COMPLEX INTERPOLATION OF COMPACT OPERATORS. AN UPDATE.

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Abstract. After 41 years it is still not known whether an operator acting on a Banach pair and which acts compactly on one or both of the "endpoint" spaces also acts compactly on the complex interpolation spaces generated by the pair. We report some recent steps towards solving this and related problems.

1. Introduction.

The year 1964 saw the appearance of two remarkable and fundamental papers in the theory of interpolation spaces. Jacques–Louis Lions and Jaak Peetre [11] introduced their "real method" spaces $(A_0, A_1)_{\theta,p}$ and Alberto Calderón [1] introduced his "complex method" spaces $[A_0, A_1]_\theta$. Both papers provide numerous important results about their respective interpolation spaces, including some compactness theorems.

Now, more than 41 years after the appearance of these papers, we are still unable to answer the following very natural question which is asked implicitly in Calderón's paper, and answered affirmatively there in an important special case.

Question C: *Suppose that A_0 and A_1 are compatible Banach spaces, i.e., they form a Banach pair, and that so are B_0 and B_1. Suppose that $T : A_0 + A_1 \to B_0 + B_1$ is a linear operator such that $T : A_0 \to B_0$ compactly and $T : A_1 \to B_1$ boundedly. Does it follow that T maps the complex interpolation space $[A_0, A_1]_\theta$ into the compact interpolation space $[B_0, B_1]_\theta$ compactly for each $\theta \in (0, 1)$?*

We cannot even answer the following question which could be expected to be somewhat easier.

Question C_2: *This is the same as Question C, but under the stronger hypothesis that $T : A_1 \to B_1$ is also compact.*

In addition to the case considered in [1], there are also quite a number of other special cases in which Question C, and therefore also Question C_2 have since been discovered to have affirmative answers. Most of the relevant papers for such results are mentioned on p. 262 of [7] and p. 353 of [8]. We refer also to [2], [12], and the website [5].

By contrast, the analogues of Questions C and C_2 in which the spaces $[A_0, A_1]_\theta$ and $[B_0, B_1]_\theta$ are replaced by $(A_0, A_1)_{\theta,p}$ and $(B_0, B_1)_{\theta,p}$ are apparently somewhat easier to answer. In fact the analogue of Question C_2 was answered affirmatively [9]

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already in 1969, and the analogue of Question C was answered affirmatively [3, 6] in 1992.

In this short note we will report on some recent developments related to Questions C and C_2 and consider possible approaches towards answering them. We will assume that the reader is familiar with most earlier papers treating this topic, and also with the alternative definitions in [10] of complex interpolation spaces via minimal (orbit) and maximal (co-orbit) functors applied to pairs of weighted sequence spaces (FL_0^p, FL_1^p), $p = 1, \infty$.

Since notation varies slightly from paper to paper, we should specify that here, for $p \in [1, \infty]$, we shall let FL^p denote the space of complex sequences $\{\lambda_k\}_{k\in\mathbb{Z}}$ which arise as Fourier coefficients of some element of $L^p(\mathbb{T})$ with the norm induced by the norm of $L^p(\mathbb{T})$. Analogously FC is the closed subspace of FL^∞ of sequences of Fourier coefficients of continuous functions. Then, for each $\alpha \in \mathbb{R}$, we let FL_α^p denote the space of sequences $\{\lambda_k\}_{k\in\mathbb{Z}}$ such that $\left\{e^{\alpha k}\lambda_k\right\}_{k\in\mathbb{Z}} \in FL_\alpha^p$ with the obvious norm, and FC_α is defined analogously. Finally, for any Banach space A, we use the usual notation $\ell^p(A)$ for the space of all A valued sequences $\{a_n\}_{n\in\mathbb{N}}$ for which the norm $\left\|\{a_n\}_{n\in\mathbb{N}}\right\|_{\ell^p(A)} := \left\|\{\|a_n\|_A\}_{n\in\mathbb{N}}\right\|_{\ell^p}$ is finite.

