ON THE LENGTH OF THE LONGEST INCREASING SUBSEQUENCE IN A RANDOM PERMUTATION

Béla Bollobás and Svante Janson

Dedicated to Paul Erdős on his eightieth birthday.

Abstract. Complementing the results claiming that the maximal length L_n of an increasing subsequence in a random permutation of $\{1, 2, \dots, n\}$ is highly concentrated, we show that L_n is not concentrated in a short interval: $\sup_l \mathrm{P}(l \le L_n \le l + n^{1/16} \log^{-3/8} n) \to 0$ as $n \to \infty$.

1. Introduction

Ulam [9] proposed the study of L_n, the maximal length of an increasing subsequence of a random permutation of the set $[n] = \{1, 2, \dots, n\}$. Hammersley [4], Logan and Shepp [7], and Veršik and Kerov [10] proved that $\mathrm{E}\, L_n \sim 2\sqrt{n}$ and

$$L_n/\sqrt{n} \xrightarrow{\mathrm{p}} 2 \quad \text{as} \quad n \to \infty. \tag{1.1}$$

Frieze [3] showed that the distribution of L_n is sharply concentrated about its mean; his result was improved by Bollobás and Brightwell [2], who in particular proved that

$$\mathrm{Var}(L_n) = O\big(n^{1/2} (\log n / \log\log n)^2\big). \tag{1.2}$$

(The log factors have recently been removed by Talagrand [8].) Somewhat surprisingly, it is not known that the distribution of L_n is not much more concentrated than claimed by (1.2). In fact, it has not even been ruled out that if $w(n) \to \infty$ then $\mathrm{P}(|L_n - \mathrm{E}\, L_n| < w(n)) \to 0$ as $n \to \infty$. Our aim in this paper is to rule out this possibility for a fairly fast-growing function $w(n)$, and to give a lower bound for $\mathrm{Var}(L_n)$, complementing (1.2).

Theorem 1.

$$\mathrm{P}(|L_n - \mathrm{E}\, L_n| \le n^{1/16} \log^{-3/8} n) \to 0 \quad \text{as} \quad n \to \infty.$$

More generally, if a_n and b_n are any numbers such that

$$\inf \mathrm{P}(a_n \le L_n \le b_n) > 0, \quad \text{then} \quad (b_n - a_n)/n^{1/16} \log^{-3/8} n \to \infty.$$

In particular, for sufficiently large n,

$$\mathrm{Var}\, L_n \ge n^{1/8} \log^{-3/4} n.$$

There is still a wide gap between the upper and lower bound, and there is no reason to believe that the bounds given here are the best possible. In fact, a boot-strap argument

Second author supported by the Göran Gustafsson Foundation for Research in Natural Sciences and Medicine

suggests that the range of variation is at least about $n^{1/10}$, see Theorem 2 below, and it is quite possible that the upper bound in (1.2) is sharp up to logarithmic factors, as conjectured in [2].

It is well-known that L_n also can be defined as the height of the random partial order defined as follows. Consider the unit square $Q = [0,1]^2$ with the coordinate order. Thus for (x, y), $(x', y') \in Q$ set $(x, y) \le (x', y')$ if and only if $x \le x'$ and $y \le y'$, let $(\xi_i)_{i=1}^{\infty}$ be independent, uniformly distributed random points in Q and consider the induced partial order on the set $(\xi_i)_{i=1}^{n}$.

Let $\mu > 0$ be a constant and let m be the Lebesque measure in Q. Let us regard a Poisson process with intensity μdm in Q as a random subset of Q. Equivalently, let N be independent of $(\xi_i)_1^{\infty}$, with distribution $\mathrm{Po}(\mu)$, and take the set $\{\xi_i : 1 \le i \le N\}$. Write H_μ for the height of the induced partial order on this set.

