GRAPH LIMITS AND HEREDITARY PROPERTIES

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Abstract. We give a survey of some general results on graph limits associated to hereditary classes of graphs. As examples, we consider some classes defined by forbidden subgraphs and some classes of intersection graphs, including triangle-free graphs, chordal graphs, cographs, interval graphs, unit interval graphs, threshold graphs, and line graphs.

1. Introduction

We use standard concepts from the theory of graph limits, see e.g. [25; 5; 6; 10] and the recent book by Lovász [24]. (We use here mainly the notation of [10].) $\mathcal{U}$ is the set of all unlabelled graphs (all our graphs are finite and simple), and this is embedded (as a countable, discrete, dense, open) subset of a compact metric space $\overline{\mathcal{U}}$; the complement $\widehat{\mathcal{U}} := \overline{\mathcal{U}} \setminus \mathcal{U}$ is the set of graph limits, which thus itself is a compact metric space. If F and G are graphs, then the *homomorphism number* $t(F,G)$ is the probability that a uniformly random mapping φ of the vertex set $V(F)$ into $V(G)$ is a graph homomorphism, i.e., $u \sim v \implies \varphi(u) \sim \varphi(v)$ for $u, v \in V(F)$; similarly, the *induced subgraph number* $t_{\mathrm{ind}}(F,G)$ is (when $|F| \le |G|$) the probability that a uniformly random injective mapping $V(F) \to V(G)$ is a graph isomorphism onto an induced subgraph, i.e., $u \sim v \iff \varphi(u) \sim \varphi(v)$, The subgraph numbers $t(F,G)$ and the induced subgraph numbers $t_{\mathrm{ind}}(F,G)$ extend from graphs $G \in \mathcal{U}$ to general $G \in \overline{\mathcal{U}}$ by continuity (see the references above for details, and note that F always is a graph, which we regard as fixed). Moreover, the topology of $\overline{\mathcal{U}}$ can be described by these numbers $t(F,G)$ or $t_{\mathrm{ind}}(F,G)$, and a sequence of graphs (G_n) converges to a graph limit $\Gamma \iff |G_n| \to \infty$ and $t(F,G_n) \to t(F,\Gamma)$ for every graph $F \iff |G_n| \to \infty$ and $t_{\mathrm{ind}}(F,G_n) \to t_{\mathrm{ind}}(F,\Gamma)$ for every graph F. Furthermore, a graph limit Γ is uniquely determined by the numbers $t(F,\Gamma)$ (or $t_{\mathrm{ind}}(F,\Gamma)$) for $F \in \mathcal{U}$. (Note that we define the set of graph limits as disjoint from the set of finite graphs, and that a limit of a sequence of graphs with $|G_n| \to \infty$

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thus never is a graph; this is a technical difference from e.g. [25] where some graph limits are identified with graphs. See [10] for further discussion.)

A *graph class* is a subset of the set $\mathcal{U}$ of unlabelled graphs, i.e., a class of graphs closed under isomorphisms. Similarly, a *graph property* is a property of graphs that does not distinguish between isomorphic graphs; there is an obvious 1–1 correspondence between graph classes and graph properties and we will not distinguish between a graph property and the corresponding class. A graph class or property $\mathcal{P}$ is *hereditary* if whenever a graph G has the property $\mathcal{P}$, then every induced subgraph of G also has $\mathcal{P}$; this can be written

$$G \in \mathcal{P} \text{ and } t_{\mathrm{ind}}(F,G) > 0 \implies F \in \mathcal{P}. \tag{1.1}$$

Many examples of hereditary graph classes are given in e.g. [7] and [15].

Example 1.1 (intersection graphs)**.** Many interesting hereditary graph classes are given by various classes of intersection graphs. In general, we consider a collection $\mathcal{A}$ of subsets of some universe and say that a graph G is an $\mathcal{A}$-intersection graph if there exists a collection of sets $\{A_i\}_{i\in V(G)} \in \mathcal{A}$ such that there is an edge $ij \in E(G)$ if and only if $A_i \cap A_j \neq 0$. The class $\mathcal{P}_{\mathcal{A}}$ of all $\mathcal{A}$-intersection graphs is a hereditary graph class, for any $\mathcal{A}$.

Specific examples are the classes of interval graphs, unit interval graphs, circular-arc graphs, circle graphs and permutation graphs studied in [9]. See e.g. [7] and [15] for several further examples.

Example 1.2 (forbidden subgraphs)**.** If $\mathcal{F}$ is a (finite or infinite) family of (unlabelled) graphs, let $\mathcal{U}_{\mathcal{F}}$ be the class of all graphs that do not contain any graph from $\mathcal{F}$ as an induced subgraph: Then $\mathcal{U}_{\mathcal{F}}$ is a hereditary graph class. We will study this type of graph classes in Section 4, where also several examples are given.

Example 1.3 (monotone properties)**.** A property $\mathcal{P}$ is *monotone* if whenever a graph G has the property $\mathcal{P}$, then so does every (not necessarily induced) subgraph of G. In analogy with (1.1), this can be written

$$G \in \mathcal{P} \text{ and } t_{\mathrm{inj}}(F,G) > 0 \implies F \in \mathcal{P}, \tag{1.2}$$

where $t_{\mathrm{inj}}(F,G)$ is the probability that a uniformly random injective mapping $V(F) \to V(G)$ is a graph homomorphism (defined as 0 unless $|F| \leq |G|$).

Obviously, every monotone property is hereditary.

Let $\mathcal{P} \subseteq \mathcal{U}$ be a graph class. We let $\overline{\mathcal{P}} \subseteq \overline{\mathcal{U}}$ be the closure of $\mathcal{P}$ in $\overline{\mathcal{U}}$, and $\widehat{\mathcal{P}} := \overline{\mathcal{P}} \cap \widehat{\mathcal{U}}$ the set of graph limits of graphs in $\mathcal{P}$. Explicitly, $\widehat{\mathcal{P}}$ is the set of graph limits Γ such that there exists a sequence of graphs G_n in $\mathcal{P}$ with $G_n \to \Gamma$. (Note that we also use $\overline{G}$ to denote the complement of a graph G; this should not cause any confusion.)

Remark 1.4. Since $\mathcal{U}$ is open and discrete in $\overline{\mathcal{U}}$, i.e. every element of $\mathcal{U}$ is isolated in $\overline{\mathcal{U}}$, we trivially have $\overline{\mathcal{P}} \cap \mathcal{U} = \mathcal{P}$; thus $\overline{\mathcal{P}} = \mathcal{P} \cup \widehat{\mathcal{P}}$. If Γ is a graph limit, then $\Gamma \in \widehat{\mathcal{P}}$ and $\Gamma \in \overline{\mathcal{P}}$ are equivalent, and we will use both formulations interchangeably.

It seems to be of interest to study the classes $\widehat{\mathcal{P}}$ of graph limits defined by various graph properties. General results are given in Lovász and Szegedy [27] and Hatami, Janson and Szegedy [17]. Some examples have been studied, on a case-by-case basis, in [8] (threshold graphs) and [9] (interval graphs and some related graph classes), and there are many other classes that could be studied; apart from the intrinsic interest of various graph classes, some further general patterns might emerge from the study of individual classes. The purpose of this survey is to collect some general results and remarks and to give a number of examples, in order to stimulate further research. (Most results are known and appear either explicitly or implicitly in the literature; some of the examples seem to be new.)

Some further notions and facts from graph limit theory are recalled in Section 2. In Section 3 we characterize graph limits of hereditary classes of graphs using random graphs. Section 4 studies the case of classes defined by forbidding certain subgraphs, and Section 5 studies classes of intersection graphs. Section 6 treats the notion of random-free graph classes, introduced by Lovász and Szegedy [27]. Sections 7–11 consider some further (rather simple) examples, including (Section 9) graph limit versions of Ramsey's theorem.

We note a simple but useful fact about a trivial case.

Theorem 1.5. *Let $\mathcal{P}$ be an arbitrary graph class. Then the following are equivalent.*

(i) *$\mathcal{P}$ is finite.*
(ii) *There exists n_0 such that $|G| \geq n_0 \implies G \notin \mathcal{P}$.*
(iii) *$\widehat{\mathcal{P}} = \emptyset$.*

Proof. (i) $\iff$ (ii) is obvious, since the set $\{G \in \mathcal{U} : |G| = n\}$ is finite for every n.

(i) $\implies$ (iii): If $\mathcal{P}$ is finite, then $\mathcal{P}$ is a closed set in $\overline{\mathcal{U}}$, so $\overline{\mathcal{P}} = \mathcal{P} \subset \mathcal{U}$ and thus $\widehat{\mathcal{P}} = \overline{\mathcal{P}} \setminus \mathcal{U} = \emptyset$.

(iii) $\implies$ (ii): If (ii) does not hold, then there is a sequence $G_n \in \mathcal{P}$ with $|G_n| \to \infty$. Then some subsequence converges, and its limit is an element of $\widehat{\mathcal{P}}$, so $\widehat{\mathcal{P}} \neq \emptyset$. □

Remark 1.6. We have so far discussed the set $\widehat{\mathcal{P}}$ of all possible limits of sequences of graphs in a class $\mathcal{P}$. Another interesting problem is to take a uniformly random graph G_n in $\mathcal{P}_n := \{G \in \mathcal{P} : |G| = n\}$ and study its asymptotic behaviour. Is there a random graph limit Γ such that $G_n \xrightarrow{\mathrm{d}} \Gamma$, where $\xrightarrow{\mathrm{d}}$ denotes convergence in distribution (as random elements of the metric space $\overline{\mathcal{P}}$)? In particular, does there exist a single graph limit Γ (necessarily in $\widehat{\mathcal{P}}$) such that $G_n \to \Gamma$ in probability? There are actually two versions of this problem, since one may take G_n either labelled or unlabelled;

the classes of graphs is the same but the distribution of a uniform unlabelled graph (with n vertices) in $\mathcal{P}$ differs in general from the distribution of a uniform labelled graph (with vertex set $[n]$) in $\mathcal{P}$, and it is possible that the limits might differ. (An example is given in Example 7.9. Nevertheless we usually expect the same limit, since most graphs have a trivial automorphism group.) We give some remarks on this problem in a few examples. The problem of limits of random graphs is studied further in [17] where it is connected to the entropy of graph limits, see also [21], [22].

