The sum of powers of subtrees sizes for random trees

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70 Years of Percolation
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References

This talk is mainly based on joint work with

Jim Fill: Electronic Journal of Probability 27 (2022);
Jim Fill and Stephan Wagner: arXiv:2212.10871.

General problem

Additive functional: Let $f(T)$ (the toll function) be a given functional of rooted trees, and define

$$F(T) := \sum_{v \in T} f(T_v),$$

where T_v is the fringe tree rooted at v, i.e. the subtree consisting of v and all its descendants.

Problem: Study asymptotics of $F(\mathcal{T}_n)$ (mean, variance, distribution, ...) when $\mathcal{T}_n$ is some random tree of "size" n, and $n \to \infty$.

Today, the random tree $\mathcal{T}_n$ will be a conditioned Galton–Watson tree with $|\mathcal{T}_n| = n$ (the number of vertices); the offspring distribution ξ will be critical with finite variance $0 < \sigma^2 < \infty$. (Higher moments assumed only occasionally.)

The toll function will be simply

$$f_\alpha(T) := |T|^\alpha$$

for a constant α.

Examples.

$\alpha = 0$ is trivial: $F_0(T) = |T|$.
(The derivative at 0 is the "shape functional". No time today.)

$\alpha = 1$ gives $F_1(T) =$ the total pathlength.

We allow α to be complex, and we consider $F_\alpha(T)$ as a function of $\alpha \in \mathbb{C}$. We write

$$X_n(\alpha) := F_\alpha(\mathcal{T}_n) = \sum_{v \in \mathcal{T}_n} |(\mathcal{T}_n)_v|^\alpha$$

$$\widetilde{X}_n(\alpha) := X_n(\alpha) - \mathbb{E}\, X_n(\alpha).$$

Remark

Why complex α?

- Useful in proofs (also for real α) since powerful methods of analytic functions can be used.

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- Useful in proofs (also for real α) since powerful methods of analytic functions can be used.
- Gives us new problems to study. How do the phase transitions look in the complex plane?

There are two phase transitions for real α: $\alpha = 0$ and $\alpha = \frac{1}{2}$.

Thus three phases in the complex plane:

$$\mathrm{Re}(\alpha) < 0, \qquad 0 < \mathrm{Re}(\alpha) < \tfrac{1}{2}, \qquad \mathrm{Re}(\alpha) > \tfrac{1}{2}.$$

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Thus three phases in the complex plane:

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What happens at the boundaries $\text{Re}(\alpha) = 0$ and $\text{Re}(\alpha) = \frac{1}{2}$?

Let $\mathcal{T}_{n,*}$ be a random fringe tree, i.e. $(\mathcal{T}_n)_v$ for a random vertex $v \in \mathcal{T}_n$. Then

$$\mathbb{E}\, X_n(\alpha) = \sum_{k=0}^{\infty} k^{\alpha} n\, \mathbb{P}(|\mathcal{T}_{n,*}| = k).$$

Let $\mathcal{T}$ be an (unconditioned) Galton–Watson tree with the given offspring distribution. Then $\mathcal{T}_{n,*}$ has asymptotically the distribution of $\mathcal{T}$ (Aldous, 1991). Recall that

$$\mathbb{P}(|\mathcal{T}| = k) \sim ck^{-3/2}.$$

Consequently, the number of fringe trees of size k in $\mathcal{T}_n$ is $\approx cnk^{-3/2}$.

Hence, $\mathbb{E}\, X_n(\alpha)$ is dominated by small fringe trees for $\operatorname{Re}\alpha < 1/2$, and by large fringe trees for $\operatorname{Re}\alpha > 1/2$.

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Similarly, the variance, and distribution, are dominated by small fringe trees for $\operatorname{Re}\alpha < 0$, and by large fringe trees for $\operatorname{Re}\alpha > 0$.

Let

$$\mu(\alpha) := \mathbb{E}\,|\mathcal{T}|^{\alpha} = \sum_{k=1}^{\infty} k^{\alpha}\,\mathbb{P}(|\mathcal{T}| = k).$$

This converges for $\mathrm{Re}(\alpha) < \frac{1}{2}$, and defines an analytic function in this half-plane.