In [2] the proof of (1.2) was based on a study of H_n. In particular they proved that

$$\mathrm{P}\Big(|H_n - \mathrm{E}\, H_n| > K_1 \lambda \frac{n^{1/4} \log n}{\log\log n}\Big) \le e^{-\lambda^2} \tag{1.3}$$

for some constant K_1, every $n \ge 3$ and every λ with $1 \le \lambda \le n^{1/4}/\log\log n$. For larger λ their proof yields

$$\mathrm{P}(|H_n - \mathrm{E}\, H_n| > K_2 \lambda^2 \log \lambda) \le e^{-\lambda^2}. \tag{1.4}$$

These inequalities hold for non-integer n as well: and that if $n \ge 3$ and $1 \le \lambda \le n^{1/4}/\log\log n$, then for every $\mu \le n$, we have

$$\mathrm{P}\Big(|H_\mu - \mathrm{E}\, H_\mu| > K_3 \lambda \frac{n^{1/4} \log n}{\log\log n}\Big) \le e^{-\lambda^2}. \tag{1.5}$$

It is rather curious that our proof of a lower bound will use these results together with, as well as the following estimate from [2]:

$$0 \le 2n^{1/2} - \mathrm{E}\, H_n \le K_4 n^{1/4} \log^{3/2} n / \log\log n. \tag{1.6}$$

Remark. It is shown in [2] that (1.3) holds for L_n as well. (The same is true for (1.4) and (1.5).) Similarly, Theorem 1 holds for H_n too; this follows from the proof of Theorem 1 below, with a few simplifications.

The variables L_n and H_n may be defined, more generally, for random subsets of the d-dimensional cube $[0,1]^d$. The results in [2] include this generalization, and it would be interesting to find lower bounds for the variance. Unfortunately, and somewhat surprisingly, the method used here does not work when $d \ge 3$. We try to explain this failure at the end of the paper.

2. Proof of Theorem 1

The idea behind the proof is that L_n essentially depends only on the points in a strip of measure $n^{-\alpha}$ for some $\alpha > 0$ ($\alpha = 1/8$ if we ignore logarithmic factors). The number of points in this strip is approximately Poisson distributed with expectation $n^{1-\alpha}$; hence the random variation of this number is of order $n^{(1-\alpha)/2}$ and the relative variation is $n^{-(1-\alpha)/2}$. This ought to correspond to a relative variation in the height of the same order $n^{-(1-\alpha)/2}$, ignoring the further variation due to the random position of the points, which would give a variation of order at least $n^{1/2} \cdot n^{-(1-\alpha)/2} = n^{\alpha/2}$.

We introduce some notation. For a Borel set $S \subset Q$, let

$$N_n(S) = |\{i \le n : \xi_i \in S\}|$$

be the number of those of our n random points that lie in S, and let $L_n(S)$ be the height of the partial order defined by these $N_n(S)$ points; similarly, let $H_n(S)$ be the height of the partial order defined by the restriction of our Poisson process to S. Finally, let $S_\delta = \{(x,y) \in Q : |x-y| \le \delta\}$ be the strip of width 2δ along the diagonal. We shall deduce our theorem from two lemmas. The first of these claims that the height only depends on the points in S_δ for a fairly small value of δ.

Lemma 1. *If K is sufficiently large, then with $\delta_n = Kn^{-1/8}\log^{3/4}n\,(\log\log n)^{-1/2}$ we have*

$$\mathrm{P}\big(L_n \ne L_n(S_{\delta_n})\big) \to 0 \quad \text{as} \quad n \to \infty.$$

Proof. We claim that $K = \max(3{K_3}^{1/2}, 2K_4^{1/2})$ will do, where K_3 and K_4 are the constants in (1.5) and (1.6). In fact, we shall prove slightly more than claimed, namely that the probability that the set $\{\xi_i : 1 \le i \le n\}$ contains a point $\xi_i \notin S_{\delta_n}$ that belongs to a maximal chain is $o(1)$. Since the probability that a Poisson process Ξ in Q with intensity n has exactly n points with probability at least $e^{-1}n^{-1/2}$, it suffices to show that the corresponding probability for the Poisson process Ξ is $o(n^{-1/2})$.