2. Graphons and random graphs

A graph limit Γ can be represented by a *graphon*, which is a symmetric measurable function $W : \mathcal{S}^2 \to [0,1]$ for some probability space $(\mathcal{S},\mu)$. (Often, but not always, taken as $([0,1],\lambda)$, where λ is Lebesgue measure.) Note that the representation is far from unique, see e.g. [4] and [20]. One defines, for a graph F and a graphon W,

(2.1)
$$t(F,W) := \int_{\mathcal{S}^{|F|}} \prod_{ij\in E(F)} W(x_i,x_j)\,\mathrm{d}\mu(x_1)\cdots\,\mathrm{d}\mu(x_{|F|}),$$

(2.2)
$$t_{\mathrm{ind}}(F,W) := \int_{\mathcal{S}^{|F|}} \prod_{ij\in E(F)} W(x_i,x_j) \prod_{ij\notin E(F)} (1-W(x_i,x_j))\,\mathrm{d}\mu(x_1)\cdots\,\mathrm{d}\mu(x_{|F|}),$$

and the graphon W represents the graph limit Γ that has $t(F,\Gamma)=t(F,W)$ for every graph F (or, equivalently, $t_{\mathrm{ind}}(F,\Gamma)=t_{\mathrm{ind}}(F,W)$ for every F).

Unlike graph limits, graphons are not uniquely determined by the homomorphism numbers $t(F,W)$; we say that two graphons W and W' (possibly defined on different probability spaces) are *equivalent* if they represent the same graph limit, i.e., if $t(F,W)=t(F,W')$ for all graphs F. (For other characterizations of equivalent graphons, see e.g. Borgs, Chayes and Lovász [4], Bollobás and Riordan [3], Janson [20].) If F has connected components $F_1,\dots,F_m$, then

$$t(F,W)=\prod_{i=1}^{m} t(F_i,W); \tag{2.3}$$

hence it suffices here (and for many other purposes) to consider connected F.

Graph limits may thus be regarded as equivalence classes of graphons. We write (following [28]) $[W]$ for the graph limit represented by a graphon W. It is often convenient to represent graph limits by graphons, and we may, for example, write $G_n \to W$ for $G_n \to [W]$.

Let $X_1, X_2, \dots$ be an i.i.d. sequence of random elements of $\mathcal{S}$ with distribution μ. Then (2.1)–(2.2) can be written more concisely as

$$t(F,W) = \mathbb{E} \prod_{ij\in E(F)} W(X_i,X_j), \tag{2.4}$$

$$t_{\mathrm{ind}}(F,W) = \mathbb{E}\left(\prod_{ij\in E(F)} W(X_i,X_j) \prod_{ij\notin E(F)} (1-W(X_i,X_j))\right). \tag{2.5}$$

A graphon defines a random graph $G(n,W)$ with vertex set $[n] := \{1,\dots,n\}$ for every $n \ge 1$ by a standard construction: let $X_1, X_2, \dots$ be as above, and given $X_1,\dots,X_n$, let ij be an edge with probability $W(X_i,X_j)$, independently for all pairs (i,j) with $1 \le i < j \le n$. It follows by (2.5) that if F is any graph with vertex set $[n]$, then

$$\mathbb{P}\bigl(G(n,W) = F\bigr) = t_{\mathrm{ind}}(F,W); \tag{2.6}$$

equivalently, see (2.4),

$$\mathbb{P}\bigl(G(n,W) \supseteq F\bigr) = t(F,W). \tag{2.7}$$

This shows that the random graph $G(n,W)$ is the same (in the sense that the distribution is the same) for all graphons representing the same graph limit Γ. Thus every graph limit Γ defines a random graph $G(n,\Gamma)$ with vertex set $[n]$ for every $n \ge 1$, and this random graph can by (2.6) be defined directly by the formula

$$\mathbb{P}\bigl(G(n,\Gamma) = F\bigr) = t_{\mathrm{ind}}(F,\Gamma) \tag{2.8}$$

for every graph F on $[n]$, which gives the distribution.

It is shown by Borgs, Chayes, Lovász, Sós and Vesztergombi [5, Theorem 4.5] that as $n \to \infty$, the random graph $G(n,W)$ converges a.s. to W. Thus,

$$G(n,\Gamma) \xrightarrow{\mathrm{p}} \Gamma, \qquad \text{as } n \to \infty. \tag{2.9}$$

3. Graph limits and random graphs

Graph limits in $\overline{\mathcal{P}}$ (or equivalently, in $\widehat{\mathcal{P}} := \overline{\mathcal{P}} \cap \widehat{\mathcal{U}}$) can be characterized by the random graphs $G(n,\Gamma)$.

Theorem 3.1. *Let $\mathcal{P}$ be a hereditary graph class and let Γ be a graph limit. Then $\Gamma \in \overline{\mathcal{P}}$ if and only if $G(n,\Gamma) \in \mathcal{P}$ a.s. for every $n \ge 1$.*

This is an immediate consequence of the following more detailed result.

Theorem 3.2. *Let $\mathcal{P}$ be a hereditary graph class and let Γ be a graph limit. Then one of the following alternatives hold:*

(i) *$\Gamma \in \overline{\mathcal{P}}$ and $G(n,\Gamma) \in \mathcal{P}$ a.s. for every $n \ge 1$.*
(ii) *$\Gamma \notin \overline{\mathcal{P}}$ and $\mathbb{P}\bigl(G(n,\Gamma) \in \mathcal{P}\bigr) \to 0$ as $n \to \infty$.*

Proof. Let $\Gamma \in \overline{\mathcal{P}}$ and suppose that $G_n \to \Gamma$ with $G_n \in \mathcal{P}$. If $F \notin \mathcal{P}$, then $t_{\mathrm{ind}}(F, G_n) = 0$ for every n by (1.1), and thus, by (2.8),

$$\mathbb{P}(G(n,\Gamma) = F) = t_{\mathrm{ind}}(F,\Gamma) = \lim_{n\to\infty} t_{\mathrm{ind}}(F, G_n) = 0.$$

Conversely, if $\Gamma \notin \overline{\mathcal{P}}$, then there is an open neighbourhood V of Γ in $\overline{\mathcal{U}}$ such that $V \cap \mathcal{P} = \emptyset$. As said above, $G(n,\Gamma) \to \Gamma$ a.s., which implies convergence in probability. Thus $\mathbb{P}\big(G(n,\Gamma) \in V\big) \to 1$ and $\mathbb{P}\big(G(n,\Gamma) \in \mathcal{P}\big) \le \mathbb{P}\big(G(n,\Gamma) \notin V\big) \to 0$ as $n \to \infty$. □

We obtain a couple of easy corollaries of Theorem 3.1.

Theorem 3.3. *Let $\mathcal{P}$ be a hereditary graph class and let Γ be a graph limit. Then the following are equivalent:*

(i) $\Gamma \in \overline{\mathcal{P}}$.
(ii) $t_{\mathrm{ind}}(F,\Gamma) > 0 \implies F \in \mathcal{P}$.
(iii) *If $F \notin \mathcal{P}$, then $t_{\mathrm{ind}}(F,\Gamma) = 0$.*

Proof. Immediate by Theorem 3.1 and (2.8). □

Theorem 3.4. *Let $\{\mathcal{P}_\alpha\}$ be a finite or infinite family of hereditary graph classes and let $\mathcal{P} = \bigcap_\alpha \mathcal{P}_\alpha$. Then $\overline{\mathcal{P}} = \bigcap_\alpha \overline{\mathcal{P}}_\alpha$.*

Thus, for example, if a graph limit is the limit of some sequence G_n of graphs in $\mathcal{P}_1$, and also of another such sequence G_n' in $\mathcal{P}_2$, then it is the limit of some sequence G_n'' in $\mathcal{P}_1 \cap \mathcal{P}_2$. (This is not true in general, without the assumption that the classes are hereditary. For example, let $\mathcal{P}_2 = \mathcal{U} \setminus \mathcal{P}_1$, with, say, $\mathcal{P}_1$ the class of interval graphs.)

Proof. Suppose that $\Gamma \in \bigcap_\alpha \overline{\mathcal{P}}_\alpha$. If $t_{\mathrm{ind}}(F,\Gamma) > 0$, then Theorem 3.3(i)⇒(ii) shows that $F \in \mathcal{P}_\alpha$ for every α; hence $F \in \mathcal{P}$. Consequently, Theorem 3.3 in the opposite direction shows that $\Gamma \in \overline{\mathcal{P}}$. The converse is obvious. (Alternatively, one can use Theorem 3.1 directly, with a little care if the family $\{\mathcal{P}_\alpha\}$ is uncountable.) □

Remark 3.5. Conversely, we may ask whether every graph in $\mathcal{P}$ can be obtained (with positive probability) as $G(n,\Gamma)$ for some $\Gamma \in \overline{\mathcal{P}}$ and some n. By (2.8), the class of graphs obtainable in this way equals $\bigcup_{\Gamma\in\overline{\mathcal{P}}} \mathcal{I}(\Gamma)$ where

$$\mathcal{I}(\Gamma) := \{F \in \mathcal{U} : t_{\mathrm{ind}}(F,\Gamma) > 0\}. \tag{3.1}$$

By Theorem 3.3 (see also [27]), $\bigcup_{\Gamma\in\overline{\mathcal{P}}} \mathcal{I}(\Gamma) \subseteq \mathcal{P}$ for every hereditary graph class $\mathcal{P}$. Lovász and Szegedy [27] have shown that equality holds if and only if $\mathcal{P}$ has the following property:

(P1) *If $G \in \mathcal{P}$ and v is a vertex in G, and we enlarge G by adding a twin v' to v, i.e., a new vertex with the same neighbours as v, then at least one of the two graphs obtained by further either adding or not adding an edge vv' belongs to $\mathcal{P}$.*

4. Forbidden subgraphs

If $\mathcal{F}$ is a (finite or infinite) family of (unlabelled) graphs, we let $\mathcal{U}_{\mathcal{F}}$ be the class of all graphs that do not contain any graph from $\mathcal{F}$ as an induced subgraph, i.e.,

$$\mathcal{U}_{\mathcal{F}} := \{G \in \mathcal{U} : t_{\text{ind}}(F, G) = 0 \text{ for } F \in \mathcal{F}\}. \tag{4.1}$$

This is evidently a hereditary class.

