To fixate or not to fixate in two-type annihilating branching random walks

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15 October, 2020; revised 9 February, 2021

Abstract

We study a model of competition between two types evolving as branching random walks on $\mathbb{Z}^d$. The two types are represented by red and blue balls respectively, with the rule that balls of different colour annihilate upon contact. We consider initial configurations in which the sites of $\mathbb{Z}^d$ contain one ball each, which are independently coloured red with probability p and blue otherwise. We address the question of *fixation*, referring to the sites eventually settling for a given colour, or not. Under a mild moment condition on the branching rule, we prove that the process will fixate almost surely for $p \neq 1/2$, and that every site will change colour infinitely often almost surely for the balanced initial condition $p = 1/2$.

1 Introduction

Position, on each site of a connected graph G, an urn. Each urn may contain either red or blue balls, but not both at once. At the dawn of time ($t = 0$), red and blue balls are distributed in the urns according to some rule. The balls come equipped with unit-rate Poisson clocks, and when a clock rings, the corresponding ball immediately sends an independent copy of itself to each of the urns at neighbouring sites (while the ball with the clock remains where it is). As red and blue balls may not exist together in the same urn, they annihilate on a one-to-one basis.

In the case that G is connected and *finite*, the authors together with Morris [2] have proved that the system of urns will eventually almost surely contain balls of only one colour. In the current paper we examine the process on the d-dimensional integer lattice $\mathbb{Z}^d$, for $d \geqslant 1$, evolving from an initial configuration with a ball at each site, which independently from one another are coloured red with probability p and blue otherwise. Our main result shows that for $p \neq 1/2$ the colouring of the lattice induced by the urn process eventually *fixates* almost surely on a single colour, and that for $p = 1/2$ each site almost surely switches colour infinitely many times. We shall prove our results for a general family of branching mechanisms, further described below, of which the above mentioned nearest-neighbour rule is merely one example.

Models for systems of particles annihilating upon contact have a long history. The question of site recurrence in a one-dimensional system of (non-branching) random walkers annihilating upon contact was raised in the mid 1970s by Erdős and Ney [16]. Higher dimensional versions of the same problem was soon after considered by Griffeath [18] and Arratia [3]. These problems concern a system of particles of a single type. Analogous models consisting of two types of particles have been suggested in the physics literature as descriptive for the inert chemical reaction $A + B \to \emptyset$, see e.g. [24, 28]. These models tend to require a different set of techniques for their analysis. In this setting, Bramson and Lebowitz [12, 13] derived the rate of decay of the density of particles in such a two-type model where particles perform simple random walks, and particles of different type annihilate upon contact. In more recent work, Cabezas, Rolla and Sidoravicius [14] addressed site recurrence in a similar setting, and proved that the origin is visited at arbitrarily large times. A related model has been considered by Damron, Gravner, Junge, Lyu and Sivakoff [15].

The (discrete time) branching random walk first arose as a geometric interpretation of the evolution of generations in an age-dependent branching process, in work of Kingman [22], Biggins [5, 6] and Bramson [10]. Later work has explored important connections between branching random walks and their continuum counterpart, branching Brownian motion, to central objects in statistical physics such as spin glasses and the discrete planar Gaussian free field. For a more detailed discussion on these models and connections, we refer the reader to the monographs [9, 27, 29].

Survival for a version of the branching random walk, where any two particles annihilate upon contact, was studied by Bramson and Gray [11]. Very much in spirit of their and other authors' work (cited above), and further motivated by the corresponding question for Glauber dynamics of the Ising model (see, e.g., [17, 23]), we here address fixation for the two-type annihilating urn system on $\mathbb{Z}^d$ starting from a stationary random initial configuration. In the monochromatic setting, in which all balls have the same colour, so there are no annihilations, the process we study corresponds to a continuous time branching random walk on the integer lattice. For this reason, we shall interchangeably refer to the model we consider as a *competing urn scheme* and as a *two-type annihilating branching random walk*. Our analysis of this process will in large parts be based on a combination of martingale techniques, and elements of Fourier analysis.

1.1 Model and results

We proceed with a somewhat more formal description of the model we consider, and introduce some notation. We shall encode the presence of a red ball with the value $+1$ and the presence of a blue ball by the value -1. This encoding produces a bijection between particle configurations and integer-valued vectors indexed by $\mathbb{Z}^d$. Below, a *configuration* on $\mathbb{Z}^d$ will refer to a vector $\boldsymbol{\zeta} = (\zeta_{\mathbf{z}})_{\mathbf{z}\in\mathbb{Z}^d}$ of integers. A configuration $\boldsymbol{\zeta}$ is said to be *locally finite* if $|\zeta_{\mathbf{z}}| < \infty$ for all $\mathbf{z} \in \mathbb{Z}^d$, and *finite* if the total number of particles $\|\boldsymbol{\zeta}\| := \sum_{\mathbf{z}\in\mathbb{Z}^d} |\zeta_{\mathbf{z}}|$ is finite. (For typographical convenience, we occasionally write $\zeta(\mathbf{z})$ for $\zeta_{\mathbf{z}}$.)

Let Φ be a probability measure on finite non-negative configurations on $\mathbb{Z}^d$, and let φ denote a generic random configuration distributed according to Φ. Given a locally finite initial configuration $\boldsymbol{\zeta}$, we assign to each ball in $\boldsymbol{\zeta}$ a clock independent from everything else. At the ring of a clock at position $\mathbf{z}$, the corresponding ball makes an independent draw from the distribution Φ, and positions new balls accordingly, translated by $\mathbf{z}$. (The ball with the clock is assumed to remain where it is, although we shall comment on this restriction below, in Remark 1.4.) As before, all children have the same colour as their parent, and if one or several balls are positioned in an urn with balls of opposite colour, then they immediately annihilate one for one until all remaining balls in the urn are of the same colour. The nearest-neighbour rule (from [2]) described above thus corresponds to Φ being the degenerate measure supported on the configuration consisting of one ball at each of the $2d$ neighbours to the origin.

Since brevity is the soul of wit, we have here chosen to be brief; we shall at later occasions in the text have more to say about the construction of the process as the need arises.

We shall henceforth impose two restrictions on Φ. We say that Φ is *irreducible*[1] if it is not supported on a proper subgroup of $\mathbb{Z}^d$. We will assume throughout that Φ is irreducible and that $\|\varphi\|$ has finite mean, so that

$$0 < \lambda := \mathbb{E}[\|\varphi\|] < \infty. \tag{1.1}$$

Remark 1.1. We will for simplicity only consider initial configurations with at most one ball at each site. (Although more general cases might also be interesting, see Section 8.) In this case, and assuming (1.1), the process described above is well-defined and without explosions; more precisely, for any finite box $B(\mathbf{0}, r) := [-r, r]^d$ and any finite T, there is a.s. (almost surely) only a finite number of balls appearing in $B(\mathbf{0}, r)$ at some time in $[0, T]$, and as a consequence there is only a finite number of nucleations (branching events) and annihilations at any given site in a finite time interval. It is straightforward to verify these claims for any finite initial configuration. For monochromatic (possibly infinite) initial configurations these claims follow since the expected number of balls at any given site at time t is at most $e^{\lambda t} < \infty$, cf. (3.4) with $p = 1$. In general, as we detail in Section 4, we may formally define the annihilating process for arbitrary initial configurations as the a.s. limit of processes with finite initial configurations; the claimed properties are shown to carry over from the finite setting. In addition to the above, since the process has only a finite number of jumps at each site in each finite time interval, we may assume the standard convention that the process is right-continuous with left limits.

We aim in this paper to understand the evolution of the annihilating system on $\mathbb{Z}^d$ starting from a stationary random initial configuration. To be precise, given $p \in [0, 1]$ and $d \geqslant 1$, define a *p-random Bernoulli colouring* of $\mathbb{Z}^d$ as follows: for each $\mathbf{z} \in \mathbb{Z}^d$, the corresponding urn initially contains a single red ball with probability p, and a single blue ball otherwise, all independently. Hence, the resulting configuration corresponds to an element in $\{-1, 1\}^{\mathbb{Z}^d}$. We say that the two-type annihilating branching random walk *fixates*[2] if there exists a colour c such that every urn eventually contains only balls of colour c.

[1] Assuming that the offspring distribution is irreducible means no loss of generality, since we otherwise could consider the process on the subgroup G of $\mathbb{Z}^d$ generated by the support of Φ; note that $G \cong \mathbb{Z}^{d_1}$ for some $d_1 \leqslant d$ and that the process on $\mathbb{Z}^d$ then decomposes into independent copies of the process on G, supported on different translates (cosets) of G. We ignore the trivial case when the support of Φ is $\{\mathbf{0}\}$; then the urns are independent continuous-time branching processes, each with a fixed colour.

[2] The times the different urns fixate are random and different; we do not claim that there is a single time when all urn have the same colour. (Indeed, since the system is infinite, we cannot expect this.)

As a measure on the displacement of balls in each nucleation, we define for $r > 0$,

$$|||\varphi|||_r := \sum_{\mathbf{z}\in\mathbb{Z}^d} |\mathbf{z}|^r \varphi(\mathbf{z}). \tag{1.2}$$

Our main theorem is the following. (By symmetry, it suffices to consider $p \geqslant \frac{1}{2}$.)

Theorem 1.2. *Let $d \in \mathbb{N}$, and let Φ be an irreducible probability measure on finite configurations on $\mathbb{Z}^d$ such that $\mathbb{E}\left[|||\varphi|||_1^2\right] < \infty$, $\mathbb{E}\left[|||\varphi|||_2\right] < \infty$, and either $\mathbb{E}[\|\varphi\|^3] < \infty$ if $d = 1$ or $\mathbb{E}[\|\varphi\|^2] < \infty$ if $d \geqslant 2$. Then, for the competing urn scheme on $\mathbb{Z}^d$ starting from a p-random Bernoulli colouring, almost surely:*

(i) *For $p > \frac{1}{2}$ the system fixates; each urn is eventually red.*

(ii) *For $p = \frac{1}{2}$ every site changes colour infinitely often.*

Furthermore, we define the density of red sites at time t as the limit (if it exists)

$$\rho(t) := \lim_{n\to\infty} \frac{1}{(2n+1)^d} \sum_{\mathbf{z}\in[-n,n]^d} \mathbf{1}\{\mathbf{z} \text{ is red at time } t\}, \tag{1.3}$$

and show, in Section 6, that almost surely the limit exists for all $t \geqslant 0$ and satisfies $\rho(t) = \mathbb{P}(\text{origin red at time } t)$. In particular, for $p > \frac{1}{2}$, it follows that the density of red urns a.s. tends to 1 as $t \to \infty$. It is remarkable that for $p = \frac{1}{2}$ our arguments do not show that $\rho(t) \to \frac{1}{2}$ as $t \to \infty$; see Section 8 for a more precise conjecture.

Remark 1.3. For part (ii) of Theorem 1.2 the condition $\mathbb{E}[\|\varphi\|^2] < \infty$ suffices for all (irreducible) offspring distributions in all dimensions $d \geqslant 1$. We do not know if the same condition suffices also for $d = 1$ in part (i) of the theorem, or for Theorem 1.5 below, on which the proof of part (i) is based. For L^2 convergence in Theorem 1.5, the finite second moment condition suffices for all (irreducible) offspring distributions in all dimensions $d \geqslant 1$. We do not know whether the conditions

$$\mathbb{E}\left[|||\varphi|||_1^2\right] < \infty \quad \text{and} \quad \mathbb{E}\left[|||\varphi|||_2\right] < \infty \tag{1.4}$$

on the spatial displacement that figure in our theorems are necessary.

Remark 1.4. In the definition of the model, as offspring is produced, the parent is assumed to remain where it is. As a result, in the monochromatic version of the process, once a ball is born it remains in the same place at all future times. More generally we could assume that each ball lives for an exponentially distributed life time, at the end of which it reproduces according to Φ and disappears. This is certainly more general, as Φ could be specified to produce a copy of the parent in its place with probability one. In addition, this allows us, for instance, to consider models where the balls move according to continuous time random walks, which in each step branch with a non-zero probability. We shall in Section 7 describe how our results can be extended to cover also this setting.

Central in order to understand the annihilating process will be to closely examine the evolution of the monochromatic process (without annihilations), in which each site of $\mathbb{Z}^d$ independently is initially occupied by a particle with probability $p \in (0,1]$ and otherwise empty. Let $\mathcal{Y}^p(t) = (Y^p_{\mathbf{z}}(t))_{\mathbf{z}\in\mathbb{Z}^d}$ be the configuration at time $t \geqslant 0$ of this process, where thus $(Y^p_{\mathbf{z}}(0))_{\mathbf{z}\in\mathbb{Z}^d}$ are i.i.d. Bernoulli with parameter p.

We shall prove the following result on the asymptotics of the monochromatic system.

Theorem 1.5. *Let $d \in \mathbb{N}$, and let Φ be an irreducible probability measure on finite configurations on $\mathbb{Z}^d$ such that $\mathbb{E}\left[|||\varphi|||_1^2\right] < \infty$, $\mathbb{E}\left[|||\varphi|||_2\right] < \infty$, and either $\mathbb{E}[\|\varphi\|^3] < \infty$ if $d = 1$ or $\mathbb{E}[\|\varphi\|^2] < \infty$ if $d \geqslant 2$. Then, for every $p \in [0,1]$ and every $\mathbf{z} \in \mathbb{Z}^d$, we have, with λ given by* (1.1),

$$\lim_{t\to\infty} e^{-\lambda t} Y^p_{\mathbf{z}}(t) \;=\; p \tag{1.5}$$

almost surely and in L^2.

1.2 Outlines of proof and paper

In the monochromatic process balls do not interact with each other, and in order to understand its asymptotics it will suffice to examine the evolution of each ball initially present in the system separately. For this purpose we let $\mathcal{X}_{\mathbf{z}}(t) = (X_{\mathbf{z},\mathbf{x}}(t))_{\mathbf{x}\in\mathbb{Z}^d}$ be the configuration at time t of the process started with a single ball at $\mathbf{z}$, i.e., $X_{\mathbf{z},\mathbf{x}}(0) = \delta_{\mathbf{x},\mathbf{z}}$. (These processes are obviously just translates of $\mathcal{X}_{\mathbf{0}}(t)$, the evolution of a single ball started at the origin, but the collection of all of them will be useful in our arguments.) Note that the process $(\mathcal{X}_{\mathbf{z}}(t))_{t\geqslant 0}$ is a multi-type continuous time Markov branching process with type space $\mathbb{Z}^d$; see e.g. [4, Section V.7]. Moreover, the dynamics of the process is translation invariant, which in particular implies that $\|\mathcal{X}_{\mathbf{z}}(t)\|$, the total number of balls in the system, evolves as a (single-type) continuous time Markov branching process in which each individual gets $\|\varphi\|$ children with rate 1. The finite moment condition (1.1) is well-known to imply that the process is almost surely finite at all times, see [4, Section III.2]; in fact, it is easily seen that $e^{-\lambda t}\|\mathcal{X}_{\mathbf{z}}(t)\|$ is a martingale, and thus, in particular,

$$\mathbb{E}\|\mathcal{X}_{\mathbf{z}}(t)\| = e^{\lambda t}\,\mathbb{E}\|\mathcal{X}_{\mathbf{z}}(0)\| = e^{\lambda t}, \tag{1.6}$$

see e.g. [4, Section III.4 and Theorem III.7.1] or Lemma 2.1 below.

A large part of our work will consist in exploring the evolution of the process $\mathcal{X}_{\mathbf{0}}(t)$ starting with a single ball in Section 2, and its implications for the monochromatic process $\mathcal{Y}^p(t)$, which is studied in Section 3, leading to a proof of Theorem 1.5. The analysis will be based on martingale techniques and elements of Fourier analysis. An important step in the argument is a precise variance estimate, which is stated as Proposition 3.1(ii).

We then return to the two-colour competition process, starting from a p-random Bernoulli colouring, which we describe by the vector $\mathcal{Z}(t) = (Z_{\mathbf{x}}(t))_{\mathbf{x}\in\mathbb{Z}^d}$. Although our main interest lies in the case of a random initial configuration, we will in the proofs consider various versions, and we thus allow an arbitrary initial configuration $\boldsymbol{\zeta} = (\zeta_{\mathbf{x}})_{\mathbf{x}\in\mathbb{Z}^d} \in \{-1,0,1\}^{\mathbb{Z}^d}$, deterministic or random. (Thus $\zeta_{\mathbf{x}} = -1$ means a blue ball at $\mathbf{x}$, 1 means a red ball and 0 means no ball.) We let $\mathcal{Z}(t,\boldsymbol{\zeta})$, for $t \geqslant 0$, denote the process started from $\boldsymbol{\zeta}$. In particular, the monochromatic process $\mathcal{Y}^p(t)$ equals (as a process) $\mathcal{Z}(t,\boldsymbol{\zeta})$ with

$\boldsymbol{\zeta} = (\zeta_{\mathbf{x}})_{\mathbf{x}\in\mathbb{Z}^d}$ independent Bernoulli with parameter p, and $\mathcal{X}_{\mathbf{z}}(t)$ corresponds to $\mathcal{Z}(t,\boldsymbol{\zeta})$ with $\boldsymbol{\zeta} = (\delta_{\mathbf{x},\mathbf{z}})_{\mathbf{x}\in\mathbb{Z}^d}$, whereas $\mathcal{Z}(t)$ itself corresponds to $\mathcal{Z}(t,\boldsymbol{\zeta})$ with $\boldsymbol{\zeta}$ being the p-random Bernoulli colouring whose entries are ±1-valued and independent from one another.

We describe in Section 4 how the annihilating process can be defined in a formal fashion, and derive along the way some properties that will be used in the proof of our main theorem. One such preliminary result is a coupling, previously employed in [2], that enables us to ignore annihilations and instead study a pair of (dependent) monochromatic processes, to which we can apply results from previous sections. Theorem 1.2(i) then is as an easy consequence of Theorem 1.5. The balanced case, Theorem 1.2(ii), will require a finer analysis of the order of fluctuations of the monochromatic process, which suitably comes out as a side while proving Theorem 1.5, together with a decoupling argument showing that the states of any finite set of sites are irrelevant for the long term evolution. Details are given in Section 5.

At the end of this paper, in Section 6, we show that the density as defined in (1.3) is well-defined, and in Section 7, we describe how to adapt our arguments to cover the more general version of our process where balls are assumed to die as they reproduce, cf. Remark 1.4. Finally, Section 8 contains some further directions and open problems.

2 The evolution of a single ball

In this section we analyze the evolution of a single ball, i.e., the process $\mathcal{X}_{\mathbf{z}}(t)$. By translation invariance, we may without loss of generality assume $\mathbf{z} = \mathbf{0}$. Furthermore, in this section (only), we drop the index $\mathbf{z}$ indicating the starting position and use the notation $\mathcal{X}(t) = (X_{\mathbf{x}}(t))_{\mathbf{x}\in\mathbb{Z}^d}$ for $\mathcal{X}_{\mathbf{0}}(t)$. The analysis is based on a combination of Fourier analysis and a martingale approach. We assume throughout that (1.1) holds.

