Patterns in random permutations avoiding some patterns

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Aofa 2018
Uppsala, 28 June, 2018

Patterns in a permutation

Let $\mathfrak{S}_n$ be the set of permutations of $[n] := \{1, \ldots, n\}$.

If $\sigma = \sigma_1 \cdots \sigma_k \in \mathfrak{S}_k$ and $\pi = \pi_1 \cdots \pi_n \in \mathfrak{S}_n$, then an *occurrence* of σ in π is a subsequence $\pi_{i_1} \cdots \pi_{i_k}$, with $1 \leq i_1 < \cdots < i_k \leq n$, that has the same relative order as σ. σ is called a *pattern*.

Example: $\underline{3}14\underline{25}$ is an occurence of 213 in 31425

Let $n_\sigma(\pi)$ be the number of occurrences of σ in π.
For example, $n_{21}(\pi)$ is the number of inversions in π.

A permutation π *avoids* a pattern σ if there is no occurence of σ in π, i.e., if $n_\sigma(\pi) = 0$.

Let $\mathfrak{S}_n(\tau) := \{\pi \in \mathfrak{S}_n : n_\tau(\pi) = 0\}$, the set of permutations of length n that avoid τ.

Similarly, let $\mathfrak{S}_n(\tau_1, \ldots, \tau_k) := \bigcap_i \mathfrak{S}_n(\tau_i)$, the set of permutations of length n that avoid $\tau_1, \ldots, \tau_k$.

Example

Donald Knuth, *The Art of Computer Programming, vol. 1*, Exercise 2.2.1-5:

A permutation π can be obtained by a stack if and only if π is 312-avoiding, i.e., $\pi \in \mathfrak{S}_n(312)$.

Example

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A permutation π can be obtained by a stack if and only if π is 312-avoiding, i.e., $\pi \in \mathfrak{S}_n(312)$.

Equivalently:

A permutation π is stack-sortable if and only if π is 231-avoiding.

Example

A permutation π can be sorted by 2 parallel queues if and only if π is 321-avoiding, i.e., $\pi \in \mathfrak{S}_n(321)$. [Tarjan (1972)]

Example

A permutation π is deque-sortable if and only if π is $\{2431, 4231\}$-avoiding, i.e., $\pi \in \mathfrak{S}_n(2431, 4231)$. [West (1995)]

Further examples, properties and references:
See Stanley, *Enumerative Combinatorics*,
Exercises 6.19 x (321), y (312), ee (321), ff (312), ii (231), oo (132), xx (321); 6.25 g (321); 6.39 k, l ($\{2413, 3142\}$), m ($\{1342, 1324\}$); 6.47 a ($\{4231, 3412\}$); 6.48 (1342).

(Or Stanley, *Catalan Numbers*)

One fundamental question, studied by many authors, is the size of these classes $|\mathfrak{S}_n(\tau)|$ and $|\mathfrak{S}_n(\tau_1, \ldots, \tau_k)|$.

Theorem

If $|\tau| = 3$, then

$$|\mathfrak{S}_n(\tau)| = C_n = \frac{1}{n+1}\binom{2n}{n},$$

the nth Catalan number.

The cases with $|\tau| \geq 4$ are much more complicated. See e.g. Bóna (2004).

Some results are also known for $|\mathfrak{S}_n(\tau_1, \ldots, \tau_k)|$ with $k \geq 2$.

Example All cases with all $|\tau_i| = 3$ are treated by Simion and Schmidt (1995). For example, several such cases yield 2^{n-1}.

Example $|\mathfrak{S}_n(2431, 4231)| = r_{n-1}$, a Schröder number.

A related problem is to study properties of a random permutation chosen uniformly from a class $\mathfrak{S}_n(\tau_1, \dots, \tau_k)$.

Several properties of such restricted random permutations have been studied by a number of authors. For example: consecutive patterns, descents, major index, number of fixed points, position of fixed points, exceedances, longest increasing subsequence, shape and distribution of individual values π_i.