We similarly define

$$\overline{\mathcal{U}}_{\mathcal{F}} := \{\Gamma \in \overline{\mathcal{U}} : t_{\text{ind}}(F, \Gamma) = 0 \text{ for } F \in \mathcal{F}\}, \tag{4.2}$$

and have the following simple result ([8, Theorem 3.2]).

Theorem 4.1. *Let $\mathcal{U}_{\mathcal{F}}$ be given by* (4.1). *Then $\overline{\mathcal{U}_{\mathcal{F}}} = \overline{\mathcal{U}}_{\mathcal{F}}$. In other words, if $\Gamma \in \widehat{\mathcal{U}}$ is a graph limit, then Γ is a limit of a sequence of graphs in $\mathcal{U}_{\mathcal{F}}$ if and only if $t_{\text{ind}}(F, \Gamma) = 0$ for $F \in \mathcal{F}$.*

Proof. If $G_n \to \Gamma$ with $G_n \in \mathcal{U}_{\mathcal{F}}$, then $t_{\text{ind}}(F, \Gamma) = \lim_{n\to\infty} t_{\text{ind}}(F, G_n) = 0$ for every $F \in \mathcal{F}$, by (4.1) and the continuity of $t_{\text{ind}}(F, \cdot)$. Thus $\Gamma \in \overline{\mathcal{U}}_{\mathcal{F}}$.

Conversely, suppose that $\Gamma \in \widehat{\mathcal{U}}$ and $t_{\text{ind}}(F, \Gamma) = 0$ for $F \in \mathcal{F}$. It follows from (2.8) that if $F \in \mathcal{F}$ then, for any $n \geq 1$, $G(n, \Gamma) \neq F$ a.s.; thus $G(n, \Gamma) \notin \mathcal{F}$ a.s. Moreover, every induced subgraph of $G(n, \Gamma)$ has the same distribution as $G(m, \Gamma)$ for some $m \leq n$; hence a.s. no induced subgraph belongs to $\mathcal{F}$ and thus $G(n, \Gamma) \in \mathcal{U}_{\mathcal{F}}$. Hence $\Gamma \in \overline{\mathcal{U}_{\mathcal{F}}}$ by Theorem 3.1. ∎

Remark 4.2. Every hereditary class of graphs $\mathcal{P}$ is of the form $\mathcal{U}_{\mathcal{F}}$ for some $\mathcal{F}$; we can simply take $\mathcal{F} := \mathcal{U} \setminus \mathcal{P}$. (In this case, Theorem 4.1 reduces to Theorem 3.3.) However, we are mainly interested in cases when $\mathcal{F}$ is a small family.

Example 4.3. The class of *triangle-free graphs* is $\mathcal{U}_{\{K_3\}}$. Theorem 4.1 shows that the triangle free graph limits (i.e., the limits of triangle free graphs) are the graph limits Γ with $t_{\text{ind}}(K_3, \Gamma) = 0$.

Example 4.4. The class of *chordal graphs* or *triangulated graphs* is the class of all graphs not containing any induced C_k with $k \geq 4$, i.e., $\mathcal{U}_{\{C_4, C_5, \dots\}}$. (See Example 5.4 below and [7, Section 1.2] and [15, Chapter 4] for other equivalent characterizations (and further names).) By Theorem 4.1, a graph limit Γ is a chordal graph limit if and only if $t_{\text{ind}}(C_k, \Gamma) = 0$ for every $k \geq 4$.

Example 4.5. The class $\mathcal{CR}$ of *cographs* equals $\mathcal{U}_{\{P_4\}}$, see [7, in particular Theorem 11.3.3] where several alternative characterizations are given. Thus, if Γ is a graph limit, $\Gamma \in \overline{\mathcal{CR}}$ if and only if $t_{\text{ind}}(P_4, \Gamma) = 0$. Such graph limits are studied in Lovász and Szegedy [26].

Example 4.6 (Diaconis, Holmes and Janson [8])**.** The class $\mathcal{T}$ of *threshold graphs* equals $\mathcal{U}_{\{2K_2, P_4, C_4\}}$ Thus, if Γ is a graph limit, then $\Gamma \in \overline{\mathcal{T}}$ if and only if $t_{\text{ind}}(P_4, \Gamma) = t_{\text{ind}}(C_4, \Gamma) = t_{\text{ind}}(2K_2, \Gamma) = 0$.

Example 4.7. The class $\mathcal{UI}$ of *unit interval graphs* equals the class of graphs that contain no induced subgraph isomorphic to C_k for any $k \geq 4$, $K_{1,3}$, S_3 or $\overline{S}_3$, where S_3 is the graph on 6 vertices $\{1, \dots, 6\}$ with edge set $\{12, 13, 23, 14, 25, 36\}$, and $\overline{S}_3$ is its complement [7, Theorem 7.1.9]. Thus, if Γ is a graph limit, $\Gamma \in \overline{\mathcal{UI}}$ if and only if $t_{\mathrm{ind}}(F, \Gamma) = 0$ for every $F \in \{C_k\}_{k \geq 4} \cup \{K_{1,3}, S_3, \overline{S}_3\}$.

Further examples are studied in Sections 7–11.

We obtain just as easily a corresponding result for subclasses of a given hereditary graph class obtained by forbidding induced subgraphs.

Theorem 4.8. *Let $\mathcal{P}$ be a hereditary graph class and define, for a family $\mathcal{F}$ of graphs,*

$$\mathcal{P}_{\mathcal{F}} := \{G \in \mathcal{P} : t_{\mathrm{ind}}(F, G) = 0 \text{ for } F \in \mathcal{F}\} = \mathcal{P} \cap \mathcal{U}_{\mathcal{F}}. \tag{4.3}$$

Then

$$\overline{\mathcal{P}_{\mathcal{F}}} = \overline{\mathcal{P}}_{\mathcal{F}} := \{\Gamma \in \overline{\mathcal{P}} : t_{\mathrm{ind}}(F, \Gamma) = 0 \text{ for } F \in \mathcal{F}\}. \tag{4.4}$$

Proof. An immediate consequence of Theorems 4.1 and 3.4. □

Example 4.9. The class $\mathcal{UI}$ of *unit interval graphs* equals also the class of interval graphs that contain no induced subgraph isomorphic to $K_{1,3}$ [7, p. 111]. Thus, if $\mathcal{I}$ is the class of all interval graphs, then $\mathcal{UI} = \mathcal{I}_{\{K_{1,3}\}}$ and hence $\overline{\mathcal{UI}} = \overline{\mathcal{I}}_{\{K_{1,3}\}}$. In other words, if Γ is a graph limit, then $\Gamma \in \overline{\mathcal{UI}}$ if and only if $\Gamma \in \overline{\mathcal{I}}$ and $t_{\mathrm{ind}}(K_{1,3}, \Gamma) = 0$.

We have here considered induced subgraphs. We obtain similar results if we forbid general subgraphs. Let $\mathcal{U}^*_{\mathcal{F}}$ be the class of all graphs that do not contain any graph from $\mathcal{F}$ as a subgraph, i.e.,

$$\mathcal{U}^*_{\mathcal{F}} := \{G \in \mathcal{U} : t(F, G) = 0 \text{ for } F \in \mathcal{F}\}. \tag{4.5}$$

This is evidently a hereditary class (and a monotone class, see Example 1.3). In fact, this can be seen as a special case of forbidding induced subgraphs, since $\mathcal{U}^*_{\mathcal{F}} = \mathcal{U}_{\mathcal{F}^*}$, where $\mathcal{F}^*$ is the family of all graphs H that contain a spanning subgraph F (i.e., a subgraph $F \subseteq H$ with $|F| = |H|$) with $F \in \mathcal{F}$.

Theorem 4.10. *Let $\mathcal{U}^*_{\mathcal{F}}$ be given by* (4.5)*. Then*

$$\overline{\mathcal{U}^*_{\mathcal{F}}} = \{\Gamma \in \overline{\mathcal{U}} : t(F, \Gamma) = 0 \text{ for } F \in \mathcal{F}\}. \tag{4.6}$$

Proof. By the argument in the proof of Theorem 4.1, or by Theorem 4.1 applied to $\mathcal{F}^*$. □

There is also a version corresponding to Theorem 4.8.

Example 4.11. The class of *triangle-free graphs* in Example 4.3 can also be defined as $\mathcal{U}^*_{\{K_3\}}$. Thus Theorem 4.10 applies and shows that the triangle free graph limits are the graph limits Γ with $t(\Gamma, K_3) = 0$. (This is in accordance with Example 4.3 since $t_{\mathrm{ind}}(K_3, \Gamma) = t(K_3, \Gamma)$ for any graph limit Γ.)

Example 4.12. The class of *bipartite graphs* equals $\mathcal{U}^*_{\{C_3,C_5,\dots\}}$. Thus a graph limit Γ is a limit of bipartite graphs if and only if $t(C_k,\Gamma)=0$ for every odd $k\geq 3$. (We treat here bipartite graphs as a special case of simple graphs. Bipartite graphs, with an explicit bipartition, can also be treated separately, with a corresponding but distinct limit theory, see e.g. [10].)

5. Intersection graphs

Consider the class $\mathcal{P}_\mathcal{A}$ of $\mathcal{A}$-intersection graphs defined by a collection $\mathcal{A}$ of sets as in Example 1.1. We define $W=W_\mathcal{A}:\mathcal{A}\times\mathcal{A}\to\{0,1\}$ by

$$W(A,B)=\begin{cases}1 & \text{if } A\cap B\neq\emptyset,\\ 0 & \text{if } A\cap B=\emptyset.\end{cases} \tag{5.1}$$

Hence, a graph G is an $\mathcal{A}$-intersection graph if and only if there is a function $i\mapsto A_i$ mapping $V(G)$ into $\mathcal{A}$ such that the adjacency matrix of G equals $(W(A_i,A_j))_{ij}$, except on the diagonal.