However,

$$\mu(\alpha) \to \infty \qquad \text{as } \alpha \nearrow \tfrac{1}{2}.$$

Theorem

(i). *If* $\mathrm{Re}(\alpha) < \frac{1}{2}$, *then*

$$\mathbb{E}\, X_n(\alpha) = \mu(\alpha) n + o(n)$$

(ii). *If* $\mathrm{Re}(\alpha) > \frac{1}{2}$, *then*

$$\mathbb{E}\, X_n(\alpha) = \frac{1}{\sqrt{2}\sigma} \frac{\Gamma(\alpha - \frac{1}{2})}{\Gamma(\alpha)} n^{\alpha+\frac{1}{2}} + o(n^{\alpha+\frac{1}{2}})$$

(iii). *If* $\alpha = \frac{1}{2}$, *then*

$$\mathbb{E}\, X_n(1/2) = \frac{1}{\sqrt{2\pi\sigma^2}} n \log n + o(n \log n).$$

Critical line $\mathrm{Re}(\alpha) = \frac{1}{2}$

Recall that $\mu(\alpha) \to \infty$ as $\alpha \nearrow \frac{1}{2}$.

Theorem

The function $\mu(\alpha)$ has a continuous extension to $\mathrm{Re}(\alpha) = \frac{1}{2}$, $\alpha \neq 1/2$.

Theorem

If $-\frac{1}{2} < \mathrm{Re}(\alpha) \leq \frac{1}{2}$ and $\alpha \neq \frac{1}{2}$, then

$$\mathbb{E}\, X_n(\alpha) = \mu(\alpha) n + \frac{1}{\sqrt{2}\sigma} \frac{\Gamma(\alpha - \frac{1}{2})}{\Gamma(\alpha)} n^{\alpha + \frac{1}{2}} + o\big(n^{(\mathrm{Re}\,\alpha)_+ + \frac{1}{2}}\big).$$

Let ξ be the offspring distribution.

Theorem

Suppose that $\mathbb{E}\,\xi^{2+\delta} < \infty$ *where* $0 < \delta \leq 1$.

(i). *Then* $\mu(\alpha)$ *can be analytically continued to a meromorphic function in* $\mathrm{Re}(\alpha) < \frac{1}{2} + \frac{\delta}{2}$ *with a simple pole at* $\frac{1}{2}$.

(ii). *Moreover, the estimate above*

$$\mathbb{E}\,X_n(\alpha) = \mu(\alpha)n + \frac{1}{\sqrt{2}\sigma}\frac{\Gamma(\alpha-\frac{1}{2})}{\Gamma(\alpha)}n^{\alpha+\frac{1}{2}} + o(n^{(\mathrm{Re}\,\alpha)_+ + \frac{1}{2}})$$

holds for $-\frac{1}{2} < \mathrm{Re}(\alpha) < \frac{1}{2} + \frac{\delta}{2}$ *with* $\alpha \neq \frac{1}{2}$.

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With more moments, $\mu(\alpha)$ can be extended further. In particular, if $\mathbb{E}\,\xi^r < \infty$ for all $r > 0$, then $\mu(\alpha)$ is meromorphic in $\mathbb{C}$, with (simple) poles only at $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots$.

The additional moment assumption is really needed here.

Theorem

There exists ξ *with* $\mathbb{E}\,\xi = 1$ *and* $\mathbb{E}\,\xi^2 < \infty$ *such that* $\mu(\alpha)$ *cannot be extended analytically across* $\mathrm{Re}(\alpha) = \frac{1}{2}$ *(at any point).*

Asymptotic distribution

Recall that $X_n(\alpha) := \widetilde{X}_n(\alpha) + \mathbb{E}\, X_n(\alpha)$.

Hence it suffices to consider $\widetilde{X}_n(\alpha)$ and then combine with the results above for $\mathbb{E}\, X_n(\alpha)$.

$\mathrm{Re}(\alpha) < 0$

Let $H_- := \{\alpha : \mathrm{Re}(\alpha) < 0\}$.

