

# Depth-First Search performance in a random digraph with geometric degree distribution

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Scaling limits: from statistical mechanics to manifolds.  
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# Background and abstract

Donald Knuth asked about properties of Depth-First Search (DFS) in a random digraph.

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Donald Knuth asked about properties of Depth-First Search (DFS) in a random digraph.

We give some answers.

# The model

## DFS in a digraph:

- ▶ Start with an arbitrary vertex.
- ▶ Explore the arcs from that vertex one by one
- ▶ When an arc is found that leads to a vertex that has not been seen before, explore all arcs from this vertex in the same way, recursively, before proceeding
- ▶ When there are no more arcs found, we have created a tree containing all descendants of the first vertex. If there is any vertex left, start again with a new vertex, and repeat until all vertices are explored.

This generates a spanning forest (the *depth-first forest*) in the digraph.

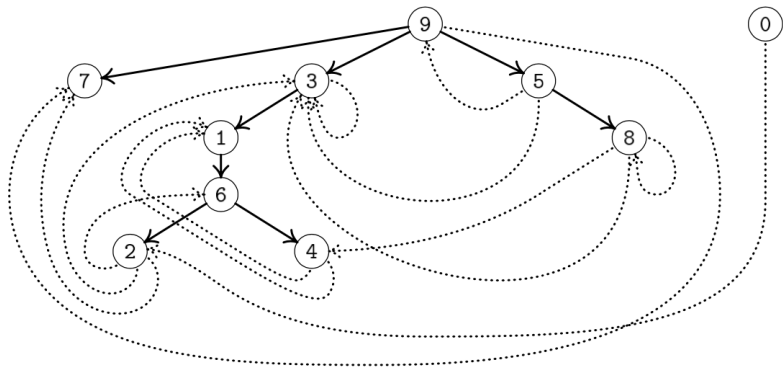


Figure: Example of a depth-first forest (solid). (Courtesy of Donald Knuth.)

# The random digraph:

- ▶ The random digraph has  $n$  vertices. Each vertex  $v$  has a random outdegree  $\eta_v$ ; these are i.i.d. (independent and identically distributed). We denote their distribution by  $\mathbf{P}$ .
- ▶ The endpoint of the  $\eta_v$  arcs from  $v$  are chosen randomly, uniformly among all vertices (including  $v$ ) and independently.
- ▶ We consider asymptotics as  $n \rightarrow \infty$  and the degree distribution  $\mathbf{P}$  is fixed. (Extensions to  $\mathbf{P} = \mathbf{P}_n$ ?)

**Remark.** The digraph is really a multi-digraph, since loops and parallel arc may occur. (But they are few, and unimportant.)

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Knuth asked in particular about the case when  $\mathbf{P}$  is a geometric distribution  $\text{Ge}(1 - p)$  ( $0 < p < 1$ ). In this case, the process is almost memory-free; each time we arrive or return to a vertex, we toss a coin and:

- ▶ with probability  $p$  construct and follow a new arc to a random vertex;
- ▶ with probability  $1 - p$  leave the vertex and return to its parent.



## Geometric outdegrees

Assume that the outdegree distribution is geometric  $\text{Ge}(1 - p)$  for some fixed  $0 < p < 1$ , and thus has mean

$$\lambda := \mathbb{E} \eta = \frac{p}{1 - p}.$$

Let  $v_t$  be the  $t$ -th vertex discovered by the DFS ( $t = 1, \dots, n$ ), and let  $d(t)$  be the depth of  $v_t$  in the resulting depth-first forest, i.e., the number of tree edges that connect the root of the current tree to  $v_t$ . The first found vertex  $v_1$  is a root, and thus  $d(1) = 0$ .