Equip $\mathcal{A}$ with some suitable σ-field making W measurable and let μ be any probability measure on $\mathcal{A}$. Then W can be regarded as a graphon defined on the probability space $(\mathcal{A},\mu)$, and defines thus, see Section 2, random graphs $G(n,W)$ and a graph limit Γ such that $G(n,W)\to\Gamma$ a.s. By construction, the random graph $G(n,W)\in\mathcal{P}_\mathcal{A}$ for every n, and is thus a random $\mathcal{A}$-intersection graph; it follows that the graph limit $\Gamma\in\overline{\mathcal{P}_\mathcal{A}}$.

This defines a graph limit $\Gamma\in\overline{\mathcal{P}_\mathcal{A}}$ for every probability measure μ on $\mathcal{A}$. Note that we here use a fixed W and vary μ to obtain different graph limits, in contrast to the more common situation when μ is a fixed measure on some space $\mathcal{S}$ (usually Lebesgue measure on [0,1]) and we vary W. We therefore denote the graph limit obtained in this way from μ by Γ_μ.

It is easily seen that every $\mathcal{A}$-intersection graph G can be obtained with positive probability as $G(n,W)$ for a suitable μ; if G is defined by a family $A_i\in\mathcal{A}$, $i\in V(G)$, and $|V(G)|=n$, take $\mu=\frac{1}{n}\sum_i\delta_{A_i}$, the normalized sum of point masses at the points A_i.

The problem whether every $\mathcal{A}$-intersection graph limit can be represented as Γ_μ for some probability measure μ on $\mathcal{A}$ is more subtle. We do not know any general results and give only some examples. Note that a class of intersection graphs typically can be defined by several different families $\mathcal{A}$, and it is conceivable that the answer depends on the precise choice of the family $\mathcal{A}$, and not just on the resulting class of graphs. (Although we do not know any such examples.) We have not investigated this further, and we consider only some natural choices of $\mathcal{A}$.

Example 5.1. The class of *interval graphs* can be defined as the $\mathcal{A}$-intersection graphs where $\mathcal{A}$ is the family $\{[a,b]:0\leq a\leq b\leq 1\}$ of closed intervals of [0,1]. This family $\mathcal{A}$ can be identified with a closed subset of $\mathbb{R}^2$ (a triangle), and is then a compact metric space, which we equip with the usual Borel σ-field. It is shown in [9] that, with this choice of $\mathcal{A}$, every interval graph limit is Γ_μ for some (non-unique) probability measure μ on $\mathcal{A}$.

Furthermore, [9] also gives similar results, with similar natural classes $\mathcal{A}$, for *circular-arc graphs*, *circle graphs* and *permutation graphs*.

Example 5.2. However, it is also shown in [9] that the corresponding result fails for the class of *unit interval graphs*. This is the class of $\mathcal{A}$-intersection graphs where $\mathcal{A} = \{[x, x+1] : x \in \mathbb{R}\}$, the family of closed unit intervals in $\mathbb{R}$, but not every unit interval graph limit can be represented as Γ_μ for a probability measure μ on $\mathcal{A}$. (It is an open problem whether there exists another family $\mathcal{A}'$ defining the same class of graphs such that every unit interval graph limit can be represented as Γ_μ for the family $\mathcal{A}'$.)

Example 5.3. For a more trivial counterexample, let $\mathcal{A}$ be the countable family of all finite subsets of $\mathbb{N}$; then every graph is an $\mathcal{A}$-intersection graph. (If G is a graph, label the edges by integers and for each vertex v, let A_v be the set of the edges incident to v.) However, a graph limit of the type Γ_μ is always random-free, see Section 6 below; since there are many graph limits that are not random-free, not every graph limit can be represented as Γ_μ.

Example 5.4. The class of *chordal graphs* defined in Example 4.4 can also be defined as the class of intersection graphs of subtrees in a tree [15, Section 4.5]. (In order to make this fit the formulation in Example 1.1, we take a countably infinite universal tree T, containing all finite trees as subtrees, for example constructed by taking disjoint copies of all finite trees and joining them to a common root. We then let $\mathcal{A}$ be the family of all finite subtrees of T.)

We shall see in Example 6.3 that not every chordal graph limit is random-free, and thus not every chordal graph limit can be represented as Γ_μ (for this or any other family $\mathcal{A}$).

We do not know any natural representation of all chordal graph limits, and leave that as an open problem.

6. Random-free graph limits and classes

A graphon W is said to be *random-free* if it is $\{0,1\}$-valued a.e. (In this case, the random graph $G(n, W)$ depends only on the random points $X_1, X_2, \dots$, without further randomization, which is a reason for the name.) If W_1 and W_2 are two graphons that represent the same graph limit, and one of them is random-free, then both are, see [20] for a detailed proof. Consequently, we can define a graph limit Γ to be random-free if some representing graphon is random-free, and in this case every representing graphon is random-free.

Random-free graphons are studied in [20, Section 10], where it is shown, for example, that a graph limit Γ is random-free if and only if the random graph $G(n, \Gamma)$ has entropy $o(n^2)$ (thus quantifying a sense in which there is less randomness than otherwise). Another result, see [28] and [20], is that

Γ is random-free if and only if it is a limit of a sequence of graphs in the stronger metric δ_1.

Lovász and Szegedy [27] define a graph property to be *random-free* if every graph limit $\Gamma \in \overline{\mathcal{P}}$ is random-free. They show some consequences of this. Moreover, they show that a hereditary graph property $\mathcal{P}$ is random-free if and only if the following property holds:

(P2) *There exists a bipartite graph F with bipartition (V_1, V_2) such that no graph obtained from F by adding edges within V_1 and V_2 is in $\mathcal{P}$.*

The representation theorems in [9] show that the class of *interval graphs* is random-free, together with the classes of *circular-arc graphs* and *circle graphs*. (Of course, then every subclass is also random-free, for example the classes of *unit interval graphs* and *permutation graphs* also studied in [9] and the class of *threshold graphs* for which random-free graphons were found in [8].)

Remark 6.1. We do not know explicit examples of graphs F satisfying (P2) for the random-free classes of graphs just mentioned, but we guess that small examples exist.

However, not every class of intersection graphs is random-free.

Example 6.2. Let $\mathcal{A}$ be the family of finite subsets of some infinite set (for example $\mathbb{N}$). Then, as said in Example 5.3, every graph is an $\mathcal{A}$-intersection graph. Consequently, every graph limit is an $\mathcal{A}$-intersection graph limit, and the class of $\mathcal{A}$-intersection graphs is not random-free.

Example 6.3. The class of *chordal graphs* can be defined in several ways, see Example 4.4 and Example 5.4; it is an example both of a graph class defined by forbidden induced subgraphs and a class of intersection graphs.

We show that this class does not satisfy (P2); thus it is *not* random-free. The class of chordal graphs is thus a non-trivial class of intersection graphs that is not random-free.

Let F be a bipartite graph with bipartition (V_1, V_2), and let F_1 be the graph obtained by adding all edges inside V_1 to F. We shall show that F_1 is chordal, which shows that (P2) does not hold.

Assume that a cycle C is an induced subgraph of F_1. If C has two vertices in V_1, then these are adjacent in F_1 so they have to be adjacent in C. Thus, C has at most 3 vertices in V_1; if it has 3, then C has no other vertices, and if it has 2, they have to be adjacent in C. Hence, there are at most 2 edges in C than go between V_1 and V_2. Since F_1, and thus C, has no edges inside V_2, it follows that C has at most one vertex in V_2. Consequently, C has in every case at most 3 vertices.

We have shown that F_1 has no induced cycle of length greater than 3, i.e., F_1 is chordal. Thus no graph F as in (P2) exists, so by the result of Lovász and Szegedy [27] stated above, the class of chordal graphs is *not* random-free.

In fact, we can easily construct a graphon that defines a chordal graph limit but is not random-free. Let $\mathcal{S} = \{0,1\}$ be a space with two points, say with measure $\frac{1}{2}$ each, and let $W^{\mathcal{CH}}$ be the graphon $W^{\mathcal{CH}}(x,y) := (x+y)/2$ defined on $\mathcal{S}$. (Alternatively, one can take $\mathcal{S} = [0,1]$ and $W^{\mathcal{CH}}(x,y) := (\lfloor 2x \rfloor + \lfloor 2y \rfloor)/2$.) The random graph $G(n, W^{\mathcal{CH}})$ is of the type of F_1 above (with V_1 the set of vertices i with $X_i = 1$); thus the argument above shows that $G(n, W^{\mathcal{CH}})$ is chordal. By Theorem 3.1, the graph limit $[W^{\mathcal{CH}}]$ generated by the graphon $W^{\mathcal{CH}}$ is a chordal graph limit, and it is evidently not random-free. (See also Theorem 10.3.)

Problem 6.4. Investigate for other classes of intersection graphs whether they are random-free or not.

We turn to graph classes determined by forbidden subgraphs, noting that the class of chordal graphs in Example 6.3 is one example of such a graph class that is not random-free. Again, we have no general theorem and give just one more example.

Example 6.5. The class $\mathcal{CR}$ of *cographs*, see Example 4.5, is random-free. This is shown in Lovász and Szegedy [26] for regular graphons in $\overline{\mathcal{CR}}$; we show the general case by verifying (P2), using the result of Lovász and Szegedy [27] stated above. We take the graph F as the bipartite graph with 12 vertices $A, B, C, D, E, F, a, b, c, d, e, f$ and edge set

$$\{Aa, Bb, Cc, Dd, Ee, Ff, Ac, Bd, Ae, Bf, Ca, Cb, Db, Fa\}.$$

Suppose that there exists a graph F_1 obtained by adding edges within $V_1 = \{A, B, C, D, E, F\}$ and $V_2 = \{a, b, c, d, e, f\}$ to F such that $F_1 \in \mathcal{CR}$, i.e., F_1 contains no induced P_4. If exactly one of AB and ab is an edge in F_1, then $aABb$ or $AabB$ is an induced P_4; hence either both are edges or neither is; let us write this as $AB \iff ab$. Similarly, $CD \iff cd$, $EF \iff ef$, $AB \iff cd$, $AB \iff ef$. Consequently, if AB is an edge, then so are ab, ef and EF, and then $EFab$ is an induced P_4; conversely, if AB is not an edge then neither is ab nor CD and then $aCbD$ is an induced P_4. Hence no such graph F_1 without induced P_4 exists, so (P2) holds.