2. Some significant choices of the "range" pair (B_0, B_1).

As proved in [8] (cf. also [6]), the problem of answering Question C can be reduced to the problem of answering any one of a number of special cases of Question C. Among those reductions, the following one, suggested by various ideas in [4] and [10] (cf. Proposition 3 of [8, Proposition 3, p. 356] and Step 1 of the proof of Theorem 2.1 of [6, pp. 339–340]), is particularly relevant for our discussion here:

Proposition 2.1. *In order to answer Question C, it suffices to resolve it in the special case where the "domain" pair* (A_0, A_1) *is* $\ell^1(FL_0^1), \ell^1(FL_1^1))$ *and the "range" pair* (B_0, B_1) *is either* $(\ell^\infty(FL_0^\infty), \ell^\infty(FL_1^\infty))$ *or* $(\ell^\infty(FC_0), \ell^\infty(FC_1))$.

Let us put this into perspective with what is known so far. The case where (A_0, A_1) is an arbitrary pair and (B_0, B_1) is (FC_0, FC_1) can be resolved affirmatively as an immediate corollary of known results. In fact, using Fejér's classical theorem about Fourier series, it is clear that the pair (FC_0, FC_1) satisfies the special approximation condition required for Calderón's partial answer to Question C in [1].

Some months ago we reasoned that, until such time as someone sees how to resolve the case where (B_0, B_1) is enlarged from (FC_0, FC_1) to $(\ell^\infty(FC_0), \ell^\infty(FC_1))$ or to $(\ell^\infty(FL_0^\infty), \ell^\infty(FL_1^\infty))$, a reasonable intermediate step would be to consider the case where $(B_0, B_1) = (FL_0^\infty, FL_1^\infty)$. This pair apparently does not have any of the properties which would enable it to be treated by known theorems which resolve various special cases of Question C.

Recently we have been able to resolve this intermediate case:

Theorem 2.2. *Suppose that* (A_0, A_1) *is an arbitrary Banach pair and that* $T : A_0 + A_1 \to FL_0^\infty + FL_1^\infty$ *is a bounded operator such that* $T : A_0 \to FL_0^\infty$ *compactly and* $T : A_1 \to FL_1^\infty$ *boundedly. Then* $T : [A_0, A_1]_\theta \to [FL_0^\infty, FL_1^\infty]_\theta$ *compactly for each* $\theta \in (0, 1)$.

A paper containing a proof of this result is currently in preparation. We remark that $[FL_0^\infty, FL_1^\infty]_\theta = FC_\theta$. (Cf. [10, 8]).

3. NECESSARY AND SUFFICIENT CONDITIONS IN TERMS OF INFINITE MATRICES MAPPING $(\ell^1(FL_0^1), \ell^1(FL_1^1))$ INTO $(\ell^\infty(FL_0^\infty), \ell^\infty(FL_1^\infty))$.

In the light of Proposition 2.1, one way of trying to answer Question C is to study various properties of the operators which map $\ell^1(FL_j^1)$ into $\ell^\infty(FL_j^\infty)$ for $j = 0, 1$. These operators can be realized as infinite matrices, or rather as infinite matrices each of whose entries is itself an infinite matrix. These matrices were used in Theorem 2.1 of [6, p. 339] to show that if, for any given pairs (A_0, A_1) and (B_0, B_1), Question C has an affirmative answer for one particular value of $\theta \in (0, 1)$, then this implies an affirmative answer for all $\theta \in (0, 1)$ for those pairs.

As time passes it seems that we should give more consideration also to the possibility of a negative answer to Question C. Indeed this possibility is also raised by the result to be mentioned in the next section. In this section we briefly describe how a more careful examination of the above mentioned infinite matrices enables one to formulate questions about them which may ultimately provide an affirmative *or negative* answer to Question C.

