TAIL BOUNDS FOR SUMS OF GEOMETRIC AND EXPONENTIAL VARIABLES

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Abstract. We give explicit bounds for the tail probabilities for sums of independent geometric or exponential variables, possibly with different parameters.

1. Introduction and notation

Let $X = \sum_{i=1}^n X_i$, where $n \geqslant 1$ and X_i, $i = 1, \dots, n$, are independent geometric random variables with possibly different distributions: $X_i \sim \mathrm{Ge}(p_i)$ with $0 < p_i \leqslant 1$, i.e.,

$$\mathbb{P}(X_i = k) = p_i(1-p_i)^{k-1}, \qquad k = 1, 2, \dots. \tag{1.1}$$

Our goal is to estimate the tail probabilities $\mathbb{P}(X \geqslant x)$. (Since X is integer-valued, it suffices to consider integer x. However, it is convenient to allow arbitrary real x, and we do so.)

We define

$$\mu := \mathbb{E}\, X = \sum_{i=1}^n \mathbb{E}\, X_i = \sum_{i=1}^n \frac{1}{p_i}, \tag{1.2}$$

$$p_* := \min_i p_i. \tag{1.3}$$

We shall see that p_* plays an important role in our estimates, which roughly speaking show that the tail probabilities of X decrease at about the same rate as the tail probabilities of $\mathrm{Ge}(p_*)$, i.e., as for the variable X_i with smallest p_i and thus fattest tail.

Recall the simple and well-known fact that (1.1) implies that, for any non-zero z such that $|z|(1-p_i) < 1$,

$$\mathbb{E}\, z^{X_i} = \sum_{k=1}^\infty z^k\, \mathbb{P}(X_i = k) = \frac{p_i z}{1-(1-p_i)z} = \frac{p_i}{z^{-1} - 1 + p_i}. \tag{1.4}$$

For future use, note that since $x \mapsto -\ln(1-x)$ is convex on $(0,1)$ and 0 for $x = 0$,

$$-\ln(1-x) \leqslant -\frac{x}{y}\ln(1-y), \qquad 0 < x \leqslant y < 1. \tag{1.5}$$

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Remark 1.1. The theorems and corollaries below hold also, with the same proofs, for infinite sums $X = \sum_{i=1}^{\infty} X_i$, provided $\mathbb{E}\, X = \sum_i p_i^{-1} < \infty$.

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2. Upper bounds for the upper tail

We begin with a simple upper bound obtained by the classical method of estimating the moment generating function (or probability generating function) and using the standard inequality (an instance of Markov's inequality)

$$\mathbb{P}(X \geqslant x) \leqslant z^{-x}\, \mathbb{E}\, z^X, \qquad z \geqslant 1, \tag{2.1}$$

or equivalently

$$\mathbb{P}(X \geqslant x) \leqslant e^{-tx}\, \mathbb{E}\, e^{tX}, \qquad t \geqslant 0. \tag{2.2}$$

(Cf. the related "Chernoff bounds" for the binomial distribution that are proved by this method, see e.g. [3, Theorem 2.1], and see e.g. [1] for other applications of this method. See also e.g. [2, Chapter 2] or [4, Chapter 27] for more general large deviation theory.)

Theorem 2.1. *For any* $p_1, \dots, p_n \in (0, 1]$ *and any* $\lambda \geqslant 1$,

$$\mathbb{P}(X \geqslant \lambda\mu) \leqslant e^{-p_*\mu(\lambda-1-\ln\lambda)}. \tag{2.3}$$

Proof. If $0 \leqslant t < p_i$, then $e^{-t} - 1 + p_i \geqslant p_i - t > 0$, and thus by (1.4),

$$\mathbb{E}\, e^{tX_i} = \frac{p_i}{e^{-t} - 1 + p_i} \leqslant \frac{p_i}{p_i - t} = \Bigl(1 - \frac{t}{p_i}\Bigr)^{-1}. \tag{2.4}$$

