CAN TIME-HOMOGENEOUS DIFFUSIONS PRODUCE ANY DISTRIBUTION?

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Abstract. Given a centred distribution, can one find a time-homogeneous martingale diffusion starting at zero which has the given law at time 1? We answer the question affirmatively if generalized diffusions are allowed.

1. Introduction

Consider a distribution with finite mean on the real line. Can this distribution be recovered as the distribution of a diffusion at time 1? Clearly not if the support of the distribution is disconnected since diffusions are continuous, but the answer is yes, and in many ways, if the target distribution is sufficiently regular. What if we also require our process to be a martingale or to be time-homogeneous? Again, easy constructions exist (see Remark 2.4 below) to show that a suitable process exists. But, what if we require our process to be both time-homogeneous and a martingale? The original motivation for us to study this problem comes from a calibration problem in mathematical finance.

Our approach involves the speed measure of a diffusion and time-changes of Brownian motion, and makes no assumptions on the regularity of the target law. Indeed, to allow for target distributions with arbitrary support the natural class of processes to consider is the class of generalized diffusions (sometimes referred to as gap diffusions) discussed below. Our main result is to show that within this class of processes there is a time-homogeneous martingale with the given distribution at time 1.

In Section 2 we introduce the class of generalised diffusions as time-changes of a Brownian motion. We also formulate our main result, Theorem 2.3, which states that given a distribution on $\mathbb{R}$ with finite mean, there exists a generalised diffusion that is a martingale and that has this distribution at time 1. In Section 3 we collect some general results about time-changes and generalised diffusions. In Section 4 we study a discrete version of the inverse problem. For a given distribution with mass only in a finite number of points, we show that there exists a time homogeneous martingale Markov chain with the given distribution at time 1. In Section 5 we consider the case of a general distribution on the real axis. By approximating the distribution with finitely supported distributions, the existence of a

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solution to the inverse problem is obtained, thereby proving Theorem 2.3. In Section 6, we apply our results to solve a calibration problem in mathematical finance. In Section 7 we characterize the speed measures for which the corresponding generalised diffusion is a (local) martingale. Finally, Section 8 concludes with some open problems.

2. Construction of generalised diffusions

In this section we construct, following [6] (see also [11] and [12]), time-homogeneous generalised diffusion processes as time-changes of Brownian motion. The time-change is specified in terms of the so-called *speed measure*.

Let ν be a nonnegative Borel measure on the real line; ν may be finite or infinite. Let B_t be a Brownian motion starting at x_0, and let L_u^x be its local time at the point x up to time u. Recall that (a.s.) L_u^x is continuous in $(u,x) \in [0,\infty) \times \mathbb{R}$ and increasing in u. Define the increasing process

$$\Gamma_u := \int_{\mathbb{R}} L_u^x \nu(dx), \tag{2.1}$$

noting that $\Gamma_u \in [0,\infty]$, and let

$$A_t := \inf\{u : \Gamma_u > t\} \tag{2.2}$$

be its right-continuous inverse. The process

$$X_t := B_{A_t}$$

will be called a generalised diffusion with speed measure ν.

Example 2.1. If $\nu(dx) = \frac{dx}{\sigma^2(x)}$ for some continuous non-vanishing function σ, then X_t is a weak solution of

$$dX_t = \sigma(X_t)\, dW_t.$$

In this case, X_t is a diffusion. This example is the motivation for calling the measure ν 'speed measure', but note that ν rather measures the inverse of the speed.

Remarks 2.2.
(i) Almost surely, $L_u^x \nearrow \infty$ for every x as $u \to \infty$; hence if ν is non-zero, then $\Gamma_u \to \infty$ a.s. as $u \to \infty$, and thus $A_t < \infty$ and X_t is well-defined for every $t \in [0,\infty)$ a.s. However, we have to exclude the exceptional case $\nu = 0$, when $\Gamma_u = 0$ for every u and $A_t = \infty$ for every $t \geq 0$, so X_t is not defined. (For technical reasons, we allow $\nu = 0$ when discussing A_t, but we always assume $\nu \neq 0$ when considering X_t, sometimes without explicitly saying so.)

(ii) Γ_u is left-continuous (by monotone convergence) and continuous at every u such that $\Gamma_{u+} < \infty$ (by dominated convergence); hence $u \mapsto \Gamma_u$ is continuous everywhere except that there might exist a single (random) finite u_0 where Γ_u jumps to $+\infty$: $\Gamma_{u_0} < \infty$ but $\Gamma_u = \infty$ for all $u > u_0$. (For example, this happens if ν is an infinite point mass, see Example 3.6 below.)

(iii) By (2.2) and the left-continuity of Γ_u, for all $t, u \geq 0$,

$$A_t < u \iff \Gamma_u > t. \tag{2.3}$$

Equivalently, $u \leq A_t \iff t \geq \Gamma_u$. It follows that $\Gamma_{A_t} \leq t \leq \Gamma_{A_t+}$ for all t, and thus, by (ii), $\Gamma_{A_t} = t$ for all t such that $\Gamma_{A_t+} < \infty$; see further Lemma 7.4.

(iv) Γ_u is finite for all u a.s. if and only if ν is a locally finite measure (also called a *Radon measure*), i.e., $\nu(K) < \infty$ for every compact K. In this case, Γ_u is continuous by (ii), and Γ_u is a *continuous additive functional (CAF)* of B; conversely, every CAF of Brownian motion is of this type, see [10, Theorem 22.25] and [14, Chapter X].

(v) Although $B_0 = x_0$, A_0 may be strictly positive, and in general $X_0 \neq x_0$, see Lemma 3.3 below for a precise result. In [6], X_0 is defined to be x_0, but this has the disadvantage of making X_t possibly not right-continuous at $t = 0$. We follow here instead the standard practise of considering right-continuous processes. We define $A_{0-} = 0$ and $X_{0-} = B_{A_{0-}} = x_0$, and thus allow the possibility that $X_{0-} \neq X_0$.

(vi) We let $(\mathcal{F}_t)_{t\geq 0}$ denote the standard completed Brownian filtration. Then each A_t is a $(\mathcal{F}_t)$-stopping time, and X_t is adapted to the filtration $(\mathcal{G}_t) = (\mathcal{F}_{A_t})$, $t \geq 0-$. (In particular, $\mathcal{G}_{0-} = \mathcal{F}_0$ is trivial.)

In the sequel, we let "stopping time" mean $(\mathcal{F}_t)$-stopping time unless we say otherwise.

(vii) Even though the process X_t is constructed as a time change of Brownian motion, it is in general not necessarily a martingale or even a local martingale; see Section 7 for a detailed discussion. However, we are mainly interested in cases in which X_t is a local martingale, and preferably an integrable martingale. Recall the convention just made that $X_{0-} = x_0$ while X_0 may be different. We say that $(X_t)_{t\geq 0-}$ is a (local) martingale if $(X_t)_{t\geq 0}$ is a (local) martingale (for the filtration $(\mathcal{G}_t)$) and, further, $\mathbb{E}\, X_0 = X_{0-}$. (This is equivalent to the standard definition interpreted for the index set $\{0-\} \cup [0, \infty)$.)

Our main result is that any given distribution with finite mean can be obtained as the distribution of X_1 for some such generalised diffusion with a suitable choice of speed measure ν.

Theorem 2.3. *Let μ be a distribution on the real axis with finite mean $\overline{\mu} = \int_{\mathbb{R}} x\mu(dx)$. Then there exists a generalised diffusion X such that $X_{0-} = \overline{\mu}$, $(X_t)_{0-\leq t\leq 1}$ is a martingale, and the distribution of X_1 is μ. Furthermore:*

(i) *$X_0 = \overline{\mu}$ if and only if $\overline{\mu} \in \operatorname{supp}\mu$.*

(ii) *$\mathbb{E}\, A_1 = \operatorname{Var}(\mu) := \int_{\mathbb{R}}(x - \overline{\mu})^2\mu(dx) \leq \infty$. In particular, $\mathbb{E}\, A_1 < \infty$ if and only if $\operatorname{Var}(\mu) < \infty$.*

It follows from Theorem 7.9 that, actually, $(X_t)_{0-\leq t<\infty}$ is a martingale.

Remark 2.4. Theorem 2.3 guarantees the existence of a time-homogeneous (generalised) diffusion which is a martingale and has a certain distribution at time 1. Note that the problem is much easier if some of these requirements are dropped. For example, for a given distribution μ, one can find a non-decreasing function $f : \mathbb{R} \to \mathbb{R}$ such that $f(B_1)$ has law μ (in particular, $f = F_\mu^{-1} \circ \Phi$, where F_μ is the cumulative distribution function associated with μ and Φ is the normal CDF) and then the process $X_t = f(B_t)$ is time-homogeneous with distribution μ at time 1. Further, if μ has a density which is strictly positive and differentiable, then X is a time-homogeneous diffusion with $dX_t = a(X_t)\, dB_t + b(X_t)dt$ where $a := f' \circ f^{-1}$ and $2b := f'' \circ f^{-1}$. Typically, however, it is not a martingale.

Similarly, with f as above, the process $M_t = \mathbb{E}(f(B_1) \mid \mathcal{F}_t)$ is a martingale with distribution μ at time 1, but typically it is not time-homogeneous. Again, in the regular case, we have $M_t = g(B_t, t)$ for some function g solving the heat equation, and $dM_t = g'(g^{-1}(M_t, t), t)\, dB_t$ is a time-inhomogeneous martingale diffusion, where $g^{-1}(\cdot, t)$ is the space-inverse of $g(\cdot, t)$, and $g'(\cdot, t)$ denotes the spatial derivative.

Remark 2.5. In [9], a similar inverse problem is studied. Instead of allowing for generalised diffusions, the authors consider the case of diffusions with regular diffusion coefficients obtaining an approximate solution to the inverse problem using variational techniques. In [3] the authors solve a related inverse problem in which the goal is to construct a generalised diffusion Y such that Y is a martingale and such that the distribution of Y_{τ_E} is μ, where τ_E is an independent random exponential time. Stopping at an independent exponential time is more tractable than stopping at a fixed time.

Remark 2.6. One interpretation of our results is that we construct a stopping time $\tau \equiv A_1$ such that $B_\tau \equiv B_{A_1} \equiv X_1$ has law μ. The general problem of finding a stopping time T such that B_T has a given distribution is known as the *Skorokhod stopping problem*, see e.g. [13, Section 5.3], where other constructions of such stopping times are given.

Example 2.7. As a very simple case, if μ is the normal distribution $N(x_0, \sigma^2)$, we may take $d\nu = \sigma^{-2} dx$, a constant multiple of the Lebesgue measure. Then $\Gamma_u = \sigma^{-2} u$ and $A_t = \sigma^2 t$ (both non-random), so $X_t = B_{\sigma^2 t}$, cf. Example 2.1. In general, however, it seems difficult to find ν explicitly.

3. Preliminaries

Recall that the *support* of a measure ν on $\mathbb{R}$ is

$$\operatorname{supp} \nu := \mathbb{R} \setminus \bigcup \bigl\{ U \subseteq \mathbb{R} : U \text{ is open and } \nu(U) = 0 \bigr\}.$$

In other words, $x \in \operatorname{supp} \nu$ if and only if every neighbourhood of x has positive measure.

Similarly, we define the *infinity set* $\operatorname{supp}_\infty \nu$ by

$$\operatorname{supp}_\infty \nu := \mathbb{R} \setminus \bigcup \bigl\{ U \subseteq \mathbb{R} : U \text{ is open and } \nu(U) < \infty \bigr\}.$$

In other words, $x \in \operatorname{supp}_\infty \nu$ if and only if every neighbourhood of x has infinite measure. Thus, $\operatorname{supp}_\infty \nu = \emptyset$ if and only if ν is locally finite. By definition, $\operatorname{supp} \nu$ and $\operatorname{supp}_\infty \nu$ are closed subsets of $\mathbb{R}$.

If S is any Borel set in $\mathbb{R}$, we let $H_S := \inf\{t \geq 0 : B_t \in S\}$ denote the *hitting time* of S (for the Brownian motion B_t). Note that if $S \neq \emptyset$, then $H_S < \infty$ a.s. We will only consider cases when S is closed, and then $B_{H_S} \in S$. For $x \in \mathbb{R}$, we write H_x for $H_{\{x\}}$.

Lemma 3.1. *If $\nu \neq 0$, then a.s. $X_t = B_{A_t} \in \operatorname{supp} \nu$ for all $t \geq 0$.*

Proof. If $s > 0$ and $B_s \notin \operatorname{supp} \nu$, then there exists $\varepsilon > 0$ such that $B_u \notin \operatorname{supp} \nu$ for $u \in [s-\varepsilon, s+\varepsilon]$, and then L_u^x is constant for $u \in [s-\varepsilon, s+\varepsilon]$ for each $x \in \operatorname{supp} \nu$; hence Γ_u is constant for $u \in [s-\varepsilon, s+\varepsilon]$, and thus $A_t \neq s$ for all $t \geq 0$.