Let M be the number of points in $\Xi \setminus S_\delta$ that belong to a maximal chain in Ξ. Then

$$M = \sum_{\xi\in\Xi} f(\xi,\Xi),$$

where

$$f(\xi,\Xi) = I(\xi \notin S_\delta)\cdot I(\xi \text{ belongs to a maximal chain in } \Xi).$$

Hence, using an easily proved formula for Poisson processes (see, e.g., [5, Lemma 2.1], and [6, Lemma 10.1 and Exercise 11.1]),

$$\begin{aligned}\mathrm{E}\,M &= \mathrm{E}\sum_{\xi\in\Xi} f(\xi,\Xi) = \int_Q \mathrm{E}\,f(z,\Xi\cup\{z\})n\,dm(z)\\ &= \int_{Q\setminus S_\delta} \mathrm{P}(z \text{ belongs to a maximal chain in } \Xi\cup\{z\})n\,dm(z). \qquad (2.1)\end{aligned}$$

Fix $z = (x,y) \notin S_\delta$ and let $s = (x+y)/2$, $t = (x-y)/2$, $Q_1 = [0,x]\times[0,y]$ and $Q_2 = [x,1]\times[y,1]$. Then, writing $|R|$ for the area of a set $R \subset Q$, we have

$$\begin{aligned}|Q_1|^{1/2} + |Q_2|^{1/2} &= (s^2-t^2)^{1/2} + ((1-s)^2-t^2)^{1/2}\\ &\le s - \frac{t^2}{2s} + 1 - s - \frac{t^2}{2(1-s)}\\ &= 1 - \frac{t^2}{2s(1-s)}\\ &\le 1-2t^2\\ &\le 1-\frac{1}{2}\delta^2.\end{aligned}$$

The random variables $H_n(Q_1)$ and $H_n(Q_2)$ have the same distributions as H_{μ_1} and H_{μ_2}, respectively, with $\mu_i = n|Q_i|$, $i=1,2$. Setting $\delta = \delta_n$, inequality (1.6) implies that if n is large enough,

$$\mathrm{E}\,H_{\mu_1} + \mathrm{E}\,H_{\mu_2} \le 2\mu_1^{1/2} + 2\mu_2^{1/2} \le 2n^{1/2} - n^{1/2}\delta_n^2 \le \mathrm{E}\,H_n - 1 - \tfrac{1}{2}n^{1/2}\delta_n^2.$$

Hence, by applying (1.5) with $\lambda = (2\log n)^{1/2}$, we find that

$$\begin{aligned}
&\mathrm{P}(z \text{ belongs to a maximal chain in } \Xi \cup \{z\}) = \mathrm{P}(H_n(Q_1) + H_n(Q_2) + 1 \geq H_n) \\
&\quad \leq \mathrm{P}(H_{\mu_1} \geq \mathrm{E}\, H_{\mu_1} + \tfrac{1}{6}n^{1/2}\delta_n^2) + \mathrm{P}(H_{\mu_2} \geq \mathrm{E}\, H_{\mu_2} + \tfrac{1}{6}n^{1/2}\delta_n^2) \\
&\qquad + \mathrm{P}(H_n \leq \mathrm{E}\, H_n - \tfrac{1}{6}n^{1/2}\delta_n^2) \\
&\quad \leq 3\exp(-2\log n) = 3n^{-2}.
\end{aligned}$$

Consequently, (2.1) yields $\mathrm{E}\, M \leq 3n^{-1}$, and the result follows. $\square$

Lemma 2. *Suppose that* $\delta_n \searrow 0$ *and that* $\mathrm{P}\big(L_n \neq L_n(S_{\delta_n})\big) \to 0$ *as* $n \to \infty$. *If* (α_n) *is any sequence with* $\alpha_n = o({\delta_n}^{-1/2})$ *then*

$$\sup_x \mathrm{P}(|L_n - x| \leq \alpha_n) \to 0 \quad \text{as} \quad n \to \infty.$$

Proof. It is convenient to use couplings, and we begin by recalling the relevant definitions. A *coupling* of two random variables X and Y (possibly defined on different probability spaces), is a pair of random variables (X', Y') defined on a common probability space such that $X' \overset{\mathrm{d}}{=} X$ and $Y' \overset{\mathrm{d}}{=} Y$. The notion of coupling depends only on the distributions of X and Y, so we may as well talk about a coupling of two distributions (which can be formulated as finding a joint distribution with given marginals).