Hence, if $0 \leqslant t < p_* = \min_i p_i$, then

$$\mathbb{E}\, e^{tX} = \prod_{i=1}^{n} \mathbb{E}\, e^{tX_i} \leqslant \prod_{i=1}^{n} \Bigl(1 - \frac{t}{p_i}\Bigr)^{-1} \tag{2.5}$$

and, by (2.2),

$$\mathbb{P}(X \geqslant \lambda\mu) \leqslant e^{-t\lambda\mu}\, \mathbb{E}\, e^{tX} \leqslant \exp\Bigl(-t\lambda\mu + \sum_{i=1}^{n} - \ln\Bigl(1 - \frac{t}{p_i}\Bigr)\Bigr). \tag{2.6}$$

By (1.5) and $0 < p_*/p_i \leqslant 1$, we have, for $0 \leqslant t < p_*$,

$$-\ln\Bigl(1 - \frac{t}{p_i}\Bigr) \leqslant -\frac{p_*}{p_i} \ln\Bigl(1 - \frac{t}{p_*}\Bigr). \tag{2.7}$$

Consequently, (2.6) yields

$$\begin{aligned} \mathbb{P}(X \geqslant \lambda\mu) &\leqslant \exp\Bigl(-t\lambda\mu - \ln\Bigl(1 - \frac{t}{p_*}\Bigr) \sum_{i=1}^{n} \frac{p_*}{p_i}\Bigr) \\ &= \exp\Bigl(-t\lambda\mu - p_*\mu \ln\Bigl(1 - \frac{t}{p_*}\Bigr)\Bigr). \end{aligned} \tag{2.8}$$

Choosing $t = (1 - \lambda^{-1})p_*$ (which is optimal in (2.8)), we obtain (2.3). □

As a corollary we obtain a bound that is generally much cruder, but has the advantage of not depending on the p_i's at all.

Corollary 2.2. *For any $p_1, \dots, p_n \in (0, 1]$ and any $\lambda \geqslant 1$,*

$$\mathbb{P}(X \geqslant \lambda\mu) \leqslant \lambda e^{1-\lambda} = e\lambda e^{-\lambda}. \tag{2.9}$$

Proof. Use $\mu \geqslant 1/p_i$ for each i, and thus $\mu p_* \geqslant 1$ in (2.3). (Alternatively, use $t = (1 - \lambda^{-1})/\mu$ in (2.8).) □

The bound in Theorem 2.1 is rather sharp in many cases. Also the cruder (2.9) is almost sharp for $n = 1$ (a single X_i) and small $p_* = p_1$; in this case $\mu = 1/p_1$ and

$$\mathbb{P}(X \geqslant \lambda\mu) = (1 - p_1)^{\lceil \lambda\mu \rceil - 1} = \exp\bigl(\lambda + O(\lambda p_1)\bigr). \tag{2.10}$$

Nevertheless, we can improve (2.3) somewhat, in particular when $p_* = \min_i p_i$ is not small, by using more careful estimates.

Theorem 2.3. *For any $p_1, \dots, p_n \in (0, 1]$ and any $\lambda \geqslant 1$,*

$$\mathbb{P}(X \geqslant \lambda\mu) \leqslant \lambda^{-1}(1 - p_*)^{(\lambda - 1 - \ln \lambda)\mu}. \tag{2.11}$$

The proof is given below. We note that Theorem 2.3 implies a minor improvement of Corollary 2.2:

Corollary 2.4. *For any $p_1, \dots, p_n \in (0, 1]$ and any $\lambda \geqslant 1$,*

$$\mathbb{P}(X \geqslant \lambda\mu) \leqslant e^{1-\lambda}. \tag{2.12}$$

Proof. Use (2.11) and $(1 - p_*)^\mu \leqslant e^{-p_*\mu} \leqslant e^{-1}$. □

We begin the proof of Theorem 2.3 with two lemmas yielding a minor improvement of (2.1) using the fact that the variables are geometric. (The lemmas actually use only that one of the variables is geometric.)

