If F_H denotes the face of H containing the face F in G, then the size of F_H is greater than or equal to the size of F. [Since the Archimedean and Laves lattices are all periodic, one only needs to check a sufficiently large bounded region to verify this property.] Note that deleting a set of edges which includes an edge of a face need not strictly increase the size of the face: A triangular face may be obtained by deleting edges in the $(3, 12^2)$ lattice.

Lemma ϕ: Let $G, H \in \mathcal{A} \cup \mathcal{L}$. If $\phi(H) < \phi(G)$, then $H \not\subseteq G$.

Proof: If $H \subseteq G$, then every face of G is entirely contained in some face of H. Since $G \in \mathcal{A} \cup \mathcal{L}$, deletion of vertices or edges does not decrease the polygon size, and the face of H has a larger size. Taking the minimum over all faces of G, $\phi(H) \geq \phi(G)$.

Lemma Φ: Let $G, H \in \mathcal{A} \cup \mathcal{L}$. If $\Phi(H) < \Phi(G)$, then $H \not\subseteq G$.

Proof: If $H \subseteq G$, then every face of G is contained in a face of H. Since $G \in \mathcal{A} \cup \mathcal{L}$, deletion of vertices or edges does not decrease the polygon size, and the face of H has an equal or larger size. Taking the maximum over all faces of G, $\Phi(H) \geq \Phi(G)$.

We will say that two polygonal faces are *incident* if they share a common vertex but have no common edge. Let $I(G)$ denote the maximum number of $\phi(G)$-gons incident to any $\phi(G)$-gon in G.

Lemma I: Let $G, H \in \mathcal{A} \cup \mathcal{L}$, $G \neq D(3, 12^2)$. If $\phi(H) = \phi(G)$ and $I(H) > I(G)$, then $H \not\subseteq G$.

Proof: Let F be a face of size $\phi(G)$ in G which has $I(G)$ incident faces of size $\phi(G)$. Deletion of vertices and edges cannot create any additional faces of size $\phi(G)$. If H is obtained by deletion of vertices or edges of G, the face F may not remain in H, or the face F may remain, in which case the number of faces of size $\phi(H) = \phi(G)$ is less than or equal to $I(G)$. Taking the maximum over all

faces of size $\phi(H)$, noting that there may be more faces of size $\phi(G)$ in G than in the subgraph H, we have $I(H) \leq I(G)$.

Remark: The reasoning above is not valid when $G = (3, 12^2)$, since it contains 3-cycles which may become 3-gons in a subgraph. However, since $I(G) = 18$, the hypothesis $I(H) > I((3, 12^2))$ is not satisfied for any $H \in \mathcal{A} \cup \mathcal{L}$, so no conclusion could be drawn anyway.

Examples: Each 3-gon in $(3^3, 4^2)$ is incident to 2 others, but is incident to 3 in $(3, 6, 3, 6)$, so

$$(3, 6, 3, 6) \not\subseteq (3^3, 4^2).$$

Each 3-gon in $(3^3, 4^2)$ is incident to 2 others, but is incident to 3 in $(3^4, 6)$, so

$$(3^4, 6) \not\subseteq (3^3, 4^2).$$

Each 3-gon in $(3, 4, 6, 4)$ is incident to no others, but is incident to 3 in $(3, 6, 3, 6)$, so

$$(3, 6, 3, 6) \not\subseteq (3, 4, 6, 4).$$

Each 3-gon in $(3^3, 4^2)$ is incident to 2, but is incident to 4 in $(3^2, 4, 3, 4)$, so

$$(3^2, 4, 3, 4) \not\subseteq (3^3, 4^2).$$

We will say that two polygonal faces are adjacent if they share a common edge. Let $A(G)$ denote the maximum number of $\phi(G)$-gons that are adjacent to a $\phi(G)$-gon in G.