2.1 Elements of Fourier analysis

We proceed with the study of the process $(\mathcal{X}(t))_{t\geqslant 0}$, evolving from a single ball initially at the origin. Recall that $\mathcal{X}(t) = (X_{\mathbf{x}}(t))_{\mathbf{x}\in\mathbb{Z}^d}$, for each $t \geqslant 0$, is an almost surely finite configuration on $\mathbb{Z}^d$. From a harmonic analysis point of view, the dual group of $\mathbb{Z}^d$ is the cycle group $\mathbb{T}^d$, which we identify with $(-\pi,\pi]^d$. Hence, there is a natural correspondence between configurations on $\mathbb{Z}^d$ and certain complex-valued functions on $\mathbb{T}^d$. More precisely, we define the *Fourier transform* of $\mathcal{X}(t)$ as

$$\widehat{\mathcal{X}}_{\mathbf{u}}(t) := \sum_{\mathbf{x}\in\mathbb{Z}^d} e^{i\mathbf{u}\cdot\mathbf{x}} X_{\mathbf{x}}(t), \quad \mathbf{u} \in \mathbb{T}^d. \tag{2.1}$$

We note that for $t = 0$ this definition yields $\widehat{\mathcal{X}}_{\mathbf{u}}(0) = 1$, and for $\mathbf{u} = \mathbf{0}$ we obtain

$$\widehat{\mathcal{X}}_{\mathbf{0}}(t) = \sum_{\mathbf{x}\in\mathbb{Z}^d} X_{\mathbf{x}}(t) = \|\mathcal{X}(t)\|, \tag{2.2}$$

the total number of balls at time t. In general, the inequality $|\widehat{\mathcal{X}}_{\mathbf{u}}(t)| \leqslant \|\mathcal{X}(t)\|$ remains valid. $\mathcal{X}(t)$ may be recovered via the *inversion formula*:

$$X_{\mathbf{x}}(t) = \int_{\mathbb{T}^d} e^{-i\mathbf{u}\cdot\mathbf{x}} \widehat{\mathcal{X}}_{\mathbf{u}}(t)\,\mathrm{d}\mathbf{u}, \tag{2.3}$$

where $\mathrm{d}\mathbf{u}$ denotes the normalized Lebesgue measure $(2\pi)^{-d}\,\mathrm{d}\mathbf{u}_1\cdots\mathrm{d}\mathbf{u}_d$ on $\mathbb{T}^d$. (This is easily checked; plug in (2.1) and compute the integral.)

Denote by $\mu = (\mu(\mathbf{x}))_{\mathbf{x}\in\mathbb{Z}^d}$ the coordinate-wise expectation of Φ, i.e. $\mu(\mathbf{x}) := \mathbb{E}[\varphi(\mathbf{x})]$. Then, by (1.1),

$$\|\mu\| := \sum_{\mathbf{x}\in\mathbb{Z}^d} \mu(\mathbf{x}) = \sum_{\mathbf{x}\in\mathbb{Z}^d} \mathbb{E}\,\varphi(\mathbf{x}) = \mathbb{E}\sum_{\mathbf{x}\in\mathbb{Z}^d} \varphi(\mathbf{x}) = \mathbb{E}\|\varphi\| = \lambda < \infty. \tag{2.4}$$

Hence, also μ has a well-defined Fourier transform $\widehat{\mu}(\mathbf{u}) := \sum_{\mathbf{x}\in\mathbb{Z}^d} e^{i\mathbf{u}\cdot\mathbf{x}}\mu(\mathbf{x})$; note that

$$\widehat{\mu}(\mathbf{0}) = \|\mu\| = \lambda. \tag{2.5}$$

(The Fourier transform $\widehat{\zeta}(\mathbf{u})$ of any finite configuration ζ on $\mathbb{Z}^d$ is defined analogously.)

As said in the introduction, it is well-known that $e^{-\lambda t}\widehat{\mathcal{X}}_{\mathbf{0}}(t) = e^{-\lambda t}\|\mathcal{X}(t)\|$ is a continuous time martingale. We extend this to arbitrary $\mathbf{u}\in\mathbb{T}^d$ in the next lemma. Let

$$M_{\mathbf{u}}(t) := e^{-\widehat{\mu}(\mathbf{u})t}\widehat{\mathcal{X}}_{\mathbf{u}}(t). \tag{2.6}$$

In particular, by (2.5) and (2.2),

$$M_{\mathbf{0}}(t) := e^{-\lambda t}\|\mathcal{X}(t)\|. \tag{2.7}$$

Lemma 2.1. *The process $(M_{\mathbf{u}}(t))_{t\geqslant 0}$ is a martingale for each $\mathbf{u}\in\mathbb{T}^d$. In particular,*

$$\mathbb{E}[\widehat{\mathcal{X}}_{\mathbf{u}}(t)] = e^{\widehat{\mu}(\mathbf{u})t}, \quad \mathbf{u}\in\mathbb{T}^d. \tag{2.8}$$

Taking $\mathbf{u} = \mathbf{0}$ in Lemma 2.1 we recover, using (2.7), the fact noted above that $e^{-\lambda t}\|\mathcal{X}(t)\|$ is a martingale, and in particular, since also $\|\mathcal{X}(0)\| = 1$, that

$$\mathbb{E}[\|\mathcal{X}(t)\|] = e^{\lambda t}, \tag{2.9}$$

i.e., that (1.6) holds.

Proof of Lemma 2.1. We prove first (2.8). Once (2.8) has been proven the martingale property will follow from the Markov and branching properties together with homogeneity in time and space. Hence, it will suffice to prove (2.8).

Note that, almost surely, no two clocks ever ring at the same time. If the clock rings for a ball at $\mathbf{z}$, then $\mathcal{X}(t)$ jumps by (a copy of) φ translated by the vector $\mathbf{z}$. Hence, $\widehat{\mathcal{X}}_{\mathbf{u}}(t)$ then jumps by

$$\Delta\widehat{\mathcal{X}}_{\mathbf{u}}(t) = \sum_{\mathbf{y}\in\mathbb{Z}^d} e^{i\mathbf{u}\cdot(\mathbf{z}+\mathbf{y})}\varphi(\mathbf{y}) = e^{i\mathbf{u}\cdot\mathbf{z}}\widehat{\varphi}(\mathbf{u}), \tag{2.10}$$

and the expected jump of $\widehat{\mathcal{X}}_{\mathbf{u}}(t)$, given that the clock rings for a ball at $\mathbf{z}$, is

$$e^{i\mathbf{u}\cdot\mathbf{z}}\,\mathbb{E}[\widehat{\varphi}(\mathbf{u})] = e^{i\mathbf{u}\cdot\mathbf{z}}\widehat{\mu}(\mathbf{u}), \tag{2.11}$$

which by (2.4) is finite. Since the number of balls at $\mathbf{z}$ is $X_{\mathbf{z}}(t)$, and each rings with intensity 1, this implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbb{E}[\widehat{\mathcal{X}}_{\mathbf{u}}(t)] = \sum_{\mathbf{z}\in\mathbb{Z}^d} \mathbb{E}[X_{\mathbf{z}}(t)]e^{i\mathbf{u}\cdot\mathbf{z}}\widehat{\mu}(\mathbf{u}) = \widehat{\mu}(\mathbf{u})\,\mathbb{E}[\widehat{\mathcal{X}}_{\mathbf{u}}(t)], \tag{2.12}$$

and (2.8) follows by the initial condition $\widehat{\mathcal{X}}_{\mathbf{u}}(0) = 1$. □

As another consequence of Lemma 2.1, we obtain a formula for the expected number of balls at a given position. Since $|\widehat{\mathcal{X}}_{\mathbf{u}}(t)| \leqslant \|\mathcal{X}(t)\|$ and $\mathbb{E}\|\widehat{\mathcal{X}}(t)\| < \infty$, we may combine the inversion formula (2.3), Fubini's theorem and (2.8) to obtain the expression

$$\mathbb{E}[X_{\mathbf{z}}(t)] = \int_{\mathbb{T}^d} e^{-i\mathbf{u}\cdot\mathbf{z}}\, \mathbb{E}[\widehat{\mathcal{X}}_{\mathbf{u}}(t)]\, \mathrm{d}\mathbf{u} = \int_{\mathbb{T}^d} e^{-i\mathbf{u}\cdot\mathbf{z}} e^{\widehat{\mu}(\mathbf{u})t}\, \mathrm{d}\mathbf{u}. \tag{2.13}$$

2.2 Second moment analysis

To obtain higher moments of the process we will require a stronger assumption on the moments of Φ. This is also where condition (1.4) on the displacement of Φ comes in.

We begin by noting that the condition $\mathbb{E}[\|\varphi\|^2] < \infty$ implies that $\mathbb{E}[\|\mathcal{X}(t)\|^2] < \infty$ for all $t \geqslant 0$, see [4, Corollary III.6.1] or [20, Theorem 6.3.6]. Since $|\widehat{\mathcal{X}}_{\mathbf{u}}(t)| \leqslant \|\mathcal{X}(t)\|$ we have as a consequence that $\mathbb{E}[|M_{\mathbf{u}}(t)|^2] < \infty$ for all $\mathbf{u} \in \mathbb{T}^d$ and $t \geqslant 0$; in other words, $M_{\mathbf{u}}(t)$ is a square-integrable martingale for every $\mathbf{u} \in \mathbb{T}^d$. The following proposition shows that under the condition $\operatorname{Re}\widehat{\mu}(\mathbf{u}) > \frac{1}{2}\lambda$, this martingale is L^2-bounded.

Proposition 2.2. *Assume that $\mathbb{E}[\|\varphi\|^2] < \infty$, and let $\mathbf{u} \in \mathbb{T}^d$ be such that $\operatorname{Re}\widehat{\mu}(\mathbf{u}) > \frac{1}{2}\lambda$. Then the process $(M_{\mathbf{u}}(t))_{t\geqslant 0}$ is an L^2-bounded martingale; in particular, the limit $M^*_{\mathbf{u}} := \lim_{t\to\infty} M_{\mathbf{u}}(t)$ exists almost surely and in L^2. Furthermore, there exists a constant $C(\mathbf{u})$, which is uniformly bounded for $\operatorname{Re}\widehat{\mu}(\mathbf{u}) - \frac{1}{2}\lambda \geqslant c$ for any $c > 0$, such that for all $t \geqslant 0$*

$$\mathbb{E}\left[|M_{\mathbf{u}}(t) - M^*_{\mathbf{u}}|^2\right] \;\leqslant\; C(\mathbf{u})\, \mathbb{E}[\|\varphi\|^2]\, e^{-(2\operatorname{Re}\widehat{\mu}(\mathbf{u})-\lambda)t}, \tag{2.14}$$

and if, in addition, $\mathbb{E}\left[\|\!|\varphi|\!\|_1^2\right] < \infty$ and $\mathbb{E}\left[\|\!|\varphi|\!\|_2\right] < \infty$, then for all $t \geqslant 0$

$$\mathbb{E}\left[|M_{\mathbf{u}}(t) - M_{\mathbf{0}}(t)|^2\right] \;\leqslant\; C(\mathbf{u})|\mathbf{u}|^2. \tag{2.15}$$

The following lemma will be the first step towards the above proposition.

Lemma 2.3. *Suppose $\mathbb{E}[\|\varphi\|^2] < \infty$. For every $\mathbf{u}, \mathbf{v} \in \mathbb{T}^d$ and $t \geqslant 0$ we have*

$$\mathbb{E}\left[M_{\mathbf{u}}(t)M_{\mathbf{v}}(t)\right] = 1 + \mathbb{E}\left[\widehat{\varphi}(\mathbf{u})\widehat{\varphi}(\mathbf{v})\right] \int_0^t e^{\widehat{\mu}(\mathbf{u}+\mathbf{v})x - [\widehat{\mu}(\mathbf{u})+\widehat{\mu}(\mathbf{v})]x}\, \mathrm{d}x. \tag{2.16}$$

Proof. The two processes $(M_{\mathbf{u}}(t))_{t\geqslant 0}$ and $(M_{\mathbf{v}}(t))_{t\geqslant 0}$ are square-integrable martingales. Hence, their quadratic covariation $[M_{\mathbf{u}}, M_{\mathbf{v}}](t)$ is well-defined; see e.g. [25, Section II.6]. (Although not needed here, we note that it may be defined as the following limit, in probability [25, Theorem II.23],

$$[M_{\mathbf{u}}, M_{\mathbf{v}}](t) := M_{\mathbf{u}}(0)M_{\mathbf{v}}(0) + \lim_{|P_n|\to 0} \sum_{k=1}^{n} \big(M_{\mathbf{u}}(t_k) - M_{\mathbf{u}}(t_{k-1})\big)\big(M_{\mathbf{v}}(t_k) - M_{\mathbf{v}}(t_{k-1})\big), \tag{2.17}$$

where $(P_n)_{n\geqslant 1}$ is some sequence of partitions $0 = t_0 < t_1 < \cdots < t_n = t$ of $[0, t]$ with mesh $\max_k |t_k - t_{k-1}|$ tending to zero.)

The process $\{M_{\mathbf{u}}(t)M_{\mathbf{v}}(t) - [M_{\mathbf{u}}, M_{\mathbf{v}}](t) : t \geqslant 0\}$ is again a martingale [25, Corollary 2 to Theorem II.27], which vanishes at $t = 0$ by definition, and thus

$$\mathbb{E}\left[M_{\mathbf{u}}(t)M_{\mathbf{v}}(t)\right] \;=\; \mathbb{E}\left[[M_{\mathbf{u}}, M_{\mathbf{v}}](t)\right] \quad \text{for all } t \geqslant 0. \tag{2.18}$$

Furthermore, $M_{\mathbf{u}}(t)$ and $M_{\mathbf{v}}(t)$ have finite variation on each compact time interval (since each realisation has piece-wise smooth trajectories). This implies [25, Theorems II.26 and II.28] that $[M_{\mathbf{u}}, M_{\mathbf{v}}](t)$ is a pure jump process with jumps given by

$$\Delta[M_{\mathbf{u}}, M_{\mathbf{v}}](t) \;=\; \Delta M_{\mathbf{u}}(t)\Delta M_{\mathbf{v}}(t) \;=\; e^{-(\widehat{\mu}(\mathbf{u})+\widehat{\mu}(\mathbf{v}))t}\Delta\widehat{\mathcal{X}}_{\mathbf{u}}(t)\Delta\widehat{\mathcal{X}}_{\mathbf{v}}(t). \tag{2.19}$$

Similarly to the proof of Lemma 2.1, if the clock of a ball at $\mathbf{z}$ rings, then by (2.10)

$$\Delta\widehat{\mathcal{X}}_{\mathbf{u}}(t)\Delta\widehat{\mathcal{X}}_{\mathbf{v}}(t) \;=\; e^{i(\mathbf{u}+\mathbf{v})\cdot\mathbf{z}}\widehat{\varphi}(\mathbf{u})\widehat{\varphi}(\mathbf{v}). \tag{2.20}$$

Since the number of balls at $\mathbf{z}$ is $X_{\mathbf{z}}(t)$ and each rings with intensity 1, we obtain from the above and (2.8) that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\left[[M_{\mathbf{u}}, M_{\mathbf{v}}](t)\right] \;&=\; \sum_{\mathbf{z}\in\mathbb{Z}^d}\mathbb{E}[X_{\mathbf{z}}(t)]e^{-(\widehat{\mu}(\mathbf{u})+\widehat{\mu}(\mathbf{v}))t}e^{i(\mathbf{u}+\mathbf{v})\cdot\mathbf{z}}\,\mathbb{E}\left[\widehat{\varphi}(\mathbf{u})\widehat{\varphi}(\mathbf{v})\right]\\ &=\; \mathbb{E}[\widehat{\mathcal{X}}_{\mathbf{u}+\mathbf{v}}(t)]e^{-(\widehat{\mu}(\mathbf{u})+\widehat{\mu}(\mathbf{v}))t}\,\mathbb{E}\left[\widehat{\varphi}(\mathbf{u})\widehat{\varphi}(\mathbf{v})\right]\\ &=\; e^{(\widehat{\mu}(\mathbf{u}+\mathbf{v})-\widehat{\mu}(\mathbf{u})-\widehat{\mu}(\mathbf{v}))t}\,\mathbb{E}\left[\widehat{\varphi}(\mathbf{u})\widehat{\varphi}(\mathbf{v})\right].\end{aligned} \tag{2.21}$$

Integrating (2.21) over the interval $[0,t]$, recalling that $[M_{\mathbf{u}}, M_{\mathbf{v}}](0) = M_{\mathbf{u}}(0)M_{\mathbf{v}}(0) = 1$, and then using (2.18) completes the proof. □

Proof of Proposition 2.2. Note that the complex conjugates of $\widehat{\mathcal{X}}_{\mathbf{u}}(t)$ and $\widehat{\mu}(\mathbf{u})$ are given by $\widehat{\mathcal{X}}_{-\mathbf{u}}(t)$ and $\widehat{\mu}(-\mathbf{u})$, and consequently that

$$|M_{\mathbf{u}}(t)|^2 = M_{\mathbf{u}}(t)\overline{M_{\mathbf{u}}(t)} = M_{\mathbf{u}}(t)M_{-\mathbf{u}}(t). \tag{2.22}$$

By Lemma 2.3 we find that

$$\begin{aligned}\mathbb{E}\left[|M_{\mathbf{u}}(t)|^2\right] \;&=\; 1+\mathbb{E}\left[|\widehat{\varphi}(\mathbf{u})|^2\right]\int_0^t e^{(\widehat{\mu}(\mathbf{0})-\widehat{\mu}(\mathbf{u})-\widehat{\mu}(-\mathbf{u}))x}\,\mathrm{d}x\\ &=\; 1+\mathbb{E}\left[|\widehat{\varphi}(\mathbf{u})|^2\right]\int_0^t e^{(\lambda-2\operatorname{Re}\widehat{\mu}(\mathbf{u}))x}\,\mathrm{d}x.\end{aligned} \tag{2.23}$$

Hence, for $\mathbf{u}\in\mathbb{T}^d$ such that $2\operatorname{Re}\widehat{\mu}(\mathbf{u}) > \lambda$ the complex-valued martingale $(M_{\mathbf{u}}(t))_{t\geqslant 0}$ is bounded in L^2. The existence of an almost sure and L^2 limit $M^*_{\mathbf{u}}$ is now a consequence of the martingale convergence theorem. Moreover, as increments over disjoint time intervals for square-integrable martingales are uncorrelated, we have for any $s \geqslant t$ that

$$\mathbb{E}\left[|M_{\mathbf{u}}(s)|^2\right] = \mathbb{E}\left[|M_{\mathbf{u}}(s)-M_{\mathbf{u}}(t)|^2\right] + \mathbb{E}\left[|M_{\mathbf{u}}(t)|^2\right] \tag{2.24}$$

and hence by (2.23), since $\mathbb{E}[|\widehat{\varphi}(\mathbf{u})|^2] \leqslant \mathbb{E}[\|\varphi\|^2]$, that

$$\mathbb{E}\left[|M_{\mathbf{u}}(s)-M_{\mathbf{u}}(t)|^2\right] \;=\; \mathbb{E}\left[|M_{\mathbf{u}}(s)|^2\right]-\mathbb{E}\left[|M_{\mathbf{u}}(t)|^2\right] \;\leqslant\; \mathbb{E}\left[\|\varphi\|^2\right]\int_t^s e^{(\lambda-2\operatorname{Re}\widehat{\mu}(\mathbf{u}))x}\,\mathrm{d}x. \tag{2.25}$$

Sending $s\to\infty$ thus yields (2.14).