Problem 6.6. Investigate for other classes of graphs with forbidden (induced) subgraphs whether they are random-free or not.

7. Disjoint cliques

A *clique* is a complete graph K_n, $n \geq 1$. A *disjoint clique graph* is a graph that is a disjoint union of cliques. (Note that this includes the possibility of isolated vertices.) We let $\mathcal{DC}$ be the class of disjoint clique graph.

A graph is a disjoint clique graph if and only if the extended adjacency relation ($x \sim y$ or $x = y$) is transitive, and thus an equivalence relation. (We may thus also call these graphs *transitive*.) Consequently, a graph is a disjoint clique graph if and only if it has no induced subgraph P_3, i.e.,

$$\mathcal{DC} = \mathcal{U}_{\{P_3\}} := \{G \in \mathcal{U} : t_{\text{ind}}(P_3, G) = 0\}. \tag{7.1}$$

Note also that $\mathcal{DC}$ is the class of $\mathcal{A}$-intersection graphs where $\mathcal{A}$ is any infinite family of disjoint sets, for example $\mathcal{A} = \{\{i\} : i \in \mathbb{N}\}$.

By Theorem 4.1, (7.1) yields immediately the corresponding characterization of disjoint clique graph limits.

Theorem 7.1. $\overline{\mathcal{DC}} = \overline{\mathcal{U}}_{\{P_3\}}$. *Hence, a graph limit* Γ *is a disjoint clique graph limit if and only if* $t_{\text{ind}}(P_3, \Gamma) = 0$. □

We can obtain another, more explicit, description of the disjoint clique graph limits as follows. Note that a disjoint clique graph is determined by the partition of the vertex set into connected components; an unlabelled disjoint clique graph is thus determined by the sequence of component sizes. In particular, the number of labelled disjoint clique graphs of order n is the Bell number $B(n)$ and the number of unlabelled disjoint clique graphs of order n is the number $p(n)$ of partitions of n, see e.g. [13]. (Asymptotics are given in [13, Example VIII.6 and Section VIII.6].)

We denote for any graph G the component sizes by $C_1(G) \geq C_2(G) \geq \dots$, ordered in decreasing order and extended by $G_k(G) = 0$ when k is larger than the number of components.

A *mass-partition* is a sequence $\mathbf{s} = (s_i)_{i=1}^{\infty}$ of non-negative real numbers such that

$$s_1 \geq s_2 \geq \cdots \geq 0 \qquad \text{and} \qquad \sum_{i=1}^{\infty} s_i \leq 1.$$

(We think of breaking a unit mass into pieces of sizes $s_1, s_2, \dots$, together with "dust", i.e., infinitesimal pieces, of mass $1 - \sum_{i=1}^{\infty} s_i \geq 0$.) The set $\mathcal{M}$ of all mass-partitions is given the metric $d(\mathbf{s}, \mathbf{s}') := \max_i |s_i - s'_i|$ and is then a compact metric space; the topology equals the topology of pointwise convergence (i.e., convergence of each s_i separately); for these and other properties, see [2, Section 2.1].

Note that each graph G defines a mass-partition $(C_i(G)/|G|)_{i=1}^{\infty}$.

We define, for a mass-partition $\mathbf{s}$,

$$S_k(\mathbf{s}) := \sum_{i=1}^{\infty} s_i^k, \qquad k \geq 1. \tag{7.2}$$

Lemma 7.2. (i) *If* $\boldsymbol{s}$ *and* $\boldsymbol{s}'$ *are mass-partitions, then* $\boldsymbol{s} = \boldsymbol{s}'$ *if and only if* $S_k(\boldsymbol{s}) = S_k(\boldsymbol{s}')$ *for every* $k \geq 2$.

(ii) *If* $\boldsymbol{s}^{(n)}$ *is a sequence of mass-partitions, then* $\boldsymbol{s}^{(n)} \to \boldsymbol{s}$ *in* $\mathcal{M}$ *if and only if* $S_k(\boldsymbol{s}^{(n)}) \to S_k(\boldsymbol{s})$ *for every* $k \geq 2$.

Proof. (i): We have $s_1 = \sup_i s_i = \lim_{k\to\infty} S_k(\mathbf{s})^{1/k}$. Hence $S_k(\mathbf{s}) = S_k(\mathbf{s}')$ for all large k implies $s_1 = s'_1$, and it follows by induction that $s_i = s'_i$ for all i. The converse is obvious.

(ii): Since $\mathbf{s} \in \mathcal{M}$ implies $s_i \leq 1/i$, it follows by dominated convergence that if $\mathbf{s}^{(n)} \to \mathbf{s}$, then $S_k(\mathbf{s}^{(n)}) \to S_k(\mathbf{s})$ for each $k \geq 2$.

For the converse we note that this means that the map $\Phi : \mathbf{s} \mapsto (S_k(\mathbf{s})_{k=2}^{\infty}$ is a continuous map of $\mathcal{M}$ into $[0,1]^\infty$. Moreover, by (i), this map is one-to-one. Since $\mathcal{M}$ is compact, it follows that Φ is a homeomorphism. □

Remark 7.3. In Lemma 7.2(i), we then also have $S_1(\mathbf{s}) = S_1(\mathbf{s}')$, but in (ii), it does not follow that $S_1(\mathbf{s}) \to S_1(\mathbf{s}')$; for an example, let $s_i^{(n)} = 1/n$ for $i \le n$; then $\mathbf{s}^{(n)} \to 0$ but $S_1(\mathbf{s}^{(n)}) = 1$ for each n.

Given a mass-partition $\mathbf{s}$ we define a graphon $W_{\mathbf{s}}^{\mathcal{M}}$ by taking disjoint subsets $(A_i)_{i=1}^\infty$ of a probability space $(\mathcal{S},\mu)$ such that $\mu(A_i) = s_i$ and defining $W_{\mathbf{s}}^{\mathcal{M}} := \sum_{i=1}^\infty \mathbf{1}_{A_i \times A_i}$. It is easily seen from the definition (2.1) (or (2.4)) that if F is connected and $|F| = k \ge 2$, then

$$t(F, W_{\mathbf{s}}^{\mathcal{M}}) = S_k(\mathbf{s}) := \sum_{i=1}^{\infty} s_i^k. \tag{7.3}$$

Since $t(K_1, W) = 1$ for any graphon W, it follows by (2.3) that the graphons $W_{\mathbf{s}}^{\mathcal{M}}$ defined in this way all are equivalent and thus define a unique graph limit $\Gamma_{\mathbf{s}}^{\mathcal{M}}$. (This is an example of a direct sum of (infinitely many) graph limits, see [18].)

Remark 7.4. One canonical choice of $W_{\mathbf{s}}^{\mathcal{M}}$ is to take $\mathcal{S} := \mathbb{N} \cup \{\infty\}$ with $\mu\{i\} = s_i$, $i < \infty$, and thus $\mu\{\infty\} = 1 - \sum_{i=1}^\infty s_i$, letting $A_i = \{i\}$. Then $W_{\mathbf{s}}^{\mathcal{M}}(x,y) = \mathbf{1}\{x = y < \infty\}$. (A related construction is to take $W_{\mathbf{s}}^{\mathcal{M}}(x,y) := \mathbf{1}\{x = y\}$ on a probability space with point masses $s_1, s_2, \dots$, and the rest of the mass in a continuous part.)

Another canonical choice is to take $(\mathcal{S},\mu) = ([0,1],\lambda)$ and let $A_i = [\sum_1^{i-1} s_j, \sum_1^i s_j)$.

Theorem 7.5. *The set $\widehat{\mathcal{DC}}$ of disjoint clique graph limit is the set $\{\Gamma_{\boldsymbol{s}}^{\mathcal{M}} : \boldsymbol{s} \in \mathcal{M}\}$, and the mapping $\boldsymbol{s} \mapsto \Gamma_{\boldsymbol{s}}^{\mathcal{M}}$ is a homeomorphism of $\mathcal{M}$ onto $\widehat{\mathcal{DC}}$.*

If G_n is a sequence of disjoint clique graphs with $|G_n| \to \infty$ and Γ is a graph limit, then $G_n \to \Gamma$ if and only if $\Gamma = \Gamma_{\boldsymbol{s}}^{\mathcal{M}}$ for some $\boldsymbol{s} \in \mathcal{M}$, and $(C_i(G_n)/|G_n|)_i \to \boldsymbol{s}$ in $\mathcal{M}$.

Proof. Lemma 7.2, (7.3) and (2.3) show that $\mathbf{s} \mapsto t(F, W_{\mathbf{s}}^{\mathcal{M}})$ is continuous for every graph F, which is the same as saying that $\mathbf{s} \to \Gamma_{\mathbf{s}}^{\mathcal{M}}$ is a continuous map $\mathcal{M} \to \widehat{\mathcal{U}}$. Since $\mathcal{M}$ is compact and the map is one-to-one by (7.3) and Lemma 7.2, it is a homeomorphism onto a closed subset of $\widehat{\mathcal{U}}$, which we temporarily denote by K.