Lemma A: Let $G, H \in \mathcal{A} \cup \mathcal{L}$, $G \neq D(3, 12^2)$. If $\phi(H) = \phi(G)$ and $A(H) > A(G)$, then $H \not\subseteq G$.

Proof: Let F be a face of size $\phi(G)$ in G which has $A(G)$ adjacent faces of size $\phi(G)$. Deletion of vertices and edges cannot create any additional faces of size $\phi(G)$. If H is obtained by deletion of vertices or edges of G, the face F may not remain in H, or the face F may remain, in which case the number of adjacent faces of size $\phi(H) = \phi(G)$ is less than or equal to $A(G)$. Taking the maximum over all faces of size $\phi(H)$, noting that there may be more faces of size $\phi(G)$ in G than in the subgraph H, we have $A(H) \leq A(G)$.

Examples: Each 3-gon in $(3^2, 4, 3, 4)$ is adjacent to one other, but is adjacent to 2 in $(3^3, 4^2)$, so

$$(3^3, 4^2) \not\subseteq (3^2, 4, 3, 4).$$

Each 3-gon in $(3^2, 4, 3, 4)$ is adjacent to one other, but is adjacent to 2 or more in $(3^4, 6)$, so

$$(3^4, 6) \not\subseteq (3^2, 4, 3, 4).$$

Our next condition involves the possible sizes of unions of polygons.

Lemma C : Suppose H contains k-gons and G does not. If deleting edges from any n-gons in G with $n < k$ produces only n-gons with $n > k$, then $H \not\subseteq G$.

Proof: Suppose H is a subgraph of G which contains a face F of size k. F is not a face of G, so it must be a union of faces $F_1, F_2, F_3, \ldots, F_l$, where $l \geq 2$. However, by hypothesis, a polygon created by such a union has at least $n > k$ edges. Thus, no subgraph of G can have a face of size k.

Examples: The following examples illustrate the application of Lemma C: Deleting edges of 3-gons in $(3, 6, 3, 6)$ gives 7-gons or larger, not 4- or 6-gons, so

$$(4, 6, 12) \not\subseteq (3, 6, 3, 6).$$

Deleting edges of 3-gons in $(3, 6, 3, 6)$ gives 7-gons or larger, not 4-gons, so

$$(4, 8^2) \not\subseteq (3, 6, 3, 6).$$

Deleting edges of 3-gons in $(3, 6, 3, 6)$ gives 7-gons or larger, and not 6-gons, so

$$(6^3) \not\subseteq (3, 6, 3, 6).$$

Deleting edges of 3-gons in $(3, 6, 3, 6)$ gives 7-gons or larger, and not 4- or 6-gons, so

$$(3, 4, 6, 4) \not\subseteq (3, 6, 3, 6).$$

Deleting edges in (6^3) gives 10-gons or larger, not 5-gons, so

$$(6^3) \not\subseteq D(3^2, 4, 3, 4),$$

$$(6^3) \not\subseteq D(3^3, 4^2),$$

and

$$(6^3) \not\subseteq D(3^4, 6).$$

Lemma L: Let $G, H \in \mathcal{L}$, $G, H \neq D(3, 12^2)$. If $\phi(G) = \phi(H)$, then $G \not\subseteq H$ and $H \not\subseteq G$.

Proof: Since G and H are Laves lattices, all their faces are the same size, say k. Deletion of vertices or edges produces only n-gons, for $n > k$. So neither lattice can be obtained from the other by deletion. Thus, if they are not isomorphic, G and H are incomparable. It is easily checked that all pairs of Laves lattices with a common face size are non-isomorphic.

Remark: Note that $D(3, 12^2)$ could not be included in the group of fully-triangulated lattices in the previous paragraph, because vertices and edges can be deleted to obtain different lattices which are still fully-triangulated. In fact, $D(3, 12^2)$ does contain (3^6) as a subgraph!

6 Non-Inclusion Proofs for Special Cases

In this section, we provide non-inclusion proofs for several cases which are not covered by the methods in Section 5. These cases are indicated by an "S" in Table 1.

