Let us start with a classic result due to Bollobás and Erdős (1976) and Matula (1976). Its proof, based on the second moment method, can be found also in Bollobás (1985, Chapter XI).

Theorem 7.1. *For $\varepsilon > 0$ and $b = 1/(1-p)$, set*

$$\hat{k}_{\pm\varepsilon} = \lfloor 2\log_b n - 2\log_b \log_b np + 2\log_b(e/2) + 1 \pm \varepsilon/p \rfloor. \tag{7.1}$$

Then, for $p = p(n)$ such that $p > n^{-\delta}$ for every $\delta > 0$ but $p \le c$ for some $c < 1$, a.a.s.

$$\hat{k}_{-\varepsilon} \le \alpha(\mathbb{G}(n,p)) \le \hat{k}_{\varepsilon}.$$

■

Remark 7.2. In fact, Bollobás and Erdős (1976) and Matula (1976) proved that in the above range of $p(n)$, the stability number $\alpha(\mathbb{G}(n,p))$ is asymptotically concentrated on at most two points, that is, there is a sequence $\hat{k}(n)$ such that *a.a.s.* $\hat{k}(n) \le \alpha(\mathbb{G}(n,p)) \le \hat{k}(n) + 1$.

In this section we will concentrate on the case when $p = p(n) \le \log^{-2} n$. Then, in order to avoid dealing with logarithms of base b, instead of $\hat{k}_{\pm\varepsilon}$ it is convenient to use the functions $k_{\pm\varepsilon}$, defined as

$$k_{\pm\varepsilon} = \left\lfloor \frac{2}{p}(\log np - \log\log np + 1 - \log 2 \pm \varepsilon) \right\rfloor. \tag{7.2}$$

Elementary calculations show (Exercise!) that for $p \le \log^{-2} n$, $\varepsilon > 0$, and n large enough, we have $\hat{k}_{-3\varepsilon} \le k_{-\varepsilon} \le \hat{k}_{-\varepsilon}$ and $\hat{k}_{\varepsilon} \le k_{\varepsilon} \le \hat{k}_{3\varepsilon}$, and so it does not matter very much whether we use $\hat{k}_{\pm\varepsilon}$ or $k_{\pm\varepsilon}$ to estimate $\alpha(\mathbb{G}(n,p))$.

Let $X(k) = X(k; n, p)$ denote the number of stable sets of size k in $\mathbb{G}(n,p)$. Since $\alpha(\mathbb{G}(n,p)) \ge k$ if and only if $X(k) > 0$, the most natural way of handling $\alpha(\mathbb{G}(n,p))$ is to study the behavior of $X(k)$. First we will estimate the probability $\mathbb{P}(X(k) > 0)$ for $k_{-\varepsilon} \le k \le k_{\varepsilon}$, using the second moment method. The following lemma shows that this approach works well for $p = p(n)$ which does not tend to 0 too fast.

Lemma 7.3. *Let $\varepsilon > 0$, and $k_{\pm\varepsilon}$ be defined as in (7.2). Then there exists a constant $C_\varepsilon > 0$ such that for $C_\varepsilon/n \le p = p(n) \le \log^{-2} n$, we have*

$$\mathbb{P}(X(k_\varepsilon) > 0) \le \mathbb{E}\, X(k_\varepsilon) \to 0 \tag{7.3}$$

and

$$\mathbb{E}\, X(k_{-\varepsilon}) \to \infty$$

as $n \to \infty$. Furthermore, if $\log^2 n/\sqrt{n} \le p \le \log^{-2} n$, then

$$\mathbb{P}(X(k_{-\varepsilon}) > 0) = 1 - o(1) \tag{7.4}$$

and if $C_\varepsilon/n \le p \le \log^2 n/\sqrt{n}$, then for large n

$$\mathbb{P}(X(k_{-\varepsilon}) > 0) \ge \exp\left(-\frac{k_{-\varepsilon}}{\log^3 np}\right) \ge \exp\left(-\frac{2}{p\log^2 np}\right). \tag{7.5}$$

In particular, if $\log^2 n/\sqrt{n} \le p \le \log^{-2} n$, then a.a.s.

$$k_{-\varepsilon} \le \alpha(\mathbb{G}(n,p)) \le k_{\varepsilon}. \tag{7.6}$$

Proof. The first moment of $X(k_\varepsilon)$ is rather easy to handle. For instance, for np large enough,

$$\mathbb{E}\, X(k_\varepsilon) = \binom{n}{k_\varepsilon}(1-p)^{\binom{k_\varepsilon}{2}} \le \left(\frac{en}{k_\varepsilon}\exp\left(-\frac{p(k_\varepsilon - 1)}{2}\right)\right)^{k_\varepsilon}$$

$$\le \left(\frac{enp}{2(\log np - \log\log np)}\exp\left(-\frac{p(k_\varepsilon - 1)}{2}\right)\right)^{k_\varepsilon} \le \exp(-\varepsilon k_\varepsilon/2) \to 0.$$

We leave to the reader an elementary verification (Exercise!) that if $np \ge C_\varepsilon$, where C_ε is a sufficiently large constant, then for large n

$$\mathbb{E}\, X(k_{-\varepsilon}) \ge \exp(\varepsilon k_{-\varepsilon}/2) \to \infty, \tag{7.7}$$

and concentrate on the proof of (7.4) and (7.5).

Let us set, for convenience, $k = k_{-\varepsilon}$ and $X = X(k)$, and assume that $C_\varepsilon/n \le p \le \log^{-2} n$ with C_ε large enough. As we have already mentioned, our proof is based on a standard second moment argument, that is, we will estimate $\mathbb{E}\, X^2$ and then deduce (7.4) and (7.5) from (3.3). Note first that

$$\begin{aligned}\frac{\mathbb{E}\, X^2}{(\mathbb{E}\, X)^2} - 1 &= \frac{\binom{n}{k}(1-p)^{\binom{k}{2}}\sum_{i=0}^{k}\binom{k}{i}\binom{n-k}{k-i}(1-p)^{\binom{k}{2}-\binom{i}{2}}}{\left[\binom{n}{k}(1-p)^{\binom{k}{2}}\right]^2} - 1 \\ &\le \sum_{i=1}^{k}\frac{\binom{k}{i}\binom{n-k}{k-i}}{\binom{n}{k}}(1-p)^{-\binom{i}{2}} = \sum_{i=1}^{k} a_i,\end{aligned} \tag{7.8}$$

where

$$a_i = \frac{\binom{k}{i}\binom{n-k}{k-i}}{\binom{n}{k}}(1-p)^{-\binom{i}{2}} \quad \text{for} \quad i = 1, 2, \ldots, k.$$

Furthermore, let

$$b_i = \frac{a_{i+1}}{a_i} = \frac{(k-i)^2}{(i+1)(n-2k+i+1)}(1-p)^{-i}.$$

It is not hard to see that for small i, the sequence b_i decreases with i because of the factor $i+1$ in the denominator, for intermediate i it grows due to the factor $(1-p)^{-i}$ and, finally, when the difference $k-i$ becomes small, b_i declines