Lemma 2.5. (i) *For any integers j and k with $j \geqslant k$,*

$$\mathbb{P}(X \geqslant j) \geqslant (1 - p_*)^{j-k}\, \mathbb{P}(X \geqslant k). \tag{2.13}$$

(ii) *For any real numbers x and y with $x \geqslant y$,*

$$\mathbb{P}(X \geqslant x) \geqslant (1 - p_*)^{x-y+1}\, \mathbb{P}(X \geqslant y). \tag{2.14}$$

Proof. (i). We may without loss of generality assume that $p_* = p_1$. Then, for any integers i, j, k with $j \geqslant k$,

$$\mathbb{P}(X \geqslant j \mid X - X_1 = i) = \mathbb{P}(X_1 \geqslant j - i) = (1 - p_*)^{(j-i-1)_+}, \tag{2.15}$$

and similarly for $\mathbb{P}(X \geqslant k \mid X - X_1 = i)$. Since $(j - i - 1)_+ \leqslant j - k + (k - i - 1)_+$, it follows that

$$\mathbb{P}(X \geqslant j \mid X - X_1 = i) \geqslant (1 - p_*)^{j-k}\, \mathbb{P}(X \geqslant k \mid X - X_1 = i) \tag{2.16}$$

for every i, and thus (2.13) follows by taking the expectation.

(ii). For real x and y we obtain from (2.13)

$$\begin{aligned}\mathbb{P}(X \geqslant x) &= \mathbb{P}(X \geqslant \lceil x \rceil) \geqslant (1-p_*)^{\lceil x \rceil - \lceil y \rceil}\, \mathbb{P}(X \geqslant \lceil y \rceil) \\ &\geqslant (1-p_*)^{x-y+1}\, \mathbb{P}(X \geqslant y). \qquad (2.17)\end{aligned}$$

□

Lemma 2.6. *For any $x \geqslant 0$ and $z \geqslant 1$ with $z(1-p_*) < 1$,*

$$\mathbb{P}(X \geqslant x) \leqslant \frac{1 - z(1-p_*)}{p_*} z^{-x}\, \mathbb{E}\, z^X. \qquad (2.18)$$

Proof. Since $z \geqslant 1$, (2.13) implies that for every $k \geqslant 1$,

$$\begin{aligned}\mathbb{E}\, z^X &\geqslant \mathbb{E}(z^X \cdot \mathbf{1}\{X \geqslant k\}) = \mathbb{E}\Bigl(\Bigl(z^k + (z-1)\sum_{j=k}^{X-1} z^j\Bigr)\mathbf{1}\{X \geqslant k\}\Bigr) \\ &= \mathbb{E}\Bigl(z^k \mathbf{1}\{X \geqslant k\} + (z-1)\sum_{j=k}^{\infty} z^j \mathbf{1}\{X \geqslant j+1\}\Bigr) \\ &= z^k\, \mathbb{P}(X \geqslant k) + (z-1)\sum_{j=k}^{\infty} z^j\, \mathbb{P}(X \geqslant j+1) \\ &\geqslant z^k\, \mathbb{P}(X \geqslant k)\Bigl(1 + (z-1)\sum_{j=k}^{\infty} z^{j-k}(1-p_*)^{j+1-k}\Bigr) \\ &= z^k\, \mathbb{P}(X \geqslant k)\Bigl(1 + \frac{(z-1)(1-p_*)}{1-z(1-p_*)}\Bigr) \\ &= z^k\, \mathbb{P}(X \geqslant k)\frac{p_*}{1-z(1-p_*)}. \qquad (2.19)\end{aligned}$$

The result (2.18) follows when $x = k$ is a positive integer. The general case follows by taking $k = \max(\lceil x \rceil, 1)$ since then $\mathbb{P}(X \geqslant x) = \mathbb{P}(X \geqslant k)$. □

Proof of Theorem 2.3. We may assume that $p_* < 1$. (Otherwise every $p_i = 1$ and $X_i = 1$ a.s., so $X = n = \mu$ a.s. and the result is trivial.) We then choose

$$z := \frac{\lambda - p_*}{\lambda(1-p_*)}, \qquad (2.20)$$

i.e.,

$$z^{-1} = \frac{\lambda(1-p_*)}{\lambda - p_*} = 1 - \frac{(\lambda-1)p_*}{\lambda - p_*}; \qquad (2.21)$$

note that $z^{-1} \leqslant 1$ so $z \geqslant 1$ and $z^{-1} > 1 - p_* \geqslant 1 - p_i$ for every i. Thus, by (1.4),

$$\mathbb{E}\, z^X = \prod_{i=1}^{n} \mathbb{E}\, z^{X_i} = \prod_{i=1}^{n} \frac{p_i}{z^{-1} - 1 + p_i} = \prod_{i=1}^{n} \frac{1}{1 - (1 - z^{-1})/p_i}. \qquad (2.22)$$