I consider here instances of the following general problem:

Fix patterns $\tau_1, \dots, \tau_k$ and σ. What is the asymptotic distribution, as $n \to \infty$, of $n_\sigma(\boldsymbol{\pi})$ for $\boldsymbol{\pi} \in \mathfrak{S}_n(\tau_1, \dots, \tau_k)$, chosen uniformly at random?

Example

Take $\sigma = 21$. (Recall that $n_{21}(\pi)$ is the number of inversions in π.) What is the asymptotic distribution of the number of inversions in a random $\boldsymbol{\pi} \in \mathfrak{S}_n(\tau_1, \dots, \tau_k)$?

I consider today only the cases with $|\tau_i| = 3$, when I can give more or less complete results.

Also these simple cases are treated case by case, by different methods.
No general method is known for these problems, even in the comparatively simple case $|\tau| = 3$.

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PLEASE HELP!

Remark. Some impressive results for $\mathfrak{S}_n(2413, 3142)$ (separable permutations) are recently given by Bassino, Bouvel, Féray, Gerin, Pierrot (2017), with generalizations by Bassino, Bouvel, Féray, Gerin, Maazoun, Pierrot (2017).

Trivial cases

There are some trivial cases, with $|\mathfrak{S}_n(\tau_1, \dots, \tau_k)| = 0$, 1 or 2.

For example, $\mathfrak{S}_n(123, 321) = \emptyset$. ($n \geq 5$)

All cases with $|\tau_1| = \cdots = |\tau_k| = 3$ and $k \geq 4$ are trivial.

We ignore trivial cases.

Symmetries

There are many cases, even with all $|\tau_i| = 3$, but the number is reduced by obvious symmetries:

inverse: $25341 \leftrightarrow 51342$

reflection left-right: $25431 \leftrightarrow 13452$

reflection up-down: $25431 \leftrightarrow 41235$

Remark. These generate a dihedral group of 8 symmetries.
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If we represent permutations by square 0–1-matrices, then these symmetries are the usual 8 symmetries of a square.

These symmetries reduce the 37 non-trivial cases $\mathfrak{S}_n(\tau_1, \ldots, \tau_k)$ with $|\tau_i| = 3$ to
1 with $k = 0$ (unrestricted permutations in $\mathfrak{S}_n$)
2 with $k = 1$
4 with $k = 2$
4 with $k = 3$

Unrestricted permutations

As a background, consider random permutations without restrictions.

Theorem (Bóna (2007, 2010), Janson, Nakamura and Zeilberger (2015))

Consider a random unrestricted permutation $\boldsymbol{\pi}_n \in \mathfrak{S}_n$*. Then* $n_\sigma(\boldsymbol{\pi}_n)$ *is asymptotically normally distributed, for any* σ*: if* $k := |\sigma|$ *then*

$$\frac{n_\sigma(\boldsymbol{\pi}_n) - n^k/k!^2}{n^{k-1/2}} \xrightarrow{\mathrm{d}} N(0, \gamma_\sigma^2)$$

for some constant $\gamma_\sigma > 0$*.*

Proof.

A random permutation $\boldsymbol{\pi}_n$ can be obtained by taking i.i.d. random variables $X_1, \dots, X_n \sim U(0,1)$ and considering their ranks. Then

$$n_\sigma(\boldsymbol{\pi}_n) = \sum_{i_1 < \cdots < i_m} f\left(X_{i_1}, \dots, X_{i_m}\right)$$

for a suitable (indicator) function f.

This sum is an asymmetric U-statistic, and the result follows by general results on U-statistics [in principle Hoeffding (1948), see e.g. Janson (1997, 2018+)] □

The 11 cases all have asymptotic distributions of one of the following two types. Let $\boldsymbol{\pi}_n \in \mathfrak{S}_n(\tau_1, \dots, \tau_k)$ be uniformly random.

I. Normal limits: For every $\sigma \in \mathfrak{S}_*(\tau_1, \dots, \tau_k)$, there exists constants α, β, γ such that, as $n \to \infty$,

$$\frac{n_\sigma(\boldsymbol{\pi}_n) - \beta n^\alpha}{n^{\alpha - 1/2}} \xrightarrow{\mathrm{d}} N(0, \gamma^2),$$

with convergence of all moments.