We also define the total variation distance of two random variables X and Y (or, more properly, of their distributions $\mathcal{L}(X)$ and $\mathcal{L}(Y)$) as

$$d_{TV}(X, Y) = \sup_A |\,\mathrm{P}(X \in A) - \mathrm{P}(Y \in A)|, \tag{2.2}$$

taking the supremum over all Borel sets A. If (X', Y') is a coupling of X and Y then, clearly, $d_{TV}(X, Y) = d_{TV}(X', Y') \leq \mathrm{P}(X' \neq Y')$. Conversely, it is easy to construct a coupling of X and Y such that equality holds (such couplings are known as *maximal* couplings). Thus

$$d_{TV}(X, Y) = \min \mathrm{P}(X' \neq Y'), \tag{2.3}$$

where the minimum ranges over all couplings of X and Y. Moreover, provided the probability space where X is defined is rich enough, there exists a maximal coupling (X', Y') of X and Y with $X' = X$.

We may assume that $\delta_n < 1$ and $\alpha_n \geq {\delta_n}^{-1/4} \to \infty$. (All limits in the proof are taken as $n \to \infty$.)

Let $m = m(n) = \lceil 6\alpha_n\sqrt{n}\rceil \leq 7\alpha_n\sqrt{n}$, and let $\mu = \mu(n) = |S_{\delta_n}|$; thus

$$\delta_n \leq \mu \leq 2\delta_n.$$

We use the facts that, for any $n, p, \lambda_1, \lambda_2$,

$$d_{TV}\big(\mathrm{Bi}(n,p), \mathrm{Po}(np)\big) \leq p$$

and

$$d_{TV}\big(\mathrm{Po}(\lambda_1), \mathrm{Po}(\lambda_2)\big) \leq |\lambda_1 - \lambda_2| / \max(\lambda_1, \lambda_2)^{1/2},$$

see e.g. [1, Theorems 2.M and 1.C]. Hence

$$\begin{aligned}
d_{TV}\bigl(N_n(S_{\delta_n}),N_{n+m}(S_{\delta_n})\bigr) &= d_{TV}\bigl(\mathrm{Bi}(n,\mu),\mathrm{Bi}(n+m,\mu)\bigr)\\
&\le d_{TV}\bigl(\mathrm{Bi}(n,\mu),\mathrm{Po}(n\mu)\bigr)+d_{TV}\bigl(\mathrm{Po}(n\mu),\mathrm{Po}((n+m)\mu)\bigr)\\
&\qquad + d_{TV}\bigl(\mathrm{Po}((n+m)\mu),\mathrm{Bi}(n+m,\mu)\bigr)\\
&\le \mu + m\mu/(n\mu)^{1/2}+\mu\\
&= mn^{-1/2}\mu^{1/2}+2\mu \le 7\sqrt{2}\alpha_n\delta_n^{1/2}+4\delta_n \le 14\alpha_n\delta_n^{1/2}.
\end{aligned}$$

Choose a maximal coupling (N_n', N_{n+m}') of $N_n(S_{\delta_n})$ and $N_{n+m}(S_{\delta_n})$, and let $(\xi_i')_{i=1}^\infty$ be a sequence of independent random points, uniformly distributed in S_{δ_n}; assume also that (ξ_i') is independent of (N_n', N_{n+m}'). Let $L'(N)$ be the height of the partial order defined by $\{\xi_i' : i \le N\}$. Then $\bigl(L'(N_n'), L'(N_{n+m}')\bigr)$ is a coupling of $L_n(S_{\delta_n})$ and $L_{n+m}(S_{\delta_n})$, and thus

$$\begin{aligned}
d_{TV}\bigl(L_n(S_{\delta_n}),L_{n+m}(S_{\delta_n})\bigr) &\le \mathrm{P}\bigl(L'(N_n')\ne L'(N_{n+m}')\bigr)\\
&\le \mathrm{P}(N_n'\ne N_{n+m}') = d_{TV}\bigl(N_n(S_{\delta_n}),N_{n+m}(S_{\delta_n})\bigr)\\
&\le 14\alpha_n\delta_n^{1/2}.
\end{aligned}$$