By (2.22), (2.7) (with $t = 1 - z^{-1} < p_*$) and (2.21),

$$\begin{aligned}\ln \mathbb{E}\, z^X &= -\sum_{i=1}^n \ln\Bigl(1 - \frac{1-z^{-1}}{p_i}\Bigr) \leqslant -\sum_{i=1}^n \frac{p_*}{p_i} \ln\Bigl(1 - \frac{1-z^{-1}}{p_*}\Bigr) \\ &= -\sum_{i=1}^n \frac{p_*}{p_i} \ln\Bigl(1 - \frac{\lambda-1}{\lambda-p_*}\Bigr) = -\mu p_* \ln\frac{1-p_*}{\lambda-p_*} = \mu p_* \ln\frac{\lambda-p_*}{1-p_*}.\end{aligned} \tag{2.23}$$

Furthermore, by (2.20),

$$\frac{1-z(1-p_*)}{p_*} = \frac{1-(\lambda-p_*)/\lambda}{p_*} = \frac{1}{\lambda}. \tag{2.24}$$

Hence, Lemma 2.6, (2.20) and (2.23) yield

$$\begin{aligned}\ln \mathbb{P}(X \geqslant \lambda\mu) &\leqslant -\ln\lambda - \lambda\mu \ln z + \ln \mathbb{E}\, z^X \\ &\leqslant -\ln\lambda - \lambda\mu \ln\frac{\lambda-p_*}{\lambda(1-p_*)} + \mu p_* \ln\frac{\lambda-p_*}{1-p_*} \\ &= -\ln\lambda + \lambda\mu \ln(1-p_*) + \mu f(\lambda),\end{aligned} \tag{2.25}$$

where

$$\begin{aligned}f(\lambda) &:= -\lambda \ln\frac{\lambda-p_*}{\lambda} + p_* \ln\frac{\lambda-p_*}{1-p_*} \\ &= -(\lambda-p_*)\ln(\lambda-p_*) + \lambda\ln\lambda - p_*\ln(1-p_*).\end{aligned} \tag{2.26}$$

We have $f(1) = -\ln(1-p_*)$ and, for $\lambda \geqslant 1$, using (1.5),

$$f'(\lambda) = -\ln(\lambda-p_*) + \ln\lambda = -\ln\Bigl(1-\frac{p_*}{\lambda}\Bigr) \leqslant -\frac{1}{\lambda}\ln(1-p_*). \tag{2.27}$$

Consequently, by integrating (2.27), for all $\lambda \geqslant 1$,

$$f(\lambda) \leqslant -\ln(1-p_*) - \ln\lambda \cdot \ln(1-p_*), \tag{2.28}$$

and the result (2.11) follows by (2.25). □

Remark 2.7. Note that for large λ, the exponents above are roughly linear in λ, while for $\lambda = 1+o(1)$ we have $\lambda - 1 - \ln\lambda \sim \frac{1}{2}(\lambda-1)^2$ so the exponents are quadratic in $\lambda - 1$. The latter is to be expected from the central limit theorem. However, if $\lambda = 1+\varepsilon$ with ε very small and the central limit theorem is applicable, then $\mathbb{P}(X \geqslant (1+\varepsilon)\mu)$ is roughly $\exp(-\varepsilon^2\mu^2/(2\sigma^2))$, where $\sigma^2 = \operatorname{Var} X = \sum_{i=1}^n \operatorname{Var} X_i = \sum_{i=1}^n \frac{1-p_i}{p_i^2}$. Hence, in this case the exponents in (2.3) and (2.11) are asymptotically too small by a factor of rougly, for small p_i,

$$\frac{p_*\mu}{\mu^2/\sigma^2} \approx \frac{p_*\sum_{i=1}^n p_i^{-2}}{\sum_{i=1}^n p_i^{-1}}, \tag{2.29}$$

which may be much smaller than 1. (For example if $p_2 = \dots = p_n$ and $p_1 = p_2/n^{1/3}$.)