In particular, $\mathbb{E}\, n_\sigma(\boldsymbol{\pi}_n) \sim \beta n^\alpha$, and we have concentration:

$$\frac{n_\sigma(\boldsymbol{\pi}_n)}{\mathbb{E}\, n_\sigma(\boldsymbol{\pi}_n)} \xrightarrow{\mathrm{p}} 1.$$

II. Non-normal limits without concentration: For every $\sigma \in \mathfrak{S}_*(\tau_1, \dots, \tau_k)$, there exists a constant α such that

$$\frac{n_\sigma(\boldsymbol{\pi}_n)}{n^\alpha} \xrightarrow{\mathrm{d}} W_\sigma,$$

with convergence of all moments, for some random variable $W_\sigma > 0$.

T	$\lvert\mathfrak{S}_n(T)\rvert$	type I	type II	as. variance = 0
$\emptyset$	$n!$	$\lvert\sigma\rvert$		
$\{132\}$	C_n		$(\lvert\sigma\rvert + D(\sigma))/2$	$m \cdots 1$
$\{321\}$	C_n		$(\lvert\sigma\rvert + B(\sigma))/2$	$1 \cdots m$
$\{132, 312\}$	2^{n-1}	$\lvert\sigma\rvert$		
$\{231, 312\}$	2^{n-1}	$B(\sigma)$		$1 \cdots m$
$\{231, 321\}$	2^{n-1}	$B(\sigma)$		$1 \cdots m$
$\{132, 321\}$	$\binom{n}{2} + 1$		$\lvert\sigma\rvert$	
$\{231, 312, 321\}$	F_{n+1}	$B(\sigma)$		$1 \cdots m$
$\{132, 231, 312\}$	n		$\lvert\sigma\rvert$	
$\{132, 231, 321\}$	n		$\lvert\sigma\rvert - 1$ or $\lvert\sigma\rvert$	$1 \cdots m$
$\{132, 213, 321\}$	n		$\lvert\sigma\rvert$	
$\{2413, 3142\}$	s_{n-1}		$\lvert\sigma\rvert$	

This table shows whether $n_\sigma(\boldsymbol{\pi}_n)$ has limits of type I (normal) or II (non-normal). The exponent $\alpha = \alpha(\sigma)$ is given in the column for the type. (The mean is of order n^α.)

$C_n := \frac{1}{n+1}\binom{2n}{n}$ is a Catalan number; F_{n+1} is a Fibonacci number; s_{n-1} is a Schröder number; $D(\sigma)$ is the number of descents and $B(\sigma)$ is the number of blocks in σ.

A block in σ is a minimal interval $[i, j]$ such that π maps $[1, i-1]$, $[i, j]$ and $[j+1, n]$ to themselves.

Remark. We do not know whether a general set of forbidden permutations T has limits in distribution of $n_\sigma(\boldsymbol{\pi}_n)$ (after normalization) at all.
Even if limits exist, no reason is known that they have to be of type I or II above.

Remark. The non-normal limits in the cases $\{132\}$, $\{321\}$ and $\{2413, 3142\}$ can all be expressed as functionals of a Brownian excursion $\mathbf{e}(t)$. However, the expressions in these three cases are, in general, quite different (and obtained by quite different arguments), so there is no obvious hope for a unification.
(The other cases of non-normal limits in the table are different, and of a more elementary kind.)

132-avoiding permutations (or 213, 231, 312)

Theorem

Let $\sigma \in \mathfrak{S}_*(132)$ *and let* $\lambda(\sigma) := |\sigma| + D(\sigma)$*,where* $D(\sigma)$ *is the number of descents in* σ*, i.e., indices* i *such that* $\sigma_i > \sigma_{i+1}$ *or* $i = |\sigma|$*.*

If $\boldsymbol{\pi}_n \in \mathfrak{S}_n(132)$ *is uniformly random, then*

$$n_\sigma(\boldsymbol{\pi}_n)/n^{\lambda(\sigma)/2} \xrightarrow{\mathrm{d}} \Lambda_\sigma$$

for some strictly positive random variable Λ_σ*.*

We have $1 \leq D(\sigma) \leq |\sigma|$, and thus

$$|\sigma| + 1 \leq \lambda(\sigma) \leq 2|\sigma|,$$

with the extreme values $\lambda(\sigma) = |\sigma| + 1$ if and only if $\sigma = 1 \cdots k$, and $\lambda(\sigma) = 2|\sigma|$ if and only if $\sigma = k \cdots 1$, for some $k = |\sigma|$.