Arguing for (2.15) we first observe that

$$|M_{\mathbf{u}}(t) - M_{\mathbf{0}}(t)|^2 \;=\; M_{\mathbf{u}}(t)M_{-\mathbf{u}}(t) + M_{\mathbf{0}}(t)^2 - M_{\mathbf{u}}(t)M_{\mathbf{0}}(t) - M_{-\mathbf{u}}(t)M_{\mathbf{0}}(t). \tag{2.26}$$

Hence, Lemma 2.3 gives that

$$\begin{aligned}\mathbb{E}\big[|M_{\mathbf{u}}(t) - M_{\mathbf{0}}(t)|^2\big] \;&=\; \mathbb{E}\left[|\widehat{\varphi}(\mathbf{u})|^2\right]\int_0^t e^{(\lambda - 2\operatorname{Re}\widehat{\mu}(\mathbf{u}))x}\,\mathrm{d}x + \mathbb{E}\left[|\widehat{\varphi}(\mathbf{0})|^2\right]\int_0^t e^{-\lambda x}\,\mathrm{d}x \\ &\qquad - 2\operatorname{Re}\mathbb{E}\left[\widehat{\varphi}(\mathbf{u})\widehat{\varphi}(\mathbf{0})\right]\int_0^t e^{-\lambda x}\,\mathrm{d}x \\ &=\; \mathbb{E}\left[|\widehat{\varphi}(\mathbf{u})|^2\right]\int_0^t \left(e^{(\lambda-2\operatorname{Re}\widehat{\mu}(\mathbf{u}))x} - e^{-\lambda x}\right)\mathrm{d}x \\ &\qquad + \mathbb{E}\left[|\widehat{\varphi}(\mathbf{u}) - \widehat{\varphi}(\mathbf{0})|^2\right]\int_0^t e^{-\lambda x}\,\mathrm{d}x. \end{aligned} \tag{2.27}$$

Since $\mathbb{E}[|\widehat{\varphi}(\mathbf{u})|^2] \leqslant \mathbb{E}[\|\varphi\|^2]$, estimating the integrals leads to the upper bound

$$\begin{aligned}\mathbb{E}\big[|M_{\mathbf{u}}(t) - M_{\mathbf{0}}(t)|^2\big] &\leqslant \mathbb{E}\left[\|\varphi\|^2\right]\left(\frac{1}{2\operatorname{Re}\widehat{\mu}(\mathbf{u}) - \lambda} - \frac{1}{\lambda}\right) + \frac{1}{\lambda}\,\mathbb{E}\left[|\widehat{\varphi}(\mathbf{u}) - \widehat{\varphi}(\mathbf{0})|^2\right] \\ &\leqslant \mathbb{E}\left[\|\varphi\|^2\right]\frac{2(\lambda - \operatorname{Re}\widehat{\mu}(\mathbf{u}))}{(2\operatorname{Re}\widehat{\mu}(\mathbf{u}) - \lambda)\lambda} + \frac{1}{\lambda}\,\mathbb{E}\left[|\widehat{\varphi}(\mathbf{u}) - \widehat{\varphi}(\mathbf{0})|^2\right]. \end{aligned} \tag{2.28}$$

In order to obtain an upper bound of order $|\mathbf{u}|^2$ we first note that

$$|e^{i\mathbf{u}\cdot\mathbf{z}} - 1| \;\leqslant\; |\mathbf{u}\cdot\mathbf{z}| \leqslant |\mathbf{u}||\mathbf{z}|, \tag{2.29}$$

using the mean-value theorem and Cauchy–Schwarz' inequality. Hence, recalling (1.2),

$$|\widehat{\varphi}(\mathbf{u}) - \widehat{\varphi}(\mathbf{0})| \;\leqslant\; \sum_{\mathbf{z}\in\mathbb{Z}^d} |e^{i\mathbf{u}\cdot\mathbf{z}} - 1|\varphi(\mathbf{z}) \;\leqslant\; |\mathbf{u}| \sum_{\mathbf{z}\in\mathbb{Z}^d} |\mathbf{z}|\varphi(\mathbf{z}) \;=\; |\mathbf{u}|\,|||\varphi|||_1. \tag{2.30}$$

Similarly we obtain, since $\mu(\mathbf{z}) = \mathbb{E}[\varphi(\mathbf{z})]$,

$$\lambda - \operatorname{Re}\widehat{\mu}(\mathbf{u}) = \widehat{\mu}(\mathbf{0}) - \operatorname{Re}\widehat{\mu}(\mathbf{u}) = \sum_{\mathbf{z}\in\mathbb{Z}^d}\big(1 - \cos(\mathbf{u}\cdot\mathbf{z})\big)\,\mathbb{E}[\varphi(\mathbf{z})] \;\leqslant\; \tfrac{1}{2}|\mathbf{u}|^2\,\mathbb{E}[|||\varphi|||_2]. \tag{2.31}$$

Hence, plugging (2.30) and (2.31) into (2.28) leaves us with

$$\mathbb{E}\left[|M_{\mathbf{u}}(t) - M_{\mathbf{0}}(t)|^2\right] \;\leqslant\; \mathbb{E}\left[\|\varphi\|^2\right]\frac{|\mathbf{u}|^2\,\mathbb{E}[|||\varphi|||_2]}{(2\operatorname{Re}\widehat{\mu}(\mathbf{u}) - \lambda)\lambda} + \frac{1}{\lambda}|\mathbf{u}|^2\,\mathbb{E}\left[|||\varphi|||_1^2\right] = O\big(|\mathbf{u}|^2\big) \tag{2.32}$$

as required. □

2.3 Bounds on the spatial displacement of balls

We next consider the mean spatial distribution of $\mathcal{X}(t)$. Recall that the expected total number of balls at time t is $\mathbb{E}\|\mathcal{X}(t)\| = e^{\lambda t}$ by (2.9). Define

$$p_{\mathbf{z}}(t) \;:=\; \frac{\mathbb{E}[X_{-\mathbf{z}}(t)]}{\mathbb{E}\,\|\mathcal{X}(t)\|} \;=\; e^{-\lambda t}\,\mathbb{E}[X_{-\mathbf{z}}(t)], \tag{2.33}$$

the proportion of the expected number of balls at time t that are expected to be at $-\mathbf{z}$, when starting (as always in this section) from a single ball at the origin. Note that $p_{\mathbf{z}}(t)$ coincides with the expected contribution to the origin of a ball started at $\mathbf{z}$. (The choice of $-\mathbf{z}$ is just for notational convenience in later sections, e.g. in (3.9) and (4.2).) Note that, trivially by the definitions, for every $t \geqslant 0$, we have $p_{\mathbf{z}}(t) \geqslant 0$ and

$$\sum_{\mathbf{z}\in\mathbb{Z}^d} p_{\mathbf{z}}(t) = 1. \tag{2.34}$$

We shall next derive some key quantitative estimates that we shall use in later sections.

Proposition 2.4. *Assume that Φ is irreducible and satisfies $\mathbb{E}[\|\varphi\|^2] < \infty$, $\mathbb{E}\left[|||\varphi|||_1^2\right] < \infty$ and $\mathbb{E}\left[|||\varphi|||_2\right] < \infty$. Then, for $t \geqslant 1$,*

(i) $\displaystyle\sup_{\mathbf{z}\in\mathbb{Z}^d} p_{\mathbf{z}}(t) = O(t^{-d/2})$,

(ii) $\displaystyle\sum_{\mathbf{z}\in\mathbb{Z}^d} p_{\mathbf{z}}(t)^2 = \Theta(t^{-d/2})$,

(iii) $\displaystyle e^{-2\lambda t}\sum_{\mathbf{z}\in\mathbb{Z}^d} \mathbb{E}\left[\big(X_{\mathbf{z}}(t) - p_{-\mathbf{z}}(t)\|\mathcal{X}(t)\|\big)^2\right] = O(t^{-(d+2)/2})$.

Proof. We start with (i), and observe that by the definition (2.33) and (2.13) we have

$$p_{\mathbf{z}}(t) \;=\; e^{-\lambda t}\,\mathbb{E}[X_{-\mathbf{z}}(t)] = \int_{\mathbb{T}^d} e^{i\mathbf{u}\cdot\mathbf{z}} e^{-(\lambda-\widehat{\mu}(\mathbf{u}))t}\,\mathrm{d}\mathbf{u} \;\leqslant\; \int_{\mathbb{T}^d} e^{-(\lambda-\operatorname{Re}\widehat{\mu}(\mathbf{u}))t}\,\mathrm{d}\mathbf{u}. \tag{2.35}$$

To further bound the integral we have the following well-known standard estimate, which highlights the importance of the irreducibility assumption. Note that a complementary upper bound (for all $\mathbf{u}$) is given in (2.31), provided $\mathbb{E}[|||\varphi|||_2] < \infty$.

Claim 1. *Assume that Φ is irreducible. Then, $\operatorname{Re}\widehat{\mu}(\mathbf{u}) < \lambda$ for all $\mathbf{u} \in \mathbb{T}^d\setminus\{\mathbf{0}\}$, and there exists $c > 0$ such that*

$$\lambda - \operatorname{Re}\widehat{\mu}(\mathbf{u}) \;\geqslant\; c|\mathbf{u}|^2 \quad \textit{for all } |\mathbf{u}| \leqslant c. \tag{2.36}$$

Proof of Claim. The statement is obtained by analyzing the identity

$$\lambda - \operatorname{Re}\widehat{\mu}(\mathbf{u}) \;=\; \widehat{\mu}(\mathbf{0}) - \operatorname{Re}\widehat{\mu}(\mathbf{u}) \;=\; \sum_{\mathbf{z}\in\mathbb{Z}^d}\big[1-\cos(\mathbf{u}\cdot\mathbf{z})\big]\mu(\mathbf{z}). \tag{2.37}$$

We omit the details. □

Since Φ is irreducible, Claim 1 gives a constant $c > 0$ such that (2.36) holds. Let $K_c := \{\mathbf{u} \in \mathbb{T}^d : |\mathbf{u}| \geqslant c\}$. Since K_c is compact, Claim 1 and continuity of $\widehat{\mu}(\mathbf{u})$ also gives a constant $\gamma > 0$ such that $\lambda - \operatorname{Re}\widehat{\mu}(\mathbf{u}) \geqslant \gamma$ on K_c. Hence, together with (2.35),

$$\sup_{\mathbf{z}\in\mathbb{Z}^d} p_{\mathbf{z}}(t) \;\leqslant\; \int_{|\mathbf{u}|<c} e^{-c|\mathbf{u}|^2 t}\,\mathrm{d}\mathbf{u} + \int_{K_c} e^{-\gamma t}\,\mathrm{d}\mathbf{u} \;=\; O(t^{-d/2}) + O(e^{-\gamma t}). \tag{2.38}$$

This proves part (i).

The upper bound in (ii) is immediate from (i) and (2.34). Due to the identity in (2.35) we may for a lower bound use Parseval's formula together with (2.31) to obtain that

$$\sum_{\mathbf{z}\in\mathbb{Z}^d} p_{\mathbf{z}}(t)^2 \;=\; \int_{\mathbb{T}^d} \Big|e^{-(\lambda-\widehat{\mu}(\mathbf{u}))t}\Big|^2 \,\mathrm{d}\mathbf{u} \;\geqslant\; \int_{\mathbb{T}^d} e^{-Ct|\mathbf{u}|^2}\,\mathrm{d}\mathbf{u} = t^{-d/2}\int_{(-\pi t^{1/2},\pi t^{1/2}]^d} e^{-C|\mathbf{u}|^2}\,\mathrm{d}\mathbf{u}, \tag{2.39}$$

where $C = \mathbb{E}[\|\!|\varphi|\!\|_2]/2$, which for $t \geqslant 1$ is bounded below by a constant times $t^{-d/2}$.

For (iii) we recall (2.7) and the definition (2.33) of $p_{\mathbf{z}}(t)$, by which

$$X_{\mathbf{z}}(t) - p_{-\mathbf{z}}(t)\|\mathcal{X}(t)\| \;=\; X_{\mathbf{z}}(t) - \mathbb{E}[X_{\mathbf{z}}(t)]M_{\mathbf{0}}(t). \tag{2.40}$$

Using the inversion formula (2.3) and (2.13) we find this equal to

$$\int_{\mathbb{T}^d} e^{-i\mathbf{u}\cdot\mathbf{z}}\big[\widehat{\mathcal{X}}_{\mathbf{u}}(t) - e^{\widehat{\mu}(\mathbf{u})t}M_{\mathbf{0}}(t)\big]\,\mathrm{d}\mathbf{u} \;=\; \int_{\mathbb{T}^d} e^{-i\mathbf{u}\cdot\mathbf{z}}e^{\widehat{\mu}(\mathbf{u})t}\big[M_{\mathbf{u}}(t) - M_{\mathbf{0}}(t)\big]\,\mathrm{d}\mathbf{u}. \tag{2.41}$$

The right-hand side of (2.41) is the Fourier transform of a function on $\mathbb{T}^d$. Hence, by Parseval's formula, we obtain

$$\sum_{\mathbf{z}\in\mathbb{Z}^d}\Big|X_{\mathbf{z}}(t) - p_{-\mathbf{z}}(t)\|\mathcal{X}(t)\|\Big|^2 \;=\; \int_{\mathbb{T}^d}\Big|e^{\widehat{\mu}(\mathbf{u})t}\big[M_{\mathbf{u}}(t) - M_{\mathbf{0}}(t)\big]\Big|^2\,\mathrm{d}\mathbf{u}. \tag{2.42}$$

Taking expectation yields

$$\sum_{\mathbf{z}\in\mathbb{Z}^d}\mathbb{E}\Big[\big|X_{\mathbf{z}}(t) - p_{-\mathbf{z}}(t)\|\mathcal{X}(t)\|\big|^2\Big] \;=\; \int_{\mathbb{T}^d} e^{2\,\mathrm{Re}\,\widehat{\mu}(\mathbf{u})t}\,\mathbb{E}\left[|M_{\mathbf{u}}(t) - M_{\mathbf{0}}(t)|^2\right]\mathrm{d}\mathbf{u}. \tag{2.43}$$

Let $c > 0$, K_c and $\gamma > 0$ be as above, so that $\lambda - \mathrm{Re}\,\widehat{\mu}(\mathbf{u}) \geqslant c|\mathbf{u}|^2$ when $|\mathbf{u}| \leqslant c$ and $\lambda - \mathrm{Re}\,\widehat{\mu}(\mathbf{u}) \geqslant \gamma$ on on the complementary set K_c. We may without loss of generality assume that $c > 0$ was chosen so that also $\lambda - \mathrm{Re}\,\widehat{\mu}(\mathbf{u}) \leqslant \lambda/4$ for $|\mathbf{u}| \leqslant c$ and that $\gamma \leqslant \lambda/4$.

For $|\mathbf{u}| \leqslant c$ we use (2.36) and (2.15), and find that for some constant C_1,

$$e^{2\,\mathrm{Re}\,\widehat{\mu}(\mathbf{u})t}\,\mathbb{E}\left[|M_{\mathbf{u}}(t) - M_{\mathbf{0}}(t)|^2\right] \;\leqslant\; C_1 e^{2(\lambda - c|\mathbf{u}|^2)t}|\mathbf{u}|^2, \qquad |\mathbf{u}| \leqslant c. \tag{2.44}$$

Next we observe that for all $\mathbf{u}$, (2.23) implies that there exists a constant C_2 such that

$$\mathbb{E}\left[|M_{\mathbf{u}}(t) - M_{\mathbf{0}}(t)|^2\right] \;\leqslant\; 2\,\mathbb{E}\left[|M_{\mathbf{u}}(t)|^2\right] + 2\,\mathbb{E}\left[|M_{\mathbf{0}}(t)|^2\right] \;\leqslant\; C_2\Big[1 + \int_0^t e^{(\lambda - 2\,\mathrm{Re}\,\widehat{\mu}(\mathbf{u}))x}\,\mathrm{d}x\Big]. \tag{2.45}$$

By distinguishing between the cases $2\,\mathrm{Re}\,\widehat{\mu}(\mathbf{u}) \geqslant \frac{5}{4}\lambda$ and $2\,\mathrm{Re}\,\widehat{\mu}(\mathbf{u}) \leqslant \frac{5}{4}\lambda$, we obtain from (2.45), rather crudely,

$$\mathbb{E}\left[|M_{\mathbf{u}}(t) - M_{\mathbf{0}}(t)|^2\right] \leqslant C_3 e^{\max\left\{0,\frac{3}{2}\lambda - 2\,\mathrm{Re}\,\widehat{\mu}(\mathbf{u})\right\}t}. \tag{2.46}$$

Hence, on K_c,

$$e^{2\operatorname{Re}\widehat{\mu}(\mathbf{u})t}\,\mathbb{E}\left[|M_{\mathbf{u}}(t)-M_{\mathbf{0}}(t)|^2\right] \leqslant C_3 e^{\max\left\{2\operatorname{Re}\widehat{\mu}(\mathbf{u}),\frac{3}{2}\lambda\right\}t} \leqslant C_3 e^{2(\lambda-\gamma)t}, \qquad \mathbf{u}\in K_c. \tag{2.47}$$

Combining (2.43) with the estimates (2.44) and (2.47) yields

$$\begin{aligned} e^{-2\lambda t}\sum_{\mathbf{z}\in\mathbb{Z}^d}\mathbb{E}\left[\left|X_{\mathbf{z}}(t)-p_{-\mathbf{z}}(t)\|\mathcal{X}(t)\|\right|^2\right] &\leqslant \int_{|\mathbf{u}|<c} C_1 e^{-2c|\mathbf{u}|^2 t}|\mathbf{u}|^2\,\mathrm{d}\mathbf{u} + \int_{K_c} C_3 e^{-2\gamma t}\,\mathrm{d}\mathbf{u} \\ &= O(t^{-(d+2)/2}) + O(e^{-2\gamma t}). \end{aligned} \tag{2.48}$$

This proves (iii) and thus completes the proof. □

3 The monochromatic process

We have defined the monochromatic process $\mathcal{Y}^p(t) = (Y^p_{\mathbf{x}}(t))_{\mathbf{x}\in\mathbb{Z}^d}$ as the process with independent Bernoulli distributed random initial values $Y^p_{\mathbf{x}}(0)$ with parameter p. Our main goal in the present section is to prove Theorem 1.5. A key step will be to derive a variance bound that will be central also later. By translation invariance, $Y^p_{\mathbf{x}}(t)$ has the same distribution for all $\mathbf{x}\in\mathbb{Z}^d$, and we may consider only $\mathbf{x}=\mathbf{0}$.