3. Upper bounds for the lower tail

We can similarly bound the probability $\mathbb{P}(X \leqslant \lambda\mu)$ for $\lambda \leqslant 1$. We give only a simple bound corresponding to Theorem 2.1. (Note that $\lambda - 1 - \ln\lambda > 0$ for both $\lambda \in (0,1)$ and $\lambda \in (1,\infty)$.)

Theorem 3.1. *For any $p_1,\dots,p_n \in (0,1]$ and any $\lambda \leqslant 1$,*

$$\mathbb{P}(X \leqslant \lambda\mu) \leqslant e^{-p_*\mu(\lambda-1-\ln\lambda)}. \tag{3.1}$$

Proof. We follow closely the proof of Theorem 2.1. If $t \geqslant 0$, then by (1.4),

$$\mathbb{E}\, e^{-tX_i} = \frac{p_i}{e^t - 1 + p_i} \leqslant \frac{p_i}{t + p_i} = \Bigl(1 + \frac{t}{p_i}\Bigr)^{-1}. \tag{3.2}$$

Hence

$$\mathbb{E}\, e^{-tX} = \prod_{i=1}^n \mathbb{E}\, e^{-tX_i} \leqslant \prod_{i=1}^n \Bigl(1 + \frac{t}{p_i}\Bigr)^{-1} \tag{3.3}$$

and, in analogy to (2.2),

$$\mathbb{P}(X \leqslant \lambda\mu) \leqslant e^{t\lambda\mu}\, \mathbb{E}\, e^{-tX} \leqslant \exp\Bigl(t\lambda\mu - \sum_{i=1}^n \ln\Bigl(1 + \frac{t}{p_i}\Bigr)\Bigr). \tag{3.4}$$

In analogy with (2.7), still by the convexity of $-\ln x$,

$$-\ln\Bigl(1 + \frac{t}{p_i}\Bigr) \leqslant -\frac{p_*}{p_i} \ln\Bigl(1 + \frac{t}{p_*}\Bigr), \tag{3.5}$$

and (3.4) yields

$$\begin{aligned}\mathbb{P}(X \leqslant \lambda\mu) &\leqslant \exp\Bigl(t\lambda\mu - \ln\Bigl(1 + \frac{t}{p_*}\Bigr) \sum_{i=1}^n \frac{p_*}{p_i}\Bigr) \\ &= \exp\Bigl(t\lambda\mu - p_*\mu \ln\Bigl(1 + \frac{t}{p_*}\Bigr)\Bigr).\end{aligned} \tag{3.6}$$

Choosing $t = (\lambda^{-1} - 1)p_*$, we obtain (3.1). □

4. A lower bound

We show also a general lower bound for the upper tail probabilities, which shows that for constant $\lambda > 1$, the exponents in Theorems 2.1 and 2.3 are at most a constant factor away from best possible.

Theorem 4.1. *For any $p_1,\dots,p_n \in (0,1]$ and any $\lambda \geqslant 1$,*

$$\mathbb{P}(X \geqslant \lambda\mu) \geqslant \frac{(1-p_*)^{1+1/p_*}}{2p_*\mu} (1-p_*)^{(\lambda-1)\mu}. \tag{4.1}$$

Lemma 4.2. *If $A \geqslant 1$ and $0 \leqslant x \leqslant 1/A$, then*

$$A\bigl(x + \ln(1-x)\bigr) \leqslant \ln\bigl(1 - Ax^2/2\bigr). \tag{4.2}$$

Proof. Let $f(x) := A\bigl(x + \ln(1-x)\bigr) - \ln\bigl(1 - Ax^2/2\bigr)$. Then $f(0) = 0$ and

$$f'(x) = A\Bigl(1 - \frac{1}{1-x}\Bigr) + \frac{Ax}{1 - Ax^2/2} = -\frac{Ax}{1-x} + \frac{Ax}{1 - Ax^2/2} \leqslant 0 \qquad (4.3)$$

for $0 \leqslant x < 1/A \leqslant 1$, since then $0 < 1 - x \leqslant 1 - Ax^2/2$. Hence $f(x) \leqslant 0$ for $0 \leqslant x \leqslant 1/A$. □