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The quantity  $d(t)$  follows a Markov chain with transitions ( $1 \leq t < n$ ):

- ▶  $d(t+1) = d(t) + 1$ . For some  $k \geq 1$ ,  $v_t$  has at least  $k$  outgoing arcs, the first  $k-1$  arcs lead to vertices already visited, and the  $k$ th arc leads to a new vertex. Probability:

$$\sum_{k=1}^{\infty} p^k \left(\frac{t}{n}\right)^{k-1} \left(1 - \frac{t}{n}\right) = \frac{(1 - t/n)p}{1 - pt/n}. \quad (1)$$

- ▶  $d(t+1) = d(t)$ , assuming  $d(t) > 0$ . All arcs from  $v_t$  lead to already visited vertices, i.e., the first case does not happen; furthermore, the parent of  $v_t$  has at least one future (not yet seen) arc leading to an unvisited vertex. These two events are independent. Moreover, by the lack-of-memory property, the second event has the probability (1). Probability:

$$\left(1 - \frac{(1 - t/n)p}{1 - pt/n}\right) \frac{(1 - t/n)p}{1 - pt/n}. \quad (2)$$

- ▶  $d(t+1) = d(t) - \ell$ , assuming  $d(t) > \ell \geq 1$ .

Similar. All arcs from  $v_t$  lead to already visited vertices, and so do all future arcs from the  $\ell$  nearest ancestors of  $v_t$ , but not from the  $(\ell + 1)$ th. Probability:

$$\left(1 - \frac{(1 - t/n)p}{1 - pt/n}\right)^{\ell+1} \frac{(1 - t/n)p}{1 - pt/n}. \quad (3)$$

- ▶  $d(t+1) = d(t) - \ell$ , assuming  $d(t) = \ell \geq 0$ .

Similar, except that the  $(\ell + 1)$ th ancestor does not exist and we ignore it. Probability:

$$\left(1 - \frac{(1 - t/n)p}{1 - pt/n}\right)^{\ell+1}. \quad (4)$$

We can summarize this in the formula

$$d(t+1) = \max(d(t) + 1 - \xi_t, 0)$$

where  $\xi_t$  is a random variable, independent of the history, with the geometric distribution  $\text{Ge}(\pi_t)$ , where

$$\pi_t := \frac{(1 - t/n)p}{1 - pt/n} = 1 - \frac{1 - p}{1 - pt/n}.$$

Define

$$\tilde{d}(t) := \sum_{i=1}^{t-1} (1 - \xi_i),$$

this is a sum of independent random variables. Induction yields

$$d(t) = \tilde{d}(t) - \min_{1 \leq j \leq t} \tilde{d}(j), \quad 1 \leq t \leq n.$$

**Remark.** Similar formulas have been used for other, related, problems with random graphs and trees, where trees have been coded as walks, see for example Aldous (1997). Note that in our case,  $\tilde{d}(t)$  may have negative jumps of arbitrary size.

Let  $\theta := t/n$ . Then, uniformly for  $0 \leq \theta \leq \theta^*$  for any  $\theta^* < 1$ ,

$$\begin{aligned}\mathbb{E}[\tilde{d}(t)] &= \sum_{i=1}^{t-1} (1 - \mathbb{E} \xi_i) = \sum_{i=1}^{t-1} \left(1 - \frac{1-p}{p(1-i/n)}\right) \\ &= n\tilde{\ell}(\theta) + O(1),\end{aligned}$$

where

$$\tilde{\ell}(\theta) := \int_0^\theta \left(1 - \frac{1}{\lambda(1-x)}\right) dx = \theta + \frac{1}{\lambda} \log(1-\theta).$$

Note that the derivative  $\tilde{\ell}'(\theta) = 1 - \lambda^{-1}/(1-\theta)$  is (strictly) decreasing on  $(0, 1)$ , i.e.,  $\tilde{\ell}$  is concave, and  $\tilde{\ell}(\theta) \rightarrow -\infty$  as  $\theta \rightarrow 1$ .

NOW TO BLACKBOARD!

## General outdegree distribution $P$

For a general outdegree distribution, the depth is no longer a Markov chain.

**Substitute:** The DFS uses a stack of unexplored arcs, for which we have seen the start vertex but not the end. The stack evolves as follows:

- S1 If the stack is empty, pick a new vertex  $v$  that has not been seen before (if there is no such vertex, we have finished). Otherwise, pop the last arc from the stack, and reveal its endpoint  $v$  (which is uniformly random over all vertices). If  $v$  already is seen, repeat.
- S2 ( $v$  is now a new vertex) Reveal the outdegree  $\eta$  of  $v$  and add to the stack  $\eta$  new arcs from  $v$ , with unspecified endpoints.  
GOTO S1

The size  $I(t)$  of the stack is a Markov chain.