Theorem

- *There exists a random analytic function* $\widetilde{X}(\alpha)$, $\alpha \in H_-$, *such that, as* $n \to \infty$,

$$n^{-1/2}\widetilde{X}_n(\alpha) \xrightarrow{\mathrm{d}} \widetilde{X}(\alpha)$$

  for each fixed $\alpha \in H_-$, *and uniformly on each compact subset of* H_-. *(I.e., in the space* $\mathcal{H}(H_-)$ *of analytic functions on* H_-.*)*
- $\widetilde{X}(\alpha)$ *is a complex centred Gaussian, for every fixed* $\alpha \in H_-$. *Also jointly.*
- *The covariance matrix of* $\widetilde{X}(\alpha)$ *depends on the offspring distribution.*

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- *The covariance matrix of* $\widetilde{X}(\alpha)$ *depends on the offspring distribution.*

In this case $X_n(\alpha) = F_\alpha(\mathcal{T}_n)$ is dominated by the many small fringe trees. Hence normality, but not universality.

$\mathrm{Re}(\alpha) > 0$

Let $H_+ := \{\alpha : \mathrm{Re}(\alpha) > 0\}$. (No problems for $\mathrm{Re}(\alpha) = \frac{1}{2}$.)

Theorem

- *There exists a random analytic function* $\widetilde{Y}(\alpha)$, $\alpha \in H_+$, *such that, as* $n \to \infty$,

  $$\widetilde{Y}_n(\alpha) := n^{-\alpha-\frac{1}{2}} \widetilde{X}_n(\alpha) \xrightarrow{\mathrm{d}} \sigma^{-1} \widetilde{Y}(\alpha)$$

  for each fixed $\alpha \in H_+$, *and uniformly on each compact subset of* H_+. *(I.e., in the space* $\mathcal{H}(H_+)$ *of analytic functions on* H_+*.)*
- $\widetilde{Y}(\alpha)$ *is not Gaussian.*
- $\widetilde{Y}(\alpha)$ *does not depend on the offspring distribution.*

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$$\widetilde{Y}_n(\alpha) := n^{-\alpha-\frac{1}{2}} \widetilde{X}_n(\alpha) \xrightarrow{\mathrm{d}} \sigma^{-1} \widetilde{Y}(\alpha)$$

  for each fixed $\alpha \in H_+$, and uniformly on each compact subset of H_+. (I.e., in the space $\mathcal{H}(H_+)$ of analytic functions on H_+.)
- *$\widetilde{Y}(\alpha)$ is not Gaussian.*
- *$\widetilde{Y}(\alpha)$ does not depend on the offspring distribution.*

In this case $\widetilde{X}_n(\alpha) = F_\alpha(\mathcal{T}_n) - \mathbb{E}\, F_\alpha(\mathcal{T}_n)$ is dominated by the large fringe trees. Therefore universality but not normality.

Critical line $\mathrm{Re}(\alpha) = 0$

Theorem

Assume $\mathbb{E}\,\xi^{2+\delta} < \infty$ *for some* $\delta > 0$. *(Conjecture: not needed.)*

- *For every real* $t \neq 0$, *as* $n \to \infty$,

$$\frac{\widetilde{X}_n(\mathrm{i}t)}{\sqrt{n \log n}} \xrightarrow{\mathrm{d}} \sigma^{-1} \widetilde{Z}(\mathrm{i}t),$$

where $\widetilde{Z}(\mathrm{i}t)$ *is a symmetric complex normal variable with variance*

$$\mathbb{E}\,|\widetilde{Z}(\mathrm{i}t)|^2 = \frac{1}{\sqrt{\pi}} \,\mathrm{Re}\, \frac{\Gamma(\mathrm{i}t - \frac{1}{2})}{\Gamma(\mathrm{i}t)} > 0. \tag{1}$$

- $\widetilde{Z}(\mathrm{i}t)$ *thus does not depend on the offspring distribution.*
- *The convergence holds jointly for any finite number of* t, *with independent limits* $\widetilde{Z}(\mathrm{i}t)$ *for all* $t > 0$.
- *Thus no convergence to a continuous random function on* $\mathrm{i}\mathbb{R}$.