We begin by introducing a useful representation. Let $\boldsymbol{\eta} = (\eta_{\mathbf{z}})_{\mathbf{z}\in\mathbb{Z}^d}$ be a vector of independent Bernoulli distributed entries with parameter p. (I.e., $\eta_{\mathbf{z}}\in\{0,1\}$ with $\mathbb{P}(\eta_{\mathbf{z}} = 1) = p$.) For each $\mathbf{z}\in\mathbb{Z}^d$, let, as above, $\mathcal{X}_{\mathbf{z}}(t) = (X_{\mathbf{z},\mathbf{x}}(t))_{\mathbf{x}\in\mathbb{Z}^d}$ be the process started with a single ball at $\mathbf{z}$, and assume further that these processes are independent of each other and of $\boldsymbol{\eta}$. Then we can construct the process $\mathcal{Y}^p$ as $\mathcal{Y}^p(t) = \sum_{\mathbf{z}\in\mathbb{Z}^d}\eta_{\mathbf{z}}\mathcal{X}_{\mathbf{z}}(t)$, i.e., the process which for $\mathbf{x}\in\mathbb{Z}^d$ and $t\geqslant 0$ is given by

$$Y^p_{\mathbf{x}}(t) = \sum_{\mathbf{z}\in\mathbb{Z}^d}\eta_{\mathbf{z}}X_{\mathbf{z},\mathbf{x}}(t). \tag{3.1}$$

(This is because in the monochromatic process there are no annihilations and balls evolve independently.)

We next use this representation to prove the following key proposition.

Proposition 3.1. *Assume that Φ is irreducible and that $p\in(0,1]$.*

(i) *If $\lambda = \mathbb{E}[\|\varphi\|] < \infty$, then for every $t\geqslant 0$,*

$$\mathbb{E}\left[e^{-\lambda t}Y^p_{\mathbf{0}}(t)\right] = p. \tag{3.2}$$

(ii) *If $\mathbb{E}[\|\varphi\|^2] < \infty$, $\mathbb{E}\left[\|\!|\varphi|\!\|_1^2\right] < \infty$ and $\mathbb{E}\left[\|\!|\varphi|\!\|_2\right] < \infty$, then for some constant $C = C(p,\Phi)$ and all $t\geqslant 1$,*

$$\operatorname{Var}\left[e^{-\lambda t}Y^p_{\mathbf{0}}(t)\right] = C\sum_{\mathbf{z}\in\mathbb{Z}^d}p_{\mathbf{z}}(t)^2 + O(t^{-(d+1)/2}) = \Theta(t^{-d/2}). \tag{3.3}$$

Proof of Proposition 3.1. (i): By (3.1), independence, translation invariance, and (2.9), using an interchange of order of summation and expectation, that is justified since all variables are non-negative,

$$\mathbb{E}[Y_{\mathbf{0}}^p(t)] = p\sum_{\mathbf{z}\in\mathbb{Z}^d}\mathbb{E}[X_{\mathbf{z},\mathbf{0}}(t)] = p\sum_{\mathbf{z}\in\mathbb{Z}^d}\mathbb{E}[X_{\mathbf{0},-\mathbf{z}}(t)] = p\,\mathbb{E}[\|\mathcal{X}_{\mathbf{0}}(t)\|] = pe^{\lambda t}, \tag{3.4}$$

which yields (3.2).

(ii): Under the assumption that $\mathbb{E}[\|\varphi\|^2] < \infty$, it follows by Proposition 2.2 (with $\mathbf{u} = \mathbf{0}$) and (2.7) that for each $\mathbf{z} \in \mathbb{Z}^d$, the process $\{e^{-\lambda t}\|\mathcal{X}_{\mathbf{z}}(t)\| : t \geqslant 0\}$ is an L^2-bounded martingale. Hence, the limit

$$W_{\mathbf{z}} := \lim_{t\to\infty} e^{-\lambda t}\|\mathcal{X}_{\mathbf{z}}(t)\| \tag{3.5}$$

exists almost surely and in L^2, and (2.9) implies

$$\mathbb{E}[W_{\mathbf{z}}] = 1. \tag{3.6}$$

Note that by (2.7),

$$W_{\mathbf{0}} = \lim_{t\to\infty} M_{\mathbf{0}}(t) = M_{\mathbf{0}}^*. \tag{3.7}$$

We decompose $Y_{\mathbf{0}}^p(t)$ in the following manner, using (3.1) and (2.34).

$$\begin{aligned} e^{-\lambda t}Y_{\mathbf{0}}^p(t) - p = &\sum_{\mathbf{z}\in\mathbb{Z}^d}\eta_{\mathbf{z}}e^{-\lambda t}\big(X_{\mathbf{z},\mathbf{0}}(t) - p_{\mathbf{z}}(t)\|\mathcal{X}_{\mathbf{z}}(t)\|\big) \\ &+ \sum_{\mathbf{z}\in\mathbb{Z}^d} p_{\mathbf{z}}(t)\eta_{\mathbf{z}}\big(e^{-\lambda t}\|\mathcal{X}_{\mathbf{z}}(t)\| - W_{\mathbf{z}}\big) \\ &+ \sum_{\mathbf{z}\in\mathbb{Z}^d} p_{\mathbf{z}}(t)\big(\eta_{\mathbf{z}}W_{\mathbf{z}} - p\big). \end{aligned} \tag{3.8}$$

We will prove below that the sums converge in L^2, and that their values as elements of L^2 are independent of the order of summation, so the decomposition is well-defined. Moreover, although we don't really need this, the proof also shows that for any given fixed order of summation (given by a fixed enumeration of $\mathbb{Z}^d$), the sums converge a.s.

Denote the three sums on the right-hand side of (3.8) by $\Sigma_1(t)$, $\Sigma_2(t)$ and $\Sigma_3(t)$. Note first that by translation invariance, (2.33) and (2.9),

$$\mathbb{E}[X_{\mathbf{z},\mathbf{0}}(t)] = \mathbb{E}[X_{\mathbf{0},-\mathbf{z}}(t)] = e^{\lambda t}p_{\mathbf{z}}(t) = p_{\mathbf{z}}(t)\,\mathbb{E}[\|\mathcal{X}_{\mathbf{z}}(t)\|]. \tag{3.9}$$

Hence, the terms in the first sum have zero mean. The same holds for the terms in the second and third sums too by (2.9) and (3.6). It follows that each of the three sums consists of independent terms with zero mean; hence a sufficient (and necessary) condition for the existence of the sum, in L^2 and almost surely (for any fixed order of summation), is that the sum of the variance of the summands is finite. We state this well-known standard result formally for easy reference.

Claim 2 (Kolmogorov)**.** *Let $\xi_1, \xi_2, \dots$ be independent zero mean random variables and let S_n denote the sum of the first n of them. If $\sum_{k=1}^{\infty} \mathrm{Var}(\xi_k) < \infty$, then $S_\infty := \lim_{n\to\infty} S_n$ exists almost surely and in L^2, and*

$$\mathrm{Var}\,(S_\infty) \;=\; \sum_{k=1}^{\infty} \mathrm{Var}(\xi_k). \tag{3.10}$$

Proof of Claim. For the existence of the limit, see e.g. [19, Lemma 6.5.2 and Theorem 6.5.2]. The formula then follows since $\mathbb{E}[S_\infty^2] = \lim_{n\to\infty} \mathbb{E}[S_n^2]$. □

We treat the three terms in (3.8) separately and in order, obtaining estimates of the variance and at the same time showing the existence of the sums $\Sigma_j(t)$ in L^2 and a.s.

First, Claim 2 and translation invariance show (assuming it is finite) that

$$\begin{aligned}\mathrm{Var}\,[\Sigma_1(t)] \;&=\; p\, e^{-2\lambda t} \sum_{\mathbf{z}\in\mathbb{Z}^d} \mathbb{E}\left[\left(X_{\mathbf{z},\mathbf{0}}(t) - p_{\mathbf{z}}(t)\|\mathcal{X}_{\mathbf{z}}(t)\|\right)^2\right].\\ &=\; p\, e^{-2\lambda t} \sum_{\mathbf{z}\in\mathbb{Z}^d} \mathbb{E}\left[\left(X_{\mathbf{0},-\mathbf{z}}(t) - p_{\mathbf{z}}(t)\|\mathcal{X}_{\mathbf{0}}(t)\|\right)^2\right].\end{aligned} \tag{3.11}$$

By Proposition 2.4(iii) the right-hand side is indeed finite, so $\Sigma_1(t)$ is well-defined and (3.11) justified. By the same proposition, we find that, for some $C_1 < \infty$,

$$\mathrm{Var}\,[\Sigma_1(t)] = \mathbb{E}\left[\Sigma_1(t)^2\right] \leqslant C_1 t^{-(d+2)/2}. \tag{3.12}$$

Similarly, Claim 2 yields

$$\mathrm{Var}\,[\Sigma_2(t)] \;=\; p \sum_{\mathbf{z}\in\mathbb{Z}^d} p_{\mathbf{z}}(t)^2\, \mathbb{E}\left[\left(e^{-\lambda t}\|\mathcal{X}_{\mathbf{z}}(t)\| - W_{\mathbf{z}}\right)^2\right], \tag{3.13}$$

where we note that by translation invariance, (2.7), (3.7), and Proposition 2.2,

$$\mathbb{E}\left[\left(e^{-\lambda t}\|\mathcal{X}_{\mathbf{z}}(t)\| - W_{\mathbf{z}}\right)^2\right] = \mathbb{E}\left[\left(e^{-\lambda t}\|\mathcal{X}_{\mathbf{0}}(t)\| - W_{\mathbf{0}}\right)^2\right] = \mathbb{E}\left[(M_{\mathbf{0}}(t) - M_{\mathbf{0}}^*)^2\right] \leqslant C_2 e^{-\lambda t}. \tag{3.14}$$

Recalling (2.34), we see from (3.14) that the sum in (3.13) converges, so $\Sigma_2(t)$ is well-defined, and furthermore

$$\mathrm{Var}\,[\Sigma_2(t)] = \mathbb{E}\left[\Sigma_2(t)^2\right] \leqslant C_2 e^{-\lambda t}. \tag{3.15}$$

Finally, by Proposition 2.2 the variables $W_{\mathbf{z}} \overset{\mathrm{d}}{=} W_{\mathbf{0}} = M_{\mathbf{0}}^*$ exist in L^2, so Claim 2 gives

$$\mathrm{Var}\,[\Sigma_3(t)] \;=\; \sum_{\mathbf{z}\in\mathbb{Z}^d} p_{\mathbf{z}}(t)^2\, \mathbb{E}\left[(\eta_{\mathbf{z}} W_{\mathbf{z}} - p)^2\right] \;=\; \left(p\,\mathbb{E}[W_{\mathbf{0}}^2] - p^2\right) \sum_{\mathbf{z}\in\mathbb{Z}^d} p_{\mathbf{z}}(t)^2. \tag{3.16}$$

Note that $\mathbb{E}[W_{\mathbf{0}}^2] \geqslant \mathbb{E}[W_{\mathbf{0}}]^2 = 1$. Furthermore, equality would imply $W_{\mathbf{0}} = 1$ a.s., and thus the martingale $e^{-\lambda t}\|\mathcal{X}_{\mathbf{z}}(t)\|$ would be constant, i.e., $\|\mathcal{X}_{\mathbf{z}}(t)\| = e^{\lambda t}$ a.s. for every

$t \geqslant 0$, which is absurd. Hence, $\mathbb{E}[W_{\mathbf{0}}^2] > 1$, and thus $p\,\mathbb{E}[W_{\mathbf{0}}^2] - p^2 > 0$ for all $p \in (0,1]$. Proposition 2.4(ii) and (3.16) thus show that

$$\operatorname{Var}[\Sigma_3(t)] = \mathbb{E}\left[\Sigma_3(t)^2\right] = \Theta(t^{-d/2}), \qquad t \geqslant 1. \tag{3.17}$$

Examining the variance estimates (3.12), (3.15) and (3.17), we conclude that $\Sigma_3(t)$ has a variance of larger order than the other two sums. We conclude that the variance of $\Sigma_3(t)$ is the dominating term in the variance of $Y_{\mathbf{0}}^p(t)$; more precisely, (3.8) and Minkowski's inequality imply, using (3.12) and (3.15),

$$\Big|\big(\operatorname{Var}[e^{-\lambda t}Y_{\mathbf{0}}^p(t)]\big)^{1/2} - \big(\mathbb{E}[\Sigma_3(t)^2]\big)^{1/2}\Big| \leqslant \big(\mathbb{E}[\Sigma_1(t)^2]\big)^{1/2} + \big(\mathbb{E}[\Sigma_2(t)^2]\big)^{1/2} = O\big(t^{-(d+2)/4}\big). \tag{3.18}$$

The first equality in (3.3) now follows by (3.16) and (3.17) (with $C = p\,\mathbb{E}[W_{\mathbf{0}}^2] - p^2 > 0$). The second equality follows for large t by Proposition 2.4(ii); it trivially extends to all $t \geqslant 1$ since $\operatorname{Var}[e^{-\lambda t}Y_{\mathbf{0}}^p(t)]$, as a simple consequence of (3.1), is bounded below by some positive number for every bounded interval $[1,T]$. □

Remark 3.2. In fact, it is easy to see, e.g. as a consequence of [4, III.4.(5)], that $\mathbb{E}[W_{\mathbf{z}}^2] = 1 + \mathbb{E}[\|\varphi\|^2]/\lambda$, and thus $C = p\,\mathbb{E}[\|\varphi\|^2]/\lambda + p - p^2$. Note also that the bounds (3.12) and (3.15) are uniform in p.

Proof of Theorem 1.5. The case $p = 0$ is trivial, so we assume $p > 0$. By translation invariance, we may assume $\mathbf{x} = 0$. Proposition 3.1 immediately yields L^2 convergence in (1.5), so it remains to establish almost sure convergence. We shall show, for every fixed $\delta > 0$, that

$$e^{-\lambda\delta n}Y_{\mathbf{0}}^p(\delta n) \xrightarrow{\text{a.s.}} p \qquad \text{as } n \to \infty. \tag{3.19}$$

Since $Y_{\mathbf{0}}^p(t)$ is non-decreasing in t, this implies that, a.s.,

$$e^{-\lambda\delta}p \leqslant \liminf_{t\to\infty} e^{-\lambda t}Y_{\mathbf{0}}^p(t) \leqslant \limsup_{t\to\infty} e^{-\lambda t}Y_{\mathbf{0}}^p(t) \leqslant e^{\lambda\delta}p. \tag{3.20}$$

Hence, a.s. (3.20) holds for all rational $\delta > 0$, which implies $\lim_{t\to\infty} e^{-\lambda t}Y_{\mathbf{0}}^p(t) = p$.

Thus, fix $\delta > 0$. In order to show (3.19), we again use the decomposition (3.8) and show that $\Sigma_1(\delta n) \xrightarrow{\text{a.s.}} 0$, $\Sigma_2(\delta n) \xrightarrow{\text{a.s.}} 0$, and $\Sigma_3(\delta n) \xrightarrow{\text{a.s.}} 0$ as $n \to \infty$.

First, (3.12) shows that

$$\mathbb{E}\left[\sum_{n=1}^{\infty}\Sigma_1(\delta n)^2\right] = \sum_{n=1}^{\infty}\mathbb{E}\left[\Sigma_1(\delta n)^2\right] \leqslant C_1\sum_{n=1}^{\infty}(\delta n)^{-(d+2)/2} < \infty. \tag{3.21}$$

In particular, a.s., $\sum_{n=1}^{\infty}\Sigma_1(\delta n)^2 < \infty$ and thus $\lim_{n\to\infty}\Sigma_1(\delta n) = 0$.

Similarly, (3.15) implies that a.s. $\lim_{n\to\infty}\Sigma_2(\delta n) = 0$.

To complete the proof of the theorem, it remains to show that also a.s. $\Sigma_3(\delta n) \to 0$ as $n \to \infty$. For $d \geqslant 3$ this follows from (3.17), under the assumption that $\mathbb{E}[\|\varphi\|^2] < \infty$, just like for $\Sigma_1(t)$ and $\Sigma_2(t)$. For $d = 1, 2$ we need to argue differently, which requires a stronger moment condition.

We will appeal to a theorem of Pruitt [26]. Note that $\Sigma_3(\delta n)$ is of the form $\sum_{k\geqslant 1} a_{n,k}X_k$, where the random variables $X_k := \eta_{\mathbf{z}_k} W_{\mathbf{z}_k} - p$ (for some arbitrary enumeration $(\mathbf{z}_k)_k$ of $\mathbb{Z}^d$) are i.i.d. with mean zero and the coefficients $a_{n,k} := p_{\mathbf{z}_k}(\delta n)$ are non-negative and satisfy, by Proposition 2.4(i),

$$\sum_{k\geqslant 1} a_{n,k} = 1 \quad \text{and} \quad \max_{k\geqslant 1} a_{n,k} = O(n^{-d/2}). \tag{3.22}$$

Let $r = \max\{1 + 2/d, 2\}$, which for $d = 1$ gives $r = 3$ and for $d \geqslant 2$ gives $r = 2$. Note that the assumption $\mathbb{E}[\|\varphi\|^r] < \infty$ implies $\mathbb{E}[W_{\mathbf{z}}^r] < \infty$, and hence that $\mathbb{E}[|X_k|^r] < \infty$; see [8, Theorems 1 and 3].[3] Consequently, if $\mathbb{E}[\|\varphi\|^r] < \infty$, then [26, Theorem 2] gives that a.s. $\lim_{n\to\infty} \Sigma_3(\delta n) = 0$.

We have shown that each of the three sums $\Sigma_j(t)$ on the right-hand side of (3.8) a.s. tends to 0 for $t = n\delta \to \infty$, which as said above yields (3.19) and completes the proof of Theorem 1.5. □

Remark 3.3. The work of Pruitt [26] was brought to our attention by Luca Avena and Conrado da Costa. Our previous proof (for the cases $d = 1, 2$) was based on Rosenthal's inequality (see [19, Theorem 3.9.1]) and required the stronger conditions $\mathbb{E}[\|\varphi\|^{4+\varepsilon}] < \infty$ when $d = 1$ and $\mathbb{E}[\|\varphi\|^{2+\varepsilon}] < \infty$ when $d = 2$, for some $\varepsilon > 0$.

4 The two-type annihilating process

We now, finally, turn to the two-colour competition process. Foremost, we shall prove that the process is well-defined in the generality that it is studied in this paper, and along the way establish some properties that we will need for the proof of our main theorem.