Similarly, if $s = 0$ and $B_s \notin \operatorname{supp} \nu$, then there exists $\varepsilon > 0$ such that $\Gamma_u = 0$ for $u \in [0, \varepsilon]$. Therefore $A_t > 0 = s$ for all $t \geq 0$. □

Lemma 3.2. *If T is any finite stopping time for B_t, then a.s., for all $\varepsilon > 0$,*

$$L_{T+\varepsilon}^{B_T} > L_T^{B_T} \geq 0.$$

Consequently, there is for every $\varepsilon > 0$ a.s. a (random) $\delta > 0$ and an open set U containing B_T such that $L_{T+\varepsilon}^x - L_T^x \geq \delta$ for all $x \in U$.

Proof. The first claim is an immediate consequence of the strong Markov property and the fact that $L_\varepsilon^x > 0$ a.s. for a Brownian motion started at x and any $\varepsilon > 0$.

The second claim follows since L is continuous. □

Lemma 3.3. *$A_0 = H_{\operatorname{supp} \nu}$, the hitting time of the support of ν, a.s. In particular, if $x_0 \in \operatorname{supp} \nu$, then $A_0 = 0$ and $X_0 = x_0$ a.s., but if $x_0 \notin \operatorname{supp} \nu$, then $A_0 > 0$ and $X_0 \neq x_0$ a.s.*

Proof. If $u < H = H_{\operatorname{supp} \nu}$, then $\Gamma_u = 0$. On the other hand, by Lemma 3.2, for any $\varepsilon > 0$, $L_{H+\varepsilon}^{B_H} > 0$ a.s. and, moreover, $L_{H+\varepsilon}^x > 0$ for x in a neighbourhood U of B_H. Since $B_H \in \operatorname{supp} \nu$, we have $\nu(U) > 0$ and thus $\Gamma_{H+\varepsilon} > 0$. In other words, a.s. $\Gamma_u > 0$ for all $u > H$.

The definition (2.2) of A_t now shows that $A_0 = H$ a.s., and the result follows, recalling Lemma 3.1. □

Lemma 3.4. *We have $\mathbb{E}\, X_0 = x_0 = X_{0-}$ if and only if neither $\operatorname{supp} \nu \subset (-\infty, x_0)$ nor $\operatorname{supp} \nu \subset (x_0, \infty)$.*

Proof. If $\operatorname{supp} \nu$ intersects both $[x_0, \infty)$ and $(-\infty, x_0]$, then $B_{u \wedge H_{\operatorname{supp} \nu}}$ is a bounded martingale and thus $\mathbb{E}\, X_0 = \mathbb{E}\, B_{H_{\operatorname{supp} \nu}} = B_0 = x_0$.

On the other hand, if $\operatorname{supp} \nu \subset (-\infty, x_0)$, then Lemma 3.1 implies $X_0 < x_0$ a.s., so $\mathbb{E}\, X_0 < x_0$. Similarly, if $\operatorname{supp} \nu \subset (x_0, \infty)$, then $\mathbb{E}\, X_0 > x_0$. □

We define, for given ν and x_0,

$$x_+ := \inf\{x \geq x_0 : x \in \operatorname{supp} \nu\}, \tag{3.1}$$
$$x_- := \sup\{x \leq x_0 : x \in \operatorname{supp} \nu\}, \tag{3.2}$$
$$x_+^\infty := \inf\{x \geq x_0 : x \in \operatorname{supp}_\infty \nu\}, \tag{3.3}$$
$$x_-^\infty := \sup\{x \leq x_0 : x \in \operatorname{supp}_\infty \nu\}. \tag{3.4}$$

Note that these may be $\pm\infty$ (when the corresponding sets are empty). In general,

$$-\infty \leq x_-^\infty \leq x_- \leq x_0 \leq x_+ \leq x_+^\infty \leq \infty.$$

It follows from Lemmas 3.3 and 3.4 that we have the following cases:

(i) If $x_0 \in \operatorname{supp} \nu$ (i.e., $x_- = x_+ = x_0$), then $X_0 = x_0$.
(ii) If $\operatorname{supp} \nu \subset (x_0, \infty)$ (i.e., $x_- = -\infty$ and $x_0 < x_+ < \infty$), then $X_0 = x_+$.
(iii) If $\operatorname{supp} \nu \subset (-\infty, x_0)$ (i.e., $x_+ = +\infty$ and $-\infty < x_- < x_0$), then $X_0 = x_-$.
(iv) Otherwise (i.e., if $-\infty < x_- < x_0$ and $x_0 < x_+ < \infty$), then $X_0 \in \{x_-, x_+\}$, with the unique distribution satisfying $\mathbb{E}\, X_0 = x_0$.

Lemma 3.5. *Let $H = H_{\operatorname{supp}_\infty \nu}$ be the hitting time for B_t of the infinity set of ν. Then a.s. $\Gamma_u = \infty$ for all $u > H$, and thus $A_t \leq H$ for all $t \geq 0$.*

On the other hand, $\Gamma_u < \infty$ a.s. for all $u < H$.

Proof. If $H < \infty$, then $B_H \in \operatorname{supp}_\infty \nu$. For any $\varepsilon > 0$, by Lemma 3.2, $L^x_{H+\varepsilon} \geq \delta > 0$ for x in a neighbourhood U of B_H and some $\delta > 0$. Since $B_H \in \operatorname{supp}_\infty \nu$, we have $\nu(U) = \infty$ and thus $\Gamma_{H+\varepsilon} \geq \int_U L^x_{H+\varepsilon} \nu(dx) = \infty$. The definition (2.2) of A_t now shows that $A_t \leq H$ a.s.

If $u < H$, then $K := \{B_s : 0 \leq s \leq u\}$ is a compact interval disjoint from $\operatorname{supp}_\infty \nu$, and thus $\nu(K) < \infty$. Since $x \mapsto L^x_u$ is continuous and vanishes outside K, we have $\Gamma_u = \int_K L^x_u \nu(dx) < \infty$. ∎

Example 3.6. Let ν be an infinite point mass at $x_1 \in \mathbb{R}$. Then $\operatorname{supp} \nu = \operatorname{supp}_\infty \nu = \{x_1\}$. If H_{x_1} is the hitting time of x_1, then $\Gamma_u = 0$ for $u < H_{x_1}$, cf. Lemma 3.3 and its proof, but a.s. $\Gamma_u = \infty$ for $u > H_{x_1}$ by Lemma 3.5. Hence, a.s., $A_t = H_{x_1}$ and $X_t = x_1$ for every $t \geq 0$.

In particular, if $x_1 = x_0$, then $H_{x_1} = 0$ a.s., and thus $A_t = 0$ a.s. for all $t \geq 0$.

More generally, if ν is a measure such that $\nu(S) = 0$ or $\nu(S) = \infty$ for every Borel set S, then $\operatorname{supp}_\infty \nu = \operatorname{supp} \nu$, and Lemmas 3.5 and 3.3 imply that a.s. $A_t = H_{\operatorname{supp} \nu}$ for all $t \geq 0$, so $X_t = X_0$ is the first point of $\operatorname{supp} \nu$ hit by B_u.

Lemma 3.7. *Suppose that $\operatorname{supp}_\infty \nu \cap (-\infty, x_0]$ and $\operatorname{supp}_\infty \nu \cap [x_0, \infty)$ are both non-empty. Then $B_{A_t \wedge u} \in [x_-^\infty, x_+^\infty]$ for all $t, u \geq 0$, and thus $(B_{A_t \wedge u})_{u \geq 0}$ is a bounded martingale for each fixed t.*

Proof. Recall $x_\pm^\infty$ from (3.3)–(3.4). By assumption, $-\infty < x_-^\infty \le x_0 \le x_+^\infty < \infty$. Let $H = H_{\operatorname{supp}_\infty \nu} = H_{\{x_-^\infty, x_+^\infty\}}$. Then $x_-^\infty \le B_u \le x_+^\infty$ for all $u \le H$; thus Lemma 3.5 implies that $x_-^\infty \le B_{u \wedge A_t} \le x_+^\infty$ for any $u \ge 0$ and $t \ge 0$. Finally, $(B_{A_t \wedge u})_{u \ge 0}$ is a martingale since A_t is a stopping time. □

Lemma 3.8. *If* $\mathbb{E}\, A_t < \infty$*, then* $\mathbb{E}\, X_t = x_0$ *and* $\operatorname{Var} X_t = \mathbb{E}(X_t - x_0)^2 = \mathbb{E}\, A_t < \infty$*.*

Proof. This is an instance of Wald's lemmas, see e.g. [13, Theorems 2.44 and 2.48]. □

Lemma 3.9. *If* $\mathbb{E}\, A_{t_0} < \infty$ *for some* $t_0 < \infty$*, then* $(X_t)_{0- \le t \le t_0}$ *is a square integrable martingale.*

Proof. Since each $A_t \wedge n$ is a bounded stopping time, if $0- \le s < t$, then

$$\mathbb{E}(B_{A_t \wedge n} \mid \mathcal{F}_{A_s}) = B_{A_t \wedge n \wedge A_s} = B_{A_s \wedge n} \tag{3.5}$$

a.s. (see e.g. [10, Theorem 7.29]), so $B_{A_t \wedge n}$, $t \ge 0-$, is a martingale. Furthermore, by Wald's lemma, i.e., since $(B_t - x_0)^2 - t$ is a martingale,

$$\mathbb{E}(B_{A_t \wedge n} - x_0)^2 = \mathbb{E}\,(A_t \wedge n)\,. \tag{3.6}$$

It follows that for any fixed $t \le t_0$, $\mathbb{E}(B_{A_t \wedge n} - x_0)^2 \le \mathbb{E}\, A_t < \infty$; hence the variables $B_{A_t \wedge n}$, $n \ge 1$, are uniformly integrable, and thus $B_{A_t \wedge n} \to B_{A_t}$ in L^1. If $0- \le s \le t \le t_0$, we thus obtain, by letting $n \to \infty$ in (3.5), $\mathbb{E}(B_{A_t} \mid \mathcal{F}_{A_s}) = B_{A_s}$ a.s.

Thus $X_t = B_{A_t}$, $0- \le t \le t_0$, is an integrable martingale; it is square integrable by Lemma 3.8. □

Note that the converse to Lemma 3.8 does not always hold: we may have $\operatorname{Var} X_t < \infty$ also when $\mathbb{E}\, A_t = \infty$. For example, this happens in Example 3.6 if $x_1 \ne x_0$. We give a simple sufficient condition for $\mathbb{E}\, A_t < \infty$.

Lemma 3.10. *Suppose that* $\operatorname{supp}_\infty \nu \cap (-\infty, x_0]$ *and* $\operatorname{supp}_\infty \nu \cap [x_0, \infty)$ *both are non-empty. Then* $(X_t)_{t \ge 0-}$*, is a bounded martingale with* $\mathbb{E}\, X_t = x_0$ *and* $\mathbb{E}\, A_t = \mathbb{E}(X_t - x_0)^2 < \infty$ *for every* $t \ge 0$*.*

Proof. Lemma 3.7 shows that $x_-^\infty \le X_t = B_{A_t} \le x_+^\infty$, for any $t \ge 0$, so (X_t) is uniformly bounded.

For each n, $A_t \wedge n$ is a bounded stopping time and (3.6) holds. Letting $n \to \infty$, we find $\mathbb{E}(X_t - x_0)^2 = \mathbb{E}\, A_t$ by dominated and monotone convergence, and thus $\mathbb{E}\, A_t \le \max\{(x_0 - x_-^\infty)^2, (x_+^\infty - x_0)^2\} < \infty$. Finally, Lemma 3.9 shows that X_t is a martingale. □

We have defined A_t in (2.2) so that it is right-continuous. The corresponding left-continuous process is

$$A_{t-} := \inf\{u \ge 0 : \Gamma_u \ge t\}; \tag{3.7}$$

note that for $t > 0$, $A_{t-} = \lim_{s \nearrow t} A_s$, while $A_{0-} = 0$ (as defined in Remark 2.2(v) above), and A_{t-} is a stopping time. It is possible that $A_{t-} < A_t$, i.e. that A_t jumps; this corresponds to time intervals where Γ_u is constant, because B_u moves in the complement of $\operatorname{supp}\nu$, so unless $\operatorname{supp}\nu = \mathbb{R}$, it will a.s. happen for some t. However, the next lemma shows that there is a.s. no jump for a fixed $t > 0$. (Equivalently, for a fixed $t > 0$, there is a.s. at most one $u > 0$ such that $\Gamma_u = t$.)

Lemma 3.11. *Let t be fixed with $0 < t < \infty$. Then a.s. $A_{t-} = A_t$.*

Note that the result fails for $t = 0$, see Lemma 3.3.