Furthermore, using $L_{n+m}(S_{\delta_{n+m}}) \le L_{n+m}(S_{\delta_n}) \le L_{n+m}$, we see that

$$\begin{aligned}
&d_{TV}(L_n,L_{n+m})\\
&\quad\le \mathrm{P}(L_n\ne L_n(S_{\delta_n}))+\mathrm{P}(L_{n+m}\ne L_{n+m}(S_{\delta_n}))+d_{TV}\bigl(L_n(S_{\delta_n}),L_{n+m}(S_{\delta_n})\bigr)\\
&\quad\le \mathrm{P}(L_n\ne L_n(S_{\delta_n}))+\mathrm{P}(L_{n+m}\ne L_{n+m}(S_{\delta_{n+m}}))+14\alpha_n\delta_n^{1/2}\\
&\quad\to 0.
\end{aligned}$$

Hence a maximal coupling (L_n', L_{n+m}') of L_n and L_{n+m} satisfies $\mathrm{P}(L_n'\ne L_{n+m}')\to 0$.

We next define another coupling of L_n and L_{n+m}, now trying to push the variables apart. Observe that necessarily $n\delta_n\to\infty$, since otherwise, for some $C<\infty$ and arbitrarily large n,

$$\mathrm{E}\,L_n(S_{\delta_n}) \le \mathrm{E}\,N_n(S_{\delta_n}) = n|S_{\delta_n}| \le 2n\delta_n \le 2C,$$

which contradicts $L_n/\sqrt{n} \xrightarrow{\mathrm{p}} 2$ and $\mathrm{P}(L_n\ne L_n(S_{\delta_n}))\to 0$. Hence $m=O(\alpha_n n^{1/2}) = o(\delta_n^{-1/2}n^{1/2}) = o(n)$.

In particular, we may assume that $n>3m$. Set $Q_1=[0,\frac{m}{3n}]^2$ and $Q_2=(\frac{m}{3n},1]^2$. Then

$$L_{n+m} \ge L_{n+m}(Q_1)+L_{n+m}(Q_2). \tag{2.4}$$

Moreover, $N_{n+m}(Q_1)\sim \mathrm{Bi}(n+m,(\frac{m}{3n})^2)$ with an expectation of $(n+m)(\frac{m}{3n})^2 > \frac{m^2}{9n} \ge 4\alpha_n^2$; and it follows from Chebyshev's inequality, that

$$\mathrm{P}(N_{n+m}(Q_1)\ge 2\alpha_n^2)\to 1. \tag{2.5}$$

Since the distribution of $L_{n+m}(Q_1)$ conditional on $N_{n+m}(Q_1)=\nu$ equals the distribution of L_ν for any $\nu\ge 1$, we obtain from (1.1) that

$$\mathrm{P}(L_{n+m}(Q_1)>2\alpha_n)\to 1. \tag{2.6}$$

Similarly, $n+m-N_{n+m}(Q_2) \sim \mathrm{Bi}\big(n+m, 1-(1-\frac{m}{3n})^2\big)$ with expectation

$$(n+m)\Big(2\frac{m}{3n}-\frac{m^2}{9n^2}\Big) = \big(\tfrac{2}{3}+o(1)\big)m,$$

and thus

$$\mathrm{P}(N_{n+m}(Q_2) \ge n) = \mathrm{P}(n+m-N_{n+m}(Q_2) \le m) \to 1. \tag{2.7}$$