The form of the typical reasoning for these proofs is as follows: First, we identify a particular structure in the graph H, for instance a cycle with specified degree sequence. (The degree sequence of a cycle or path is the degrees of the vertices as the cycle or path is transversed.) We then show that this structure does not exist in the graph G, nor can be created by deletions from G. Therefore, we conclude that H is not isomorphic to a subgraph of G.

6.1 $D(4,6,12) \not\subseteq D(3,12^2)$

In $D(4,6,12)$, all 3-paths with degree sequence (6,4,6) are in the interior of 8-cycles. In $D(3,12^2)$, there are only 3-vertices and 12-vertices, so we must delete edges from 12-vertices to obtain 4-vertices and 6-vertices. However, the shortest cycle in $D(3,12^2)$ that surrounds any three 12-vertices has length 9, so $D(4,6,12)$ cannot be a subgraph of $D(3,12^2)$.

6.2 $D(4,8^2) \not\subseteq D(3,12^2)$

In $D(4,8^2)$, each 4-vertex is adjacent to every vertex of a surrounding 4-cycle. In $D(3,12^2)$, the only vertices in the interior of a 4-cycle are 3-vertices. Therefore, $D(4,8^2)$ cannot be a subgraph of $D(3,12^2)$.

6.3 $D(3,4,6,4) \not\subseteq (3^6)$

In $D(3,4,6,4)$, there are 8-cycles which surround 3 vertices. The shortest cycle in 3^6 which has 3 vertices in the interior has length 9, so $D(3,4,6,4)$ cannot be included in (3^6).

6.4 $(6^3) \not\subseteq (3,4,6,4)$

We claim that we cannot create vertices of type 6^3 by deleting edges from $(3,4,6,4)$. Edges of a 6-gon in $(3,4,6,4)$ cannot be deleted, since an n-gon of larger size would be created. New 6-gons can only be created by deleting two opposite edges of a 4-gon. A vertex of a new 6-gon can only be of type $(6^2,4)$, $(6^2,5)$, $(6^2,7)$, or $(6^2,k)$ where $k>7$, but not of type 6^3. See Figure 6.

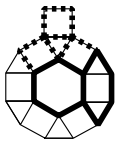

Figure 6: $(6^3) \not\subseteq (3,4,6,4)$

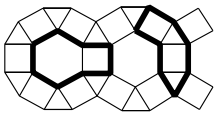

Figure 7: $(4,8^2) \not\subseteq (3,4,6,4)$

6.5 $(4, 8^2) \not\subseteq (3, 4, 6, 4)$

In $(4, 8^2)$, every other edge of an 8-gon is adjacent to a 4-gon. New 4-gons cannot be created in $(3, 4, 6, 4)$ by deletion. There are only two ways to create an 8-gon from $(3, 4, 6, 4)$, either by deleting an edge in a 6-gon, or by combining 2 3-gons and 2 4-gons. See Figure 7. In neither case does the resulting 8-gon have the property that every other edge is adjacent to a 4-gon.

6.6 $(3, 6, 3, 6) \not\subseteq (3^4, 6)$

In $(3, 6, 3, 6)$ each edge in a 6-gon is adjacent to a 3-gon. If any edge in $(3^4, 6)$ is deleted, either this property is violated for an existing 6-gon or a polygon with more than 6 edges is created.

6.7 $(3, 6, 3, 6) \not\subseteq (3^2, 4, 3, 4)$

In $(3, 6, 3, 6)$, each edge in a 6-gon is adjacent to a 3-gon. To obtain a 6-gon in $(3^2, 4, 3, 4)$, a 4-gon must be enlarged by deletion of 2 edges, or by deletion of one edge and combination with two 3-gons adjacent to each other. However, the resulting 6-gons have at least one edge which is adjacent to a 4-gon or larger.