Let G_n be a sequence of disjoint clique graphs with $G_n \to \infty$ and let $\mathbf{c}_n$ be the mass-partition $(C_i(G_n)/|G_n|)_i$ defined by G_n. It is easy to see that if F is connected and $|F| = k \ge 2$, then, using (7.3),

$$t(F, G_n) = S_k(\mathbf{c}_n) + O(1/|G_n|) = S_k(\mathbf{c}_n) + o(1) = t(F, \Gamma_{\mathbf{c}_n}^{\mathcal{M}}) + o(1). \tag{7.4}$$

Hence, using (2.3), for any graph limit Γ,

$$G_n \to \Gamma \iff \Gamma_{\mathbf{c}_n}^{\mathcal{M}} \to \Gamma. \tag{7.5}$$

Since $\Gamma^{\mathcal{M}}_{\mathbf{c}_n} \in K$ and K is a closed subset of $\widehat{\mathcal{U}}$, it follows that if $G_n \to \Gamma$, then $\Gamma \in K$, so $\Gamma = \Gamma^{\mathcal{M}}_{\mathbf{s}}$ for some $\mathbf{s} \in \mathcal{M}$. Furthermore, since we have shown that $\mathbf{s} \mapsto \Gamma^{\mathcal{M}}_{\mathbf{s}}$ is an homeomorphism, (7.5) also implies that $G_n \to \Gamma^{\mathcal{M}}_{\mathbf{s}}$ if and only if $\mathbf{c}_n \to \mathbf{s}$. It follows also that $\widehat{\mathcal{DC}} = K$. □

Example 7.6. The extreme cases of disjoint clique graphs are the complete graph K_n and the empty graph $E_n = \overline{K_n}$; they converge as $n \to \infty$ to the graph limits [1] and [0] defined by the constant graphons 1 and 0, respectively. Note that [1] and [0] are the graph limits $\Gamma^{\mathcal{M}}_{\mathbf{s}}$ defined by the mass-partitions $(1, 0, 0, \dots)$ and $(0, 0, \dots,)$. (The families $\{K_n\}$ and $\{E_n\}$ are themselves examples of hereditary classes, although quite trivial.)

We can easily obtain limit results for uniformly random clique graphs. We give both an unlabelled and a labelled version.

Theorem 7.7. (i) *If G_n is a uniformly random unlabelled disjoint clique graph of order n, then $G_n \xrightarrow{\mathrm{p}} [0]$ as $n \to \infty$.*

(ii) *If H_n is a uniformly random labelled disjoint clique graph of order n, then $H_n \xrightarrow{\mathrm{p}} [0]$ as $n \to \infty$.*

Proof. (i): In this case, $(C_i(G_n))_i$ is a uniformly random partition of n, and it is well-known that $C_1(G_n)$ is of the order $\sqrt{n}\log n$ (see Erdős and Lehner [12] and Fristedt [14] for more precise results). Hence $C_1(G_n)/n \xrightarrow{\mathrm{p}} 0$ and thus $(C_i(G_n)/n)_i \xrightarrow{\mathrm{p}} (0, 0, \dots)$ in $\mathcal{M}$. Consequently, $G_n \xrightarrow{\mathrm{p}} [0]$ by Theorem 7.5.

(ii): Similarly, $C_1(H_n)$ is the size of the largest part in a uniformly random set partition of $[n]$. It follows from the asymptotics of the Bell numbers that $C_1(H_n)/n \xrightarrow{\mathrm{p}} 0$, see Sachkov [29] for a much more precise result. Thus, similarly to (i), $H_n \xrightarrow{\mathrm{p}} [0]$ by Theorem 7.5. □

Remark 7.8. For any graphs G_n with $|G_n| \to \infty$, $G_n \to [0]$ if and only if the number of edges $e(G_n) = o(n^2)$. Hence, these results for random disjoint clique graphs are equivalent to $e(G_n)/n^2 \xrightarrow{\mathrm{p}} 0$ and $e(H_n)/n^2 \xrightarrow{\mathrm{p}} 0$.

Example 7.9. Let $\mathcal{DC}_1$ be the subclass of $\mathcal{DC}$ consisting of disjoint clique graphs with at most one non-trivial clique, i.e., the graphs that are the disjoint union of a K_m and an E_{n-m}, with $1 \le m \le n$. Thus, if $G \in \mathcal{DC}_1$, then $C_2(G) \le 1$. It follows easily from Theorem 7.5 that the set $\widehat{\mathcal{DC}_1}$ is the subset of $\widehat{\mathcal{DC}}$ consisting of the graph limits $\Gamma^{\mathcal{M}}_t := \Gamma^{\mathcal{M}}_{\mathbf{s}_t}$, $t \in [0,1]$, where $\mathbf{s}_t$ is the mass-partition $(t, 0, \dots)$; note that $\Gamma^{\mathcal{M}}_{\mathbf{s}_t}$ is represented by the graphon $\mathbf{1}_{[0,t]\times[0,t]}$ on $([0,1], \lambda)$. (Thus, $\Gamma^{\mathcal{M}}_0 = [0]$ and $\Gamma^{\mathcal{M}}_1 = [1]$, cf. Example 7.6.)

An unlabelled graph in $\mathcal{DC}_1$ is determined by the numbers n and m above. Hence, if we let G_n be a uniformly random unlabelled graph of order n in $\mathcal{DC}_1$, then m is uniformly distributed over $\{1, \dots, n\}$. It follows that $G_n \xrightarrow{\mathrm{d}} \Gamma^{\mathcal{M}}_T$, where the random graph limit $\Gamma^{\mathcal{M}}_T$ has $T \in [0,1]$ random and uniformly distributed. This is thus an example of a hereditary class where a uniformly random graph G_n converges in distribution to some random (and non-deterministic) graph limit.

On the other hand, it is easily seen that a uniformly random labelled graph G_n in $\mathcal{DC}_1$ converges in probability (and thus in distribution) to the non-random $\Gamma^{\mathcal{M}}_{1/2}$. In fact, since the decomposition $K_m \cup E_{n-m}$ is unique when $m \geq 2$, it is easily seen that the random graph $G(n, \Gamma^{\mathcal{M}}_{1/2})$ is almost uniformly distributed, in the sense that the total variation distance $d_{\mathrm{TV}}(G_n, G(n, \Gamma^{\mathcal{M}}_{1/2})) \to 0$ as $n \to \infty$; thus $G_n \overset{\mathrm{p}}{\longrightarrow} \Gamma^{\mathcal{M}}_{1/2}$ follows from (2.9).

8. Line graphs

The *line graph* $L(G)$ of a graph G has as vertices the edges of G, with e and f adjacent in $L(G)$ if they have a common endpoint in G. Let $\mathcal{LG}$ be the class of line graphs.

There are several other, equivalent, characterizations, see [7, Theorem 7.1.8]. In particular, there is a set $\mathcal{F}_L$ of 9 graphs such that a graph is a line graph if and only if it does not have any induced subgraph in $\mathcal{F}_L$, i.e., $\mathcal{LG} = \mathcal{U}_{\mathcal{F}_L}$. (See also Šoltés [30] and Lai and Šoltés [23].) Theorem 4.1 yields immediately the corresponding characterization of disjoint clique graph limits.

Theorem 8.1. $\overline{\mathcal{LG}} = \overline{\mathcal{U}}_{\mathcal{F}_L}$. *Hence, a graph limit* Γ *is a line graph limit if and only if* $t_{\mathrm{ind}}(F, \Gamma) = 0$ *for* $F \in \mathcal{F}_L$. □

The line graph of a star is a complete graph, and therefore every disjoint clique graph is a line graph (viz. the line graph of a disjoint union of stars). There are many other line graphs, but we shall see that the line graph limits are the same as the limits of the subclass of disjoint clique graphs.

Lemma 8.2. *If* G *is a line graph of order* n*, then* G *has a subgraph* H *that is a disjoint clique graph with* $V(H) = V(G)$ *and* $|E(G) \setminus E(H)| \leq 4n^{5/3} = o(n^2)$.

Proof. The line graph G is a union of edge-disjoint cliques $\mathcal{C}_i$, with each vertex in at most two $\mathcal{C}_i$. (If $G = L(G')$, then $\mathcal{C}_i$ is the set of edges in G' incident to a vertex i in G'.) Note that thus $\sum_i |\mathcal{C}_i| \leq 2n$.

Let $H_1 := V(G) \cup \bigcup\{\mathcal{C}_i : |\mathcal{C}_i| > n^{2/3}\}$, where $V(G)$ is seen as an empty graph; i.e., H equals G with all edges in cliques $\mathcal{C}_i$ with $|\mathcal{C}_i| \leq n^{2/3}$ removed. The number of removed edges is

$$|E(G) \setminus E(H_1)| = \sum_{|\mathcal{C}_i| \leq n^{2/3}} \binom{|\mathcal{C}_i|}{2} \leq \sum_{|\mathcal{C}_i| \leq n^{2/3}} |\mathcal{C}_i|^2 \leq n^{2/3} \sum_{|\mathcal{C}_i| \leq n^{2/3}} |\mathcal{C}_i| \\ \leq 2n \cdot n^{2/3} = 2n^{5/3}.$$

Since $\sum_i |\mathcal{C}_i| \leq 2n$, there are at most $2n/n^{2/3} = 2n^{1/3}$ remaining cliques. Since two cliques have at most one common vertex (they are edge-disjoint), the number of vertices that belong to 2 cliques is at most $\binom{2n^{1/3}}{2} \leq 2n^{2/3}$.

Delete also all edges incident to any these vertices. This leaves a graph $H \subseteq H_1$ that is a union of disjoint cliques, and

$$|E(H_1) \setminus E(H)| \leq 2n^{2/3} \cdot n = 2n^{5/3}.$$

Hence $|E(G) \setminus E(H)| \leq 4n^{5/3}$. □

Theorem 8.3. *$\widehat{\mathcal{LG}} = \widehat{\mathcal{DC}}$, i.e., a graph limit is a line graph limit if and only if it is a disjoint clique graph limit. Hence, the set $\widehat{\mathcal{LG}}$ of line graph limits is the set $\{\Gamma^{\mathcal{M}}_{\boldsymbol{s}} : \boldsymbol{s} \in \mathcal{M}\}$.*

Proof. Let G_n be a sequence of line graphs with $G_n \to \Gamma$. By Lemma 8.2, we may select disjoint clique graphs $H_n \subseteq G_n$ such that $V(H_n) = V(G_n)$ and $|E(G_n) \setminus V(H_n)| = o(|G_n|^2)$. It follows easily that for any graph F, $t(F; G_n) = t(F, H_n) + o(1)$, and thus $t(F, H_n) = t(F, \Gamma) + o(1)$; hence $H_n \to \Gamma$. This shows that any line graph limit Γ is a disjoint clique graph limit. The converse is obvious. □

Corollary 8.4. *If Γ is a graph limit, then $t_{\text{ind}}(F, \Gamma) = 0$ for every $F \in \mathcal{F}_L$ if and only if $t_{\text{ind}}(P_3, \Gamma) = 0$.*

Proof. By Theorems 7.1, 8.1 and 8.3. □

Remark 8.5. Note that Corollary 8.4 holds for graph limits but not for graphs. (There are graphs G that are line graphs, and thus $t_{\text{ind}}(F, G) = 0$ for every $F \in \mathcal{F}_L$, but $t_{\text{ind}}(P_3, G) > 0$; for example $G = P_3$.)