Proof. Let $\varepsilon > 0$. Since A_{t-} is a stopping time, Lemma 3.2 shows that a.s. there exists a neighbourhood U of $B_{A_{t-}}$ and some $\delta > 0$ such that $L^x_{A_{t-}+\varepsilon} \geq L^x_{A_{t-}} + \delta$ for $x \in U$. Further, since $A_{t-1/n} \to A_{t-}$ as $n \to \infty$, Lemma 3.1 implies that a.s. $B_{A_{t-}} \in \operatorname{supp}\nu$ and thus $\nu(U) > 0$; hence either $\Gamma_{A_{t-}} = \infty$ or

$$\Gamma_{A_{t-}+\varepsilon} \geq \Gamma_{A_{t-}} + \delta\nu(U) > \Gamma_{A_{t-}}. \tag{3.8}$$

If $\Gamma_{A_{t-}+\varepsilon} < \infty$, then Γ_u is continuous at $u = A_{t-}$, see Remarks 2.2(ii), and it follows from (3.7) that $\Gamma_{A_{t-}} \geq t$ (and actually $\Gamma_{A_{t-}} = t$); thus (3.8) yields $\Gamma_{A_{t-}+\varepsilon} > t$. This is trivially true also when $\Gamma_{A_{t-}+\varepsilon} = \infty$.

Thus, a.s., $\Gamma_{A_{t-}+\varepsilon} > t$, which implies that that $A_t \leq A_{t-} + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, and $A_{t-} \leq A_t$, the result follows. ∎

When considering several speed measures ν_n, we use n as a superscript to denote the corresponding Γ^n_u, A^n_t and X^n_t; we use always the same B_t.

If S is a topological space (in our case $\mathbb{R}$ or an interval in $\mathbb{R}$), we let $C_{\mathrm{c}}(S)$ denote the space of continuous functions $S \to \mathbb{R}$ with compact support, and $C^+_{\mathrm{c}}(S)$ the subset of such functions $S \to [0, \infty)$.

Lemma 3.12. *Let $\nu, \nu_1, \nu_2, \dots$ be a sequence of measures on $\mathbb{R}$. Assume either*

(i) $\int_{\mathbb{R}} \varphi \, \mathrm{d}\nu_n \to \int_{\mathbb{R}} \varphi \, \mathrm{d}\nu$ *as $n \to \infty$ for every $\varphi \in C^+_{\mathrm{c}}(\mathbb{R})$,*

or, more generally,

(ii) *there exists an interval (a, b) with $-\infty \leq a < x_0 < b \leq \infty$ such that $\int \varphi \, \mathrm{d}\nu_n \to \int \varphi \, \mathrm{d}\nu$ as $n \to \infty$ for every $\varphi \in C^+_{\mathrm{c}}(a, b)$, and also for every $\varphi \in C^+_{\mathrm{c}}(\mathbb{R})$ such that $\varphi(a) > 0$ or $\varphi(b) > 0$.*

Then, for each $t > 0$, $A^n_t \to A_t$ a.s., and thus, if $\nu, \nu_1, \nu_2, \dots$ are non-zero, $X^n_t \to X_t$ a.s., where $X, X^1, X^2, \dots$ are the corresponding generalised diffusions constructed from the same Brownian motion.

Proof. The local time $L^x_u \in C^+_{\mathrm{c}}(\mathbb{R})$, as a function of x, for every $u \geq 0$. In (i) we thus have, for every $u \geq 0$,

$$\Gamma^n_u = \int_{\mathbb{R}} L^x_u \nu_n(dx) \to \int_{\mathbb{R}} L^x_u \nu(dx) = \Gamma_u. \tag{3.9}$$

In (ii), let $H = H_{\{a,b\}}$ be the hitting time of $\{a, b\}$. If $u < H$, then the support of L^x_u is contained in (a, b), so $L^x_u \in C^+_{\mathrm{c}}(a, b)$ and $\Gamma^n_u \to \Gamma_u$ as in (3.9).

If $u > H$, then a.s. $L_u^{B_H} > 0$ by Lemma 3.2. Since $\varphi(x) := L_u^x$ is continuous and $B_H = a$ or $B_H = b$, the assumption shows that (3.9) holds in this case too.

Hence, in both (i) and (ii), $\Gamma_u^n \to \Gamma_u$ for all $u \geq 0$ except possibly when $u = H$.

Let $s = A_t$. By (2.2), $\Gamma_u > t$ for $u > s$ and $\Gamma_u \leq t$ for $u < s$. Further, if $\Gamma_u = t$ for some $u < s$, then $A_{t-} \leq u < s = A_t$, which has probability 0 by Lemma 3.11. Consequently, a.s. $\Gamma_u < t$ if $u < s$.

Assume for simplicity that $s = A_t < \infty$. (The case $A_t = \infty$ is similar.) If $\varepsilon > 0$ and $s + \varepsilon \neq H$, then thus $\Gamma_{s+\varepsilon}^n \to \Gamma_{s+\varepsilon} > t$. Hence, for sufficiently large n, $\Gamma_{s+\varepsilon}^n > t$ and thus $A_t^n \leq s + \varepsilon$. Since $\varepsilon > 0$ is almost arbitrary, it follows that $\limsup_{n\to\infty} A_t^n \leq s$ a.s. Similarly, considering $s - \varepsilon \neq H$, it follows that $\liminf_{n\to\infty} A_t^n \geq s$ a.s. Consequently, $A_t^n \to A_t$ a.s. as $n \to \infty$.

Finally, if $A_t < \infty$, then $X_t^n = B_{A_t^n} \to B_{A_t} = X_t$ a.s. follows, since B_u is continuous. □

Lemma 3.13. *Let $a > b > x_0$ and $\delta > 0$. Then there exists a constant $C = C(a, b, x_0, \delta)$ such that if $\mathbb{P}(X_1 \geq a) \geq \delta$, then $\nu[x_0, b] \leq C$.*

Proof. Assume for convenience $x_0 = 0$. By replacing a by $(a+b)/2$, we may also assume that $\mathbb{P}(X_1 > a) \geq \delta$.

Let $H = H_a = \inf\{u : B_u = a\}$. By definition, $X_1 = B_{A_1}$, so if $X_1 > a$, then $H < A_1$ and thus $\Gamma_H \leq 1$. Consequently, $\mathbb{P}(\Gamma_H \leq 1) \geq \delta$.

The local time L_H^x is a continuous function of $x \in \mathbb{R}$, and it is a.s. strictly positive on $[0, a)$ by Ray's theorem [13, Thm 6.38] (a consequence of the Ray–Knight theorem which gives its distribution). Hence, $Y = \inf_{x\in[0,b]} L_H^x > 0$ a.s., and thus there is a constant $c > 0$ such that $\mathbb{P}(Y < c) < \delta$.

Hence, with positive probability $\Gamma_H \leq 1$ and $Y \geq c$. However, then

$$1 \geq \Gamma_H = \int_{\mathbb{R}} L_H^x \nu(dx) \geq \int_0^b L_H^x \nu(dx) \geq Y \nu[0, b] \geq c\nu[0, b],$$

so $\nu[0, b] \leq 1/c$. □

Lemma 3.14. *For every $K > 0$ there exists $\kappa = \kappa(K) > 0$ such that if ν is a speed measure such that $\mathbb{E}\,|X_1| \leq K$, and further $\operatorname{supp}_\infty \nu \cap (-\infty, x_0]$ and $\operatorname{supp}_\infty \nu \cap [x_0, \infty)$ both are non-empty, then $\nu[x_0 - 2K, x_0 + 2K] \geq \kappa$.*

Proof. We may assume that $x_0 = 0$. By Lemma 3.7, $(B_{u\wedge A_1})_{u\geq 0}$ is a bounded, and thus uniformly integrable, martingale, closed by $B_{A_1} = X_1$.

Let $\tilde{H} = H_{\{\pm 2K\}}$ be the hitting time of $\pm 2K$. Then

$$\mathbb{P}(\Gamma_{\tilde{H}} \leq 1) = \mathbb{P}(\tilde{H} \leq A_1) \leq \mathbb{P}\bigl(\sup_u |B_{u\wedge A_1}| \geq 2K\bigr) \leq \frac{\mathbb{E}\,|B_{A_1}|}{2K} \leq \frac{K}{2K} = \frac{1}{2}. \tag{3.10}$$

Let $Y = \max_x L_{\tilde{H}}^x$; this is a finite random variable so there exists $c > 0$ such that $\mathbb{P}(Y > c) < 1/2$. (Note that Y and c depend on K but not on ν.) With positive probability we thus have both $\Gamma_{\tilde{H}} > 1$ and $Y \leq c$. This implies, since $L^x_{\tilde H} = 0$ when $|x| > 2K$,

$$1 < \Gamma_{\tilde H} = \int_{-2K}^{2K} L^x_{\tilde H}\, \nu(dx) \le \int_{-2K}^{2K} Y\, \nu(dx) \le c\nu[-2K, 2K],$$

and the result follows with $\kappa = c^{-1}$. □

4. The discrete case

In this section we treat the inverse problem in a discrete setting. We fix points $y_0 < y_1 < y_2 < \cdots < y_{n+1}$ and consider discrete speed measures $\nu = \sum_{i=0}^{n+1} b_i \delta_{y_i}$, where δ_a is a unit point mass at the point a and b_i takes values in $[0, \infty]$. We assume that $b_0 = b_{n+1} = \infty$. We also fix a starting point $x_0 \in (y_0, y_{n+1})$. (We could for simplicity assume that $x_0 = y_{i_0}$ for some $i_0 \in \{1, 2, \dots, n\}$, but that is not necessary.)

Given such a speed measure ν and x_0, we construct a generalised diffusion X as described in Section 2 above. By Lemma 3.1, the process $(X_t)_{t\ge 0}$ only takes values in the set $\{y_i\}_{i=0}^{n+1}$. Moreover, since $b_0 = b_{n+1} = \infty$, the states y_0 and y_{n+1} are absorbing, so X is bounded, and it follows from Lemma 3.9 (or Theorem 7.3) that X is a martingale; in particular $\mathbb{E}\, X_t = x_0$. Let $p_i = \mathbb{P}(X_1 = y_i)$ be the probability that X at time 1 is in state y_i. Then $0 \le p_i \le 1$, $\sum_{i=0}^{n+1} p_i = 1$, and we also have $\sum_{i=0}^{n+1} y_i p_i = \mathbb{E}\, X_1 = x_0$.

This defines a mapping G from the set of speed measures above to the set of distributions with mean x_0. More precisely, we write $G(b_1, \dots, b_n) := (p_0, \dots, p_{n+1})$, and note that $G : B^n \to \Pi^n$, where $B^n := [0, \infty]^n$ and

$$\Pi^n := \Big\{\pi = (\pi_0, \dots, \pi_{n+1}) \in [0, 1]^{n+2} : \sum_{i=0}^{n+1} \pi_i = 1 \text{ and } \sum_{i=0}^{n+1} y_i \pi_i = x_0\Big\}.$$

Lemma 4.1. *The function $G : B^n \to \Pi^n$ is continuous.*

Proof. This is a direct consequence of Lemma 3.12. (Note that the possibility that one or several $b_i = \infty$ is no problem when we verify condition (i).) □

We will use algebraic topology to show that G is surjective; see e.g. [1, Chapter IV] for standard definitions and results used below. We begin by studying the sets B^n and Π^n. The set B^n is homeomorphic to the unit cube $[0, 1]^n$ with the boundary $\partial B^n := [0, \infty]^n \setminus (0, \infty)^n$ corresponding to the boundary $\partial[0, 1]^n = [0, 1]^n \setminus (0, 1)^n$. We write

$$\partial B^n = \bigcup_{j=1}^{n} \big(\partial_{j0} B^n \cup \partial_{j\infty} B^n\big) \tag{4.1}$$

where $\partial_{js} B^n := \{(b_i)_1^n \in B^n : b_i = s\}$.

The set Π^n is the intersection $\Delta_{n+1} \cap M_{x_0}$ of the simplex

$$\Delta_{n+1} := \Big\{(\pi_i)_0^{n+1} : \pi_i \ge 0 \text{ and } \sum_{i=0}^{n+1} \pi_i = 1\Big\}$$

and the hyperplane

$$M_{x_0} := \Big\{ (\pi_i)_0^{n+1} : \sum_{i=0}^{n+1} y_i \pi_i = x_0 \Big\}$$

in $\mathbb{R}^{n+2}$. Further, Δ_{n+1} lies in the hyperplane $L := \big\{ (\pi_i)_0^{n+1} : \sum_{i=0}^{n+1} \pi_i = 1 \big\}$. Thus Π^n is a compact convex set in the n-dimensional plane $L \cap M_{x_0}$, which can be identified with $\mathbb{R}^n$. Since $y_0 < x_0 < y_{n+1}$, M_{x_0} contains an interior point of Δ_{n+1}, so $\Pi^n = \Delta_{n+1} \cap M_{x_0}$ has a non-empty relative interior in $L \cap M_{x_0}$. Thus Π^n is an n-dimensional compact convex set in $L \cap M_{x_0}$ and its boundary is

$$\partial \Pi^n = \Pi^n \cap \partial \Delta_{n+1} = \bigcup_{j=0}^{n+1} \partial_j \Pi^n,$$

where $\partial_j \Pi^n := \big\{ (\pi_i)_0^{n+1} \in \Pi^n : \pi_j = 0 \big\}$.