Proof.

- A natural bijection between $\mathfrak{S}_n(132)$ and binary trees of order n.
- the standard bijection between the latter and Dyck paths.
- A random Dyck path converges (after scaling) in distribution to a Brownian excursion.

□

The limit variables Λ_σ above can be expressed as functionals of a Brownian excursion $\mathbf{e}(x)$. (This is a random non-negative function on $[0, 1]$.) The description is, in general, rather complicated, but some cases are simple.

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Moments of the variables Λ_σ can be calculated by a recursion formula. (Proved separately from convergence in distribution.)

Example

In the special case $\sigma = 12$, $\Lambda_{12} = \sqrt{2}\int_0^1 \mathbf{e}(x)\,\mathrm{d}x$, this is (apart from the factor $\sqrt{2}$) the well-known *Brownian excursion area*.

For the number n_{21} of inversions in $\mathfrak{S}_n(132)$, we thus have

$$\frac{\binom{n}{2} - n_{21}(\boldsymbol{\pi}_n)}{n^{3/2}} = \frac{n_{12}(\boldsymbol{\pi}_n)}{n^{3/2}} \xrightarrow{\mathrm{d}} \Lambda_{12} = \sqrt{2}\int_0^1 \mathbf{e}(x)\,\mathrm{d}x.$$

By symmetries, the left-hand side can also be seen as the number of inversions normalized by $n^{3/2}$, if we instead avoid 231 or 312.

The bijection with binary trees

Given a permutation $\pi = \pi_1 \cdots \pi_n \in \mathfrak{S}_n(132)$ find the maximum $\pi_k = n$ and make it the root. Construct recursively the left subtree from $\pi_1 \cdots \pi_{k-1}$ and the right subtree from $\pi_{k+1} \cdots \pi_n$.

Note that if π_i is in the left subtree and π_j in the right, then $\pi_i > \pi_j$ since π avoids 132. Hence the tree determines the permutation.

Example

If $i < j$, then $\pi_i < \pi_j$ only if i is a descendant of j (in its left subtree).

Hence, $n_{12}(\pi)$ equals the total left path length in the binary tree.

$$\Lambda_{12} = \sqrt{2} \int_0^1 \mathbf{e}(x)\, \mathrm{d}x.$$

$$\Lambda_{123} = \int_0^1 \mathbf{e}(x)^2\, \mathrm{d}x.$$

$$\Lambda_{1\dots m} = \frac{2^{(m-1)/2}}{(m-1)!} \int_0^1 \mathbf{e}(x)^{m-1}\, \mathrm{d}x.$$

$$\Lambda_{213} = \sqrt{2} \iint_{0 \le x < y \le 1} \mathbf{e}([x,y])\, \mathrm{d}x\, \mathrm{d}y$$

$$\Lambda_{231} = \sqrt{2} \iint_{0 \le x < y \le 1} \big(\mathbf{e}(x) - \mathbf{e}([x,y])\big)\, \mathrm{d}x\, \mathrm{d}y$$

where

$$\mathbf{e}([x,y]) := \min_{z \in [x,y]} \mathbf{e}(z).$$

321-avoiding permutations (or 123)

Theorem

Suppose $\sigma \in \mathfrak{S}_*(321)$*. Let* m *be the number of blocks in* σ*. Then, as* $n \to \infty$*, for a random* $\boldsymbol{\pi}_n \in \mathfrak{S}_n(321)$*,*

$$n_\sigma(\boldsymbol{\pi}_n)/n^{(|\sigma|+m)/2} \xrightarrow{\mathrm{d}} W_\sigma,$$

for some random variable $W_\sigma > 0$*.*

Example The number of inversions.

$$n_{21}(\boldsymbol{\pi}_n)/n^{3/2} \xrightarrow{\mathrm{d}} \Lambda_{21} = 2^{-1/2}\int_0^1 e(t)\,\mathrm{d}t,$$

where the random function $e(t)$ is a Brownian excursion.