Thus, if Γ is a line graph limit, then $t_{\text{ind}}(P_3, \Gamma) = 0$. Hence, if $G_n \to \Gamma$, then $t_{\text{ind}}(P_3, G_n) \to t_{\text{ind}}(P_3, \Gamma) = 0$, and it follows easily (using compactness and subsequences; we omit the details) that if G_n is any sequence of line graphs with $|G_n| \to \infty$, then $t_{\text{ind}}(P_3, G_n) \to 0$. Lemma 8.2 implies that $t_{\text{ind}}(P_3, G_n) = O(|G_n|^{-1/3})$; we do not know whether this rate is the best possible.

We conjecture that a uniformly random line graph (labelled or unlabelled) converges in probability to [0], just as random disjoint clique graphs do by Theorem 7.7, but we have not investigated this further.

9. Ramsey's theorem

Ramsey's theorem says that for every $r \geq 1$, every sufficiently large graph contains either K_r or its complement E_r as an induced subgraph. (See further [16].) In the notation of Section 4, the theorem says that the graph class $\mathcal{U}_{\{K_r, E_r\}}$ is finite, which by Theorems 1.5 and 4.1 are equivalent to the following:

Theorem 9.1 (Ramsey's theorem for graph limits)**.** *For every $r \geq 1$,*

$$\widehat{\mathcal{U}}_{\{K_r, E_r\}} := \{\Gamma \in \widehat{\mathcal{U}} : t_{\text{ind}}(K_r, \Gamma) = t_{\text{ind}}(E_r, \Gamma) = 0\} = \emptyset.$$

Proof. We have just given a proof from the classical Ramsey theorem. We find it instructive to also give a direct proof using graph limits; this thus yields a graph limit proof of Ramsey's theorem. (This is a simple analogue of more advanced results such as the hypergraph removal lemma that can be proved using (hyper)graph limits, see e.g. [11] and [31].)

By (2.2), the statement can be formulated as: there is no graphon W such that

$$\prod_{1\le i<j\le r} W(x_i,x_j) = \prod_{1\le i<j\le r} (1-W(x_i,x_j)) = 0 \tag{9.1}$$

for a.e. $x_1,\dots,x_r \in [0,1]$. To see that (9.1) yields a contradiction, the problem is the "a.e.", which we handle by Lemma 9.2 below, which shows that we may modify W on a null set such that (9.1) holds for all $x_1,\dots,x_r$ such that $(x_i,x_j)\in E$ for some set E of full measure and containing the diagonal $\{(x,x)\}$. We may then take $x_1=\dots=x_r=x$ for any fixed $x\in[0,1]$, and obtain $W(x,x)^{\binom{r}{2}} = (1-W(x,x))^{\binom{r}{2}} = 0$, and thus $W(x,x)=0$ and $W(x,x)=1$, a contradiction. ☐

The required lemma is a version of [19, Lemma 5.3] for several polynomials simultaneously. Since the version in [19] is stated for a single polynomial, we give a detailed statement. A *multiaffine* polynomial is a polynomial in several variables $\{x_\nu\}_{\nu\in\mathcal{I}}$, for some (finite) index set $\mathcal{I}$, such that each variable has degree at most 1; it can thus be written as a sum $\sum_{\mathcal{J}\subseteq\mathcal{I}} a_{\mathcal{J}} \prod_{\nu\in\mathcal{J}} x_\nu$.

Lemma 9.2. *Suppose that $W:[0,1]^2\to[0,1]$ is a graphon. Then there is a version W' of W, i.e. a graphon W' such that $W'=W$ a.e., and a symmetric set $E\subseteq[0,1]^2$ such that $\lambda([0,1]^2\setminus E)=0$ and $E\supseteq\{(x,x):x\in[0,1]\}$, and, moreover, if $\Phi\big((w_{ij})_{i<j}\big)$ is a multiaffine polynomial in the $\binom{m}{2}$ variables w_{ij}, $1\le i<j\le m$, for some $m\ge 2$, such that $\Phi\big((W(x_i,x_j))_{i<j}\big)=\gamma$ for a.e. $x_1,\dots,x_m\in[0,1]$ and some $\gamma\in\mathbb{R}$, then $\Phi\big((W'(x_i,x_j))_{i<j}\big)=\gamma$ for all $x_1,\dots,x_m$ such that $(x_i,x_j)\in E$ for every pair (i,j) with $1\le i<j\le m$.*

Proof. This is proved in [19, Appendix] for a single polynomial Φ. However, an inspection of the proof shows that W' and E are constructed independently of Φ, so the same choices work for any such Φ. ☐

Furthermore, it is easy to see that Ramsey's theorem is equivalent to the following:

Theorem 9.3 (Ramsey's theorem in disguise)**.** *If $\mathcal{P}$ is an infinite hereditary graph class, then either $\mathcal{P}$ contains every complete graph K_n, $n\ge1$, or $\mathcal{P}$ contains every empty graph E_n, $n\ge1$.*

Proof. Since $\mathcal{P}$ is hereditary and contains arbitrarily large graphs, Ramsey's theorem implies that for every r, $K_r\in\mathcal{P}$ or $E_r\in\mathcal{P}$. Thus at least one of $K_r\in\mathcal{P}$ or $E_r\in\mathcal{P}$ holds for arbitrarily large r, and thus for all r since $\mathcal{P}$ is hereditary. ☐

As a corollary we immediately obtain the following simple result. (Compare the weaker but more general Theorem 1.5.)

Theorem 9.4. *If $\mathcal{P}$ is an infinite hereditary graph class, then $\widehat{\mathcal{P}}$ contains either* [0] *or* [1] *(or both).*

Proof. By Theorem 9.3, since $K_n \to [1]$ and $E_n \to [0]$ as $n \to \infty$. □

Conversely, Theorem 9.4 implies Theorem 9.3 by Theorem 3.3, so Theorem 9.4 may also be regarded as a graph limit version of Ramsey's theorem.

Theorem 9.4 exhibits a minimum content of $\widehat{\mathcal{P}}$ for a hereditary class $\mathcal{P}$. It is best possible; the hereditary classes $\{K_n\}$ and $\{E_n\}$ show that $\widehat{\mathcal{P}} = \{[1]\}$ and $\widehat{\mathcal{P}} = \{[0]\}$ both are possible. (Cf. Example 7.6.) Furthermore, Theorem 11.1 gives an example with $\widehat{\mathcal{P}} = \{[0], [1]\}$ and the class $\mathcal{Q}$ defined after it is another example with $\widehat{\mathcal{Q}} = \{[0]\}$.

10. Split graphs

A *split graph* is a graph whose vertex set can be partitioned as $V_0 \cup V_1$ such that the subgraph induced by V_0 is empty and the subgraph induced by V_1 is complete, see [1]. Let $\mathcal{SP}$ denote the class of split graphs; this is evidently a hereditary class.

Theorem 10.1. *A graph limit is a split graph limit if and only if it can be represented by a graphon W such that, for some $a \in [0,1]$, $W = 0$ on $[0,a] \times [0,a]$ and $W = 1$ on $(a,1] \times (a,1]$.*

Proof. If W is a graphon of this type, then $G(n,W)$ is a.s. a split graph. (Take $V_0 = \{i : X_i \in [0,a]\}$ in the construction in Section 2.) Thus the graph limit $[W]$ generated by W belongs to $\widehat{\mathcal{SP}}$ by Theorem 3.1.

Conversely, let G_n be split graphs with $|G_n| \to \infty$ and $G_n \to W$ for some graphon W. Let $V(G_n)$ have the partition $V(G_n)_0 \cup V(G_n)_1$ as above, and let $a_n := |V(G_n)_0|/|V(G_n)|$. Order the vertices of G_n with $V(G_n)_0$ first, and let $W_{G_n}(x,y) := A_{G_n}(\lceil nx \rceil, \lceil ny \rceil)$ be the corresponding graphon, where A_{G_n} is the adjacency matrix of G_n (see e.g. [5] or [20] for this standard construction of a graphon corresponding to a graph). Then $\delta_\square(W_{G_n}, W) \to 0$ as $n \to \infty$, where $\delta_\square$ is the cut metric, see e.g. [5] or [20]. Furthermore, $\iint_{[0,a_n]^2} W_{G_n} = 0$ and $\iint_{(a_n,1]^2}(1 - W_{G_n}) \to 0$, and it follows from the definition of the cut metric that there exist sets $A_n \subseteq [0,1]$ with $\lambda(A_n) = a_n$ such that $\iint_{A_n^2} W \to 0$ and $\iint_{([0,1]\setminus A_n)^2}(1 - W) \to 0$.

Consider a subsequence such that the indicator functions $\mathbf{1}_{A_n}$ converge in the weak* topology in $L^\infty([0,1])$ (as the dual of $L^1([0,1])$) to some function $g \in L^\infty([0,1])$; this is possible by the compactness of the unit ball in the weak* topology. It is easily seen that then $\iint g(x)g(y)W(x,y) = 0$ and $\iint (1-g(x))(1-g(y))(1-W(x,y)) = 0$. Hence, if $A := \{x : g(x) > 0\}$, then $W = 0$ a.e. on $A \times A$ and $W = 1$ a.e. on $([0,1] \setminus A)^2$, and W is equivalent to a graphon of the desired type by a measure-preserving rearrangement of [0,1]. ☐

We can also describe the limit of a uniformly random split graph. Recall the graphon $W^{\mathcal{CH}}$ in Example 6.3.