Consequently, both B^n and Π^n are homeomorphic to compact convex sets in $\mathbb{R}^n$ with non-empty interiors. Every such set is homeomorphic to the unit ball $D^n := \{x \in \mathbb{R}^n : |x| \le 1\}$ via a homeomorphism mapping the boundary onto the boundary $\partial D^n = S^{n-1}$. Thus there are homeomorphisms $(B^n, \partial B^n) \approx (D^n, S^{n-1})$ and $(\Pi^n, \partial \Pi^n) \approx (D^n, S^{n-1})$.

Lemma 4.2. *The function G maps ∂B^n into $\partial \Pi^n$, and thus $G : (B^n, \partial B^n) \to (\Pi^n, \partial \Pi^n)$.*

Proof. If $(b_i)_1^n \in \partial_{j0} B^n$, then $y_j \notin \operatorname{supp} \nu$, so $X_1 \neq y_j$ a.s. by Lemma 3.1 and thus $\pi_j = 0$, so $(\pi_i)_0^{n+1} \in \partial_j \Pi^n$.

If $(b_i)_1^n \in \partial_{j\infty} B^n$, then $y_j \in \operatorname{supp}_\infty \nu$. If further $y_j \le x_0$, then Lemma 3.5 implies that $X_1 \ge y_j$ a.s., and thus $\pi_i = 0$ for $0 \le i < j$ and, e.g., $(\pi_i)_0^{n+1} \in \partial_{j-1} \Pi^n$. Similarly, if $y_j \ge x_0$, then $\pi_i = 0$ for $j < i \le n+1$ and $(\pi_i)_0^{n+1} \in \partial_{j+1} \Pi^n$. □

The homeomorphisms above induce isomorphisms of the relative homology groups $H_n(B^n, \partial B^n) \approx H_n(\Pi^n, \partial \Pi^n) \approx H_n(D^n, S^{n-1}) \approx \mathbb{Z}$. The mapping degree of the function $G : (B^n, \partial B^n) \to (\Pi^n, \partial \Pi^n)$ can thus be defined as the integer $\deg(G)$ such that the homomorphism $G_* : H_n(B^n, \partial B^n) \to H_n(\Pi^n, \partial \Pi^n)$ corresponds to multiplication by $\deg(G)$ on $\mathbb{Z}$. More precisely, this defines the mapping degree up to sign; the sign depends on the orientation of the spaces, but we have no reason to care about the orientations so we ignore them and the sign of $\deg(G)$.

Lemma 4.3. *For any $n \ge 1$, any $y_0 < y_1 < \cdots < y_{n+1}$, and $x_0 \in (y_0, y_{n+1})$, $\deg(G) = \pm 1$.*

Proof. We use induction on the dimension n. We sometimes write $G = G_n$ for clarity. For the induction step, we assume $n \ge 2$. The long exact homology sequence yields the commutative diagram

$$\begin{array}{ccccccc} 0 = H_n(B^n) & \longrightarrow & H_n(B^n, \partial B^n) & \xrightarrow{\partial} & H_{n-1}(\partial B^n) & \longrightarrow & H_{n-1}(B^n) = 0 \\ & & \downarrow G_* & & \downarrow G_* & & \\ 0 = H_n(\Pi^n) & \longrightarrow & H_n(\Pi^n, \partial \Pi^n) & \xrightarrow{\partial} & H_{n-1}(\partial \Pi^n) & \longrightarrow & H_{n-1}(\Pi^n) = 0 \end{array}$$

where the rows are exact; thus the connecting homomorphisms ∂ are isomorphisms, and the degree of $G : (B^n, \partial B^n) \to (\Pi^n, \partial \Pi^n)$ equals the degree of the restriction $G : \partial B^n \to \partial \Pi^n$.

Assume $x_0 < y_n$. (Otherwise $x_0 \geq y_n > y_1$, and we may argue similarly using $\partial_{1\infty} B^n$ and $\partial_0 \Pi^n$.) We single out the faces $\partial_{n\infty} B^n$ and $\partial_{n+1}\Pi^n$ of the boundaries and define $\partial_* B^n := \bigcup_{i=1}^n \partial_{i0} B^n \cup \bigcup_{i=1}^{n-1} \partial_{i\infty} B^n$ and $\partial_* \Pi^n := \bigcup_{i=0}^n \partial_i \Pi^n$. By the proof of Lemma 4.2, $G : \partial_{n\infty} B^n \to \partial_{n+1}\Pi^n$ and $G : \partial_* B^n \to \partial_* \Pi^n$.

We claim that the degree of $G : \partial B^n \to \partial \Pi^n$ equals the degree of $G : (\partial_{n\infty} B^n, \partial_{n\infty} B^n \cap \partial_* B^n) \to (\partial_{n+1}\Pi^n, \partial_{n+1}\Pi^n \cap \partial_* \Pi^n)$. Using homeomorphisms $\partial B^n \approx S^{n-1}$ and $\partial \Pi^n \approx S^{n-1}$ that map the faces $\partial_{n\infty} B^n$ and $\partial_{n+1}\Pi^n$ onto the upper hemisphere S_+^{n-1}, this is an instance of the general fact that if $F : S^{n-1} \to S^{n-1}$ is continuous and maps $S_\pm^{n-1} \to S_\pm^{n-1}$, then the degree of $F : (S_+^{n-1}, S_+^{n-1} \cap S_-^{n-1}) \to (S_+^{n-1}, S_+^{n-1} \cap S_-^{n-1})$ equals the degree of $F : S^{n-1} \to S^{n-1}$.

If $(b_i)_1^n \in \partial_{n\infty} B^n$, then ν has infinite point masses at both y_n and y_{n+1}, with $x_0 < y_n < y_{n+1}$. By Lemma 3.5, we can ignore y_{n+1} and we obtain the same generalized diffusion X as with the speed measure $\sum_{i=0}^n b_i \delta_{y_i}$. We can thus identify $\partial_{n\infty} B^n$ with B^{n-1} (based on the points $(y_i)_0^n$). Furthermore, there is an obvious identification $\partial_{n+1}\Pi^n = \Pi^{n-1}$, and with these identifications, $G : \partial_{n\infty} B^n \to \partial_{n+1}\Pi^n$ corresponds to $G_{n-1} : B^{n-1} \to \Pi^{n-1}$. Moreover, the various boundaries correspond so that we have the commutative diagram

$$\begin{array}{ccc} H_{n-1}(\partial_{n\infty} B^n, \partial_{n\infty} B^n \cap \partial_* B^n) & = & H_{n-1}(B^{n-1}, \partial B^{n-1}) \\ \downarrow G_{n\,*} & & \downarrow G_{n-1\,*} \\ H_{n-1}(\partial_{n+1}\Pi^n, \partial_{n+1}\Pi^n \cap \partial_* \Pi^n) & = & H_{n-1}(\Pi^{n-1}, \partial \Pi^{n-1}) \end{array}$$

where the rows are the isomorphisms given by these identifications. Hence the degree of $G_n : (\partial_{n\infty} B^n, \partial_{n\infty} B^n \cap \partial_* B^n) \to (\partial_{n+1}\Pi^n, \partial_{n+1}\Pi^n \cap \partial_* \Pi^n)$ equals $\deg(G_{n-1})$.

Combining this with the equalities above, we see that $\deg(G_n) = \deg(G_{n-1})$, which completes the induction step.

It remains to treat the initial case $n = 1$. In this case B^1 and Π^1 are intervals, and can be parametrized by b_1 and p_1. It is easy to see that the mapping $G : b_1 \mapsto p_1$ is strictly increasing, and thus a homeomorphism $B^1 \to \Pi^1$; hence $G_* : H_1(B^1, \partial B^1) \to H_1(\Pi^1, \partial \Pi^1)$ is an isomorphism, so $\deg(G) = \pm 1$. Alternatively, we may use the first commutative diagram above also in the case $n = 1$, replacing the homology groups $H_{n-1} = H_0$ by the reduced homology groups $\tilde{H}_0$. The sets ∂B^1 and $\partial \Pi^1$ contain exactly two elements each, and it is easy to see that $G : \partial B^1 \to \partial \Pi^1$ is a bijection and thus $G_* : \tilde{H}_0(\partial B^1) \to \tilde{H}_0(\partial \Pi^1)$ is an isomorphism. □

Theorem 4.4. *The function G is surjective, for any $n \geq 1$ and any $y_0 < y_1 < \cdots < y_{n+1}$ and $x_0 \in (y_0, y_{n+1})$. Consequently, the discrete inverse problem has a solution.*

Proof. An immediate consequence of Lemma 4.3, since a function $(B^n, \partial B^n) \to (\Pi^n, \partial \Pi^n)$ (or, equivalently, $(D^n, S^{n-1}) \to (D^n, S^{n-1})$) that is not surjective has mapping degree 0. □

5. The general case

In this section we study the inverse problem for arbitrary distributions on the real axis. To do this, we approximate the given distribution with a sequence of discrete distributions. For each discrete distribution we can find a discrete speed measure that solves the inverse problem according to Section 4. We then show that the sequence of discrete speed measures has a convergent subsequence, and that the limit solves the inverse problem. We begin with a lemma giving the approximation that we shall use.

Lemma 5.1. *Let μ be a probability measure on $\mathbb{R}$ with finite mean $\overline{\mu} = \int_{\mathbb{R}} x\, \mu(dx)$. Then there exists a sequence μ_n, $n \geq 1$, of probability measures with finite supports such that, as $n \to \infty$,*

(i) *$\mu_n \to \mu$ weakly;*
(ii) $\inf \operatorname{supp} \mu_n \to \inf \operatorname{supp} \mu$*;*
(iii) $\sup \operatorname{supp} \mu_n \to \sup \operatorname{supp} \mu$*;*
(iv) *each μ_n has the same mean $\overline{\mu}$ as μ.*
(v) *$\int_{\mathbb{R}} |x|\, d\mu_n(x) \to \int_{\mathbb{R}} |x|\, d\mu(x)$.*

If further μ has finite variance $\operatorname{Var}(\mu) = \int_{\mathbb{R}} (x - \overline{\mu})^2 \mu(dx)$*, then μ_n can be chosen such that*

(vi) $\operatorname{Var}(\mu_n) \to \operatorname{Var}(\mu)$.

Proof. Let Y be a random variable with distribution μ. First, truncate Y at $\pm n$ by defining

$$Y_n' := (Y \wedge n) \vee (-n).$$

Then, letting all limits in this proof be for $n \to \infty$,

$$\mathbb{E}\,|Y_n' - Y| \leq \mathbb{E}(|Y|; |Y| > n) \to 0.$$

Next, discretize by defining

$$Y_n'' := \frac{1}{n} \lfloor n Y_n' \rfloor.$$

Clearly, $|Y_n'' - Y_n'| < 1/n$. It follows that

$$|\,\mathbb{E}\,Y_n'' - \mathbb{E}\,Y| \leq \mathbb{E}\,|Y_n'' - Y| \leq \frac{1}{n} + \mathbb{E}\,|Y_n' - Y| \to 0.$$

Finally, we adjust the mean by defining

$$Y_n := Y_n'' - \mathbb{E}(Y_n'' - Y). \tag{5.1}$$

Thus $\mathbb{E}\, Y_n = \mathbb{E}\, Y = \overline{\mu}$.

Let $\mu_n = \mathcal{L}(Y_n)$, the distribution of Y_n. Then μ_n has finite support and (iv) holds by the construction. Furthermore, by (5.1),

$$\mathbb{E}\,|Y_n - Y| \le 2\,\mathbb{E}\,|Y_n'' - Y| \to 0, \tag{5.2}$$

which implies $Y_n \xrightarrow{\mathrm{d}} Y$ and thus (i). From (5.2) we also have $\mathbb{E}\,|Y_n| \to \mathbb{E}\,|Y|$, which is (v).

If $\inf \operatorname{supp} \mu = -\infty$, then (i) implies that $\inf \operatorname{supp} \mu_n \to -\infty$, so (ii) holds.

Suppose now that $\inf \operatorname{supp} \mu = a > -\infty$. If $n > |a|$, then $\inf \operatorname{supp} \mathcal{L}(Y_n') = a$, and it follows that

$$|\inf \operatorname{supp} \mu_n - a| \le \frac{1}{n} + |\,\mathbb{E}\, Y_n'' - \mathbb{E}\, Y| \to 0,$$

which shows that (ii) hold in this case too.

The proof of (iii) is similar, mutatis mutandis.