Without centring

Let, for $\mathrm{Re}(\alpha) > 0$ and $\alpha \neq \frac{1}{2}$,

$$Y(\alpha) := \widetilde{Y}(\alpha) + \frac{1}{\sqrt{2}\sigma} \frac{\Gamma(\alpha - \frac{1}{2})}{\Gamma(\alpha)}.$$

Theorem

(i). *If* $\mathrm{Re}(\alpha) > \frac{1}{2}$, *then*

$$Y_n(\alpha) := n^{-\alpha - \frac{1}{2}} X_n(\alpha) \xrightarrow{\mathrm{d}} \sigma^{-1} Y(\alpha).$$

(ii). *If* $0 < \mathrm{Re}(\alpha) \leq \frac{1}{2}$ *and* $\alpha \neq \frac{1}{2}$, *then*

$$n^{-\alpha - \frac{1}{2}} [X_n(\alpha) - n\mu(\alpha)] \xrightarrow{\mathrm{d}} \sigma^{-1} Y(\alpha).$$

Moment convergence

Theorem

All moments converge in the limit theorems above. If $\mathrm{Re}(\alpha) > 0$ *and* $\alpha \neq \frac{1}{2}$*, then the limiting moments* $\kappa_\ell := \mathbb{E}\, Y(\alpha)^\ell$ *satisfy the recursion*

$$\kappa_1 = \frac{\Gamma(\alpha - \frac{1}{2})}{\sqrt{2}\,\Gamma(\alpha)},$$

and, for $\ell \geq 2$*, with* $\alpha' := \alpha + \frac{1}{2}$*,*

$$\kappa_\ell = \frac{\ell\Gamma(\ell\alpha' - 1)}{\sqrt{2}\,\Gamma(\ell\alpha' - \frac{1}{2})}\kappa_{\ell-1} + \frac{1}{4\sqrt{\pi}}\sum_{j=1}^{\ell-1}\binom{\ell}{j}\frac{\Gamma(j\alpha' - \frac{1}{2})\Gamma((\ell-j)\alpha' - \frac{1}{2})}{\Gamma(\ell\alpha' - \frac{1}{2})}\kappa_j\kappa_{\ell-j}.$$

Proofs by singularity analysis of generating functions, using properties of Hadamard products.

Disclaimer. For $\alpha = \frac{1}{2}$, our proof requires that the offspring distribution ξ satisfies $\mathbb{E}\,\xi^{2+\delta} < \infty$ for some $\delta > 0$.

Brownian excursion, $\operatorname{Re}\alpha > 1$

Let $\mathbf{e}$ be a standard Brownian excursion. Recall that this is a random continuous function $[0,1] \to [0,\infty)$.
For a function g and $s < t$, define

$$m(g; s, t) := \inf_{u \in [s,t]} g(u).$$

Theorem

If $\operatorname{Re}\alpha > 1$*, we can represent the limit* $Y(\alpha)$ *as*

$$Y(\alpha) = 2\alpha(\alpha - 1) \iint_{0<s<t<1} (t-s)^{\alpha-2} m(\mathbf{e}; s, t)\, \mathrm{d}s\, \mathrm{d}t.$$

Proof. If we replace $\mathbf{e}$ by a suitably scaled version of the contour process of $\mathcal{T}_n$, then a calculation shows that the integral equals $n^{-\alpha-\frac{1}{2}} X_n(\alpha) + o(1)$. The contour process converges to $\mathbf{e}$ (Aldous, 1993), and the integral is a continuous functional.

Brownian excursion, $\operatorname{Re}\alpha > 1/2$

Theorem

If $\operatorname{Re}\alpha > 1/2$, we can represent the limit $Y(\alpha)$ as

$$Y(\alpha) = 2\alpha \int_0^1 t^{\alpha-1}\mathbf{e}(t)\,\mathrm{d}t - 2\alpha(\alpha-1) \iint_{0<s<t<1} (t-s)^{\alpha-2}[\mathbf{e}(t) - m(\mathbf{e}; s, t)]\,\mathrm{d}s\,\mathrm{d}t.$$

Example. $\alpha = 1$ (total pathlength) yields

$$Y(1) = 2\int_0^1 \mathbf{e}(t)\,\mathrm{d}t,$$

the *Brownian excursion area*. This case was proved by Aldous (1993).