For monochromatic initial configurations the process is, as already mentioned, well-defined as there are simply no interactions between different balls. For non-monochromatic initial configurations consisting of finitely many non-zero elements it is straightforward to construct the annihilating process, since a.s. only finitely many nucleation events occur in finite time, and no two balls nucleate simultaneously, so annihilations can be carried out in chronological order. For similar reasons the process is well-defined for initial configurations in which at least one of the colours appear in finite numbers; we refer to such configurations as *quasi-monochromatic*. Also in this setting there are a.s. at most finitely many nucleation events occurring in finite time that may result in an annihilation of balls, and the annihilations can thus be carried out as before.

It is, however, less obvious that for arbitrary initial configurations the process exists as we have described it. Since there is no 'first' event of annihilation, an attempt to determine whether a potential annihilation takes place or not could (in principle) result in the tracing of an infinite sequence of potential annihilations backwards in time. This *should* not be the case. However, in order to avoid this problem, we shall take a limiting approach where we define the annihilating process for a general initial configuration as a limit of the process for a sequence of finite initial configurations. In order to do so properly,

[3]Alternatively, see [7, Corollary to Theorem 5], applied to the Galton–Watson process $\|\mathcal{X}_{\mathbf{z}}(n)\|$, and [4, Corollary III.6.1].

we shall need to detail further how the process is constructed. Throughout this section we shall limit our attention to initial configuration in $\{-1,0,1\}^{\mathbb{Z}^d}$. For configurations, we use the product order on $\mathbb{Z}^{\mathbb{Z}^d}$, and write thus $\boldsymbol{\zeta} \leqslant \boldsymbol{\zeta}'$ for configurations $\boldsymbol{\zeta} = (\zeta_{\mathbf{x}})_{\mathbf{x}}$ and $\boldsymbol{\zeta}' = (\zeta'_{\mathbf{x}})_{\mathbf{x}}$ if and only if $\zeta_{\mathbf{x}} \leqslant \zeta'_{\mathbf{x}}$ for every $\mathbf{x} \in \mathbb{Z}^d$.

For the reader who prefers to postpone the details of this section and proceed to the proof of our main theorem, we remark that Lemma 4.1 below will be used to prove fixation for $p \neq 1/2$, and Lemmas 4.2–4.4 will be used in the proof of non-fixation at $p = 1/2$. In addition, these lemmas are used in this section to justify our definition of the annihilating process for arbitrary initial configurations. Some of the lemmas will be proven first for the quasi-monochromatic case, and in the present section used only for that case; at the end of the section we extend the proofs to the general case.

4.1 A technical digression on the construction of the process

For the remainder of this paper we make (without loss of generality) the following assumptions. We label each ball (regardless of its colour) by a finite string $(\mathbf{z}, i_1, i_2, \dots, i_m)$ with $\mathbf{z} \in \mathbb{Z}^d$, $m \geqslant 0$ and $i_j \in \mathbb{N}$, such that the ball initially at $\mathbf{z}$ (if any) is labelled by $(\mathbf{z})$, and if a ball has label $(\mathbf{z}, i_1, i_2, \dots, i_m)$, then its children are labelled by $(\mathbf{z}, i_1, i_2, \dots, i_m, i)$ for $i = 1, 2, \dots$ (in some fixed order). This gives each ball a unique label. Furthermore, we assume that we have a Poisson clock for each possible label; these clocks are independent of each other and of the initial configuration. Moreover, each clock is equipped with one realization of the random offspring configuration φ for each ring of the clock. We now define the process with each ball using the corresponding clock and the copies of φ provided by that clock. (Ticks and tocks of unused clocks are ignored.) Furthermore, when a ball annihilates another, and there are several balls at that site that may be chosen for annihilation, we chose the one that comes first according to some fixed rule, for example the ball with smallest label in lexicographic order (using an arbitrary but fixed order on $\mathbb{Z}^d$). Note that all randomness in the process $\mathcal{Z}(t)$ now lies in the clocks and the initial configuration; $\mathcal{Z}(t)$ is a deterministic function of these. Moreover, all clocks may be assumed to start ticking at the dawn of time, and are thus completely independent of the initial configuration.

At occasions we will want to emphasise or compare different initial configurations, and thus write $\mathcal{Z}(t, \boldsymbol{\zeta})$ for the state at time $t \geqslant 0$ of the process with initial configuration $\boldsymbol{\zeta}$. Since clocks are independent of the presence and colour of the balls in the initial configuration, this yields a coupling $\{\mathcal{Z}(t, \boldsymbol{\zeta})\}$ of the processes for all possible initial configurations. Due to the independence between the clocks and the initial configuration, we shall throughout this section consider deterministic initial configurations; analogous statements for random initial configurations are obtained through conditioning.

4.2 A conservative version of the annihilating system

We introduce a conservative version of the process, previously explored in [2]. In this process, red and blue balls branch and get offspring as in the competition process described above, but when a red and a blue ball meet, instead of annihilating, the two balls merge to form a purple ball. Each purple ball in the system continues to branch independently and according to the same rule as red and blue. (For definiteness, purple balls inherit the label (and thus the clock) of the older of the two balls involved in the merging.) Purple balls, however, do not interact with other balls. Consequently, we recover the competition process by ignoring all purple balls.

For quasi-monochromatic initial configurations the conservative process is well-defined for the same reasons the annihilating process is well-defined. Let $R_{\mathbf{x}}(t)$, $B_{\mathbf{x}}(t)$, and $P_{\mathbf{x}}(t)$ be the numbers of red, blue, and purple balls, respectively, at site $\mathbf{x} \in \mathbb{Z}^d$ at time t in the above conservative process, and let $\mathcal{R}(t) := (R_{\mathbf{x}}(t))_{\mathbf{x}\in\mathbb{Z}^d}$, $\mathcal{B}(t) := (B_{\mathbf{x}}(t))_{\mathbf{x}\in\mathbb{Z}^d}$, and $\mathcal{P}(t) := (P_{\mathbf{x}}(t))_{\mathbf{x}\in\mathbb{Z}^d}$ be the corresponding vectors. Then the competition process is given by $\mathcal{Z}(t) = \mathcal{R}(t) - \mathcal{B}(t)$. Furthermore, we use the standard notation $x_+ := \max(x, 0)$ and $x_- := \max(-x, 0)$ for real x, and extend this component-wise to vectors $\boldsymbol{\zeta} = (\zeta_{\mathbf{x}})_{\mathbf{x}\in\mathbb{Z}^d}$. Then, in particular, $\mathcal{R}(t) = \mathcal{Z}(t)_+$ and $\mathcal{B}(t) = \mathcal{Z}(t)_-$.

The crucial facts about this conservative process are stated in the following lemma. The lemma will in the coming sections allow us to apply the previous results for the monochromatic process in order to prove Theorem 1.2.

Lemma 4.1. *The conservative process is well-defined for any initial configuration $\boldsymbol{\zeta}$. The process $\mathcal{R}(t) + \mathcal{P}(t)$ is an instance of the monochromatic process started with $\boldsymbol{\zeta}_+$, and $\mathcal{B}(t) + \mathcal{P}(t)$ is an instance of the monochromatic process started with $\boldsymbol{\zeta}_-$. Furthermore, for all $t \geqslant 0$,*

$$\mathcal{Z}(t, \boldsymbol{\zeta}) = \mathcal{R}(t) - \mathcal{B}(t) = \big[\mathcal{R}(t) + \mathcal{P}(t)\big] - \big[\mathcal{B}(t) + \mathcal{P}(t)\big]. \tag{4.1}$$

Proof when $\boldsymbol{\zeta}$ is quasi-monochromatic. Immediate from the definitions. □

As said above, the general case will be treated at the end of the section. Note that the two monochromatic processes $\mathcal{R}(t) + \mathcal{P}(t)$ and $\mathcal{B}(t) + \mathcal{P}(t)$ are *not* independent and that they are equal to the processes $\mathcal{Z}(t, \boldsymbol{\zeta}_+)$ and $\mathcal{Z}(t, \boldsymbol{\zeta}_-)$ in distribution, but not necessarily point-wise.

4.3 Comparison of initial configurations

We next state two lemmas that will help to compare versions of the process with different initial configurations; the lemmas are proved for quasi-monochromatic initial configurations in the present subsection, and in genral at the end of the section.

We first note that the expected configuration at a given time is a linear function of the initial configuration, and state this formally for the number of balls at the origin.

Lemma 4.2. *For any (deterministic) initial configuration $\boldsymbol{\zeta}$ we have*

$$\mathbb{E}[Z_{\mathbf{0}}(t, \boldsymbol{\zeta})] = e^{\lambda t} \sum_{\mathbf{z}\in\mathbb{Z}^d} p_{\mathbf{z}}(t)\zeta_{\mathbf{z}}. \tag{4.2}$$

Proof when $\boldsymbol{\zeta}$ is quasi-monochromatic. First, consider the monochromatic case; say $\boldsymbol{\zeta} \geqslant \mathbf{0}$, so all balls are red. Then all balls evolve independently, so $Z_{\mathbf{0}}(t, \boldsymbol{\zeta}) = \sum_{\mathbf{z}\in\mathbb{Z}^d} X_{\mathbf{z},\mathbf{0}}(t)\zeta_{\mathbf{z}}$, cf. (3.1), and (4.2) follows by linearity (since all terms are non-negative) and (3.9).

In general, we introduce purple balls and use Lemma 4.1. Then, by (4.1) and using (4.2) for each of the monochromatic processes $\mathcal{R}(t) + \mathcal{P}(t)$ and $\mathcal{B}(t) + \mathcal{P}(t)$,

$$\mathbb{E}[Z_{\mathbf{0}}(t, \boldsymbol{\zeta})] = \mathbb{E}[R_{\mathbf{0}}(t) + P_{\mathbf{0}}(t)] - \mathbb{E}[B_{\mathbf{0}}(t) + P_{\mathbf{0}}(t)] = e^{\lambda t} \sum_{\mathbf{z}:\zeta_{\mathbf{z}}=1} p_{\mathbf{z}}(t) - e^{\lambda t} \sum_{\mathbf{z}:\zeta_{\mathbf{z}}=-1} p_{\mathbf{z}}(t), \tag{4.3}$$

which is well-defined since both sums are finite, and (4.2) follows. □

Our next lemma states that the process is monotone in the initial configuration. Recall that each ball is represented by a unique label of the form $(\mathbf{z}, i_1, i_2, \dots, i_m)$. We may identify balls and their labels when convenient; thus when we talk about a given ball in the competition process, we mean a ball with a given label. We refer to a label as *active* at time t if the corresponding ball exists in the process $\mathcal{Z}(t, \boldsymbol{\zeta})$. Let $\mathcal{A}^R_{\mathbf{z}}(t, \boldsymbol{\zeta})$ denote the set of labels corresponding to red balls at $\mathbf{z}$ that are active at time t, and let $\mathcal{A}^B_{\mathbf{z}}(t, \boldsymbol{\zeta})$ denote ditto for labels corresponding to blue balls.

Lemma 4.3. *Let $\boldsymbol{\zeta}$ and $\boldsymbol{\zeta}'$ be two configurations with $\boldsymbol{\zeta} \leqslant \boldsymbol{\zeta}'$. Then, a.s. for all $t \geqslant 0$ and $\mathbf{z} \in \mathbb{Z}^d$ we have*

$$\mathcal{A}^R_{\mathbf{z}}(t, \boldsymbol{\zeta}) \subseteq \mathcal{A}^R_{\mathbf{z}}(t, \boldsymbol{\zeta}') \quad \textit{and} \quad \mathcal{A}^B_{\mathbf{z}}(t, \boldsymbol{\zeta}') \subseteq \mathcal{A}^B_{\mathbf{z}}(t, \boldsymbol{\zeta}). \tag{4.4}$$

In particular, a.s. $\mathcal{Z}(t, \boldsymbol{\zeta}) \leqslant \mathcal{Z}(t, \boldsymbol{\zeta}')$ for all $t \geqslant 0$.

Proof when $\boldsymbol{\zeta}$ is quasi-monochromatic. The inequality $\boldsymbol{\zeta} \leqslant \boldsymbol{\zeta}'$ means that every red ball in $\boldsymbol{\zeta}$ exists (with the same label and colour) also in $\boldsymbol{\zeta}'$, and every blue ball in $\boldsymbol{\zeta}'$ exists also in $\boldsymbol{\zeta}$. (There may also be further red balls in $\boldsymbol{\zeta}'$ and blue balls in $\boldsymbol{\zeta}$.) We claim that this holds at all later times $t \geqslant 0$ too, and thus that (4.4) holds. In fact, since balls with the same label obey the same clock, the property (4.4) is preserved at each nucleation (including accompanying annihilations, since they follow a fixed order), as we verify next.

Suppose first that $\boldsymbol{\zeta}$ and $\boldsymbol{\zeta}'$ are both finite. Then the number of nucleations is finite in every bounded time interval. We proceed by induction, and suppose that (4.4) is preserved up to the time just before the kth nucleation. We show that (4.4) is preserved also just after the kth nucleation by considering the possible cases. If the kth ring corresponds to a red ball present only in $\mathcal{Z}(t, \boldsymbol{\zeta}')$ (the process evolving from $\boldsymbol{\zeta}'$), then the accompanied nucleation will effect only this process, and (4.4) is readily seen to be preserved. The same applies if the ring corresponds to a blue ball present only in $\mathcal{Z}(t, \boldsymbol{\zeta})$. The case when the ring corresponds to a ball that is blue in $\mathcal{Z}(t, \boldsymbol{\zeta})$ and red in $\mathcal{Z}(t, \boldsymbol{\zeta}')$ is similar (in fact, a combination): $\mathcal{A}^R_{\mathbf{z}}(t, \boldsymbol{\zeta})$ and $\mathcal{A}^B_{\mathbf{z}}(t, \boldsymbol{\zeta}')$ may decrease, while $\mathcal{A}^R_{\mathbf{z}}(t, \boldsymbol{\zeta}')$ and $\mathcal{A}^B_{\mathbf{z}}(t, \boldsymbol{\zeta})$ may increase, which preserves (4.4). If the ring corresponds to a red ball common to both processes, then consider the added balls one by one: for each added red ball at a site $\mathbf{z}$, either

(i) both processes gain a red ball at $\mathbf{z}$ with the same label;

(ii) $\mathcal{Z}(t, \boldsymbol{\zeta}')$ gains a red ball at $\mathbf{z}$ while $\mathcal{Z}(t, \boldsymbol{\zeta})$ loses a blue ball at $\mathbf{z}$; or

(iii) both processes lose a blue ball at $\mathbf{z}$.

In (i), the property (4.4) is trivially preserved. In (ii), we have $\mathcal{A}_{\mathbf{z}}^B(t, \boldsymbol{\zeta}') = \emptyset$, and it follows that (4.4) is preserved. In (iii) we consider two subcases. First, if the blue ball annihilated in $\mathcal{Z}(t, \boldsymbol{\zeta})$ does not exist in $\mathcal{Z}(t, \boldsymbol{\zeta}')$, then (4.4) is preserved. On the other hand, if the annihilated blue ball exist also in $\mathcal{Z}(t, \boldsymbol{\zeta}')$, then it must be annihilated also there, since the ball to be annihilated is chosen as the minimal one in some fixed order among all balls present, and by induction $\mathcal{A}_{\mathbf{z}}^B(t, \boldsymbol{\zeta}') \subseteq \mathcal{A}_{\mathbf{z}}^B(t, \boldsymbol{\zeta})$, so the ball is minimal also in $\mathcal{A}_{\mathbf{z}}^B(t, \boldsymbol{\zeta}')$. Hencre, (4.4) is preserved. The case then a common blue ball nucleates is analogous. Hence, in all cases (4.4) is preserved after the kth nucleation. This proves the lemma when the initial configurations are finite.

In the general quasi-monochromatic case, with initial configurations that may be infinite, the conclusion will follow just the same, since there are at most finitely many nucleations that may result in an annihilation in any given finite time interval. To see this, note that the process obtained by suppressing all annihilations is simply the monochromatic process (the process in which each entry $\zeta_{\mathbf{z}}$ of $\boldsymbol{\zeta}$ is replaced by $|\zeta_{\mathbf{z}}|$). In this process each ball initially present will result in at most finitely many descendants in a finite time interval (cf. Remark 1.1). In particular, in each of $\mathcal{Z}(t, \boldsymbol{\zeta})$ and $\mathcal{Z}(t, \boldsymbol{\zeta}')$ there are at most finitely many balls born with one of the colours in every finite time window, and thus at most a finite number of potential annihilations. □

4.4 General initial configurations

We next show that the competition process with arbitrary initial configurations can be defined as the limit of the process for finite or quasi-monochromatic configurations; we shall also see that this limit indeed satisfies the verbal description of the annihilating process given above.

We let $|\cdot|$ denote the ℓ^∞-norm on $\mathbb{R}^d$ and set

$$B(\mathbf{0}, r) := [-r, r]^d = \{\mathbf{x} : |\mathbf{x}| \leqslant r\}, \quad \text{for } r \geqslant 0. \tag{4.5}$$

Given a configuration $\boldsymbol{\zeta}$ and an integer $r \geqslant 0$, let $\boldsymbol{\zeta}^{\leqslant r}$ denote the restriction of $\boldsymbol{\zeta}$ to $B(\mathbf{0}, r)$, i.e., $\zeta_{\mathbf{z}}^{\leqslant r} := \zeta_{\mathbf{z}} \cdot \mathbf{1}\{|\mathbf{z}| \leqslant r\}$. We define also the modifications $\boldsymbol{\zeta}^{+,r}, \boldsymbol{\zeta}^{-,r}$; these are equal to $\boldsymbol{\zeta}$ in $B(\mathbf{0}, r)$, but for $\mathbf{x} \notin B(\mathbf{0}.r)$ we set

$$\zeta_{\mathbf{x}}^{+,r} := 1 \quad \text{and} \quad \zeta_{\mathbf{x}}^{-,r} := -1. \tag{4.6}$$

This means that outside $B(\mathbf{0}, r)$ we put one ball at each site, red in $\boldsymbol{\zeta}^{+,r}$ and blue in $\boldsymbol{\zeta}^{-,r}$. Note that $\boldsymbol{\zeta}^{\leqslant r}$ is finite and that $\boldsymbol{\zeta}^{+,r}$ and $\boldsymbol{\zeta}^{-,r}$ are quasi-monochromatic.

The inequalities

$$\mathcal{Z}(t, \boldsymbol{\zeta}^{-,r}) \leqslant \mathcal{Z}(t, \boldsymbol{\zeta}^{\leqslant r}) \leqslant \mathcal{Z}(t, \boldsymbol{\zeta}^{+,r}) \tag{4.7}$$

hold for $t = 0$ by definition, and a.s. for every $t \geqslant 0$ by Lemma 4.3. Similarly we have

$$\mathcal{Z}(t, \boldsymbol{\zeta}^{-,r}) \leqslant \mathcal{Z}(t, \boldsymbol{\zeta}^{-,r+1}) \quad \text{and} \quad \mathcal{Z}(t, \boldsymbol{\zeta}^{+,r}) \geqslant \mathcal{Z}(t, \boldsymbol{\zeta}^{+,r+1}) \tag{4.8}$$

for every $t \geqslant 0$.