If μ has finite variance, then $\mathbb{E}\, Y^2 < \infty$, and $\mathbb{E}\,|Y_n'|^2 = \mathbb{E}(|Y| \wedge n)^2 \to \mathbb{E}\, Y^2$. Taking square roots we find $\|Y_n'\|_2 \to \|Y\|_2$. Minkowski's inequality yields

$$\big|\|Y_n\|_2 - \|Y_n'\|_2\big| \le \|Y_n - Y_n'\|_2 \le \frac{1}{n} + \big|\mathbb{E}(Y_n'' - Y)\big| \to 0.$$

Consequently, $\|Y_n\|_2 \to \|Y\|_2$, and thus $\mathbb{E}\, Y_n^2 \to \mathbb{E}\, Y^2$. Since $\mathbb{E}\, Y_n = \mathbb{E}\, Y$, this implies $\operatorname{Var}(Y_n) \to \operatorname{Var}(Y)$, which shows (v). □

Proof of Theorem 2.3. We may for simplicity assume that $x_0 = \overline{\mu} = 0$.

Let $a_- = \inf \operatorname{supp} \mu \ge -\infty$ and $a_+ = \sup \operatorname{supp} \mu \le +\infty$. Since $\overline{\mu} = 0$, we have $a_- \le 0 \le a_+$. Moreover, if $a_- = 0$ or $a_+ = 0$, then necessarily $\mu = \delta_0$. In this case we may simply take ν as an infinite point mass at 0; then $A_t = 0$ and $X_t = 0$ for all $t \ge 0$ a.s., see Example 3.6. In the sequel we thus assume $-\infty \le a_- < 0$ and $0 < a_+ \le \infty$.

Let μ_n be a sequence of distributions satisfying (i)–(v) in Lemma 5.1. The distributions μ_n have finite supports, and thus Theorem 4.4 shows that there exist speed measures ν_n so that the corresponding generalised diffusion X^n has distribution μ_n at time 1.

If $0 < b < a_+$, choose $a \in (b, a_+)$. Then $a > b > 0$ and $a < \sup \operatorname{supp} \mu$, so $\mu(a, \infty) > 0$. Since $\mu_n \to \mu$, $\liminf_{n\to\infty} \mu_n(a, \infty) \ge \mu(a, \infty)$, so for all large n, $\mu_n(a, \infty) > \frac{1}{2}\mu(a, \infty) > 0$. Lemma 3.13 applies and implies that $\nu_n[0, b] \le C = C(b)$ for all large n, i.e., $\limsup_{n\to\infty} \nu_n[0, b] < \infty$ for every $b < a_+$.

By a symmetric argument, we also have $\limsup_{n\to\infty} \nu_n[b, 0] < \infty$ for every $b > a_-$.

Choose sequences $b_-^m \searrow a_-$ and $b_+^m \nearrow a_+$. On each interval $[b_-^m, b_+^m]$, the measures ν_n are, as we just have shown, uniformly bounded if we exclude a finite number of small n. Since $[b_-^m, b_+^m]$ is compact, we may thus choose a subsequence of ν_n such that the restrictions to $[b_-^m, b_+^m]$ converge to some measure $\hat{\nu}_m$. By a diagonal procedure, we can do this simultaneously for all m, which provides a subsequence of ν_n such that (along the subsequence) $\nu_n \to \hat{\nu}_m$ on $[b_-^m, b_+^m]$ for every m. In the sequel we consider only this subsequence.

It follows that $\Lambda(f) = \lim_{n\to\infty} \int_{\mathbb{R}} f \,\mathrm{d}\nu_n$ exists and is finite for every $f \in C_c^+(a_-, a_+)$. Clearly, Λ is a positive linear functional on $C_c^+(a_-, a_+)$, so by the Riesz representation theorem there exists a Borel measure $\widetilde{\nu}$ on (a_-, a_+) with $\Lambda f = \int_{\mathbb{R}} f \,\mathrm{d}\widetilde{\nu}$. Thus $\int_{\mathbb{R}} f \,\mathrm{d}\nu_n \to \int f \,\mathrm{d}\widetilde{\nu}$ for all $f \in C_c(a_-, a_+)$.

We define ν by adding infinite point masses at a_- and a_+, if these are finite: $\nu = \widetilde{\nu} + \infty \cdot \delta_{a_-} + \infty \cdot \delta_{a_+}$ (where $\delta_{\pm\infty} = 0$).

If a_- is finite and $f \in C_c^+(\mathbb{R})$ with $f(a_-) > 0$, then $\int f \,\mathrm{d}\nu \geq f(a_-)\nu\{a_-\} = \infty$. Furthermore, $a_-^n := \inf \operatorname{supp} \mu_n \to a_-$, so $f(a_-^n) > 0$ for all large n. The construction of ν_n in Theorem 4.4 gives ν_n an infinite point mass at a_-^n, so $\int f \,\mathrm{d}\nu_n = \infty$ for all large n. Thus $\int f \,\mathrm{d}\nu_n \to \int f \,\mathrm{d}\nu = \infty$ as $n \to \infty$. Similarly, $\int f \,\mathrm{d}\nu_n \to \int f \,\mathrm{d}\nu = \infty$ as $n \to \infty$ if $f(a_+) > 0$.

The assumptions of Lemma 3.12(ii) are satisfied, and thus a.s. $A_1^n \to A_1$ and, if $\nu \neq 0$, $X_1^n \to X_1$ as $n \to \infty$. In particular, then $X_1^n \to X_1$ in distribution, and since X_1^n has distribution μ_n and $\mu_n \to \mu$, it follows that the distribution of X_1 is μ.

It remains to verify that the measure ν is non-zero. If a_- or a_+ is finite, this is clear since ν by construction has a point mass there.

If $a_\pm = \pm\infty$, we use Lemma 5.1(v) which yields $\mathbb{E}\,|X_1^n| = \int_{\mathbb{R}} |x| \, d\mu_n(x) \to \int_{\mathbb{R}} |x| \, d\mu(x) < \infty$. Let $K := \sup_n \mathbb{E}\,|X_1^n| < \infty$. By Lemma 3.14, there exists $\kappa > 0$ such that $\nu_n[-2K, 2K] \geq \kappa$ for every n. Let $f \in C_c^+(-\infty, \infty)$ with $f = 1$ on $[-2K, 2K]$. As shown above, $\int_{\mathbb{R}} f \, d\nu_n \to \int_{\mathbb{R}} f \, d\widetilde{\nu} = \int_{\mathbb{R}} f \, d\nu$ as $n \to \infty$. Since $\int_{\mathbb{R}} f \, d\nu_n \geq \nu_n[-2K, 2K] \geq \kappa$, this implies $\int_{\mathbb{R}} f \, d\nu \geq \kappa > 0$, and thus $\nu \neq 0$.

This proves the existence of a non-zero speed measure ν such that X_1 has the desired distribution μ. We next prove that X_t is a martingale.

We have shown that $X_1^n \to X_1$ a.s. and, by Lemma 5.1(v), $\mathbb{E}\,|X_1^n| \to \mathbb{E}\,|X_1|$. This implies $\mathbb{E}\,|X_1^n - X_1| \to 0$, i.e. $X_1^n \to X_1$ in L^1 (see e.g. [10, Proposition 4.12]). For each n, $(B_{A_1^n \wedge u})_{u \geq 0}$ is by Lemma 3.7 a bounded martingale with limit $B_{A_1^n} = X_1^n$, and thus $B_{A_1^n \wedge u} = \mathbb{E}(X_1^n \mid \mathcal{F}_u)$ for every u. As $n \to \infty$, a.s. $A_1^n \to A_1$ and thus $B_{A_1^n \wedge u} \to B_{A_1 \wedge u}$. Further, we have just shown $X_1^n \to X_1$ in L^1, and this implies $B_{A_1^n \wedge u} = \mathbb{E}(X_1^n \mid \mathcal{F}_u) \to \mathbb{E}(X_1 \mid \mathcal{F}_u)$ in L^1. The two limit results both hold in probability, so the limits must coincide: $B_{A_1 \wedge u} = \mathbb{E}(X_1 \mid \mathcal{F}_u)$.

This proves that $(B_{A_1 \wedge u})_{u \geq 0}$ is a uniformly integrable martingale. Consequently, for any $(\mathcal{F}_t)$-stopping time τ, $B_{A_1 \wedge \tau} = \mathbb{E}(X_1 \mid \mathcal{F}_\tau)$. In particular, for $0- \leq t \leq 1$, $X_t = B_{A_t} = \mathbb{E}(X_1 \mid \mathcal{F}_{A_t})$, which proves that $(X_t)_{t \leq 1}$ is a martingale. This completes the main part of the proof, and we turn to (i) and (ii).

By Lemma 3.3, $X_0 = \overline{\mu}$ if and only if $\overline{\mu} \in \operatorname{supp} \nu$. If $\overline{\mu} \notin \operatorname{supp} \nu$, then there exists an open set U with $\overline{\mu} \in U$ and $\nu(U) = 0$. By Lemma 3.1, $X_1 \notin U$

a.s., so $\overline{\mu} \notin \operatorname{supp} \mu$. On the other hand, if $\overline{\mu} \in \operatorname{supp} \nu$ and U is any open set containing $\overline{\mu}$, then $\nu(U) > 0$. It is easy to see that there is a positive probability that B_t will remain in U until $\Gamma_t > 1$, and thus $X_1 \in U$. Hence $\mu(U) = \mathbb{P}(X_1 \in U) > 0$ for any such U, so $\overline{\mu} \in \operatorname{supp} \mu$.

If μ has finite variance, we may by Lemma 5.1(v) assume that $\operatorname{Var} \mu_n \to \operatorname{Var} \mu$, and thus $\sup_n \operatorname{Var} \mu_n < \infty$. Lemma 3.10 applies to every ν_n and yields

$$\mathbb{E}\, A_1^n = \mathbb{E}(X_1^n)^2 = \operatorname{Var}(\mu_n).$$

Consequently, by Fatou's lemma,

$$\mathbb{E}\, A_1 = \mathbb{E} \lim_{n\to\infty} A_1^n \le \liminf_{n\to\infty} \mathbb{E}\, A_1^n = \liminf_{n\to\infty} \operatorname{Var}(\mu_n) = \operatorname{Var}(\mu) < \infty. \tag{5.3}$$

Lemma 3.8 shows that $\mathbb{E}\, A_1 = \operatorname{Var}(\mu)$ in this case (so there is equality in (5.3)), and also if $\operatorname{Var}(\mu) = \infty$. □

Example 5.2. As an illustration of Theorem 2.3, let $Y_t = |B_t^{(3)}|^{-1}$, where $B^{(3)}$ is a 3-dimensional Brownian motion with $|B_0^{(3)}| = y_0^{-1}$ for some $y_0 > 0$. It is well-known that Y is a local martingale bounded from below, hence a supermartingale, but it is not a true martingale, compare Theorem 7.9 below. Moreover, Y can be represented as the solution of

$$\begin{cases} dY_t = Y_t^2\, dW_t, & t > 0, \\ Y_0 = y_0 \end{cases} \tag{5.4}$$

for some standard Brownian motion W. Note that Y is a diffusion by (5.4), and by Example 2.1 (together with a stopping argument), Y is the generalized diffusion with speed measure $x^{-4}dx$, $x > 0$. (Note that Y_t tends to 0 as $t \to \infty$, but never reaches 0.) Being the reciprocal of a Brownian motion, Y has an explicit density, compare [2, Example 2.2.2]. The density of Y_1 decays like Cx^{-4} for large x, so only moments of order strictly less than three exist finitely. Moreover, the expected value of Y_1 is strictly smaller than the starting value y_0.

Theorem 2.3 provides the existence of a time-homogeneous generalised diffusion X which is a true martingale such that X_1 and Y_1 have the same distribution, and $X_0 = \mathbb{E}\, Y_1 < y_0$. Consequently, there are in this case two different speed measures that give rise to the same distribution at time 1. However, only one of the corresponding processes is a martingale, and they have different starting points.

Example 5.3. A related example is when Y is the solution of

$$\begin{cases} dY_t = (1 + Y_t^2)\, dW_t, & t > 0, \\ Y_0 = 0. \end{cases}$$

By Example 2.1, the diffusion Y is the generalized diffusion with speed measure $(1+x^2)^{-2}dx$. Again, Y is a local martingale, but not a martingale, see Theorems 7.3 and 7.9 below.

Define $h(x,t) = e^{Mt}g(x)$, where g is a smooth positive function satisfying $g(x) = |x|$ for $|x| \geq 1$ and $g(x) \geq |x|$ everywhere, and M is a positive constant so that $h_t \geq \frac{1}{2}(1+x^2)^2 h_{xx}$. Let $u_N(x,t) = \mathbb{E}_x\,|Y_t| \wedge N$, where the index indicates that $Y_0 = x$. By a maximum principle argument, compare [7], $u_N(x,t) \leq h(x,t)$ independently of N. Consequently, by monotone convergence, $\mathbb{E}_0\,|Y_t| \leq h(0,t) < \infty$.