In general,

$$W_\sigma = w_\sigma \int_{t_1 < \cdots < t_m} e(t_1)^{\ell_1 - 1} \cdots e(t_m)^{\ell_m - 1} \, \mathrm{d}t_1 \cdots \mathrm{d}t_m$$

where $\ell_1, \ldots, \ell_m$ are the lengths of the blocks in σ, and w_σ is a curious combinatorial constant.

Proof.

- A bijection with Dyck paths by Billey, Jockush and Stanley (1993).
- Further developments by Hoffman, Rizzolo and Slivken (2017).
- A random Dyck path converges (after scaling) in distribution to a Brownian excursion.

□

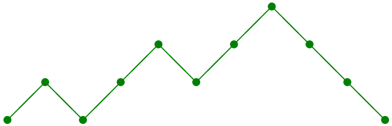
The bijection with binary trees

Fix a Dyck path γ of length $2n$, and let m be the number of increases (or decreases) in γ. Let $a_i \geq 1$ be the length of the i-th run of increases, and let $d_i \geq 1$ be the length of the i-th run of decreases in γ. Let, $A_i := \sum_{j=1}^{i} a_j$ and $D_i := \sum_{j=1}^{i} d_j$; let $\mathcal{A} := \{A_i : 1 \leq i \leq m-1\}$, $\mathcal{A}_1 := \{A_i + 1 : 1 \leq i \leq m-1\}$, $\mathcal{D} := \{D_i : 1 \leq i \leq m-1\}$, $\mathcal{A}_1^{\mathsf{c}} := [n] \setminus \mathcal{A}_1$, and $\mathcal{D}^{\mathsf{c}} := [n] \setminus \mathcal{D}$.

Finally, define the permutation $\pi_\gamma \in \mathfrak{S}_n$ as the unique permutation with $\pi : \mathcal{D} \to \mathcal{A}_1$, and therefore $\pi : \mathcal{D}^{\mathsf{c}} \to \mathcal{A}_1^{\mathsf{c}}$, such that π is increasing on $\mathcal{D}$ and on $\mathcal{D}^{\mathsf{c}}$. (In particular, $\pi_\gamma(D_i) = A_i + 1$ for $1 \leq i \leq m-1$.)

Example

The Dyck path below has $m = 3$, $(a_1, a_2, a_3) = (1, 2, 2)$, $(d_1, d_2, d_3) = (1, 1, 3)$, $(A_1, A_2, A_3) = (1, 3, 5)$, $(D_1, D_2, D_3) = (1, 2, 5)$, $\mathcal{A}_1 = \{2, 4\}$, $\mathcal{D} = \{1, 2\}$, $\mathcal{A}_1^c = \{1, 3, 5\}$, $\mathcal{D}^c = \{3, 4, 5\}$ and $\pi = 24135$.



A Dyck path of length 10 ($n = 5$).

Fact:

$$\pi_\gamma(i) \approx \begin{cases} i + \gamma(2i), & i \in \mathcal{D} \\ i - \gamma(2i), & i \in \mathcal{D}^c \end{cases}$$

Thus $\pi_\gamma(i) = i + O_p(\sqrt{n})$.

Example

If $i < j$, then ij is an inversion on π_γ if $i \in \mathcal{D}$, $j \in \mathcal{D}^c$ and

$$0 < j - i <\approx \gamma(2i) + \gamma(2j) \approx 2\gamma(2i).$$

Hence

$$n_{21}(\pi_\gamma) \approx \frac{1}{2^2} \sum_{i=1}^{n} 2\gamma(2i) \approx \frac{n\sqrt{2n}}{2} \int_0^1 \mathbf{e}(x)\,\mathrm{d}x.$$

Avoiding $\{132, 312\}$

Theorem

Let $m \geq 2$ and $\sigma \in \mathfrak{S}_m(132, 312)$. If $\boldsymbol{\pi}_n$ is random in $\mathfrak{S}_n(132, 312)$. then as $n \to \infty$,

$$\frac{n_\sigma(\boldsymbol{\pi}_n) - 2^{1-m} n^m / m!}{n^{m-1/2}} \xrightarrow{\mathrm{d}} N(0, \gamma^2).$$

Proof.