Lemma 4.4. *For any initial configuration $\boldsymbol{\zeta} \in \{-1,0,1\}^{\mathbb{Z}^d}$, $\mathbf{z} \in \mathbb{Z}^d$ and $T < \infty$ there exists a.s. a (random) $L < \infty$ such that for all $r \geqslant L$ and $t \in [0,T]$*

$$\mathcal{A}^R_{\mathbf{z}}(t,\boldsymbol{\zeta}^{+,r}) = \mathcal{A}^R_{\mathbf{z}}(t,\boldsymbol{\zeta}^{-,r}) \quad \text{and} \quad \mathcal{A}^B_{\mathbf{z}}(t,\boldsymbol{\zeta}^{+,r}) = \mathcal{A}^B_{\mathbf{z}}(t,\boldsymbol{\zeta}^{-,r}), \tag{4.9}$$

so that, in particular,

$$Z_{\mathbf{z}}(t,\boldsymbol{\zeta}^{+,r}) = Z_{\mathbf{z}}(t,\boldsymbol{\zeta}^{-,r}) \tag{4.10}$$

for all $t \in [0,T]$ and $r \geqslant L$.

Proof. Fix $T > 0$. By Lemma 4.3 we have a.s. that for all $\mathbf{z} \in \mathbb{Z}^d$, $t \in [0,T]$ and $r \geqslant 0$

$$\mathcal{A}^R_{\mathbf{z}}(t,\boldsymbol{\zeta}^{-,r}) \subseteq \mathcal{A}^R_{\mathbf{z}}(t,\boldsymbol{\zeta}^{+,r}) \quad \text{and} \quad \mathcal{A}^B_{\mathbf{z}}(t,\boldsymbol{\zeta}^{+,r}) \subseteq \mathcal{A}^B_{\mathbf{z}}(t,\boldsymbol{\zeta}^{-,r}). \tag{4.11}$$

For the reverse inclusion, we consider only the case $\mathbf{z} = \mathbf{0}$ for notational convenience; the general case is analogous. Thus, let $\mathcal{E}_r(T)$ denote the event that (4.11) holds (for all $\mathbf{z}$) but that for some $t \in [0,T]$ equation (4.9) fails for $\mathbf{z} = \mathbf{0}$. Then, on the event $\mathcal{E}_r(T)$ we have, for some $t \in [0,T]$, $Z_{\mathbf{0}}(t,\boldsymbol{\zeta}^{-,r}) + 1 \leqslant Z_{\mathbf{0}}(t,\boldsymbol{\zeta}^{+,r})$ and thus

$$\mathcal{Z}(t,\boldsymbol{\zeta}^{-,r}) + \delta_{\mathbf{x},\mathbf{0}} \leqslant \mathcal{Z}(t,\boldsymbol{\zeta}^{+,r}). \tag{4.12}$$

An induction argument, as in the proof of Lemma 4.3, shows that (4.12) implies

$$\mathcal{Z}(T,\boldsymbol{\zeta}^{-,r}) + \delta_{\mathbf{x},\mathbf{0}} \leqslant \mathcal{Z}(T,\boldsymbol{\zeta}^{+,r}). \tag{4.13}$$

In particular, $Z_{\mathbf{0}}(T,\boldsymbol{\zeta}^{-,r})+1 \leqslant Z_{\mathbf{0}}(T,\boldsymbol{\zeta}^{+,r})$. Consequently, Markov's inequality and (4.11), together with Lemma 4.2, yield

$$\mathbb{P}\big(\mathcal{E}_r(T)\big) \leqslant \mathbb{E}\left[Z_{\mathbf{0}}(T,\boldsymbol{\zeta}^{+,r}) - Z_{\mathbf{0}}(T,\boldsymbol{\zeta}^{-,r})\right] \leqslant 2e^{\lambda t} \sum_{|\mathbf{z}|>r} p_{\mathbf{z}}(T), \tag{4.14}$$

which tends to zero as $r \to \infty$. Moreover, (4.8) implies that $\mathcal{E}_r \supseteq \mathcal{E}_{r+1}$. Hence, the event that (4.11) holds but (4.9) fails for arbitrarily large r is $\mathcal{E}(T) := \limsup_{r\to\infty} \mathcal{E}_r(T) = \bigcap_{r\geqslant 1} \mathcal{E}_r(T)$, and (4.14) yields $\mathbb{P}\big(\mathcal{E}(T)\big) = 0$ as required. □

Lemma 4.5. *As $r \to \infty$, the three processes $\mathcal{Z}(t,\boldsymbol{\zeta}^{\leqslant r})$, $\mathcal{Z}(t,\boldsymbol{\zeta}^{-,r})$, and $\mathcal{Z}(t,\boldsymbol{\zeta}^{+,r})$ a.s. all converge in the product space $D[0,\infty)^{\mathbb{Z}^d}$ to a common limiting process $\mathcal{Z}^*(t,\boldsymbol{\zeta})$. Moreover, the convergence is in the strong sense that for every $\mathbf{z}$ and $T < \infty$, there exists $L = L(\mathbf{z},T)$ such that the processes at $\mathbf{z}$ all are equal to the limit $Z^*_{\mathbf{z}}(t,\boldsymbol{\zeta})$ for all $t \in [0,T]$ and $r \geqslant L$.*

Proof. Lemma 4.4 shows that given $\mathbf{z}$ and T, there exists L such that (4.10) holds for $r \geqslant L$ and all $t \leqslant T$. By (4.8), this implies $Z_{\mathbf{z}}(t,\boldsymbol{\zeta}^{+,r}) = Z_{\mathbf{z}}(t,\boldsymbol{\zeta}^{-,r}) = Z^{+,L}_{\mathbf{z}} = Z^{-,L}_{\mathbf{z}}$ for $r \geqslant L$. The result follows by (4.7). □

We will henceforth take $\mathcal{Z}^*(t,\boldsymbol{\zeta})$ as the formal definition of $\mathcal{Z}(t,\boldsymbol{\zeta})$ for every initial configuration $\boldsymbol{\zeta}$; in other words, we define

$$\mathcal{Z}(t,\boldsymbol{\zeta}) := \mathcal{Z}^*(t,\boldsymbol{\zeta}) = \lim_{r\to\infty} \mathcal{Z}(t,\boldsymbol{\zeta}^{\leqslant r}), \tag{4.15}$$

recalling that by Lemma 4.5, we could as well use $\mathcal{Z}(t, \boldsymbol{\zeta}^{+,r})$ or $\mathcal{Z}(t, \boldsymbol{\zeta}^{-,r})$ in (4.15). Note that if $\boldsymbol{\zeta}$ is quasi-monochromatic, then Lemma 4.3 implies that a.s.

$$\mathcal{Z}(t, \boldsymbol{\zeta}^{-,r}) \leqslant \mathcal{Z}(t, \boldsymbol{\zeta}) \leqslant \mathcal{Z}(t, \boldsymbol{\zeta}^{+,r}) \tag{4.16}$$

for all t and r, and thus $\mathcal{Z}^*(t, \boldsymbol{\zeta}) = \mathcal{Z}(t, \boldsymbol{\zeta})$ by Lemma 4.5, so there is no inconsistency.

We verify next that this process indeed behaves as the competition process that we have described verbally in the introduction and Section 4.1.

Lemma 4.6. *The process $\mathcal{Z}(t, \boldsymbol{\zeta})$, defined by* (4.15)*, has the following almost sure properties: For each site $\mathbf{x} \in \mathbb{Z}^d$ it holds that*

(i) *balls at $\mathbf{x}$ have a distinct labels and produce labeled offspring according to the corresponding clocks;*

(ii) *balls arriving at $\mathbf{x}$ are those born in nucleations at (possibly different) sites and annihilate according to the predefined rule;*

(iii) *there are no additional balls or annihilations occurring at $\mathbf{x}$.*

Proof. Fix $T < \infty$, $\mathbf{x} \in \mathbb{Z}^d$ and a configuration $\boldsymbol{\zeta} \in \{-1, 0, 1\}^{\mathbb{Z}^d}$. First, note that properties (i)–(iii) hold, by construction, for any quasi-monochromatic initial configuration. Next, by Lemma 4.4 there exists a.s. $r < \infty$ so that the labels active at $\mathbf{x}$ up to time T coincide for $\mathcal{Z}^*(t, \boldsymbol{\zeta})$ and $\mathcal{Z}(t, \boldsymbol{\zeta}^{+,r})$. So, (i) follows since it holds for $\mathcal{Z}(t, \boldsymbol{\zeta}^{+,r})$.

Let $\boldsymbol{\zeta}^{++}$ denote the monochromatic configuration consisting of one red ball at each site, and let $N_{\mathbf{x}}$ denote the set of locations where there is a nucleation sending a ball to $\mathbf{x}$ at some time $t \leqslant T$ in the process $\mathcal{Z}(t, \boldsymbol{\zeta}^{++})$. Note that for every quasi-monochromatic configuration $\boldsymbol{\zeta}'$, the locations at which there is a nucleation in $\mathcal{Z}(t, \boldsymbol{\zeta}')$ sending a ball to $\mathbf{x}$ (immediately annihilated or not) is a subset of $N_{\mathbf{x}}$.

Since $\mathcal{Z}(T, \boldsymbol{\zeta}^{++})$ is a.s. locally finite, the set $N_{\mathbf{x}}$ is a.s. finite. According to Lemma 4.4 there exists a.s. $r < \infty$ so that the labels active up to time T at each $\mathbf{z} \in N_{\mathbf{x}} \cup \{\mathbf{x}\}$ coincide for $\mathcal{Z}^*(t, \boldsymbol{\zeta})$ and $\mathcal{Z}(t, \boldsymbol{\zeta}^{+,r})$. Consequently, the nucleations in $\mathcal{Z}^*(t, \boldsymbol{\zeta})$ resulting in a ball att $\mathbf{x}$ are the same as those in $\mathcal{Z}(t, \boldsymbol{\zeta}^{+,r})$, and these are the only balls that appear at $\mathbf{x}$. It follows that the annihilations at $\mathbf{x}$ caused by these nucleations are the same for $\mathcal{Z}^*(t, \boldsymbol{\zeta})$ and $\mathcal{Z}(t, \boldsymbol{\zeta}^{+,r})$, and that there are no annihilations occurring other than these. This proves (ii) and (iii). □

Remark 4.7. Let us note that the limiting process $\mathcal{Z}^*(t, \boldsymbol{\zeta})$ does not depend on the order in which the initial configuration is being 'revealed' in the limiting procedure. Indeed, fix any enumeration of $\mathbb{Z}^d$ and let $\boldsymbol{\zeta}^{(k)}$ denote the configuration whose coordinates equal $\zeta_{\mathbf{z}}$ for the first k entries $\mathbf{z}$ of the enumeration and 0 otherwise. Then, for every r we may choose $K < \infty$ so that $\boldsymbol{\zeta}^{-,r} \leqslant \boldsymbol{\zeta}^{(k)} \leqslant \boldsymbol{\zeta}^{+,r}$ for all $k \geqslant K$. By Lemma 4.3 it follows that for all $k \geqslant K$ and $t \geqslant 0$

$$\mathcal{Z}(t, \boldsymbol{\zeta}^{-,r}) \leqslant \mathcal{Z}(t, \boldsymbol{\zeta}^{(k)}) \leqslant \mathcal{Z}(t, \boldsymbol{\zeta}^{+,r}). \tag{4.17}$$

By Lemma 4.5 we conclude that also $\mathcal{Z}(t, \boldsymbol{\zeta}^{(k)})$ converges to the limiting process $\mathcal{Z}^*(t, \boldsymbol{\zeta})$ as $k \to \infty$. (The same argument applies to the corresponding versions of $\mathcal{Z}(t, \boldsymbol{\zeta}^{+,r})$ and $\mathcal{Z}(t, \boldsymbol{\zeta}^{-,r})$.) In particular, the distribution of the limiting process is invariant with respect to translations.

4.5 Epilogue

Having properly defined the process we may now extend the lemmas proven above for quasi-monochromatic initial configurations to arbitrary configurations.

Proof of Lemma 4.1, general case. As in the proof of Lemma 4.6, the annihilations occurring at $\mathbf{x}$ up to time T are the same for $\mathcal{Z}^*(t, \boldsymbol{\zeta})$ and $\mathcal{Z}(t, \boldsymbol{\zeta}^{+,r})$ when r is large. Consequently, the process consisting of purple balls is well-defined for arbitrary initial configurations, as the limit as $r \to \infty$ of the purple process for $\boldsymbol{\zeta}^{+,r}$. This defines the conservative process for any $\boldsymbol{\zeta}$, and it is immediate that the claims in the lemma hold, since they hold for each $\boldsymbol{\zeta}^{+,r}$. □

Proof of Lemma 4.2, general case. The proof above now applies to arbitrary $\boldsymbol{\zeta}$. □

Proof of Lemma 4.3, general case. If $\boldsymbol{\zeta} \leqslant \boldsymbol{\zeta}'$, then $\boldsymbol{\zeta}^{+,r} \leqslant (\boldsymbol{\zeta}')^{+,r}$ for every r, and the already proven case applies. The result follows by Lemmas 4.4 and 4.5. □

5 Fixation versus non-fixation

The goal of this section is to prove Theorem 1.2. The first part of the theorem, stating that the competing urn scheme started from an unbalanced initial Bernoulli colouring will eventually fixate at red, is an immediate consequence of Theorem 1.5 together with the conservative coupling, see Section 5.1; the coupling decomposes the competition process into the difference between two monochromatic processes, one of which is larger by a factor.

The second part of the theorem, stating that the competing urn scheme started from a balanced initial Bernoulli colouring does not fixate, will require more work. Let us first present a brief sketch of the proof. We start with the intuition that the state of the origin at time $t = 1$ is unlikely to dictate the state of the origin at time $t \gg 1$. Taking this intuition to its logical conclusion we should be able to choose a fast growing sequence of times $t_1, t_2, \dots$ such that the state of the origin at time t_n is approximately independent from its states at times $t_1, \dots, t_{n-1}$. One would therefore expect the origin to be red for infinitely many of the times t_n and blue for infinitely many of the times t_n, which would complete the proof.

In order to make this rigorous we show that the state of the origin at time $t = 1$ mostly depends on the descendants of balls which start near the origin. On the contrary, at time $t \gg 1$ the state at the origin depends on descendants of balls from a much larger region, while balls originating near the origin contribute little. This will allow us to define a growing sequence of scales $r_1, r_2, \dots$ such that the state of the origin at time t_n may be well approximated by considering *only* the descendants of balls initially in the annulus $[-r_{n+1}, r_{n+1}]^d \setminus [-r_n, r_n]^d$. Since these annuli are disjoint, this will allow us to 'decouple' the state of the origin at times $t_1, t_2, \dots$.

We implement this approach in Section 5.3. An important ingredient will be to understand the likely order of magnitude of the number of balls at the origin. While the order of magnitude does not grows as fast as $e^{\lambda t}$ (as in the case $p \neq 1/2$), it is still likely to be at the order of its standard deviation, which is $t^{-d/4} e^{\lambda t}$. This will be obtained via a second moment approach, resting on the quantitative bounds for the monochromatic process obtained in Proposition 3.1; see Section 5.2 for details.

5.1 Fixation for $p \neq 1/2$

We first consider the annihilating process $\mathcal{Z}(t)$ in the unbalanced setting, that is, starting from a biased Bernoulli colouring, and prove fixation.

Proof of Theorem 1.2(i). Using Lemma 4.1, we decompose $\mathcal{Z}(t)$ into the difference of two monochromatic processes as follows:

$$\mathcal{Z}(t) = [\mathcal{R}(t) + \mathcal{P}(t)] - [\mathcal{B}(t) + \mathcal{P}(t)]. \tag{5.1}$$

Next, fix $\varepsilon > 0$ such that $2p - 1 > 3\varepsilon$. By Theorem 1.5 there exists a.s. a (random) finite T_0 such that, for all $t \geqslant T_0$,

$$e^{-\lambda t}\big(R_{\mathbf{0}}(t) + P_{\mathbf{0}}(t)\big) > p - \varepsilon \quad \text{and} \quad e^{-\lambda t}\big(B_{\mathbf{0}}(t) + P_{\mathbf{0}}(t)\big) < 1 - p + \varepsilon, \tag{5.2}$$

and hence (5.1) yields $Z_{\mathbf{0}}(t) > \varepsilon e^{\lambda t} > 0$, as required. That is, the origin fixates a.s. to red, and by translation invariance, every site a.s. fixates. □

5.2 A second moment analysis for $p = 1/2$

In this subsection we consider the annihilating process $\mathcal{Z}(t)$ in the balanced setting, that is, starting from a symmetric Bernoulli colouring in which each site is given a ball whose colour is determined by a fair coin flip. We aim to prove the following bound on deviations of $\mathcal{Z}(t)$.

Proposition 5.1. *Consider the competing urn scheme $\mathcal{Z}(t)$ in the balanced setting, and assume that $\mathbb{E}\left[|||\varphi|||_1^2\right] < \infty$, $\mathbb{E}\left[|||\varphi|||_2\right] < \infty$ and $\mathbb{E}[\|\varphi\|^2] < \infty$ holds. Then there exists a constant $c > 0$ such that for all $t > 1/c$ we have*

$$\mathbb{P}\left(e^{-\lambda t} Z_{\mathbf{0}}(t) > ct^{-d/4}\right) > c. \tag{5.3}$$

To show this, we shall use the following generic lemma, which is a conditional version of the Paley–Zygmund inequality.

Lemma 5.2. *Let X be a random variable and $\mathcal{F}$ a sub-σ-field (on some probability space). Suppose that $\mathbb{E}[X^2] \leqslant K$ and $\mathbb{P}\big(\mathbb{E}[X \mid \mathcal{F}] \geqslant 1/K\big) \geqslant 1/K$ for some constant K. Then*

$$\mathbb{P}\big(X > 1/(2K)\big) \geqslant 1/(4K^5). \tag{5.4}$$

Proof. Let F be the event that $\mathbb{E}[X \mid \mathcal{F}] \geqslant 1/K$ and let E be the event that both $\mathbb{E}[X \mid \mathcal{F}] \geqslant 1/K$ and $X > 1/(2K)$. Cauchy–Schwartz gives

$$\mathbb{E}[X\, \mathbf{1}_E] \leqslant \|X\|_2 \|\, \mathbf{1}_E\,\|_2 \leqslant K^{1/2}\, \mathbb{P}\,(E)^{1/2}\,. \tag{5.5}$$

On the other hand we have that

$$\mathbb{E}[X\, \mathbf{1}_F] \geqslant \frac{\mathbb{P}\,(F)}{K} \tag{5.6}$$

and, since $X \leqslant 1/2K$ on $F \setminus E$,

$$\mathbb{E}[X \, \mathbf{1}_{F\setminus E}] \leqslant \frac{\mathbb{P}(F \setminus E)}{2K} \leqslant \frac{\mathbb{P}(F)}{2K}. \tag{5.7}$$

Thus, since $\mathbb{P}(F) \geqslant 1/K$ by assumption,

$$\mathbb{E}[X \, \mathbf{1}_{E}] \geqslant \frac{\mathbb{P}(F)}{2K} \geqslant \frac{1}{2K^2}, \tag{5.8}$$

which combined with (5.5) gives

$$K^{1/2}\, \mathbb{P}(E)^{1/2} \geqslant \frac{1}{2K^2}. \tag{5.9}$$

It follows that $\mathbb{P}(E) \geqslant 1/4K^5$, and the result follows. □

We also need an estimate of the variance.