Theorem 2.3 thus applies and provides a generalised diffusion X_t which is a martingale started at 0 such that X_1 has the same distribution as Y_1. Consequently, we have in this case two different generalised diffusions with the same starting point that give rise to the same distribution at time 1. However, only one of these processes is a martingale.

6. An application to Mathematical Finance

In this section we study an inverse problem in Mathematical Finance. Let X be a non-negative martingale with $X_0 = x_0$, and consider the expected values

$$C(K,T) := \mathbb{E}(X_T - K)^+. \tag{6.1}$$

Here X_t has the interpretation as a price process of a stock, and the expected value $C(K,T)$ is the price of a call option with strike K and maturity T (for the sake of simplicity, we assume that interest rates are zero). The function C is non-increasing and convex as a function of K, and it satisfies $C(0,T) = x_0$, $C(\infty,T) = 0$ and $C(K,T) \geq (x_0 - T)^+$.

In Mathematical Finance, the corresponding inverse problem is of great interest. Option prices $C(K,T)$ are observable in the market, at least for a large collection of strikes K and maturities T, whereas the stock price model is not. Under some regularity conditions (often neglected in the literature), Dupire [5] determines a local volatility model

$$dX_t = \sigma(X_t,t)\,dW_t$$

such that (6.1) holds for all K and T. To do this, naturally one needs call option prices $C(K,T)$ given for all strikes $K \geq 0$ and all maturities $T > 0$.

The assumption that call option prices are known for all maturities T is often unnatural in applications - indeed, there is typically only one maturity a month. We therefore consider the situation where $C(K,T)$ is given for all strikes K but for *one* fixed maturity $T > 0$. A natural question is then whether there exists a time-homogeneous diffusion process

$$dX_t = \sigma(X_t)\,dW_t$$

such that (6.1) holds for all strikes. We have the following result.

Theorem 6.1. *Let $c : [0,\infty) \to [0,\infty)$ be a convex and non-increasing function satisfying $c(0) = x_0$, $c(\infty) = 0$ and $c(K) \geq (x_0 - K)^+$, and let $T > 0$ be fixed. Then there exists a time-homogeneous (generalised) diffusion such that X is a martingale with $X_{0-} = x_0$, and such that $c(K) = \mathbb{E}(X_T - K)^+$.*

Proof. We construct a probability distribution μ on $[0,\infty)$ as follows. On $(0,\infty)$, we let μ be defined as the measure given by the second derivative of the convex function c. Since $c(\infty)=0$, this measure has mass $-c'(0)$, where $c'(0)$ is the (right) derivative of c at 0. Moreover, we let a point mass $\epsilon = 1+c'(0)$ be located at 0; note that the assumptions $c(0)=x_0$ and $c(K)\geq (x_0-K)^+\geq x_0-K$ imply that $c'(0)\geq -1$ so $\epsilon\geq 0$.

Integration by parts shows that the expected value of the distribution μ is given by

$$\int_0^\infty x\mu(dx) = \int_0^\infty \mu(x,\infty)\,dx = -\int_0^\infty c'(x)\,dx = c(0) = x_0.$$

Now, by Theorem 2.3 (and an obvious scaling to general T) there exists a time-homogeneous generalised diffusion X with $X_{0-}=x_0$ such that X is a martingale and X_T has distribution μ. Finally, integration by parts gives

$$\mathbb{E}(X_T-K)^+ = \int_K^\infty (x-K)\mu(dx) = \int_K^\infty \mu(x,\infty)\,dx = c(K),$$

thus finishing the proof. □

7. Martingality of generalised diffusions

The speed measure constructed in the proof of Theorem 2.3 is such that the generalized diffusion (X_t) is a martingale, but as mentioned above, this is not the case for every speed measure ν. We characterize here the speed measures for which X_t is a (local) martingale.

Example 7.1. If the speed measure is given by

$$\nu(dx) = \begin{cases} dx & \text{for } x\geq 0 \\ 0 & \text{for } x<0, \end{cases}$$

then the corresponding process X is a Brownian motion reflected at 0, and, in particular, not a local martingale.

This type of reflection at an extreme point of $\operatorname{supp}\nu$ is the only thing that can prevent X from being a local martingale. This is shown in the theorem below from [6], here somewhat extended.

We first give a lemma, essentially saying that stopping X at a point $a\in\operatorname{supp}\nu$ is the same as stopping B at a. (Note that we cannot stop X at $a\notin\operatorname{supp}\nu$ since $X_t\in\operatorname{supp}\nu$ for all $t\geq 0$ by Lemma 3.1.)

Lemma 7.2. *If $a\in(x_-^\infty,x_+^\infty)\cap\operatorname{supp}\nu$ and $\tau:=\inf\{t: X_t=a\}$, then $\tau=\Gamma_{H_a}$. If $H_a<H_{\operatorname{supp}_\infty\nu}$, then also $\tau<\infty$ and $A_\tau=H_a$, while $\tau=\infty$ if $H_a>H_{\operatorname{supp}_\infty\nu}$. Furthermore, $\mathbb{P}(H_a<H_{\operatorname{supp}_\infty\nu})>0$ and $\mathbb{P}(\tau<t_0)>0$ for any $t_0>0$.*

Proof. If $H_a<H_{\operatorname{supp}_\infty\nu}$, then $\Gamma_{H_a}<\infty$ by Lemma 3.5 and thus Lemma 3.2 and the assumption $a\in\operatorname{supp}\nu$ imply that $\Gamma_{H_a+\varepsilon}>\Gamma_{H_a}$ for every $\varepsilon>0$. Thus, if $t=\Gamma_{H_a}$, then $A_t=H_a$ and $X_t=B_{A_t}=B_{H_a}=a$. On the other hand, if $t<\Gamma_{H_a}$, then $A_t<H_a$ by (2.3) so $X_t=B_{A_t}\neq a$. Hence, $\tau=\Gamma_{H_a}$.

If $H_a > H_{\mathrm{supp}_\infty \nu}$, then Lemma 3.5 yields $\Gamma_{H_a} = \infty$. Lemma 3.5 yields further, for any $t < \infty$, $A_t \le H_{\mathrm{supp}_\infty \nu} < H_a$ so $X_t = B_{A_t} \neq a$. Hence, $\tau = \infty = \Gamma_{H_a}$.

For the final statement, assume that $a \le x_0$, say. Note that the assumption implies $x_0 \notin \mathrm{supp}_\infty \nu$, so $x_-^\infty < x_0 < x_+^\infty$. Choose $b \in (x_0, x_+^\infty)$. Then $\nu[a,b] < \infty$. Let $\varepsilon := t_0/\nu[a,b]$. (The case $\nu[a,b] = 0$ is easy.) It is easy to see (e.g. using excursion theory) that with positive probability both $\sup_x L^x_{H_a} < \varepsilon$ and $H_a < H_b$. Thus $B_s \in [a,b)$ for all $s \le H_a$, and then $\tau = \Gamma_{H_a} = \int_a^b L^x_{H_a} \nu(dx) < \varepsilon\nu[a,b] = t_0$.

The claim $\mathbb{P}(H_a < H_{\mathrm{supp}_\infty \nu}) > 0$ follows from this, using Lemma 3.5, but it is easier to see it directly since $H_{\mathrm{supp}_\infty \nu} = H_{\{x_-^\infty, x_+^\infty\}}$. ∎

Theorem 7.3. $(X_t)_{t\ge 0-}$ *is a local martingale if and only if either* $\mathrm{supp}\,\nu = \{x_0\}$ *(then, trivially,* $X_t = x_0$ *for all* t*) or both the following conditions hold:*

(i) $\mathrm{supp}_\infty \nu \cap (-\infty, x_0] \neq \emptyset$ *or* $\mathrm{supp}\,\nu \cap (-\infty, b] \neq \emptyset$ *for all* $b < x_0$*,*
(ii) $\mathrm{supp}_\infty \nu \cap [x_0, \infty) \neq \emptyset$ *or* $\mathrm{supp}\,\nu \cap [b, \infty) \neq \emptyset$ *for all* $b > x_0$*.*

Moreover, if $(X_t)_{0-\le t\le t_0}$ *is a local martingale for some* $t_0 > 0$*, then* $(X_t)_{t\ge 0-}$ *is a local martingale.*

Proof. If $\mathrm{supp}\,\nu = \{x_0\}$, then $X_t = x_0$ a.s. for all t by Lemma 3.1.

Suppose now that (i) and (ii) hold. If $\mathrm{supp}_\infty \nu \cap (-\infty, x_0] \neq \emptyset$, let $a_n^- := x_-^\infty = \sup\{x \le x_0 : x \in \mathrm{supp}_\infty \nu\}$ for each n; otherwise let a_n^- be a sequence of points in $\mathrm{supp}\,\nu \cap (-\infty, x_0)$ with $a_n^- \searrow -\infty$ as $n \to \infty$. Define a sequence $a_n^+ \ge x_0$ similarly, using (ii).

Let $H_n := H_{\{a_n^-, a_n^+\}}$. For each n, $(B_{t\wedge H_n})_{t\ge 0}$ is a bounded martingale. Thus, if $s < t$, then $\mathbb{E}(B_{A_t \wedge H_n} \mid \mathcal{F}_{A_s}) = B_{A_s\wedge H_n}$, so $(B_{A_t\wedge H_n})_{t\ge 0-}$ is a martingale for the filtration $(\mathcal{G}_t) = (\mathcal{F}_{A_t})$.

Set $T_n := \Gamma_{H_n+} = \inf\{\Gamma_u : u > H_n\}$. T_n is a stopping time for the filtration $(\mathcal{G}_t)$. Further, since $H_n \le H_{n+1}$ we have $T_n \le T_{n+1}$.

If $T_n < \infty$, then $\Gamma_{H_n} = T_n < \infty$ and it follows from Lemma 3.2 that a.s. then $\Gamma_u > T_n$ for all $u > H_n$, so $A_{T_n} = H_n$; hence, $B_{A_t\wedge H_n} = B_{A_t\wedge A_{T_n}} = B_{A_{t\wedge T_n}} = X_{t\wedge T_n}$. On the other hand, it $T_n = \infty$, then $A_t \le H_n$ for every t, and thus $B_{A_t\wedge H_n} = B_{A_t} = X_t = X_{t\wedge T_n}$. In any case, thus $B_{A_t\wedge H_n} = X_{t\wedge T_n}$ a.s., so $(X_{t\wedge T_n})_{t\ge 0-}$ is a martingale.

Finally, we verify that $T_n \to \infty$ a.s. as $n \to \infty$. Recall that $B_{H_n} \in \{a_n^-, a_n^+\}$; we consider for definiteness the case when $B_{H_n} = a_n^-$, so $H_n = H_{a_n^-}$. By construction, either $a_n^- \in \mathrm{supp}_\infty \nu$, and then Lemma 3.5 implies that $\Gamma_{H_{a_n^-}+} = \infty$ a.s., or else $a_n^- \to -\infty$ and then $H_{a_n^-} \to \infty$ and thus $\Gamma_{H_{a_n^-}} \to \infty$ a.s. In both cases we obtain $T_n = \Gamma_{H_{a_n^-}+} \to \infty$ a.s. This completes the proof that $(X_t)_{t\ge 0-}$ is a local martingale.

For the converse, we assume that $\mathrm{supp}\,\nu \neq \{x_0\}$. Assume that (i) fails. (The case when (ii) fails is symmetric.) Let $\xi_- := \inf\{x : x \in \mathrm{supp}\,\nu\}$. Since (i) fails, $\xi_- > -\infty$. We may assume that $\xi_- \le x_0$, since otherwise Lemma 3.4 shows that X_t is not a local martingale at $0-$.

Let $T_1 = H_{\xi_-}$, the hitting time for B_u of ξ_-. Since (i) fails, $[\xi_-, x_0] \cap \operatorname{supp}_\infty \nu = \emptyset$ so $\xi_- \in (x_-^\infty, x_+^\infty)$ and Lemma 7.2 shows that with positive probability $T_1 = H_{\xi_-} < H_{\operatorname{supp}_\infty \nu}$, i.e., B_u hits ξ_- before it hits $\operatorname{supp}_\infty \nu$; denote this event by $\mathcal{E}$. Lemma 7.2 further shows that if $\mathcal{E}$ holds, then there exists $s < \infty$ (viz. $s = \Gamma_{T_1}$) such that $X_s = \xi_-$. We want to show that in this case there also exists $t > s$ with $X_t \neq \xi_-$.

If $\operatorname{supp} \nu = \{\xi_-\}$, then $\xi_- < x_0$ by our assumption $\operatorname{supp} \nu \neq \{x_0\}$, but then X_t is not a local martingale at $0-$ by Lemma 3.4. We may thus assume that there exists another point $x_2 \neq \xi_-$ in $\operatorname{supp} \nu$. Let T_2 be the first hitting time of x_2 after T_1. We study two cases separately.