Since our exposition here needs to be short, we will refer the reader to Section 2 of [6] as a point of departure and source of notation and more details for much of what we want to say. (Given more space we would have preferred to present these things in a slightly different way.) Thus we are dealing with the spaces E_α and F_α defined, for each $\alpha \in \mathbb{R}$, by $E_\alpha = \ell_U^1(FL_\alpha^1)$ and $F_\alpha = \ell_V^\infty(FL_\alpha^\infty)$, and we are considering an operator T which maps E_j into F_j boundedly for $j = 0, 1$. We also assume that T maps E_θ into F_θ compactly for at least one value θ_* of θ. But here, in contrast to [6], we take $\theta_* = 0$. In [6] the index sets U and V may be uncountable. But, as is clear from [8] (see the proof of Proposition 3 on pp. 356–357), we in fact only need to consider the case where U and V are countable.

As explained in [6], we can realize T via a matrix of operators $\{T_{uv}\}_{u\in U, v\in V}$ where each T_{uv} is a bounded map of FL_α^1 into FL_α^∞ for all $\alpha \in [0, 1]$. More explicitly, we have

$$Tx = \left\{ \sum_{u\in U} T_{uv} x_u \right\}_{v\in V} \quad \text{for each element } x = \{x_u\}_{u\in U} \text{ in } E_\alpha = \ell_U^1(FL_\alpha^1). \tag{1}$$

(Of course there seems to be a typographical error in (1). But this is only because our notation here has been kept consistent with some slightly unsuccessful notation used in [6].) Furthermore, for each fixed $u \in U$ and $v \in V$, the operator T_{uv} can be represented as an infinite matrix (of complex numbers) $\{t_{jk}(u, v)\}_{j,k\in\mathbb{Z}}$. In other words, we can completely specify the action of T in terms of the "matrix of matrices" $\{t_{jk}(u, v)\}_{j,k\in\mathbb{Z}, u\in U, v\in V}$.

Now let T_{uv0} be the operator represented by the diagonal matrix obtained by replacing all non diagonal elements of the preceding matrix by 0, i.e. the matrix $\{\delta_{jk} t_{jk}(u, v)\}_{j,k\in\mathbb{Z}}$. More generally, for each $n \in \mathbb{Z}$, let T_{uvn} be the operator represented by the "n-displaced" diagonal matrix $\{\delta_{j,k+n} t_{jk}(u, v)\}_{j,k\in\mathbb{Z}}$. The next step is to introduce the "diagonal" operator T_n for each fixed $n \in \mathbb{Z}$, which is specified, analogously to above, via the matrix of operators $\{T_{uvn}\}_{u\in U, v\in V}$, i.e. by the "matrix of matrices" $\{\delta_{j,k+n} t_{jk}(u, v)\}_{j,k\in\mathbb{Z}, u\in U, v\in V}$.

It can be shown (cf. [6]) that T_n maps E_α boundedly into F_α for each $\alpha \in \mathbb{R}$ and each $n \in \mathbb{Z}$. Furthermore, if $0 < \alpha < 1$, then the series $\sum_{n\in\mathbb{Z}} T_n$ converges in the norm topology of the Banach space of bounded operators mapping E_α into F_α, and the sum of this series is our original operator T. Thus, if we wish to deduce that $T : E_\theta \to F_\theta$ is compact for some $\theta \in (0,1)$, it will suffice to show that

$$T_n \text{ maps } E_\theta \text{ into } F_\theta \text{ compactly for each } n \in \mathbb{Z}. \qquad (2)$$

