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CONVERSE HYPERCONTRACTIVITY

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1. Introduction

Consider the number operator $N = -d^2/dx^2 + xd/dx$ in $L^2(\mu)$, where $d\mu = (2\pi)^{-1/2}e^{-x^2/2}dx$. Then, for each $t > 0$,

$$(1.1) \qquad \|e^{-tN}f\|_{1+e^{2t}(p-1)} \leq \|f\|_p \, , \quad f \in L_2 \, , \quad 1 < p < +\infty \, ,$$

and

$$(1.2) \qquad \|e^{-tN}f\|_{1+e^{2t}(p-1)} \geq \|f\|_p \, , \quad f \in L_2^+ \, , \quad -\infty < p < 1 \, .$$

Here (1.1) is the famous Nelson hypercontractivity inequality [5] and (1.2) was noticed by one of the authors [1].

In [6] Neveu gives a very simple and beautiful proof of (1.1) by using the Itô calculus of Brownian motion and, not surprisingly, the same line of reasoning works for $p < 1$ as well [1]. Another perhaps more elementary proof of (1.2) is obtained by extending the so-called "two-point inequality" to the parameter interval $p < 1$ [2].

The purpose of the present note is to point out that (1.1) $\Leftrightarrow$ (1.2), and that this proof also proves the analogue of (1.2) for the Poisson integral on a sphere in $\mathbb{R}^2$ or $\mathbb{R}^3$, a result which we have not been able to show in a more direct way. We begin with the number operator and indicate the minor modifications for the Poisson integral later.

2. Proof of (1.1) ⇔ (1.2)

We will prove that (1.1) and (1.2) both are equivalent to (with $Df = \frac{df}{dx}$)

$$(2.1)\qquad \int f^p \ln f^p d\mu - \|f\|_p^p \ln \|f\|_p^p \le \frac{p^2}{2(p-1)} \langle f^{p-1}, Nf\rangle = \frac{p^2}{2(p-1)} \langle D f^{p-1}, Df\rangle ,$$

$$0 < \inf f(x) \le \sup f(x) < \infty , \quad Df \in L^2 , \quad p \ne 0,1 .$$

More precisely, (1.1) is equivalent to (2.1) for $1 < p < \infty$, (1.2) is equivalent to (2.1) for $-\infty < p < 1$, and (2.1) for any value of $p \ne 0,1$ is equivalent to (2.1) for any other p. The equivalence of (1.1) and (2.1) is due to Gross [3] and we follow his ideas.

By an approximation argument, (1.1) and (1.2) are equivalent to

$$(2.2)\qquad \|e^{-tN} f\|_{1+e^{2t}(p-1)} \quad \text{is a decreasing (increasing) function of}$$

$$t > 0, \text{ for } p > 1 \ (p < 1) \text{ and } 0 < \inf f \le \sup f < \infty .$$

For such a function f, set $f_t = e^{-tN} f$.

For $t > 0$ and $-\infty < p < \infty$, $\|f_t\|_p = \exp \frac{1}{p} \ln \int f_t(x)^p d\mu(x)$ is an analytic function of p (the apparent singularity at $p = 0$ is removable) and

$$\frac{\partial}{\partial p} \|f_t\|_p = \frac{1}{p} \|f_t\|_p^{1-p} \left(\int f_t^p \ln f_t - \|f_t\|_p^p \ln \|f_t\|_p \right) \quad (p \ne 0).$$

Since $\frac{\partial}{\partial t} f_t = -Nf_t$,

$$\frac{\partial}{\partial t} \|f_t\|_p = -\|f_t\|_p^{1-p} \int f_t^{p-1} Nf_t .$$

$\frac{\partial}{\partial p} \|f_t\|_p$ and $\frac{\partial}{\partial t} \|f_t\|_p$ are bounded on compact subsets of

$\{t > 0, -\infty < p < \infty\}$. Thus $\|f_t\|_p$ is continuous. Hence $\frac{\partial}{\partial t}\|f_t\|_p$ is continuous and $\|f_t\|_p$ is differentiable. Consequently, with $p_t = 1 + e^{2t}(p-1)$, we have for $p_t \neq 0$,

$$\frac{d}{dt}\|f_t\|_{p_t} = \frac{\partial}{\partial t}\|f_t\|_{p_t} + \frac{\partial}{\partial p}\|f_t\|_{p_t}\frac{dp_t}{dt} =$$

$$= -\|f_t\|_{p_t}^{1-p_t}\int f_t^{p_t-1} Nf_t + \frac{2(p_t-1)}{p_t}\|f_t\|_{p_t}^{1-p_t}\left(\int f_t^{p_t}\ell n\, f_t - \|f_t\|_{p_t}^{p_t}\ell n\|f_t\|_{p_t}\right).$$

Since (2.2) is equivalent to $\frac{p_t^2}{p_t-1}\frac{d}{dt}\|f_t\|_{p_t} \leq 0$, it is equivalent to

$$-\frac{p^2}{p-1}\int f_t^{p-1} N f_t + 2p\left(\int f_t^p \ell n\, f_t - \|f_t\|_p^p\, \ell n\|f_t\|_p\right) \leq 0, \quad p \neq 0,1,$$

i.e. (2.1).