6.8 $(3, 4, 6, 4) \not\subseteq (3^2, 4, 3, 4)$

In $(3, 4, 6, 4)$, each edge in a 6-gon is adjacent to a 4-gon, and each 6-gon is incident to 6 3-gons. To obtain a 6-gon in $(3^2, 4, 3, 4)$, a 4-gon must be enlarged by deletion of 2 edges, or by deletion of one edge and combination with two 3-gons adjacent to each other. If 2 edges are deleted, the resulting hexagon can be incident to at most 4 3-gons. If the 4-gon is combined with two 3-gons adjacent to each other, and each edge of the resulting 6-gon is adjacent to a 4-gon, then there are no incident 3-gons.

6.9 $D(3^3, 4^2) \not\subseteq (3^2, 4, 3, 4)$ and $D(3^2, 4, 3, 4) \not\subseteq (3^2, 4, 3, 4)$

In $D(3^3, 4^2)$ there are vertices of type 5^4 and all faces are 5-gons. A vertex of type 5^4 can be obtained from $(3^2, 4, 3, 4)$ by deleting edges in only one way, shown in Figure 8. However, this configuration can not be extended to a graph in which all faces are 5-gons, because the 3-gon which is adjacent to 2 5-gons can be enlarged only to a 4-gon, a 6-gon, or a k-gon with $k > 6$.

The same reasoning is valid for $D(3^2, 4, 3, 4) \not\subseteq (3^2, 4, 3, 4)$, since $D(3^2, 4, 3, 4)$ also has vertices of type 5^4 and all its faces are 5-gons.

6.10 $D(3^3, 4^2) \not\subseteq (3^3, 4^2)$

Two types of 5-gons may be created in $(3^3, 4^2)$, by combining three 3-gons, or by combining a 3-gon and a 4-gon. Both types must occur in any subgraph of only 5-gons. Checking all possible arrangements of adjacent 5-gons verifies that

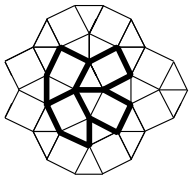

Figure 8: $D(3^3, 4^2) \not\subseteq (3^2, 4, 3, 4)$

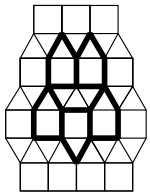

Figure 9: $D(3^3, 4^2) \not\subseteq (3^3, 4^2)$

a 5-gon of the first type cannot have degree sequence $(3, 3, 3, 4, 4)$, as 5-gons in $D(3^3, 4^2)$ have. One arrangement is shown in Figure 9.

6.11 $(4, 6, 12) \not\subseteq D(3, 4, 6, 4)$

In $(4, 6, 12)$, each 12-gon has 12 adjacent 4- and 6-gons and no incident polygons, and is connected by one edge to another 12-gon. Of all possible types of 12-cycles in $D(3, 4, 6, 4)$, only one can have 12 adjacent 4- and 6-gons and no incident polygons, but two disjoint 12-gons of this type cannot be connected by one edge. See Figure 10.

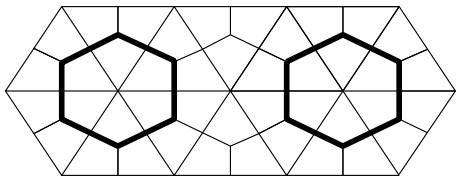

Figure 10: $(4, 6, 12) \not\subseteq D(3, 4, 6, 4)$

6.12 $(4, 8^2) \not\subseteq D(3, 4, 6, 4)$

In $(4, 8^2)$, each 8-gon has all vertices of degree 3 and edges alternatingly adjacent to 4-gons and 8-gons. There are only two types of 8-cycles in $D(3, 4, 6, 4)$ which can satisfy this property. From an 8-gon of either type, constructing the adjacent polygons to satisfy the property necessarily produces the structure shown in Figure 11 (which also illustrates the two types of 8-gons possible). However, the construction cannot be extended in a way that satisfies the property, in areas marked with dotted arrows in the Figure.