In [6] it is shown that (2) holds for *all* $\theta \in (0,1)$ whenever $T : E_\theta \to F_\theta$ is compact for (at least) one value of θ in $(0,1)$. It is also rather straightforward to see, because of the "diagonal" structure of T_n, that the condition (2) is equivalent for all real values of θ, i.e. if it holds for any particular θ then it holds for all $\theta \in \mathbb{R}$. We now point out that all this shows that the answer to Question C, regardless of whether it is yes or no, hinges inavoidably on the question of the compactness of the "diagonal" operators T_n. Furthermore the crucial things happen for values of θ which may be assumed to be arbitrarily close to 0. (Here we are also using the reiteration formula for the complex method.) In fact the answer to Question C is the same as the answer to the following question:

***Question* Δ:** *Suppose that T is an arbitrary compact operator from E_0 into F_0 with the additional property that $T : E_\alpha \to F_\alpha$ is bounded for some $\alpha \neq 0$. Does it follow that the "diagonal" operator T_n maps E_0 into F_0 compactly for all $n \in \mathbb{Z}$?*

Remark 3.1. *By considering compositions of T and of T_n with suitable shift operators, it is not hard to see that the answer to Question Δ is the same as the answer to the corresponding question about T_n for just one value of n, say $n = 0$.*

4. A RELATED QUESTION ABOUT INFINITE FAMILIES OF COMPACT SUBSETS OF ℓ^∞.

In this section we consider a different kind of question. It is a simpler version of the question appearing on page 362 of [8]. Our main reason for considering this question is that, by using arguments similar to those given in [8], it can be shown that an affirmative answer to it would suffice to imply an affirmative answer to Question C_2.

***Question CKM*$_2$:** *Suppose that, for each $\theta \in [0,2\pi)$, we are given a subset $M(e^{i\theta})$ of the unit ball of ℓ^∞. Define the set $M(0)$ to consist of all elements $\alpha = \{\alpha_n\}_{n\in\mathbb{N}}$ of ℓ^∞ which are of the form $\{f_n(0)\}_{n\in\mathbb{N}}$ for some sequence of functions f_n which are continuous on the closed unit disk and analytic in its interior and for which $\left\{f_n(e^{i\theta})\right\}_{n\in\mathbb{N}} \in M(e^{i\theta})$ for each $\theta \in [0,2\pi)$. If $M(e^{i\theta})$ is compact for every $\theta \in [0,2\pi)$, does it follow that $M(0)$ is contained in a compact subset of ℓ^∞?*

In fact an affirmative answer to Question C_2 would also follow from an affirmative answer to a special case of Question CKM_2, in the case where one makes the additional assumption that the sets $M(e^{i\theta})$ are "uniformly compact" on $[0,2\pi)$, i.e., that for each $\varepsilon > 0$ there exists an integer $N(\varepsilon)$ such that, for each $\theta \in [0,2\pi)$, the set $M(e^{i\theta})$ is contained in the union of $N(\varepsilon)$ balls in ℓ^∞ of radius ε.

If the compactness of the set $M(0)$ defined in Question CKM_2 follows when the condition imposed in Question CKM_2 is weakened so that $M(e^{i\theta})$ is assumed to be compact only for every θ in some fixed subset E of $[0,2\pi)$ with positive measure, then this suffices to give an affirmative answer to Question C. In fact we only need to consider sets E of a very particular form.

Here, in contrast to the partial affirmative result of Theorem 2.2, we report a result in a negative direction, namely the following remarkable example obtained by Fedor Nazarov which suggests the possibility of a negative answer to Question CKM_2.

For each $\varepsilon > 0$, there exists a positive integer $N(\varepsilon)$ and a collection of subsets $\left\{M(e^{it})\right\}_{t\in[0,2\pi)}$ of the unit ball such that, for each $t \in [0, 2\pi)$, the set $M(e^{it})$ is contained in the union of $N(\varepsilon)$ balls in ℓ^∞, each of radius ε, but the set $M(0)$, defined as in Question CKM_2, contains a sequence $\{e_n\}_{n\in\mathbb{N}}$ for which $\|e_n - e_m\|_{\ell^\infty} = 1$ for all $m \neq n$.

The details of this example can be found in [5].

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