If $\Gamma_{T_2} < \infty$, then Lemma 3.2 implies that a.s. $\Gamma_{T_2+\varepsilon} > \Gamma_{T_2}$ for all $\varepsilon > 0$. In this case, if $t = \Gamma_{T_2}$, then $A_t = T_2$ and $X_t = B_{T_2} = x_2 \neq \xi_-$.

If $\Gamma_{T_2} = \infty$, then $A_\infty = \inf\{u : \Gamma_u = \infty\}$ is finite. If U is a neighbourhood of B_{A_∞}, we can find $\varepsilon > 0$ such that $B_u \in U$ for $u \in (A_\infty - \varepsilon, A_\infty + \varepsilon)$, and thus $L^x_{A_\infty+\varepsilon} = L^x_{A_\infty-\varepsilon}$ for $x \notin U$. Since $\Gamma_{A_\infty+\varepsilon} - \Gamma_{A_\infty-\varepsilon} = \infty$, it follows that $\nu(U) = \infty$. Hence $B_{A_\infty} \in \operatorname{supp}_\infty \nu$, and in particular $B_{A_\infty} \neq \xi_-$. As $t \to \infty$, $A_t \to A_\infty$ and thus $X_t = B_{A_t} \to B_{A_\infty} \neq \xi_-$, so in this case, $X_t \neq \xi_-$ for all large t.

Note also that $X_t \geq \xi_-$ for all $t \geq 0$ by Lemma 3.1. If $(X_t)_{t \geq 0-}$ is a local martingale, then Y given by $Y_t = X_t - \xi_-$ is thus a non-negative local martingale, hence a supermartingale. Therefore zero is absorbing for Y, see e.g. [14, Proposition (3.4)]. But this contradicts our result above which says that with positive probability Y_t first hits zero and then takes a larger value. Hence X is not a local martingale.

For the final statement, it suffices to show that $\mathbb{P}(S_\delta \leq t_0) > 0$ for some $\delta > 0$ since the argument above then shows that $(X_t)_{t \leq t_0}$ is not a local martingale. With $\tau_1 = \inf\{t \geq 0 : X_t = \xi_-\} = \Gamma_{T_1}$ and $\tau_2 = \inf\{t > \tau_1 : X_t \neq \xi_-\}$ it thus suffices to show $\mathbb{P}(\tau_2 < t_0) > 0$.

By Lemma 7.2, $\mathbb{P}(\tau_1 \leq t_0/2) > 0$. Let $\Delta := \tau_2 - \tau_1$ and condition on $\mathcal{E}$. Then $\Delta < \infty$ a.s. and the strong Markov property of B implies that Δ is independent of τ_1 and, moreover, that $\mathbb{P}(\Delta > a + b \mid \Delta > a) = \mathbb{P}(\Delta > b)$ for any $a, b > 0$ with $\mathbb{P}(\Delta > a) > 0$; thus (conditioned on $\mathcal{E}$) either $\Delta = 0$ a.s. or Δ has an exponential distribution; in both cases $\mathbb{P}(\Delta < t_0/2) > 0$. Hence, $\mathbb{P}(\tau_2 < t_0) > 0$ which completes the proof. □

Before giving the corresponding characterization for martingales, we give some lemmas.

Lemma 7.4. *If $A_t < H_{\operatorname{supp}_\infty \nu}$, then $\Gamma_{A_t} = t$. Consequently, for all $t \geq 0$, $\Gamma_{A_t} = t \wedge \Gamma_{H_{\operatorname{supp}_\infty \nu}}$.*

Proof. If $A_t < H_{\operatorname{supp}_\infty \nu}$, then for small $\varepsilon > 0$ we have $A_t + \varepsilon < H_{\operatorname{supp}_\infty \nu}$ and thus $\Gamma_{A_t+\varepsilon} < \infty$ by Lemma 3.5, which by Remark 2.2(iii) yields $\Gamma_{A_t} = t$. Furthermore, in this case $t = \Gamma_{A_t} \leq \Gamma_{H_{\operatorname{supp}_\infty \nu}}$ so $\Gamma_{A_t} = t = t \wedge \Gamma_{H_{\operatorname{supp}_\infty \nu}}$.

Since $A_t \leq H_{\operatorname{supp}_\infty \nu}$ by Lemma 3.5, the only remaining case is $A_t = H_{\operatorname{supp}_\infty \nu}$. In this case $\Gamma_{H_{\operatorname{supp}_\infty \nu}} = \Gamma_{A_t} \leq t$ so $t \wedge \Gamma_{H_{\operatorname{supp}_\infty \nu}} = \Gamma_{H_{\operatorname{supp}_\infty \nu}} = \Gamma_{A_t}$. □

Lemma 7.5. *For every measurable function $f \geq 0$, a.s. for every $u \leq H_{\operatorname{supp}_\infty \nu}$,*

$$\int_0^u f(B_s)\,d\Gamma_s = \int_{-\infty}^{\infty} f(x) L_u^x\,\nu(dx). \tag{7.1}$$

Proof. By monotone convergence it suffices to consider $u < H_{\operatorname{supp}_\infty \nu}$, and then, by modifying ν outside the range of $\{B_s : s \leq u\}$, it suffices to consider locally finite ν. In this case, the result is a consequence of the general theory of continuous additive functionals, see [14, Corollary X.(2.13)]. (It is also easy to make a direct proof in this case, by first assuming that f is continuous, and then partitioning the interval $[0,u]$ into N subintervals, and for each subinterval estimating the change of the difference between the two sides in (7.1); we omit the details.) □

Lemma 7.6. *If $f \geq 0$ is a deterministic or random measurable function, then a.s. for every $T \leq H_{\operatorname{supp}_\infty \nu}$,*

$$\int_0^T f(s)\,d\Gamma_s = \int_0^{\Gamma_T} f(A_t)\,dt. \tag{7.2}$$

Proof. Consider a fixed ω in our probability space. By a monotone class argument (or by seeing the two sides of (7.2) as $\int f\,d\mu_L$ and $\int f\,d\mu_R$ for two finite measures μ_L, μ_R on $[0,\infty)$), it suffices to prove (7.2) when f is the indicator of an interval $[0,u)$ for some $u > 0$. In this case $\int_0^T f(s)\,d\Gamma_s = \int_0^{T\wedge u} d\Gamma_s = \Gamma_{T\wedge u}$ and, by Remarks 2.2(iii),

$$\int_0^{\Gamma_T} f(A_t)\,dt = \int_0^{\Gamma_T} \mathbf{1}\{t < \Gamma_u\}\,dt = \Gamma_T \wedge \Gamma_u = \Gamma_{T\wedge u}. \qquad \square$$

If $(X_t)_{t\geq 0-}$ is a martingale, then $\mathbb{E}\,X_{t\wedge\tau} = x_0$ for every X-stopping time τ and every $t \geq 0$. This means that $\mathbb{E}\,B_T = x_0$ for the B-stopping time $T = A_t \wedge A_\tau$. We would like to have $\mathbb{E}\,B_T = x_0$ also for other B-stopping times $T \leq A_t$, not necessarily obtained by stopping X. However, this is not always possible, as is seen by the following example.

Example 7.7. If $\operatorname{supp}\nu = \{x_0\}$, then $X_t = x_0$ for all $t \geq 0-$, so X_t is trivially a martingale. However, if further $\nu\{x_0\} < \infty$, for example if $\nu = \delta_{x_0}$, then for any $t > 0$ and $a \neq x_0$, $\mathbb{P}(B_{A_t\wedge H_a} = a) = \mathbb{P}(H_a < A_t) > 0$; since $B_{A_t\wedge H_a} \in \{a, x_0\}$, this implies $\mathbb{E}\,B_{A_t\wedge H_a} \neq x_0$.

The next lemma shows that Example 7.7 is the only counterexample. (The trivial example shows that the lemma is not as trivial as it might look.)

Lemma 7.8. *Suppose that* $\operatorname{supp}\nu \neq \{x_0\}$ *and that $(X_t)_{t\leq t_0}$ is a martingale for some $t_0 > 0$. Then, for every stopping time $T \leq A_{t_0}$ and every real a, $\mathbb{E}\,B_{T\wedge H_a} = x_0$.*

It will follow from Theorem 7.9 and its proof that, more generally, $\mathbb{E}\,B_T = x_0$ for any such T, so the H_a is not really needed but it simplifies the proof.

Proof. We suppose that $a < x_0$. (The case $a > x_0$ is similar and $a = x_0$ is trivial.)

Suppose first that $a \in \operatorname{supp}\nu$. Then Lemma 3.2 implies that a.s. $\Gamma_{H_a+\varepsilon} > \Gamma_{H_a}$ for every $\varepsilon > 0$. Hence, if $\tau := \Gamma_{H_a}$, then $A_\tau = H_a$ and $X_\tau = B_{H_a} = a$; moreover, $X_s \neq a$ for $s < H_a$. In other words, τ is the X-stopping time $\inf\{s : X_s = a\}$. The assumption that $(X_t)_{t\le t_0}$ is a martingale thus implies, for $t \le t_0$,

$$x_0 = \mathbb{E}\, X_{t\wedge\tau} = \mathbb{E}\, B_{A_t\wedge A_\tau} = \mathbb{E}\, B_{A_t\wedge H_a}. \tag{7.3}$$

If T' and T'' are two stopping times with $T' \ge T''$, then $\mathbb{E}\big(B_{T'\wedge u} | \mathcal{F}_{T''}\big) = B_{T''\wedge u}$ for every $u \ge 0$ (see e.g. [10, Theorem 7.29]). If $T' \le H_a$, then $B_s - a \ge 0$ for all $s \le T'$, and Fatou's lemma yields $\mathbb{E}(B_{T'} - a \mid \mathcal{F}_{T''}) \le B_{T''} - a$ and thus, by taking the expectation,

$$\mathbb{E}\, B_{T'} \le \mathbb{E}\, B_{T''}. \tag{7.4}$$

We apply (7.4) first to $A_{t_0} \wedge H_a$ and $T \wedge H_a$ and then to $T \wedge H_a$ and 0, obtaining

$$\mathbb{E}\, B_{A_{t_0}\wedge H_a} \le \mathbb{E}\, B_{T\wedge H_a} \le \mathbb{E}\, B_0 = x_0, \tag{7.5}$$

and (7.3) shows that we have equalities. This proves the result when $a \in \operatorname{supp}\nu$.

In general, by Theorem 7.3, either there exists some $b < a$ with $b \in \operatorname{supp}\nu$, or there exists $b \in [a, x_0]$ with $b \in \operatorname{supp}_\infty \nu$. In the first case $H_a < H_b$ and in the second case $H_a \ge H_b$ and Lemma 3.5 implies that $A_{t_0} \le H_b$ and thus $T \le A_{t_0} \le H_b \le H_a$. Consequently, in both cases $T \wedge H_a = T \wedge H_a \wedge H_b$, and the result follows from the case just proven applied to $T \wedge H_a$ and b. □

Theorem 7.9. *The following are equivalent, for any $t_0 > 0$.*

(i) $(X_t)_{t\ge 0-}$ *is a martingale.*
(ii) $(X_t)_{0-\le t\le t_0}$ *is a martingale.*
(iii) $\operatorname{supp}\nu = \{x_0\}$ *or* $x_0 \in \operatorname{supp}_\infty \nu$ *or*

$$\int_{x_0}^{\infty} (1 + |x|)\nu(dx) = \int_{-\infty}^{x_0} (1 + |x|)\nu(dx) = \infty. \tag{7.6}$$

(iv) $\operatorname{supp}\nu = \{x_0\}$ *or* $x_0 \in \operatorname{supp}_\infty \nu$ *or*

$$\int_{x_0}^{\infty} |x - x_0|\,\nu(dx) = \int_{-\infty}^{x_0} |x - x_0|\,\nu(dx) = \infty. \tag{7.7}$$

Remark 7.10. For a related result for non-negative diffusion processes, see [4].

Proof. The result is trivial if $\operatorname{supp}\nu = \{x_0\}$ or $x_0 \in \operatorname{supp}_\infty \nu$. We may thus assume $\operatorname{supp}\nu \neq \{x_0\}$ and $-\infty \le x_-^\infty < x_0 < x_+^\infty \le \infty$. The equivalence of (7.6) and (7.7) then is elementary.

Let

$$\varphi(x) := \begin{cases} 2\int_{[x_0,x)} |y|\,\nu(dy), & x \ge x_0, \\ -2\int_{[x,x_0)} |y|\,\nu(dy), & x < x_0, \end{cases} \tag{7.8}$$

and

$$\psi(x) := \begin{cases} \int_{x_0}^{x} \varphi(y)\,dy, & x \geq x_0, \\ -\int_{x}^{x_0} \varphi(y)\,dy, & x < x_0. \end{cases} \tag{7.9}$$

Then ψ is a non-negative convex function on (x_-^∞, x_+^∞) with left derivative φ and thus second derivative (in distribution sense) $\psi''(x) = 2|x|\nu(dx)$.