Theorem 10.2. *Let G_n be a random (labelled or unlabelled) split graph of order n. Then, $G_n \overset{\mathrm{p}}{\longrightarrow} W^{\mathcal{CH}}$ as $n \to \infty$.*

Proof. This follows by Theorem 10.1 and [17, Theorem 1.6], since among the graphons in Theorem 10.1, there is (up to a.e. equivalence) a unique graphon that maximizes the entropy defined in [17], viz. $W^{\mathcal{CH}}$ (regarded as a graphon on [0,1]).

It is also possible to give a direct proof; we give a sketch. Consider first the labelled case.

The partition $V_0 \cup V_1$ of a split graph is not always unique, but different such partitions can differ in at most two points. It is also easily seen that most labelled split graphs have a unique partition of this type, in the sense that the fraction of them among all labelled split graphs of order n tends to 1 as $n \to \infty$, and that $|V_0|$ is concentrated around $n/2$. Hence, we can construct a random graph G'_n, with distribution approximating G_n in the sense that the total variation distance $d_{\mathrm{TV}}(G_n, G'_n) \to 0$, by first selecting V_0 with a suitable distribution among all subsets of size in $(n/2 - n^{3/4}, n/2 + n^{3/4})$, say, and then choosing the edges between V_0 and $V_1 := [n] \setminus V_0$ at random, independently and with probability 1/2 each. It follows that $G'_n \overset{\mathrm{p}}{\longrightarrow} W^{\mathcal{CH}}$, and thus $G_n \overset{\mathrm{p}}{\longrightarrow} W^{\mathcal{CH}}$ as $n \to \infty$; we omit the details.

The unlabelled case follows from the labelled, since most split graphs have a trivial automorphism group; again we omit the details. ☐

It is easily seen that every split graph is chordal; cf. Example 6.3. It is easy to produce examples of chordal graphs that do not split; for example, any disjoint clique graph with more than two components with more than one vertex. Theorem 7.5 yields plenty of chordal graph limits that are not split graph limits, since they are not represented by any graphon as in Theorem 10.1. Nevertheless, Bender, Richmond and Wormald [1] have shown that most (labelled) chordal graphs split, and thus Theorem 10.2 yields the following.

Theorem 10.3. *Let G_n be a random (labelled or unlabelled) chordal graph of order n. Then, $G_n \overset{\mathrm{p}}{\longrightarrow} W^{\mathcal{CH}}$ as $n \to \infty$.*

Proof. The labelled case follows immediately from Theorem 10.2 and [1]. The unlabelled case follow from this because, as said above, most split graphs have a trivial automorphism group. ☐

11. Claw-free and coclaw-free graphs

The *claw* is the star $K_{1,3}$ with 4 vertices, and the *coclaw* is its complememt $\overline{K_{1,3}}$, i.e., the disjoint union of a triangle K_3 and an isolated vertex.

The *claw-free* graphs are the graphs without an induced claw, i.e., $\mathcal{U}_{\{K_{1,3}\}}$. (See e.g. [7] for this class of graphs.) By Theorem 4.1, a graph limit Γ is a claw-free graph limit (i.e., belongs to the closure $\overline{\mathcal{U}}_{\{K_{1,3}\}}$) if and only if $t(K_{1,3},\Gamma) = 0$. Similarly, a graph limit Γ is coclaw-free if and only if $t(\overline{K_{1,3}},\Gamma) = 0$. Note that a graphon W is coclaw-free if and only if $1-W$ is claw-free.

We do not know any simple characterization of claw-free graphons. (And thus not of coclaw-free graphons.) However, it is easy to characterize graph limits that are both claw-free and coclaw-free; they turn out to be trivial.

Theorem 11.1. *A graphon that is both claw-free and coclaw-free has to be either* 0 *or* 1 *a.e. Thus, the only graph limits that are both claw-free and coclaw-free are* [0] *and* [1].

Proof. Let $W : [0,1]^2 \to [0,1]$ be a claw-free graphon. By (2.2),

$$W(x_1,x_2)W(x_1,x_3)W(x_1,x_4) \times \bigl(1-W(x_2,x_3)\bigr)\bigl(1-W(x_2,x_4)\bigr)\bigl(1-W(x_3,x_4)\bigr) = 0 \tag{11.1}$$

for a.e. $x_1,x_2,x_3,x_4 \in [0,1]$. By Lemma 9.2 we may (by modifying W on a null set) assume that there exists a symmetric set $E \in [0,1]^2$ with Lebesgue measure 1, and containing the diagonal $\{(x,x)\}$, such that (11.1) holds for all x_1,x_2,x_3,x_4 such that all pairs $(x_i,x_j) \in E$.

Taking $x_1 = x_2 = x_3 = x_4 = x$, we see that $W(x,x) = 0$ or 1 for every x. Let $A_0 := \{x : W(x,x) = 0\}$ and $A_1 := \{x : W(x,x) = 1\}$. Moreover, if $x \in A_0$, we see by taking $x_1 = y$ and $x_2 = x_3 = x_4 = x$ that $W(x,y) = 0$ when $(x,y) \in E$. Thus $W(x,y) = 0$ for a.e. $(x,y) \in A_0 \times [0,1]$.

If furthermore W also is coclaw-free, the same argument applies to $1-W$ (using the same modification of W). Thus also $W(x,y) = 1$ for a.e. $(x,y) \in A_1 \times [0,1]$.

It follows that $0 = W = 1$ a.e. on $A_0 \times A_1$, and thus $A_0 \times A_1$ is a null set, so either $\lambda(A_0) = 1$ and $W = 0$ a.e., or $\lambda(A_1) = 1$ and $W = 1$ a.e. ∎

This result for graph limits translates to a corresponding result for graphs. In fact, Theorem 11.1 implies that a graph that is both claw-free and coclaw-free has to be either almost empty or almost complete. More precisely, for any sequence G_n of such graphs, with $|G_n| = n$ for simplicity, the number of edges $e(G_n)$ satisfies $e(G_n) = o(n^2)$ or $\binom{n}{2} - e(G_n) = o(n^2)$, see Remark 7.8. We give a simple direct proof of this, with an explicit (and much sharper) error bound; we find it interesting to compare the two very different proofs of the same result.

Let $\mathcal{Q}$ be the class of graphs whose components are cycles of length ≥ 4 and paths. $\mathcal{Q}$ can also be described as the class of graphs G with maximum degree $\Delta(G) \leq 2$ and no component K_3; note that $\mathcal{Q}$ is a hereditary class. Furthermore, if $G \in \mathcal{Q}$ and $|G| = n$, then $2e(G) \leq n\Delta(G) \leq 2n$, and thus $e(G) \leq n$.

Note that the estimate $e(G) \le |G|$ just given for graphs $G \in \mathcal{Q}$ implies that any sequence of graphs $G_n \in \mathcal{Q}$ with $|G_n| \to \infty$ converges to [0]; thus the only $\mathcal{Q}$ graph limit is [0], i.e., $\widehat{\mathcal{Q}} = \{[0]\}$.

Let $R(k,l)$ denote the Ramsey numbers, see [16].

Theorem 11.2. *If G is a graph with $|G| = n \ge R(8,8)$, then G is claw-free and coclaw-free if and only if $G \in \mathcal{Q}$ or $\overline{G} \in \mathcal{Q}$. In particular, then either $e(G) \le n$ or $e(G) \ge \binom{n}{2} - n$.*

Remark 11.3. It is known that $R(8,8) \le \binom{14}{7} = 3432$, see [16, Section 4.3]. This lower bound for the validity of the conclusion is presumably too high, but note that the result is not true for $3 \le n \le 6$; counterexamples are provided by a triangle with 0–3 additional vertices connected to one vertex each in the triangle.

Remark 11.4. Note that Theorem 11.1 is an immediate consequence of Theorem 11.2: By Theorem 4.1 (or Theorem 3.4) a graph limit Γ that is claw-free and coclaw-free is a limit of a sequence of graphs that are claw-free and coclaw-free, and by Theorem 11.2, this implies that Γ is the limit of a sequence G_n of graphs such that either $G_n \in \mathcal{Q}$ or $\overline{G_n} \in \mathcal{Q}$. Selecting a subsequence we thus have either $G_n \to [0]$ or $G_n \to [1]$.

Proof. It is clear that if $G \in \mathcal{Q}$, then G is claw-free and coclaw-free, and thus the same holds if $\overline{G} \in \mathcal{Q}$.

For the converse we note that since $n \ge R(8,8)$, either G or its complement $\overline{G}$ contains an empty induced subgraph E_8. We assume that E_8 is an induced subgraph of G and show that if G further is claw-free and coclaw-free, then $G \in \mathcal{Q}$. This follows from the two claims below.

Claim 1. G contains no induced K_3.

In fact, if G contains a K_3 and an E_8, then they may have at most one common vertex and thus there exist two vertex disjoint subgraphs $A \cong K_3$ and $B \cong E_7$ in G. Every vertex in B has to send at least one edge to A; otherwise it would form a $\overline{K_{1,3}}$ with A. Hence there are at least 7 edges between A and B. On the other hand, if some vertex in A is connected to 3 (or more) vertices in B, it forms a $K_{1,3}$ together with them; hence each vertex in A is connected to at most two vertices in B and the number of edges betwenn A and B is at most 6. This contradiction proves the claim.

Claim 2. The maximum degree $\Delta(G) \le 2$.

In fact, for any vertex $v \in G$, Claim 1 shows that the neighbours of v form an independent set. If G has a vertex v with degree 3 or more, then v and any three of its neighbours thus form a $K_{1,3}$. □

A graph is both claw-free and coclaw-free if and only if its complement is. Hence Theorem 11.1 and symmetry implies the following, giving another example where a uniformly random graph in a hereditary class has a limit in distribution that is random and not a single, deterministic graph limit.

Theorem 11.5. *Let G_n be a uniformly random (labelled or unlabelled) claw-free and coclaw-free graph. Then $G_n \xrightarrow{\mathrm{d}} \Gamma$ as $n \to \infty$, where Γ is the random graph limit that equals* $[0]$ *and* $[1]$ *with probability* $1/2$ *each.* □

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