To show that (2.1) for different values of p are equivalent, let $g = f^{p/2}$. Since $\langle Df^{p-1}, Df\rangle = (p-1)\langle f^{p-2}Df, Df\rangle = (p-1)(2/p)^2\langle Dg, Dg\rangle$, (2.1) for (f,p) is equivalent to (2.1) for $(g,2)$.

3. The Poisson Integral

In this section, let μ be the normalized surface measure on the unit sphere in $\mathbb{R}^d$ $(d \leq 3)$ and define N by $Nf = nf$ if f is a spherical harmonics of degree n.

Then e^{-tN} is the Poisson integral $P_{e^{-t}}$, and with a change of variables (1.1) and (1.2) correspond to

$$(3.1) \qquad \|P_r f\|_{1+r^{-2}(p-1)} \leq \|f\|_p , \quad f \in L_2 , \quad 1 < p < \infty , \quad 0 < r < 1$$

and

$$(3.2) \qquad \|P_r f\|_{1+r^{-2}(p-1)} \geq \|f\|_p , \quad f \in L_2^+ , \quad -\infty < p < 1, \; 0 < r < 1 .$$

Equivalent formulations are: Assume that u is positive and harmonic in the unit ball, then

$$(3.3) \qquad \|u(rx)\|_q \leq \|u(x)\|_p , \quad 1 < p < q < \infty , \quad 0 \leq r \leq \sqrt{\frac{p-1}{q-1}}$$

and

$$(3.4) \qquad \|u(rx)\|_q \geq \|u(x)\|_p , \quad -\infty < q < p < 1 , \quad 0 \leq r \leq \sqrt{\frac{1-p}{1-q}}$$

or

$$(3.5) \qquad \|u(rx)^{-1}\|_q \leq \|u(x)^{-1}\|_p , \quad -1 < p < q < \infty , \quad 0 \leq r \leq \sqrt{\frac{p+1}{q+1}} .$$

For $d = 1$ these reduce to the "two-point inequality" [2]. For $d = 2$ and 3, (3.1) is proved in [7] and [4] respectively. We will show that this implies (3.2) and thus (3.4) and (3.5). (The argument is independent of the dimension, but (3.1) is not known for $d \geq 4$.)

Exactly as above, (3.1) and (3.2) are equivalent to

$$(3.6) \qquad \int f^p \ln f^p d\mu - \|f\|_p^p \ln \|f\|_p^p \leq \frac{p^2}{2(p-1)} \left\langle f^{p-1}, Nf \right\rangle , \quad 0 < f \in C^\infty(S^{d-1}), \; p \neq 0,1.$$

Different values of p are no longer equivalent, but (3.6) for $p = 2$ implies (3.6) for any p. The following argument by Weissler is used for $p > 1$ in [7], [4].

Let u be the harmonic extension of f to the interior. Then $Nf(x) = \frac{\partial}{\partial r} u(rx)\Big|_{r=1}$. Let B denote the unit ball, $\partial B = S^{d-1}$ and $\Omega_d = |\partial B|$. Then

$$\Omega_d \langle f^{p-1}, Nf \rangle = \int_{\partial B} u^{p-1} \frac{\partial u}{\partial r} = \frac{1}{p} \int_{\partial B} \frac{\partial u^p}{\partial r} = \frac{1}{p} \int_B \Delta u^p =$$

$$= (p-1) \int_B u^{p-2} |\nabla u|^2 = \frac{4(p-1)}{p^2} \int_B |\nabla u^{p/2}|^2 .$$

Let $g = f^{p/2}$ and let v be the harmonic extension of g to the unit ball. Then $\Omega_d \langle g, Ng \rangle = \int_B |\nabla v|^2$ by a similar calculation. Since $u^{p/2}$ and v have the same boundary values and v is harmonic, $\int_B |\nabla v|^2 \leq \int_B |\nabla u^{p/2}|^2$ by Dirichlet's principle. Thus,

$$\frac{p^2}{4(p-1)} \langle f^{p-1}, Nf \rangle \geq \langle g, Ng \rangle ,$$

and (3.6) for (f,p) is a consequence of (3.6) for $(g,2)$. This completes the proof that (3.1) implies (3.2).

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