6.13 $D(3^4, 6) \not\subseteq D(4, 8^2)$

In $D(3^4, 6)$, there exist 6-vertices, and each pair of adjacent 5-gons share only one edge. A 6-vertex can be obtained from $D(4, 8^2)$ only by deleting 2 edges incident to an 8-vertex. There are 6 different types of pairs of edges to delete, but only one type results in each pair of adjacent 5-gons sharing only one edge:

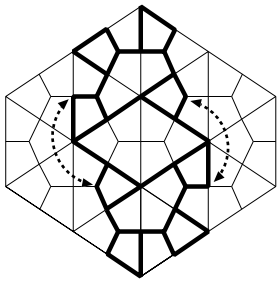

Figure 11: $(4, 8^2) \not\subseteq D(3, 4, 6, 4)$

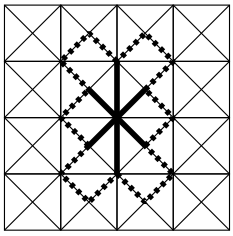

Figure 12: $D(3^4, 6) \not\subseteq D(4, 8^2)$

delete opposite edges which are incident to 8-vertices. However, since the other 6 edges must be retained, 4-gons will remain in the graph. See Figure 12.

6.14 $D(3, 4, 6, 4) \not\subseteq D(4, 8^2)$

In $D(3, 4, 6, 4)$, there are 6-vertices which are adjacent to only 4-vertices. A 6-vertex can be obtained from $D(4, 8^2)$ only by deleting 2 edges incident to an 8-vertex, while retaining the other 6 incident edges. Thus, at least two edges from the 8-vertex to 4-vertices must be retained. Since 4 additional edges must be retained, the resulting graph must contain 3-gons (See Figure 13), which do not exist in $D(3, 4, 6, 4)$.

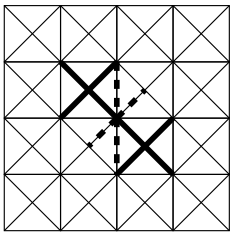

Figure 13: $D(3, 4, 6, 4) \not\subseteq D(4, 8^2)$

6.15 $(3^4, 6) \not\subseteq D(4, 8^2)$

In $(3^4, 6)$, all 3-gons contain only 5-vertices, but in $D(4, 8^2)$, the 3-gons have degree sequence $(4, 8, 8)$. Since deletions cannot create new triangles, or raise the degree of the 4-vertices in $D(4, 8^2)$, $(3^4, 6)$ cannot be a subgraph of $D(4, 8^2)$.

6.16 $D(3, 6, 3, 6) \not\subseteq D(4, 8^2)$

In $D(3, 6, 3, 6)$, each 3-vertex is in the interior of a 6-cycle with degree sequence (3,6,3,6,3,6). In $D(4, 8^2)$, no 8-vertex is in the interior of any 6-cycle. Thus,

to obtain $D(3,6,3,6)$ from $D(4,8^2)$ by deletion, the 3-vertices in $D(3,6,3,6)$ must be obtained from 4-vertices in $D(4,8^2)$, which is possible only if $D(4,8^2)$ contains 6-cycles around 4-vertices with 4-vertices at every other vertex. There are two different types of 6-cycles around 4-vertices in $D(4,8^2)$, but each type includes four 8-vertices, so $D(3,6,3,6)$ is not a subgraph of $D(4,8^2)$.

6.17 $D(3,12^2) \not\subseteq D(4,6,12)$

$D(3,12^2)$ contains a 3-gon consisting of 12-vertices. $D(4,6,12)$ does not contain such a 3-gon, and since $\Delta((4,6,12)) = 12$, none can be created by deletions.

6.18 $D(3^4,6) \not\subseteq D(4,6,12)$

In $D(3^4,6)$, around any 6-vertex there is an 18-cycle which separates it from all other 6-vertices.

