Comments on Sparse graphs using exchangeable random measures by François Caron and Emily B. Fox

Professor Svante Janson*
Uppsala University, Sweden

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I find the random graph model introduced here by Caron and Fox very interesting. Apart from its potential use in applications, it has novel and interesting mathematical properties. Moreover, it has been an inspiration of important generalizations developed after the first version of the present paper by, in particular, [1] and [2].

The relation with Kallenberg's characterization of exchangeable random measures is interesting, and presumably useful in further developments of the theory, but I would like to stress that, for the contents of the present paper, Kallenberg's highly technical theorem may serve as a (possibly important) inspiration for the model, but it is not needed for the formal construction of the model and the study of its properties.

Furthermore, the basic construction can be stated in several different, equivalent, ways. I prefer to see the basic construction in the paper as follows, including generalizations by [1] and [2]. Let $(\mathcal{S}, \mu)$ be a σ-finite measure space, and let $F(x, y)$ be a fixed symmetric measurable function $\mathcal{S} \times \mathcal{S} \to [0, 1]$. Generate a random countable point set $(w_i, \theta_i)_1^\infty$ of points in $\mathcal{S} \times \mathbb{R}_+$ by taking a Poisson point process in $\mathcal{S} \times \mathbb{R}_+$ with intensity $\mu \times \lambda$. Regard the θ_i as (labels of) vertices, and add an edge $\theta_i\theta_j$ with probability $F(w_i, w_j)$, independently for all pairs (θ_i, θ_j) with $i \leqslant j$. (Finally, eliminate isolated vertices.) The version in the present paper constructs $(w_i, \theta_i)_1^\infty$ by a CRM, which is equivalent to choosing $\mathcal{S} = \mathbb{R}_+$ with μ the Lévy measure; furthermore, F is chosen as $F(x, y) = 1 - e^{-2xy}$ (for $x \neq y$). Kallenberg's theorem yields the same random graphs by a canonical choice $(\mathcal{S}, \mu) = (\mathbb{R}_+, \lambda)$, but a different F, see Section 5.1. Other choices of F yield generalizations of the model. Other choices of $(\mathcal{S}, \mu)$ yield the same random graphs but are sometimes useful, so it seems convenient to allow an arbitrary choice and not fix it in advance.

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Finally, in connection with Theorems 3 and 5, note that, if $\int_0^\infty w\rho(dw) < \infty$, then $N_\alpha^{(e)} / \mathbb{E}\, N_\alpha^{(e)} \to 1$ a.s. as $\alpha \to \infty$. This follows easily because the loops can be ignored and, if $\bar{N}_\alpha^{(e)}$ denotes the number of non-loop edges and the edges are defined by the events $U_{ij} \leqslant F(w_i, w_j)$ for an i.i.d. array $(U_{ij})_{i \leqslant j}$, then $\bar{N}_\alpha^{(e)}/\alpha^2$ is a reverse martingale with respect to the σ-fields $\mathcal{F}_t$ generated by $(w_i)_1^\infty \cup (U_{ij})_{ij} \cup (\theta_i \mathbf{1}_{\theta_i > t})_1^\infty$.

References

[1] Christian Borgs, Jennifer T. Chayes, Henry Cohn & Nina Holden. Sparse exchangeable graphs and their limits via graphon processes. Preprint, 2016. `arXiv:1601.07134v2`

[2] Victor Veitch & Daniel M. Roy. The class of random graphs arising from exchangeable random measures. Preprint, 2015. `arXiv:1512.03099v1`