We define L''_n to be the height of the partial order defined by the first n of $\xi_1, \xi_2, \dots$ that fall in Q_2; obviously $L''_n \overset{\mathrm{d}}{=} L_n$, so (L''_n, L_{n+m}) is a coupling of L_n and L_{n+m}. Moreover, if $N_{n+m}(Q_2) \ge n$, then $L_{n+m}(Q_2) \ge L''_n$, and thus (2.4), (2.6), (2.7) yield

$$\mathrm{P}(L_{n+m} > L''_n + 2\alpha_n) \to 1. \tag{2.8}$$

Combining this coupling with a maximal coupling (L'_{n+m}, L'_n) of L_{n+m} and L_n such that $L'_{n+m} = L_{n+m}$, we obtain a coupling (L'_n, L''_n) of L_n with itself, i.e. two random variables L'_n and L''_n with $L'_n \overset{\mathrm{d}}{=} L''_n \overset{\mathrm{d}}{=} L_n$, such that

$$\mathrm{P}(L'_n > L''_n + 2\alpha_n) \ge \mathrm{P}(L_{n+m} > L''_n + 2\alpha_n) - \mathrm{P}(L_{n+m} \ne L'_n) \to 1.$$

Finally we observe that for any real x,

$$\mathrm{P}(L'_n > L''_n + 2\alpha_n) \le \mathrm{P}(L'_n > x + \alpha_n) + \mathrm{P}(L''_n < x - \alpha_n) = \mathrm{P}(|L_n - x| > \alpha_n)$$

and thus

$$\sup_x \mathrm{P}(|L_n - x| \le \alpha_n) \le 1 - \mathrm{P}(L'_n > L''_n + \alpha_n) \to 0. \qquad \square$$

Theorem 1 follows immediately from the lemmas.

3. Further remarks

Note that the proof of Theorem 1 uses the concentration results in [2], and that stronger concentration results would imply a stronger version of Theorem 1, i.e. less concentration than given above. This leads to the following result, which shows that, at least for some n, the distribution of H_n is not strictly concentrated (with, say, exponentially decreasing tails) with a variation of much less than $n^{-1/10}$. (For simplicity we consider here H_n; presumably the same result is true for L_n.)

Theorem 2. *If $\varepsilon > 0$ is sufficiently small, then there exist infinitely many n such that for some $m \le n$ we have*

$$\mathrm{P}(|H_m - \mathrm{E}\,H_m| > \varepsilon n^{1/10}) > n^{-2}.$$

Proof. Assume on the contrary, and somewhat more generally, that for some γ, $0 < \gamma < 1/2$, and all large n,

$$\mathrm{P}(|H_m - \mathrm{E}\,H_m| > n^{\gamma}) \le n^{-2}, \quad m \le n. \tag{3.1}$$

The argument in the proof of [2, Theorem 9] then yields

$$2n^{1/2} - \mathrm{E}\,H_n = O(n^{\gamma}) \tag{3.2}$$

and Lemma 1 holds for H_n, by the argument above, with

$$\delta_n = K n^{\gamma/2-1/4}, \tag{3.3}$$

provided K is large enough. Hence Lemma 2 (for H_n) shows that

$$\mathrm{P}(|H_n - \mathrm{E}\,H_n| \le \alpha_n) \to 0 \tag{3.4}$$

whenever $\alpha_n = o(\delta_n^{-1/2})$, i.e., when

$$\alpha_n n^{\gamma/4-1/8} \to 0. \tag{3.5}$$

If $\gamma < 1/10$, we may take $\alpha_n = n^\gamma$, which then satisfies (3.5), and obtain a contradiction from (3.1) and (3.4). In order to obtain the slightly stronger statement in the theorem, we let $\gamma = 1/10$ and note that if

$$\mathrm{P}(|H_n - \mathrm{E}\,H_n| > \varepsilon n^{1/10}) \le n^{-2} < 1/2 \tag{3.6}$$

for every $\varepsilon > 0$ and $n \ge n(\varepsilon)$, then there exists a sequence $\varepsilon_n \to 0$ such that

$$\mathrm{P}(|H_n - \mathrm{E}\,H_n| > \varepsilon_n n^{1/10}) < 1/2. \tag{3.7}$$