To obtain $D(3^4,6)$ from $D(4,6,12)$ by deletions, first observe that the 6-vertices in $D(4,6,12)$ cannot become 6-vertices in $D(3^4,6)$: Since all vertices adjacent to the original 6-vertex must become 3-vertices, 3-gons would be retained. Thus, 6-vertices in $D(3^4,6)$ must be obtained by deleting edges incident to 12-vertices in $D(4,6,12)$.

Next, note that a 6-vertex in $D(3^4,6)$ is connected to 6 other 6-vertices by paths of length 3. These 6-vertices in $D(3^4,6)$ must be obtained from a 12-vertex and the nearest 6 12-vertices in $D(4,6,12)$. However, as shown in Figure 14, the only cycle separating the 6-vertex from the other 6-vertices in $D(3^4,6)$ is a 12-cycle, not an 18-cycle.

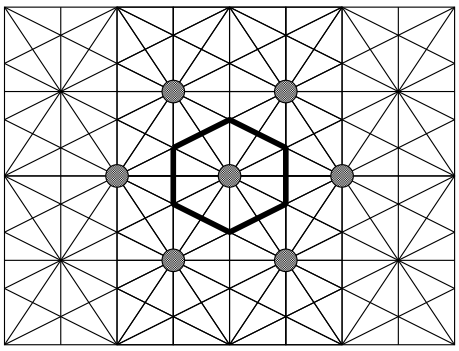

Figure 14: $D(3^4,6) \not\subseteq D(4,6,12)$

6.19 $D(3^3,4^2) \not\subseteq D(4,6,12)$

$D(3^3,4^2)$ contains doubly-infinite paths of 4-vertices. To obtain $D(3^3,4^2)$ from $D(4,6,12)$ by deletion, first note that 4-vertices in $D(4,6,12)$ cannot become 4-vertices in $D(3^3,4^2)$, since this would create either triangles or 4-gons, as

shown in Figure 15. Thus, a doubly-infinite path of 4-vertices must correspond to a path with degree sequence $(\ldots, 12, 6, 12, 6, \ldots)$ in $D(4, 6, 12)$. However, this implies that the resulting graph contains 3-gons. See Figure 15 again, in which the vertices marked with circles are required have degree 3 in $D(3^3, 4^2)$.

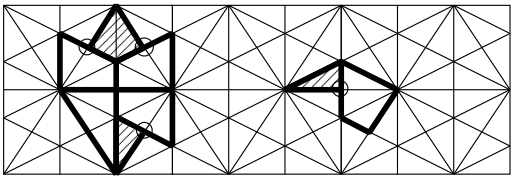

Figure 15: $D(3^3, 4^2) \not\subseteq D(4, 6, 12)$

6.20 $D(3^2, 4, 3, 4) \not\subseteq D(4, 6, 12)$

In $D(3^2, 4, 3, 4)$, each edge that connects two 3-vertices is in the interior of a 10-cycle. In $D(4, 6, 12)$, there are 3 classes of edges; those connecting vertices of degree 12 and 4, 12 and 6, and 6 and 4. Of these, only the last are in the interior of a 10-cycle. Consider the possible configurations of edges adjacent to such an edge, as shown in Figure 16, to see that all require the retention of 3-gons or 4-gons if the construction is extended.

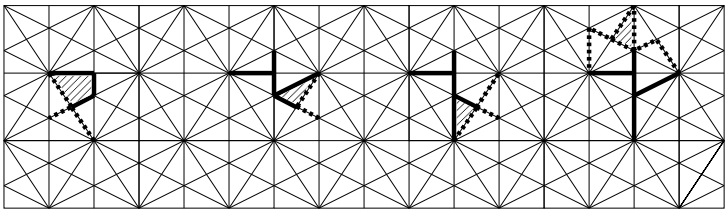

Figure 16: $D(3^2, 4, 3, 4) \not\subseteq D(4, 6, 12)$

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