A permutation π belongs to the class $\mathfrak{S}_*(132, 312)$ if and only if every entry π_i is either a maximum or a minimum.
[Simion and Schmidt, 1995].

Encode $\pi \in \mathfrak{S}_n(132, 312)$ by a sequence $\xi_2, \ldots, \xi_n \in \{\pm 1\}^{n-1}$, where $\xi_j = 1$ if π_j is a maximum in π, and $\xi_j = -1$ if π_j is a minimum. This is a bijection. Hence the code for a uniformly random $\boldsymbol{\pi}_n$ has $\xi_2, \ldots, \xi_n$ i.i.d. with $\mathbb{P}(\xi_j = 1) = \mathbb{P}(\xi_j = -1) = \frac{1}{2}$.

Let $\sigma \in \mathfrak{S}_m(132, 312)$ have the code $\eta_2, \ldots, \eta_m$. Then $\pi_{i_1} \cdots \pi_{i_m}$ is an occurrence of σ in π if and only if $\xi_{i_j} = \eta_j$ for $2 \le j \le m$. Consequently, $n_\sigma(\boldsymbol{\pi}_n)$ is an asymmetric U-statistic

$$n_\sigma(\boldsymbol{\pi}_n) = \sum_{i_1 < \cdots < i_m} f\left(\xi_{i_1}, \ldots, \xi_{i_m}\right),$$

where

$$f\left(\xi_1, \ldots, \xi_m\right) := \prod_{j=2}^{m} \mathbf{1}[\xi_j = \eta_j].$$

Note that f does not depend on the first argument.

The result follows from the theory of U-statistics. □

Example

For the number of inversions, we have $\sigma = 21$ and $m = 2$, $\eta_2 = -1$. A calculation yields $\mu = \frac{1}{2}$ and $\gamma^2 = \frac{1}{12}$, and thus

$$\frac{n_{21}(\pi_n) - n^2/4}{n^{3/2}} \xrightarrow{\mathrm{d}} N\left(0, \tfrac{1}{12}\right),$$

$\{231, 312\}$-avoiding permutations

Theorem

Let $\sigma \in \mathfrak{S}_*(231, 312)$ have m blocks. Then, for a random $\boldsymbol{\pi}_n \in \mathfrak{S}_n(231, 312)$,

$$\frac{n_\sigma(\boldsymbol{\pi}_n) - n^m/m!}{n^{m-1/2}} \xrightarrow{\mathrm{d}} N(0, \gamma^2)$$

for some constant γ^2.

Example The number of inversions.

$$\frac{n_{21}(\boldsymbol{\pi}_n) - n}{n^{1/2}} \xrightarrow{\mathrm{d}} N(0, 6).$$

Proof.

- $\pi \in \mathfrak{S}_*(231, 312) \iff$ each block is decreasing: $\ell(\ell-1)\cdots 21$ [Simion and Schmidt, 1995]. (Hence, $|\mathfrak{S}_n(231, 312)| = 2^{n-1}$.)
- If the block lengths are $\ell_1, \dots, \ell_m$, then $n_{21}(\boldsymbol{\pi}_n) = \sum_{i=1}^m \binom{\ell_i}{2}$. Similar for general σ, with multiple sum.
- $(\ell_1, \dots, \ell_m)$ is a random composition of n.
- Can be realized as the first elements, up to sum n, of an i.i.d. sequence $L_1, L_2, \dots$ of random variables with a Geometric $\mathrm{Ge}(1/2)$ distribution.
- Hence, with $\tau(n) := \min\{m : \sum_1^m L_i \geq n\}$,

$$n_{21}(\boldsymbol{\pi}_n) \overset{\mathrm{d}}{=} \sum_{i=1}^{\tau(n)} \binom{L_i}{2}$$

- Renewal theory, in a U-statistics version by Janson (2018+).