By the Itô–Tanaka formula [14, Theorem VI.(1.5)] and Lemma 7.5, for $u \leq H_{\operatorname{supp}_\infty \nu}$,

$$\psi(B_u) = \int_0^u \varphi(B_s)\,dB_s + \int_{\mathbb{R}} L_u^x\,|x|\,\nu(dx) = \int_0^u \varphi(B_s)\,dB_s + \int_0^u |B_s|\,d\Gamma_s. \tag{7.10}$$

Let $H := H_{\{a,b\}}$ where $x_-^\infty < a < x_0 < b < x_+^\infty$. Then $H < H_{\operatorname{supp}_\infty \nu}$. Further, $\varphi(B_s)$ is bounded for $s \leq H$, and consequently $\int_0^{u\wedge H} \varphi(B_s)\,dB_s$, $u \geq 0$, is a martingale. Hence, for every bounded stopping time $T \leq H$, $\mathbb{E}\int_0^T \varphi(B_s)\,dB_s = 0$, and (7.10) yields

$$\mathbb{E}(\psi(B_T)) = \mathbb{E}\int_0^T |B_s|\,d\Gamma_s. \tag{7.11}$$

Moreover, for any stopping time $T \leq H$, we can apply (7.11) to $T\wedge u$ and let $u \to \infty$. Since $T \leq H$ we have $B_{T\wedge u} \in [a,b]$, and thus $\psi(B_{T\wedge u})$ is uniformly bounded; hence dominated convergence on the left-hand side and monotone convergence on the right-hand side shows that (7.11) holds for any stopping time $T \leq H$.

We apply (7.11) first to $T = A_r \wedge H \wedge u$ for some $r, u \geq 0$. Thus, using Lemma 7.6 and $\Gamma_{A_r} \leq r$, see Remarks 2.2(iii),

$$\begin{aligned} \mathbb{E}(\psi(B_{A_r\wedge H\wedge u})) &= \mathbb{E}\int_0^{A_r\wedge H\wedge u} |B_s|\,d\Gamma_s = \mathbb{E}\int_0^{A_r\wedge H\wedge u} |B_{s\wedge H\wedge u}|\,d\Gamma_s \\ &= \mathbb{E}\int_0^{\Gamma_{A_r\wedge H\wedge u}} |B_{A_t\wedge H\wedge u}|\,dt \leq \mathbb{E}\int_0^r |B_{A_t\wedge H\wedge u}|\,dt = \int_0^r \mathbb{E}\,|B_{A_t\wedge H\wedge u}|\,dt. \end{aligned} \tag{7.12}$$

Suppose first that (7.7) holds. By translation we may assume that $x_0 = 0$. Then $\varphi(x) \to \pm\infty$ as $x \to \pm\infty$, and thus $\psi(x)/|x| \to \infty$ as $x \to \pm\infty$. In particular, $\psi(x) \geq |x|$ for large $|x|$, and thus $|x| \leq \psi(x) + C$ for some constant C. Consequently, (7.12) yields

$$\mathbb{E}\,|B_{A_r\wedge H\wedge u}| \leq C + \mathbb{E}(\psi(B_{A_r\wedge H\wedge u})) \leq C + \int_0^r \mathbb{E}\,|B_{A_t\wedge H\wedge u}|\,dt.$$

Note further that $B_{A_t\wedge H\wedge u}$ is bounded by $\max\{|a|, b\}$, so the expectations here are finite and bounded. We can thus apply Gronwall's Lemma [14, Appendix §1] and conclude

$$\mathbb{E}\,|B_{A_t\wedge H\wedge u}| \leq Ce^t. \tag{7.13}$$

Using this in (7.12) again yields the sharper estimate

$$\mathbb{E}(\psi(B_{A_t\wedge H\wedge u})) \le Ce^t. \tag{7.14}$$

Now let $a \to x_-^\infty$ and $b \to x_+^\infty$; then $H \to H_{\mathrm{supp}_\infty \nu} = H_{\{x_-^\infty, x_+^\infty\}}$. Since $A_t \le H_{\mathrm{supp}_\infty \nu}$ by Lemma 3.5, $B_{A_t\wedge H\wedge u} \to B_{A_t\wedge u}$, and Fatou's lemma yields

$$\mathbb{E}(\psi(B_{A_t\wedge u})) \le Ce^t. \tag{7.15}$$

Since $\psi(x)/|x| \to \infty$ as $x \to \pm\infty$, (7.15) implies that for a fixed t, the random variables $B_{A_t\wedge u}$, $u \ge 0$, are uniformly integrable [8, Theorem 5.4.3]. Moreover, $B_{A_t\wedge u}$, $u \ge 0$, is a martingale, and since it is uniformly integrable, $B_{A_t\wedge u} = \mathbb{E}(B_{A_t} \mid \mathcal{F}_u)$, for all $u \ge 0$, and further, see [10, Theorem 7.29], for any $s \le t$,

$$\mathbb{E}(B_{A_t} \mid \mathcal{F}_{A_s}) = B_{A_t\wedge A_s} = B_{A_s}. \tag{7.16}$$

Hence $X_t = B_{A_t}$ is a martingale when (7.7) holds, so (iv) $\Longrightarrow$ (i) $\Longrightarrow$ (ii).

Conversely, suppose that (ii) holds but (7.6) fails; by symmetry we may assume that

$$\int_{x_0}^{\infty} (1+|x|)\,\nu(dx) < \infty. \tag{7.17}$$

In particular, this shows that $\mathrm{supp}_\infty \nu \cap (x_0,\infty) = \emptyset$, so $x_+^\infty = \infty$. By translation, we may now also assume that $x_-^\infty < 0 < x_0$, and we now take $a = 0$ and any $b > x_0$. Thus $H = H_{\{a,b\}} = H_0 \wedge H_b$. We apply (7.11) with $T = A_t \wedge H$. Noting that $H \le H_0$ and $B_s \ge 0$ for $s \le H$, we obtain, using also Lemma 7.6,

$$\begin{aligned}\mathbb{E}\,\psi\big(B_{A_t\wedge H_0\wedge H_b}\big) &= \mathbb{E}\int_0^{A_t\wedge H} B_s\,d\Gamma_s = \mathbb{E}\int_0^{A_t\wedge H} B_{s\wedge H_0}\,d\Gamma_s\\ &= \mathbb{E}\int_0^{\Gamma_{A_t\wedge H}} B_{A_r\wedge H_0}\,dr.\end{aligned} \tag{7.18}$$

Since $H < H_{\mathrm{supp}_\infty \nu}$, Lemma 7.4 shows that

$$\Gamma_{A_t\wedge H} = \Gamma_{A_t} \wedge \Gamma_H = t \wedge \Gamma_{H_{\mathrm{supp}_\infty \nu}} \wedge \Gamma_H = t \wedge \Gamma_H = t \wedge \Gamma_{H_0} \wedge \Gamma_{H_b}.$$

Further, if $r \ge \Gamma_{H_0}$ then $A_r \ge H_0$ by (2.3) and thus $B_{A_r\wedge H_0} = B_{H_0} = 0$. Hence, (7.18) yields

$$\mathbb{E}\,\psi\big(B_{A_t\wedge H_0\wedge H_b}\big) = \mathbb{E}\int_0^{t\wedge\Gamma_{H_0}\wedge\Gamma_{H_b}} B_{A_r\wedge H_0}\,dr = \mathbb{E}\int_0^{t\wedge\Gamma_{H_b}} B_{A_r\wedge H_0}\,dr. \tag{7.19}$$

Let $C' := 2\int_0^\infty y\,\nu(dy)$. Then $|\varphi(x)| \le C'$ for $x \ge 0$, and thus, for any $x \ge 0$,

$$\psi(x) \le \psi(0) + C'x \le \psi(x_0) + C'x_0 + C'x = C'x_0 + C'x. \tag{7.20}$$

Thus, for $t = t_0$, the left-hand side of (7.19) is at most, using Lemma 7.8,

$$\mathbb{E}\big(C'x_0 + C'B_{A_t\wedge H_0\wedge H_b}\big) = 2C'x_0. \tag{7.21}$$

Consequently (7.19) yields

$$2C'x_0 \geq \mathbb{E}\int_0^{t_0\wedge\Gamma_{H_b}} B_{A_r\wedge H_0}\,dr. \tag{7.22}$$

Letting $b\to\infty$ we have $H_b\to\infty$ and $\Gamma_{H_b}\to\infty$, and thus (7.22) yields by monotone convergence, using Lemma 7.8 again,

$$2C'x_0 \geq \mathbb{E}\int_0^{t_0} B_{A_r\wedge H_0}\,dr = \int_0^{t_0}\mathbb{E}\,B_{A_r\wedge H_0}\,dr = t_0x_0.$$

Consequently,

$$t_0 \leq 2C' = 4\int_0^\infty y\,\nu(dy). \tag{7.23}$$

This shows that (i) $\Longrightarrow$ (iii), since we then may choose t_0 arbitrarily large in (7.23) and obtain a contradiction. To extend this to (ii) with a given t_0, we argue as follows, assuming (ii) and (7.17).

We have derived (7.23) under the assumption $x_-^\infty < 0 < x_0$. By translation, we have in general, for any x_0 and x_-^∞, and any $a\in(x_-^\infty,x_0)$,

$$t_0 \leq 4\int_a^\infty y\,\nu(dy). \tag{7.24}$$

Letting $a\to x_0$ yields

$$t_0 \leq 4\int_{x_0}^\infty y\,\nu(dy). \tag{7.25}$$

Take any $z > x_0$ with $z\in\operatorname{supp}\nu$, and let $\tau := \Gamma_{H_z}$; by Lemma 7.2, $\tau = \inf\{t : X_t = z\}$, so τ is an X-stopping time. Condition on the event $\mathcal{E} := \{\tau \leq t_0/2\}$; we have $\mathbb{P}(\mathcal{E}) > 0$ by Lemma 7.2. On the event $\mathcal{E}$, $B'_u = B_{H_z+u}$ is a Brownian motion starting at z, and the processes corresponding to Γ, A and X defined by B' are $\Gamma'_u = \Gamma_{H_z+u}-\Gamma_{H_z} = \Gamma_{H_z+u}-\tau$, $A'_t = A_{t+\tau}-H_z$ and $X'_t = B'_{A'_t} = X_{t+\tau}$. Since τ is an X-stopping time, on $\mathcal{E}$, X'_t is a martingale for $0\leq t\leq t_0/2$, and also at $t = 0-$ since $X'_0 = z = X'_{0-}$, cf. Lemma 3.3. Consequently, we may apply the result above to X', and (7.25) yields

$$\frac{t_0}{2} \leq 4\int_z^\infty y\,\nu(dy). \tag{7.26}$$

However, if (7.17) holds, then the right-hand side of (7.26) tends to 0 as $z\to\infty$, which is a contradiction. This contradiction shows that (7.17) cannot hold, and thus (ii) $\Longrightarrow$ (iii). □

Note that the case $x_0\in\operatorname{supp}_\infty\nu$ is not redundant in (iii). An example is given by $x_0 = 0$ and $\nu(dx) = dx/x$ for $x > 0$; then (7.6) fails because the second integral vanishes, nevertheless $X_t = x_0 = 0$ for all t, which trivially is a martingale.

8. Open problems

8.1. **Weakening the assumptions.** Theorem 2.3 shows the existence of a generalised diffusion solving the inverse problem provided the given distribution has finite expectation. Does the result hold without this assumption (allowing local martingales instead of only martingales)?

8.2. **Uniqueness.** Theorem 2.3 shows the existence of a generalised diffusion, defined by a speed measure ν, with a given distribution of X_1 (assuming $\mathbb{E}|X_1|$ is finite). Is the measure ν unique? Strictly speaking it is not: Lemma 3.7 shows that we will never leave the interval $[x_-^\infty, x_+^\infty]$, so any change of ν outside this interval (assuming at least one of x_-^∞ and x_+^∞ is finite) will not affect X_t at all. We may normalize ν by first taking the restriction ν_0 to (x_-^∞, x_+^∞); if x_-^∞ is finite but $x_-^\infty \notin \operatorname{supp}_\infty \nu_0$ we also add an infinite point mass at x_-^∞, and similarly for x_+^∞. The question is then: Given the distribution of X_1, is there a unique corresponding normalized speed measure such that the corresponding process $(X_t)_{t\le 1}$ is a martingale?

The problem of uniqueness is also open in the discrete case, i.e. is the mapping $G : B^n \to \Pi^n$ studied in Section 4 an injection? (As remarked in the proof, this holds for $n = 1$.)

8.3. **Relations between μ and ν.** Find relations between the speed measure ν (assumed normalized as above) and the distribution μ of X_1. For example, if μ has a point mass at x, does ν also have a point mass there? Does the converse hold? If ν is absolutely continuous, is μ too? Does the converse hold? For which ν does μ have a finite second moment?

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