Proof: Tightness

Lemma

(i). *If* $\operatorname{Re}\alpha < 0$, *then* $\mathbb{E}\,|\widetilde{X}_n(\alpha)|^2 \leq C(\alpha)n$.

(ii). *If* $\operatorname{Re}\alpha > 0$, *then* $\mathbb{E}\,|\widetilde{X}_n(\alpha)|^2 \leq C(\alpha)n^{2\operatorname{Re}\alpha+1}$, *and thus* $\mathbb{E}\,|\widetilde{Y}_n(\alpha)|^2 \leq C(\alpha)$.

In both cases $C(\alpha) = O(1 + |\alpha|^{-2})$.

This shows tightness at each fixed α.

Proof: Magic of analytic functions

Lemma

Let D be a domain in $\mathbb{C}$ and let $(Y_n(z))$ be a sequence of random analytic functions in $\mathcal{H}(D)$. Suppose that there exists a function $\gamma : D \to (0, \infty)$, bounded on each compact subset of D, such that

$$\mathbb{E}\,|Y_n(z)| \leq \gamma(z)$$

for every $z \in D$. Then the sequence (Y_n) is tight in the space $\mathcal{H}(D)$ of analytic functions on D.

Proof. Cauchy's integral formula, together with $\mathbb{E} \int = \int \mathbb{E}$.

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Proof. Cauchy's integral formula, together with $\mathbb{E} \int = \int \mathbb{E}$.

Hence, the random functions $\widetilde{Y}_n(\alpha)$ are tight in $\mathcal{H}(H_+)$.

More magic of analytic functions

Lemma

Let D be a domain in $\mathbb{C}$ and let E be a subset of D that has a limit point in D. (I.e., there exists a sequence $z_n \in E$ of distinct points and $z_\infty \in D$ such that $z_n \to z_\infty$.) Suppose that (Y_n) is a tight sequence of random elements of $\mathcal{H}(D)$ and that there exists a family of random variables $\{Y_z : z \in E\}$ such that for each $z \in E$, $Y_n(z) \xrightarrow{\mathrm{d}} Y_z$ and, moreover, this holds jointly for any finite set of $z \in E$. Then $Y_n \xrightarrow{\mathrm{d}} Y$ in $\mathcal{H}(D)$, for some random function $Y(z) \in \mathcal{H}(D)$.

Proof. Subsequences converge, and limits are determined by the restriction to E, and therefore unique.

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Proof. Subsequences converge, and limits are determined by the restriction to E, and therefore unique.

Hence, the random functions $\widetilde{Y}_n(\alpha)$ converge in distribution in $\mathcal{H}(H_+)$.

Problem: Brownian excursion, $\operatorname{Re}\alpha \le 1/2$

For $\operatorname{Re}\alpha > 1/2$, we have seen above explicit representations of $\widetilde{Y}(\alpha)$ using a Brownian excursion $\mathbf{e}(t)$.

We know that almost surely, this extends to an analytic function in the halfplane $H_+ = \{\alpha : \operatorname{Re}\alpha > 0\}$.

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We know that almost surely, this extends to an analytic function in the halfplane $H_+ = \{\alpha : \operatorname{Re}\alpha > 0\}$.

It follows, using general measure theory, that there exist a mesurable function $\psi : C[0,1] \to \mathcal{H}(H_+)$ such that

$$Y = \psi(\mathbf{e}).$$

Thus there exists a measurable function $\Psi : H_+ \times C[0,1] \to \mathbb{C}$ such that

$$Y(\alpha) = \Psi(\alpha, \mathbf{e}), \qquad \operatorname{Re}\alpha > 0.$$

However, this is only an existence statement, and we do not know any explicit representation when $\operatorname{Re}\alpha \le 1/2$.
Is there an explicit formula giving $Y(\alpha)$ in terms of $\mathbf{e}(t)$ also for $0 < \operatorname{Re}\alpha < \frac{1}{2}$?

THE END