$\{231, 321\}$-avoiding permutations

Theorem

Let $\sigma \in \mathfrak{S}_*(231, 321)$ have m blocks. Then, for a random $\boldsymbol{\pi}_n \in \mathfrak{S}_n(231, 321)$,

$$\frac{n_\sigma(\boldsymbol{\pi}_n) - \mu n^m}{n^{m-1/2}} \xrightarrow{\mathrm{d}} N(0, \gamma^2)$$

for some constants μ, γ.

Example The number of inversions.

$$\frac{n_{21}(\boldsymbol{\pi}_n) - \frac{1}{2} n}{n^{1/2}} \xrightarrow{\mathrm{d}} N(0, \tfrac{1}{4}).$$

Proof.

- $\pi \in \mathfrak{S}_*(231, 321) \iff$ each block is of the type $\ell 12 \cdots (\ell - 1)$ [Simion and Schmidt, 1995].
- Then as above.

$\square$

$\{132, 321\}$-avoiding permutations

Theorem

For a random $\boldsymbol{\pi}_n \in \mathfrak{S}_n(132, 321)$*, the number of inversions has the asymptotic distribution*

$$n^{-2} n_{21}(\boldsymbol{\pi}_n) \xrightarrow{\mathrm{d}} W := XY,$$

where (X, Y) *is uniformly distributed in the triangle* $\{x, y \geq 0,\ x + y \leq 1\}$*. The limit variable* W *has density function*

$$2 \log\bigl(1 + \sqrt{1 - 4x}\bigr) - 2 \log\bigl(1 - \sqrt{1 - 4x}\bigr), \qquad 0 < x < 1/4,$$

and moments

$$\mathbb{E}\, W^r = 2 \frac{r!^2}{(2r + 2)!}, \qquad r > 0.$$

Similar for general $\sigma \in \mathfrak{S}_*(132, 321)$.

Proof.

- $\mathfrak{S}_n(132, 321)$ has only $\binom{n}{2} + 1$ elements: the identity and $\{\pi_{k,\ell,n-k-\ell} : k, \ell \geq 1,\ k + \ell \leq n\}$, where $\pi_{k,\ell,m}$ is the permutation $(\ell+1, \ldots, \ell+k, 1, \ldots, \ell, k+\ell+1, \ldots, k+\ell+m) \in \mathfrak{S}_{k+\ell+m}$, consisting of three increasing runs of lengths k, ℓ, m (where the third run is empty when $m = 0$).
- $n_{21}(\pi_{k,\ell,n-k-\ell}) = kl$.

□

$\{231, 312, 321\}$-avoiding permutations

Theorem

Let $\sigma \in \mathfrak{S}_*(231, 312, 321)$ *have m blocks. Then, for a random* $\boldsymbol{\pi}_n \in \mathfrak{S}_n(231, 312, 321)$,

$$\frac{n_\sigma(\boldsymbol{\pi}_n) - \mu n^b}{n^{b-1/2}} \xrightarrow{\mathrm{d}} N(0, \gamma^2)$$

for some constants μ, γ.

Example The number of inversions. $\sigma = 21$. $b = 1$. A calculation yields $\mu = (3 - \sqrt{5})/2$ and $\gamma^2 = 5^{-3/2}$.

$$\frac{n_{21}(\boldsymbol{\pi}_n) - \frac{3-\sqrt{5}}{2} n}{n^{1/2}} \xrightarrow{\mathrm{d}} N\left(0, 5^{-3/2}\right).$$

Proof.

- $\pi \in \mathfrak{S}_*(231, 312, 321) \iff$ each block is of the type 1 or 21. [Simion and Schmidt, 1995].
- Thus π is determined by its sequence of block lengths $\ell_1, \dots, \ell_m$ with $\ell_i \in \{1, 2\}$ and $\sum_i \ell_i = n$.
- Let $p := (\sqrt{5} - 1)/2$, the golden ratio, so that $p + p^2 = 1$. Let $X_1, X_2, \dots$ be an i.i.d. sequence of random variables with