Lemma 5.3. *Under the conditions of Proposition 5.1, we have for all $t \geqslant 1$*

$$\operatorname{Var}[e^{-\lambda t} Z_{\mathbf{0}}(t)] = O\big(t^{-d/2}\big). \tag{5.10}$$

Proof. We use the conservative process with purple balls and (4.1), recalling that both $R_{\mathbf{0}}(t) + P_{\mathbf{0}}(t)$ and $B_{\mathbf{0}}(t) + P_{\mathbf{0}}(t)$ have the same distribution as $Y_{\mathbf{0}}^{p}(t)$ with $p = 1/2$. Hence, by Proposition 3.1(ii),

$$\operatorname{Var}[Z_{\mathbf{0}}(t)] \leqslant 4 \operatorname{Var}\big[Y_{\mathbf{0}}^{1/2}(t)\big] = O\big(t^{-d/2} e^{2\lambda t}\big), \tag{5.11}$$

as required. □

We are now in position to proceed with the proof of the proposition.

Proof of Proposition 5.1. Let $\boldsymbol{\zeta}$ be a random symmetric Bernoulli colouring and let $\mathcal{F}$ be the σ-field generated by $\boldsymbol{\zeta}$. Since the clocks are independent of $\boldsymbol{\zeta}$, Lemma 4.2 yields

$$\mathbb{E}\big[e^{-\lambda t} Z_{\mathbf{0}}(t) \mid \mathcal{F}\big] = \sum_{\mathbf{z} \in \mathbb{Z}^d} p_{\mathbf{z}}(t) \zeta_{\mathbf{z}} =: S(t). \tag{5.12}$$

$S(t)$ is a sum of independent random variables with mean 0, so using Proposition 2.4,

$$\operatorname{Var}\big[S(t)\big] = \sum_{\mathbf{z} \in \mathbb{Z}^d} p_{\mathbf{z}}(t)^2 = \Theta\big(t^{-d/2}\big). \tag{5.13}$$

We next claim that, as $t \to \infty$, we have

$$S(t)/\sqrt{\operatorname{Var}[S(t)]} \to N(0,1) \quad \text{in distribution.} \tag{5.14}$$

To see this, we use the central limit theorem with the Lyapounov condition that

$$\beta(r,t) := \big(\operatorname{Var}[S(t)]\big)^{-r/2} \sum_{\mathbf{z} \in \mathbb{Z}^d} \mathbb{E}\big[|p_{\mathbf{z}}(t) \zeta_{\mathbf{z}}|^r\big] = o(1) \quad \text{as } t \to \infty, \tag{5.15}$$

for some $r > 2$; see e.g. [19, Theorem 7.2.2 and 7.2.4]. (The central limit theorem is usually stated for finite sums, but extends immediately to L^2-convergent sums by truncation and the Cramér–Slutsky theorem.) We verify the Lyapounov condition (5.15) with $r = 3$. Then, by Proposition 2.4(i)

$$\sum_{\mathbf{z}\in\mathbb{Z}^d} \mathbb{E}\big[|p_{\mathbf{z}}(t)\zeta_{\mathbf{z}}|^3\big] = \sum_{\mathbf{z}\in\mathbb{Z}^d} p_{\mathbf{z}}(t)^3 \leqslant \sup_{\mathbf{z}} p_{\mathbf{z}}(t)^2 = O(t^{-d}). \tag{5.16}$$

Hence, (5.13) and (5.16) yield $\beta(3,t) = O(t^{-d/4})$, which verifies (5.15), so (5.14) holds.

To complete the proof, let $X := e^{-\lambda t} Z_{\mathbf{0}}(t)/\sqrt{\mathrm{Var}[S(t)]}$. Recall that $Z_{\mathbf{0}}(t)$ has mean zero, so we may from Lemma 5.3 and (5.13) deduce that $\mathbb{E}\big[e^{-2\lambda t} Z_{\mathbf{0}}(t)^2\big] \leqslant K\,\mathrm{Var}[S(t)]$ for some constant K; in other words, $\mathbb{E}[X^2] \leqslant K$. Increasing K if necessary, it follows from (5.12) and (5.14) that

$$\mathbb{P}\big(\mathbb{E}[X \mid \mathcal{F}] \geqslant 1/K\big) = \mathbb{P}\left(S(t) \geqslant \sqrt{\mathrm{Var}[S(t)]}/K\right) \geqslant 1/K \tag{5.17}$$

for all large t. Lemma 5.2 therefore shows that

$$\mathbb{P}\left(e^{-\lambda t} Z_{\mathbf{0}}(t) > \sqrt{\mathrm{Var}[S(t)]}/(2K)\right) \geqslant 1/(4K^5). \tag{5.18}$$

Since $\sqrt{\mathrm{Var}[S(t)]} = \Theta(t^{-d/4})$ by (5.13), the proof is complete. □

5.3 A decoupling argument and conclusion of the proof

We now complete the proof of part (ii) of Theorem 1.2. We began this section with an overview of the proof, including the idea that we would consider a sequence of times $t_1, t_2, \dots$ and scales $r_1, r_2, \dots$ such that the state of the origin at time t_n mostly depends on the descendants of balls which start in the annullus $B(\mathbf{0}, r_{n+1}) \setminus B(\mathbf{0}, r_n)$. We now implement this idea rigorously. In addition to the sequences of times and scales we define an auxiliary sequence $(C_i)_{i\geqslant 1}$ which controls the contribution at the origin of balls descending from within a growing sequence of regions.

Since we shall need to quantify the contribution coming from different locations we introduce some further notation. For a configuration $\boldsymbol{\zeta}$, recall that $\boldsymbol{\zeta}^{\leqslant r}$ denotes the restrictions of $\boldsymbol{\zeta}$ to $B(\mathbf{0}, r)$; we similarly let $\boldsymbol{\zeta}^{>r}$ denote the restriction of $\boldsymbol{\zeta}$ to the complement of $B(\mathbf{0}, r)$, i.e., $\zeta^{>r}_{\mathbf{z}} = \zeta_{\mathbf{z}} \cdot \mathbf{1}\{|\mathbf{z}| > r\}$. Finally, let $\boldsymbol{\zeta}^{(r,r']} := (\boldsymbol{\zeta}^{>r})^{\leqslant r'}$.

We first bound the number of balls at the origin that originate from $B(\mathbf{0}, r)$.

Lemma 5.4. *For every $r \geqslant 1$ and $\delta > 0$ there exists $C > 0$ such that for all $t \geqslant 1$ and every $\boldsymbol{\zeta} \in \{-1, 0, 1\}^{\mathbb{Z}^d}$ we have*

$$\mathbb{P}\Big(\big|Z_{\mathbf{0}}(t, \boldsymbol{\zeta}) - Z_{\mathbf{0}}(t, \boldsymbol{\zeta}^{>r})\big| > C t^{-d/2} e^{\lambda t}\Big) < \delta. \tag{5.19}$$

Proof. Recall that in $\boldsymbol{\zeta}^{>r}$, all $\boldsymbol{\zeta}_{\mathbf{x}}$ with $|\mathbf{x}| \leqslant r$ have been reset to 0. Fix r and define $\boldsymbol{\zeta}^+$ by instead letting $\boldsymbol{\zeta}^+_{\mathbf{x}} := 1$ when $|\mathbf{x}| \leqslant r$, and as before $\boldsymbol{\zeta}^+_{\mathbf{x}} = \boldsymbol{\zeta}_{\mathbf{x}}$ otherwise. Then $\boldsymbol{\zeta}^+ \geqslant \boldsymbol{\zeta}$

and $\boldsymbol{\zeta}^+ \geqslant \boldsymbol{\zeta}^{>r}$, and thus by Lemma 4.3, $Z_{\mathbf{0}}(t, \boldsymbol{\zeta}^+) \geqslant Z_{\mathbf{0}}(t, \boldsymbol{\zeta})$ and $Z_{\mathbf{0}}(t, \boldsymbol{\zeta}^+) \geqslant Z_{\mathbf{0}}(t, \boldsymbol{\zeta}^{>r})$. Consequently, the triangle inequality and Lemma 4.2 (applied four times) give

$$\begin{aligned}\mathbb{E}\big|Z_{\mathbf{0}}(t, \boldsymbol{\zeta}) - Z_{\mathbf{0}}(t, \boldsymbol{\zeta}^{>r})\big| &\leqslant \mathbb{E}\big[Z_{\mathbf{0}}(t, \boldsymbol{\zeta}^+) - Z_{\mathbf{0}}(t, \boldsymbol{\zeta})\big] + \mathbb{E}\big[Z_{\mathbf{0}}(t, \boldsymbol{\zeta}^+) - Z_{\mathbf{0}}(t, \boldsymbol{\zeta}^{>r})\big] \\ &\leqslant 3e^{\lambda t} \sum_{\mathbf{z} \in B(\mathbf{0}, r)} p_{\mathbf{z}}(t) \leqslant 3(2r+1)^d e^{\lambda t} \sup_{\mathbf{z}} p_{\mathbf{z}}(t). \end{aligned} \tag{5.20}$$

The result follows by Proposition 2.4(i) and Markov's inequality. □

We may now, finally, complete the proof of our main theorem.

Proof of Theorem 1.2(ii). Throughout the proof $\boldsymbol{\zeta}$ denotes a random symmetric Bernoulli colouring, and $c > 0$ is the constant from Proposition 5.1. Let $t_0 = 1/c$ and let $r_0 = 0$. We now define the sequences $(r_i)_{i \geqslant 1}$, $(C_i)_{i \geqslant 1}$ and $(t_i)_{i \geqslant 1}$ sequentially. For $i \geqslant 1$, choose

(i) $r_i > r_{i-1}$, using Lemma 4.5 and (4.15), such that

$$\mathbb{P}\Big(Z_{\mathbf{0}}(t_{i-1}, \boldsymbol{\zeta}) \neq Z_{\mathbf{0}}(t_{i-1}, \boldsymbol{\zeta}^{\leqslant r_i})\Big) \leqslant 2^{-i}; \tag{5.21}$$

(ii) $C_i > 0$, using Lemma 5.4, such that for every $t \geqslant 1$ and $r \geqslant r_i$,

$$\mathbb{P}\Big(\big|Z_{\mathbf{0}}(t, \boldsymbol{\zeta}^{\leqslant r}) - Z_{\mathbf{0}}(t, \boldsymbol{\zeta}^{(r_i, r]})\big| > C_i t^{-d/2} e^{\lambda t}\Big) \leqslant 2^{-i}. \tag{5.22}$$

(iii) $t_i > t_{i-1} + 1$ such that $t_i^{d/4} > 3c^{-1} C_i$, so that, by Proposition 5.1, we have

$$\mathbb{P}\Big(Z_{\mathbf{0}}(t_i, \boldsymbol{\zeta}) > 3C_i t_i^{-d/2} e^{\lambda t_i}\Big) \geqslant c. \tag{5.23}$$

In particular, (5.22) yields

$$\mathbb{P}\Big(\big|Z_{\mathbf{0}}(t_i, \boldsymbol{\zeta}^{\leqslant r_{i+1}}) - Z_{\mathbf{0}}(t_i, \boldsymbol{\zeta}^{(r_i, r_{i+1}]})\big| > C_i t_i^{-d/2} e^{-\lambda t_i}\Big) \leqslant 2^{-i}. \tag{5.24}$$

Hence, using also (5.21),

$$\mathbb{P}\Big(\big|Z_{\mathbf{0}}(t_i, \boldsymbol{\zeta}) - Z_{\mathbf{0}}(t_i, \boldsymbol{\zeta}^{(r_i, r_{i+1}]})\big| > C_i t_i^{-d/2} e^{-\lambda t_i}\Big) \leqslant 2^{-i} + 2^{-i-1} \leqslant 2^{1-i}. \tag{5.25}$$

Thus, (5.23) implies

$$\mathbb{P}\Big(Z_{\mathbf{0}}(t_i, \boldsymbol{\zeta}^{(r_i, r_{i+1}]}) > 2C_i t_i^{-d/2} e^{\lambda t_i}\Big) \geqslant c - 2^{1-i}. \tag{5.26}$$

Next, we define two 'failure' events, of which (at least) one must occur for the origin to be blue at all large times.

- Let F_1 be the event that *for infinitely many* $i \geqslant 1$ we have

$$\big|Z_{\mathbf{0}}(t_i, \boldsymbol{\zeta}) - Z_{\mathbf{0}}(t_i, \boldsymbol{\zeta}^{(r_i, r_{i+1}]})\big| > C_i t_i^{-d/2} e^{-\lambda t_i}. \tag{5.27}$$

- Let F_2 be the event that *for at most finitely many* $i \geqslant 1$ we have

$$Z_{\mathbf{0}}(t_i, \boldsymbol{\zeta}^{(r_i, r_{i+1}]}) > 2C_i t_i^{-d/2} e^{\lambda t_i}. \tag{5.28}$$

By (5.25), it follows that (5.27) occurs with probability at most 2^{1-i}. Hence, by the Borel–Cantelli lemma, we have that $\mathbb{P}(F_1) = 0$. Moreover, by (5.26), (5.28) occurs with probability at least $c/2$ for large i. Note that the processes $\mathcal{Z}(t, \boldsymbol{\zeta}^{(r_i, r_{i+1}]})$, for $i \geqslant 1$, are mutually independent by our construction. Hence the events in (5.28) are independent, and thus the other Borel–Cantelli lemma implies that $\mathbb{P}(F_2) = 0$.

Let I_1 and I_2 denote the sets of i's for which (5.27) and (5.28) occur, respectively. We have shown that $I := I_2 \setminus I_1$ is infinite almost surely. To complete the proof, we note that for each $i \in I$ we have

$$Z_{\mathbf{0}}(t_i, \boldsymbol{\zeta}) \geqslant Z_{\mathbf{0}}(t_i, \boldsymbol{\zeta}^{(r_i, r_{i+1}]}) - C_i t_i^{-d/2} e^{\lambda t_i} > C_i t_i^{-d/2} e^{\lambda t_i} > 0.$$

Hence, there are a.s. arbitrarily large times t such that $Z_{\mathbf{0}}(t, \boldsymbol{\zeta}) > 0$ and thus $\mathbf{0}$ is red. By symmetry there are a.s. also arbitrarily large t with $\mathbf{0}$ being blue. This completes the proof of part (ii) of Theorem 1.2. □

6 Existence of the density

In this section $\mathcal{Z}(t)$ will describe the evolution of the system starting from the p-random Bernoulli colouring, where $p \in [0, 1]$ is arbitrary. We aim to show that the density of red sites, as defined in (1.3), is indeed a.s. well-defined for all $t \geqslant 0$.

Theorem 6.1. *Assume $\mathbb{E}\,\|\varphi\| < \infty$. Then, for the competing urn scheme on $\mathbb{Z}^d$ starting from a p-random Bernoulli colouring, almost surely, the density $\rho(t)$ of red urns, as defined in* (1.3)*, exists for all $t \geqslant 0$ and*

$$\rho(t) = \mathbb{P}\left(Z_{\mathbf{0}}(t) > 0\right). \tag{6.1}$$

We begin with a lemma.

Lemma 6.2. *Assume $\mathbb{E}\,\|\varphi\| < \infty$. For every $T < \infty$ and $\varepsilon > 0$ there exists $\delta > 0$ such that for any $p \in [0, 1]$ and interval $I \subseteq [0, T]$ of length at most δ, we have*

$$\mathbb{P}\big(Z_{\mathbf{0}}(t) \text{ is not constant for } t \in I\big) < \varepsilon. \tag{6.2}$$

Proof. Using Lemma 4.1 (and replacing ε by $\varepsilon/2$), we see that it suffices to prove the corresponding result for the monochromatic process $\mathcal{Y}^p(t)$. Let $[a, b] \subseteq [0, T]$. Since the monochromatic process is (weakly) increasing at each site, we obtain using Markov's inequality together with (3.2) and its proof,

$$\begin{aligned} \mathbb{P}\big(Y^p_{\mathbf{0}}(t) \text{ is not constant for } t \in [a, b]\big) &= \mathbb{P}\big(Y^p_{\mathbf{0}}(b) > Y^p_{\mathbf{0}}(a)\big) \leqslant \mathbb{E}[Y^p_{\mathbf{0}}(b) - Y^p_{\mathbf{0}}(a)] \\ &= p\big(e^{\lambda b} - e^{\lambda a}\big) \leqslant (b - a)\lambda e^{\lambda T}. \end{aligned} \tag{6.3}$$

The result follows by taking δ small enough. □

Proof of Theorem 6.1. First consider a fixed $t \geqslant 0$. We use a standard type of argument. That the limit in (1.3) exists, almost surely, for a fixed $t \geqslant 0$ is a consequence of translation invariance and the (multivariate) ergodic theorem (see e.g. [21, Theorem 10.12]). Using the construction of $\mathcal{Z}(t)$ in Section 4.1, $\mathcal{Z}(t)$ is a measurable deterministic function of the clocks and the initial colouring. Furthermore, by first considering monochromatic processes and then using Lemma 4.1, we see that changing a finite number of the clocks and initial colours can only affect $Z_{\mathbf{x}}(t)$ for finitely many $\mathbf{x}$, a.s., which will not change the limit (1.3). Thus $\rho(t)$ is measurable with respect to the corresponding tail σ-field, and the Kolmogorov 0–1 law implies that $\rho(t)$ is a.s. equal to a deterministic constant. Finally, by taking expectations in (1.3) and using the bounded convergence theorem, a.s.,

$$\rho(t) = \mathbb{E}\,\rho(t) = \lim_{n\to\infty} \frac{1}{(2n+1)^d} \sum_{\mathbf{z}\in[-n,n]^d} \mathbb{P}\big(Z_{\mathbf{z}}(t) > 0\big) = \mathbb{P}\big(Z_{\mathbf{0}}(t) > 0\big). \tag{6.4}$$

This establishes (6.1) for a fixed $t \geqslant 0$.