We now choose $\alpha_n = \varepsilon_n n^{1/10}$, which satisfies (3.5), and obtain a contradiction from (3.4) and (3.7). Hence either (3.1) or (3.6), for some $\varepsilon > 0$, fails for infinitely many n, which proves the result. $\square$

Finally, let us see what happens when we try to generalize the results to the random d-dimensional order defined by random points in $Q_d = [0,1]^d$. Lemma 1 holds, with

$$\delta_n = K n^{-1/4d} \log^{3/4} n \, (\log\log n)^{-1/2}, \tag{3.8}$$

by essentially the same proof; we now define $S_\delta = \{(x_i)^d : |x_i - x_j| \le \delta,\ i < j\}$, and note that $|S_\delta| \asymp \delta^{d-1}$. For Lemma 2, however, we need

$$\alpha_n = o\big(n^{1/d-1/2} \delta_n^{-(d-1)/2}\big), \tag{3.9}$$

in which case we may take $m = K n^{1-1/d} \alpha_n$ for some large K. However, (3.8) and (3.9) imply $\alpha_n = o\big(n^{(7-3d)/8d}\big) = o(1)$ for $d \ge 3$, so we do not obtain any result at all. (We also need $\alpha_n \ge 1$). The method of Theorem 2 yields no result either: we obtain $\delta_n = K n^{\gamma/2-1/2d}$ and by (3.9) we have

$$\alpha_n = o\big(n^{(3-d)/4d-\gamma(d-1)/4}\big), \tag{3.10}$$

which again contradicts $\alpha_n \ge 1$ for any $\gamma > 0$ when $d > 3$.

We can explain this failure in terms of the heuristics at the beginning of Section 2. We still have a relative variation of the number of points in the strip S_δ of order $n^{-(1-\alpha)/2}$, for some $\alpha > 0$, but this translates to a variation of the height of order only $n^{1/d-1/2+\alpha/2}$, which does not give any non-trivial result (α is rather small). Of course, this does not preclude the possibility that there is a substantial variation of the height due to the random position of points in the strip.

References

1. A. D. Barbour, L. Holst and S. Janson, *Poisson Approximation*, Oxford Univ. Press, Oxford, 1992.
2. B. Bollobás and G. Brightwell, The height of a random partial order: concentration of measure, *Ann. Appl. Probab.* **2** (1992), 1009–1018.
3. A. Frieze, On the length of the longest monotone subsequence in a random permutation, *Ann. Appl. Probab.* **1** (1991), 301–305.
4. J.M. Hammersley, A few seedlings of research, *Proc. 6th Berkeley Symp. Math. Stat. Prob.* (1972), Univ. of California Press, 345–394.
5. S. Janson, Random coverings in several dimensions, *Acta Math.* **156** (1986), 83–118.
6. O. Kallenberg, *Random Measures*, Akademie-Verlag, Berlin, 1983.
7. B.F. Logan and L.A. Shepp, A variational problem for Young tableaux, *Advances in Mathematics* **26** (1977), 206–222.
8. Michel Talagrand, Concentration of measure and isoperimetric inequalities in product spaces (to appear).
9. S.M. Ulam, Monte Carlo calculations in problems of mathematical physics, *Modern Mathematics for the Engineer* (1961), E.F. Beckenbach Ed., McGraw Hill, New York.
10. A.M. Veršik and S.V. Kerov, Asymptotics of the Plancherel measure of the symmetric group and the limiting form of Young tableaux, *Dokl. Akad. Nauk. SSSR* **233** (1977), 1024–1028. (Russian)

Béla Bollobás, Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, 16 Mill Lane, Cambridge CB2 1SB, England

E-mail address: `B.Bollobas@pmms.cam.ac.uk`

Svante Janson, Department of Mathematics, Uppsala University, PO Box 480, S-751 06 Uppsala, Sweden

E-mail address: `svante.janson@math.uu.se`