Proof of Theorem 4.1. Let $\varepsilon := 1/(p_*\mu)$. By Theorem 3.1 (with $\lambda = 1 - \varepsilon$) and Lemma 4.2 (with $A = p_*\mu \geqslant 1$),

$$\mathbb{P}(X \leqslant (1-\varepsilon)\mu) \leqslant \exp\bigl(-p_*\mu(-\varepsilon - \ln(1-\varepsilon))\bigr) \leqslant 1 - \frac{p_*\mu\varepsilon^2}{2} = 1 - \frac{1}{2p_*\mu}. \qquad (4.4)$$

Hence, $\mathbb{P}(X \geqslant (1-\varepsilon)\mu) \geqslant 1/(2p_*\mu)$, and by Lemma 2.5(ii),

$$\mathbb{P}(X \geqslant \lambda\mu) \geqslant (1-p_*)^{(\lambda-1+\varepsilon)\mu+1}\,\mathbb{P}(X \geqslant (1-\varepsilon)\mu) \geqslant (1-p_*)^{(\lambda-1+\varepsilon)\mu+1}\frac{1}{2p_*\mu},$$

which completes the proof since $\varepsilon\mu = 1/p_*$. □

5. Exponential distributions

In this section we assume that $X = \sum_{i=1}^n X_i$ where X_i, $i = 1, \dots, n$, are independent random variables with exponential distributions: $X_i \sim \operatorname{Exp}(a_i)$, with density function $a_i x e^{-a_i x}$, $x > 0$, and expectation $\mathbb{E}\,X_i = 1/a_i$. (Thus a_i can be interpreted as a rate.) The exponential distribution is the continuous analogue of the geometric distributions, and the results above have (simpler) analogues for exponential distributions. We now define

$$\mu := \mathbb{E}\,X = \sum_{i=1}^n \mathbb{E}\,X_i = \sum_{i=1}^n \frac{1}{a_i}, \qquad (5.1)$$

$$a_* := \min_i a_i. \qquad (5.2)$$

Theorem 5.1. *Let $X = \sum_{i=1}^n X_i$ with $X_i \sim \operatorname{Exp}(a_i)$ independent.*

(i) *For any $\lambda \geqslant 1$,*

$$\mathbb{P}(X \geqslant \lambda\mu) \leqslant \lambda^{-1} e^{-a_*\mu(\lambda - 1 - \ln\lambda)}. \qquad (5.3)$$

(ii) *For any $\lambda \geqslant 1$, we have also the simpler but weaker*

$$\mathbb{P}(X \geqslant \lambda\mu) \leqslant e^{1-\lambda}. \qquad (5.4)$$

(iii) *For any $\lambda \leqslant 1$,*

$$\mathbb{P}(X \leqslant \lambda\mu) \leqslant e^{-a_*\mu(\lambda - 1 - \ln\lambda)}. \qquad (5.5)$$

(iv) *For any $\lambda \geqslant 1$,*

$$\mathbb{P}(X \geqslant \lambda\mu) \geqslant \frac{1}{2ea_*\mu} e^{-a_*\mu(\lambda-1)}. \qquad (5.6)$$

Proof. Let $X_i^{(N)} \sim \mathrm{Ge}(a_i/N)$ be independent (for $N > \max_i a_i$). Then $X_i^{(N)}/N \xrightarrow{\mathrm{d}} X_i$, where $\xrightarrow{\mathrm{d}}$ denotes convergence in distribution, and thus $X^{(N)}/N \xrightarrow{\mathrm{d}} X$, where $X^{(N)} := \sum_{i=1}^n X_i^{(N)}$. Furthermore, $\mu^{(N)} := \mathbb{E}\, X^{(N)} = M\nu$ and $p_* := \min_i(a_i/N) = a_*/N$. The results follow by taking the limit as $N \to \infty$ in (2.11), (2.12), (3.1) and (4.1). (Alternatively, we may imitate the proofs above, using $\mathbb{E}\, e^{tX_i} = a_i/(a_i - t)$ for $t < a_i$.) □

References

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