We next show how to extend this equality to all $t \geqslant 0$ simultaneously. Define the upper and lower densities $\overline{\rho}(t)$ and $\underline{\rho}(t)$ as in (1.3) but using lim sup and lim inf, respectively. These are thus always defined, and a.s. equal to each other and given by (6.1).

Given an interval I, define similarly $\overline{\rho}_+(I)$ as the upper density of sites that are red for *some* $t \in I$, and $\underline{\rho}_-(I)$ as the lower density of points that are red for *all* $t \in I$. The argument just given for $\rho(t)$ shows also that these densities a.s. exist and are equal to the corresponding probabilities at $\mathbf{0}$. Let $T < \infty$ and $\varepsilon > 0$, and let δ be as in Lemma 6.2. Then (6.2) implies that for any fixed interval $I \subseteq [0, T]$ of length at most δ we have a.s.

$$\overline{\rho}_+(I) - \underline{\rho}_-(I) < \varepsilon \tag{6.5}$$

Furthermore, for all $t \in I$,

$$\underline{\rho}_-(I) \leqslant \underline{\rho}(t) \leqslant \overline{\rho}(t) \leqslant \overline{\rho}_+(I), \tag{6.6}$$

and thus by (6.5), a.s.,

$$\sup_{t\in I}\big(\overline{\rho}(t) - \underline{\rho}(t)\big) \leqslant \overline{\rho}_+(I) - \underline{\rho}_-(I) < \varepsilon. \tag{6.7}$$

By covering $[0, T]$ by a finite number of intervals of length at most δ we conclude that a.s.

$$\sup_{t\in[0,T]} \big(\overline{\rho}(t) - \underline{\rho}(t)\big) < \varepsilon, \tag{6.8}$$

and sending $\varepsilon \to 0$ and $T \to \infty$ shows that a.s. $\overline{\rho}(t) = \underline{\rho}(t)$ for all t simultaneously.

Furthermore, write for convenience $f(t) := \mathbb{P}\big(Z_{\mathbf{0}}(t) > 0\big)$, so $\rho(t) = f(t)$ a.s. for each fixed t. Fix again an interval I as above. Then (6.2) implies that for any $s, t \in I$, $|f(s) - f(t)| < \varepsilon$. Fix $s \in I$. Since (6.6) and (6.5) imply that a.s.

$$|\overline{\rho}_+(I) - f(s)| = |\overline{\rho}_+(I) - \overline{\rho}(s)| \leqslant \overline{\rho}_+(I) - \underline{\rho}_-(I) < \varepsilon, \tag{6.9}$$

it follows that a.s.,

$$\sup_{t\in I}|\overline{\rho}_+(I)-f(t)| \leqslant |\overline{\rho}_+(I)-f(s)| + \sup_{t\in I}|f(s)-f(t)| < 2\varepsilon \tag{6.10}$$

and thus, using (6.6) and (6.5) again, a.s.

$$\sup_{t\in I}|\overline{\rho}(t)-f(t)| < 2\varepsilon + |\overline{\rho}(s)-\overline{\rho}_+(I)| < 2\varepsilon + \big(\overline{\rho}_+(I)-\underline{\rho}_-(I)\big) < 3\varepsilon. \tag{6.11}$$

By covering $[0,T]$ by a finite number of intervals of length at most δ we conclude that a.s. the same holds for I replaced by $[0,T]$, and then sending $\varepsilon\to 0$ and $T\to\infty$ shows that a.s. $\overline{\rho}(t)=f(t)$ for all $t\geqslant 0$. Hence, a.s., $\rho(t)=\overline{\rho}(t)=f(t)$ for all t simultaneously. □

Together with Theorem 1.2 it follows that for $p>\frac{1}{2}$ a.s. $\rho(t)\to 1$ as $t\to\infty$.

7 Dealing with death

As we have defined our process, at each ring of a clock, the corresponding ball produces offspring according to Φ, and remains itself where it was. Consequently, in the monochromatic version of our process, once a ball is born, it remains at its position at all future times. We shall in this section describe briefly how the results obtained for this process can be extended to allow balls to die (disappear) as they reproduce. (Recall Remark 1.4.) This is obviously more general, since we can let the parent be replaced by a copy of itself. In particular, this extension allows us to consider models where the balls move around at random, such as the standard (continuous time, discrete space) branching random walk where particles perform independent simple symmetric random walks, and in each step, with some probability, split in two or more independent copies.

So, consider the model in which particles have an exponentially distributed life time, at the end of which they are removed and replaced by a configuration φ, shifted to the position of the particle, drawn from Φ. We assume, as before, that Φ is an irreducible probability measure on finite non-negative (but not necessarily non-empty) configurations on $\mathbb{Z}^d$, satisfying $1<\mathbb{E}[\|\varphi\|]<\infty$. Under the condition that $\mathbb{E}[\|\varphi\|]>1$, then the total number of balls $\|\mathcal{X}(t)\|$, when starting from a single ball at the origin, is a supercritical branching process.[4] We outline below how our arguments may be adapted to cover this more general family of processes. (As before, we assume that $\mathbb{E}[\|\varphi\|^{4+\varepsilon}]<\infty$ and that (1.4) holds where appropriate.)

There are four places at which our arguments need modification. First, in Section 2, we need to compensate for the death of particles. Write φ' for the *change* caused as a clock rings at the origin, and by μ' its expectation. Then $\varphi'=\varphi-\delta_{\mathbf{x},\mathbf{0}}$ and $\mu'(\mathbf{x})=\mu(\mathbf{x})-\delta_{\mathbf{x},\mathbf{0}}$. Similarly, redefine $\lambda:=\mathbb{E}[\|\varphi\|]-1>0$. By replacing φ and μ by φ' and μ' throughout Section 2, then all results continue to hold for the more general family of processes. (Note, in particular, how the expression $\lambda-\operatorname{Re}\widehat{\mu}(\mathbf{u})$ is unaffected by these changes.)

[4] Note that we above have assumed, implicitly, that Φ is supported on nonempty configurations, as the contrary would simply correspond to a rescaling of time. This is no longer assumed here, resulting in the possible extinction of the process evolving from a single ball. Extinction will, of course, not be possible when starting from an infinite starting configuration.

Secondly, in Section 3, when $\mathcal{Y}^p(t)$ is no longer non-decreasing, we need an argument to deduce (3.20) from (3.19). A simple large deviation estimate will suffice, since if $Y^p_{\mathbf{0}}(t)$ changes significantly during a short time span, then a greater than expected number of clock rings must have occurred. To make this formal we introduce the events

$$A_n := \big\{e^{-\lambda\delta n} Y^p_{\mathbf{0}}(\delta n) \in (pe^{-\lambda\delta}, pe^{\lambda\delta})\big\}, \tag{7.1}$$

$$B_n := \big\{e^{-\lambda t} Y^p_{\mathbf{0}}(t) \geqslant p(1-2\delta)e^{-2\lambda\delta} \text{ for all } t \in [\delta n, \delta(n+1)]\big\}, \tag{7.2}$$

$$C_n := \big\{e^{-\lambda t} Y^p_{\mathbf{0}}(t) \leqslant p(1+2\delta)e^{2\lambda\delta} \text{ for all } t \in [\delta n, \delta(n+1)]\big\}. \tag{7.3}$$

It will thus suffice to show that for every $\delta \in (0, \frac{1}{2})$ a.s. the events B_n and C_n will occur for all but finitely many n. Let $M := Y^p_{\mathbf{0}}(\delta n)$ be the number of balls present at the origin at time δn, and let $y_n := pe^{\lambda\delta(n-1)}$. On the event $A_n \cap B_n^c$, $M > y_n$, and of these M balls at least $M - (1-2\delta)y_n$ must die before time $\delta(n+1)$. Since each balls dies with probability $1 - e^{-\delta} < \delta$, Chebyshev's inequality implies, conditioned on $M > y_n$,

$$\mathbb{P}(A_n \cap B_n^c \mid M) \leqslant \frac{M}{\big(M - (1-2\delta)y_n - \delta M\big)^2} = \frac{M}{\big((1-\delta)M - (1-2\delta)y_n\big)^2}. \tag{7.4}$$

The right-hand side is decreasing in $M \geqslant y_n$, and thus, for all such M,

$$\mathbb{P}(A_n \cap B_n^c \mid M) \leqslant \frac{y_n}{\big((1-\delta)y_n - (1-2\delta)y_n\big)^2} = Cy_n^{-1} = C'e^{-\lambda\delta n}. \tag{7.5}$$

Furthermore, this holds trivially for $M < y_n$ too, since the conditional probability then is 0. Consequently, $\mathbb{P}(A_n \cap B_n^c) \leqslant C'e^{-\lambda\delta n}$, and the Borel–Cantelli lemma shows that a.s. the event $A_n \cap B_n^c$ occurs for only finitely many n.

Similarly, with $y'_n := pe^{\lambda\delta(n+2)}$, on the event $A_{n+1} \cap C_n^c$ the number of balls at the origin exceeds $(1+2\delta)y'_n$ at some point during the time interval. Let τ be the first time that this happens, and $N \geqslant (1+2\delta)y'_n$ the number of balls at that time. At least $N - y'_n$ of these balls must die before time $\delta(n+1)$. Conditioned on τ and N, each ball dies with probability less than δ, and Chebyshev's inequality yields, for $N \geqslant y''_n := (1+2\delta)y'_n$,

$$\mathbb{P}(A_{n+1} \cap C_n^c \mid \tau, N) \leqslant \frac{N}{\big((1-\delta)N - y'_n\big)^2} \leqslant \frac{y''_n}{\big((1-\delta)y''_n - y_n\big)^2} = C''y_n^{-1} = C'''e^{-\lambda\delta n}. \tag{7.6}$$

Hence $\mathbb{P}(A_{n+1} \cap C_n^c) \leqslant C'''e^{-\lambda\delta n}$, so the event $A_{n+1} \cap C_n^c$ occurs for only finitely many n.

Since A_n a.s. occurs for all large n by (3.19), it follows that for every $\delta > 0$ a.s.

$$p(1-2\delta)e^{-2\lambda\delta} \leqslant \liminf_{t\to\infty} e^{-\lambda t} Y^p_{\mathbf{0}}(t) \leqslant \limsup_{t\to\infty} e^{-\lambda t} Y^p_{\mathbf{0}}(t) \leqslant p(1+2\delta)e^{2\lambda\delta}. \tag{7.7}$$

This completes the proof of Theorem 1.5 in this more general setting.

Next, we see how to adapt the proof of Lemma 4.4. We need a bound on the event $\mathcal{E}_r(T)$. For times t such that $Z_{\mathbf{0}}(t, \boldsymbol{\zeta}^{-,r}) < Z_{\mathbf{0}}(t, \boldsymbol{\zeta}^{+,r})$ and $Z_{\mathbf{0}}(t, \boldsymbol{\zeta}^{+,r}) > 0$, define the *excess ball* (at time t) as the red ball with smallest label (in a fixed order) that is at $\mathbf{0}$ in $\mathcal{Z}(t, \boldsymbol{\zeta}^{+,r})$ but does not exist in $\mathcal{Z}(t, \boldsymbol{\zeta}^{-,r})$; if $Z_{\mathbf{0}}(t, \boldsymbol{\zeta}^{-,r}) < Z_{\mathbf{0}}(t, \boldsymbol{\zeta}^{+,r}) \leqslant 0$ define the excess ball as the blue ball with smallest label that is at $\mathbf{0}$ in $\mathcal{Z}(t, \boldsymbol{\zeta}^{-,r})$ but does not exist in $\mathcal{Z}(t, \boldsymbol{\zeta}^{+,r})$. For completeness, if $Z_{\mathbf{0}}(t, \boldsymbol{\zeta}^{-,r}) = Z_{\mathbf{0}}(t, \boldsymbol{\zeta}^{+,r})$, define the excess ball as an extra (non-existing) ball with its own clock. Let τ denote the first time at which $Z_{\mathbf{0}}(t, \boldsymbol{\zeta}^{-,r}) < Z_{\mathbf{0}}(t, \boldsymbol{\zeta}^{+,r})$ (with $\tau = \infty$ if this never happens). Lemma 4.3 gives (4.11) as before, and since no two nucleations occur simultaneously, note that a.s. $\mathcal{E}_r(T) = \{\tau \leqslant T\}$.

Let F be the event that the excess ball does not die in the interval $(\tau, T]$. Then, on the event $\mathcal{E}_r(T) \cap F$, (4.12) holds for $t = \tau$ and an induction argument as in the proof of Lemma 4.3 implies that also (4.13) holds. Consequently, Markov's inequality yields that the right-hand side of (4.14) is a bound for $\mathbb{P}(\mathcal{E}_r(T) \cap F)$. To complete the proof it suffices to note that, since τ is a stopping time and $\mathcal{E}_r(T)$ is determined by τ,

$$\mathbb{P}(F \mid \tau) = e^{-(T-\tau)_+} \geqslant e^{-T} \tag{7.8}$$

and

$$\mathbb{P}\big(\mathcal{E}_r(T) \cap F\big) \;=\; \mathbb{E}\big[\mathbf{1}_{\mathcal{E}_r(T)}\, \mathbb{P}(F \mid \tau)\big] \;\geqslant\; e^{-T}\, \mathbb{P}\big(\mathcal{E}_r(T)\big). \tag{7.9}$$

The rest of the proof is the same as before.

Finally, we see how to prove Lemma 6.2 in the more general setting. As before, it will suffice to consider the monochromatic process $\mathcal{Y}^p(t)$. Let N denote the number of balls (in the monochromatic process) that arrive at the origin during the interval $[a, b]$, and D the number of balls already present at time a that die before time b. Then, using Markov's inequality as before,

$$\mathbb{P}\big(Y^p_{\mathbf{0}}(t) \text{ is not constant for } t \in [a, b]\big) \leqslant \mathbb{E}[N] + \mathbb{E}[D]. \tag{7.10}$$

Let D' the number of balls that arrive after time a and die before time b, and note that

$$\mathbb{E}[D] \leqslant (b-a)\, \mathbb{E}[Y^p_{\mathbf{0}}(a)] \quad \text{and} \quad \mathbb{E}[D'] \leqslant (b-a)\, \mathbb{E}[N]. \tag{7.11}$$

In addition, $N = Y^p_{\mathbf{0}}(b) - Y^p_{\mathbf{0}}(a) + D + D'$, so under the assumption that $b - a \leqslant 1/2$,

$$\mathbb{E}[N] \leqslant 2\, \mathbb{E}[Y^p_{\mathbf{0}}(b) - Y^p_{\mathbf{0}}(a) + D] \leqslant 2(e^{\lambda b} - e^{\lambda a}) + 2(b-a)e^{\lambda a} \leqslant 4(b-a)(\lambda+1)e^{\lambda T}. \tag{7.12}$$

The rest is silence.

8 Open problems and further directions

We round off with some open problems and suggested directions for further study, inspired by the results above. We give also some comments on possible extensions, some of which seem easy, but we leave them for the reader to check.

The problems may be considered for general branching rules, much like in the present paper, but in some cases (such as for the first question) it may make more sense for a specific branching rule (such as the nearest-neighbour rule, in which φ is the deterministic configuration that puts a ball at each of the $2d$ neighbours of the origin). In some cases we even expect that the answer to the question may depend on the branching rule, much opposed to the results reported in this paper.

1. For $d = 1$, what is the length of a typical monochromatic interval?

2. For $p > 1/2$, at what rate does the density of blue sites tend to zero?

3. For $p = 1/2$, at what rate does a site change colour?

4. For $p = 1/2$, how may balls are contained at the origin at a given time? Proposition 5.1 provides a partial answer, and Lemma 5.3 a matching upper bound. Is it true that $|Z_{\mathbf{0}}(t)| = \Theta(t^{-d/4}e^{\lambda t})$ with high probability, or does the density of times for which it holds tend to 1 as $t \to \infty$? Our arguments do not even seem to give the weaker conclusion that $\mathbb{P}(Z_{\mathbf{0}}(t) = 0) \to 0$, and thus (by symmetry) $\rho(t) = \mathbb{P}(Z_{\mathbf{0}}(t) > 0) \to 1/2$ as $t \to \infty$.

5. Are the moment conditions in Theorems 1.2 and 1.5 necessary? In particular, is the condition $\mathbb{E}[\|\varphi\|^2] < \infty$, instead of $\mathbb{E}[\|\varphi\|^3] < \infty$, sufficient for the conclusion of Theorem 1.5 to hold when $d = 1$?

6. We have in this paper considered competition between two types. It would be interesting to extend our results to three or more competing types. We believe that it may be challenging to find a substitute for the conservative process described in Section 4.2. Problems of a similar character were suggested also in [2].

7. We assumed throughout the paper that the initial configuration has at most one ball at each site. We can more generally consider initial configurations $(\zeta_{\mathbf{x}})_{\mathbf{x}\in\mathbb{Z}^d}$ where the $\zeta_{\mathbf{x}}$ are i.i.d. with an arbitrary distribution. We expect that the results above generalize rather easily under some moment condition on $\zeta_{\mathbf{x}}$, but we have not checked the details. We expect that it is less straightforward to adapt our techniques to allow the different types to jump at different rates, or reproduce according to different rules.

8. In the model studied by Bramson and Lebowitz [12, 13], no particles are born, and particles move according to independent continuous-time symmetric random walks. This can be regarded as an extreme case of our model (not covered above), where balls die as in Section 7 and the offspring φ consists of a single ball. For this model Cabezas, Rolla and Sidoravicius [14] have shown that, under weak assumptions, a.s. there exist arbitrarily large times when the origin is occupied. A more detailed conjecture, which seems to be open, would be that, starting from a p-random Bernoulli initial colouring, the origin is a.s. visited by both colours infinitely many times when $p = 1/2$, but not when $p > 1/2$.

9. Consider the urn process on $\mathbb{Z}^d$ run from an initial configuration with a single red and a single blue ball. Under what conditions will both red and blue balls remain in the system at all times (so-called coexistence) with positive probability? For $d = 1$, in the event of coexistence, under what conditions does an 'interface', that is a macroscopic division, between red and blue exist, and how does it evolve over time? What is the analogous higher-dimensional phenomenon? Some progress have been made to these questions for a related model by Ahlberg, Angel and Kolesnik [1].

Acknowledgements

The authors are very grateful to Robert Morris, for his encouragement to pursue this project, and his valuable input in several joint discussions. The authors are also grateful to Luca Avena and Conrado da Costa for informing about the work of Pruitt [26] (cf. Remark 3.3), to IMPA and to the Isaac Newton Institute, where parts of this work were done, and to an anonymous referee who found a gap in our argument.

This work was in part supported by grant 2016-04442 from the Swedish Research Council (DA); CNPq bolsa de produtividade Proc. 310656/2016-8 and FAPERJ Jovem cientista do nosso estado Proc. 202.713/2018 (SG); the Knut and Alice Wallenberg Foundation, the Isaac Newton Institute for Mathematical Sciences (EPSRC Grant Number EP/K032208/1), and the Simons foundation (SJ).

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