$$\mathbb{P}(X_i = 1) = p, \qquad \mathbb{P}(X_i = 2) = p^2.$$

Let $S_k := \sum_{i=1}^k X_i$ and $B(n) := \min\{k : S_k \geq n\}$. Then, the sequence $L_1, \dots, L_B$ of block lengths of a uniformly random permutation $\boldsymbol{\pi}_n \in \mathfrak{S}_*(231, 312, 321)$ has the same distribution as $(X_1, \dots, X_{B(n)})$ conditioned on $S_{B(n)} = n$.
Consequently, $n_\sigma(\boldsymbol{\pi}_n)$ can be expressed as a U-statistic based on $X_1, \dots, X_B$, conditioned as above. Use general results for U-statistics.

$\{132, 231, 312\}$-avoiding permutations

Theorem

For a random $\boldsymbol{\pi}_n \in \mathfrak{S}_n(132, 231, 312)$, the number of inversions has the asymptotic distribution

$$n^{-2} n_{21}(\boldsymbol{\pi}_n) \xrightarrow{\mathrm{d}} W := U^2/2$$

with $U \sim U(0,1)$. Thus, $2W \sim B(\frac{1}{2}, 1)$, and W has moments

$$\mathbb{E}\, W^r = \frac{1}{2^r(2r+1)}, \qquad r > 0.$$

Similar for general $\sigma \in \mathfrak{S}_*(132, 231, 312)$.

Proof.

- $\mathfrak{S}_n(132, 231, 312)$ has only the n elements

$$\pi_{k,n-k} := (k, \dots, 1, k+1, \dots, n), \qquad 1 \le k \le n$$

- Thus a random $\boldsymbol{\pi}_n = \pi_{K,n-K}$ with $K \in \{1, \dots, n\}$ uniformly random. As $n \to \infty$, $K/n \xrightarrow{\mathrm{d}} U$.
- $n_{21}(\pi_{K,n-K}) = \binom{K}{2}$.

□

$\{132, 231, 321\}$-avoiding permutations

Theorem

For a random $\boldsymbol{\pi}_n \in \mathfrak{S}_n(132, 231, 321)$*, the number of inversions has a uniform distribution on* $\{0, \dots, n-1\}$*, and thus the asymptotic distribution*

$$n^{-1} n_{21}(\boldsymbol{\pi}_n) \xrightarrow{\mathrm{d}} U \sim U(0,1).$$

Similar for general $\sigma \in \mathfrak{S}_*(132, 231, 321)$.

Proof.

- $\mathfrak{S}_n(132, 231, 321)$ has only the n elements

$$\pi_{k,n-k} := (k, 1, \dots, k-1, k+1, \dots, n), \qquad 1 \le k \le n$$

- Thus a random $\boldsymbol{\pi}_n = \pi_{K,n-K}$ with $K \in \{1, \dots, n\}$ uniformly random. As $n \to \infty$, $K/n \xrightarrow{\mathrm{d}} U$.
- $n_{21}(\pi_{K,n-K}) = K - 1$.

□

$\{132, 213, 321\}$-avoiding permutations

Theorem

For a random $\pi_n \in \mathfrak{S}_n(132, 213, 321)$, the number of inversions has the asymptotic distribution

$$n^{-2} n_{21}(\pi_n) \xrightarrow{\mathrm{d}} W := U(1-U)$$

with $U \sim U(0,1)$. Thus, $4W \sim B(1, \frac{1}{2})$, and W has moments

$$\mathbb{E}\, W^r = \frac{\Gamma(r+1)^2}{\Gamma(2r+2)}, \qquad r > 0.$$

Similar for general $\sigma \in \mathfrak{S}_*(132, 213, 321)$.

Proof.

- $\mathfrak{S}_n(132, 213, 321)$ has only the n elements

$$\pi_{k,n-k} := (k+1, \dots, n, 1, \dots, k), \qquad 1 \le k \le n$$

- Thus a random $\boldsymbol{\pi}_n = \pi_{K,n-K}$ with $K \in \{1, \dots, n\}$ uniformly random. As $n \to \infty$, $K/n \xrightarrow{\mathrm{d}} U$.
- $n_{21}(\pi_{K,n-K}) = (n-K)K$.

□