ON CERTAIN QUOTIENTS OF HARDY SPACES

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For any real number η, let Ω_η be the half–plane $\{z : \operatorname{Im} z > \eta\}$ in the complex plane and let, for $1 \le p \le \infty$, $H^p(\Omega_\eta)$ denote the usual Hardy space

$$H^p(\Omega_\eta) = \{F \in \mathcal{H}(\Omega_\eta) : ||F||_{H^p(\Omega_\eta)} = \sup_{y>\eta} ||F_y||_{L^p(\mathbf{R})} < \infty\}, \tag{1}$$

where $\mathcal{H}(\Omega)$ denotes the space of holomorphic functions in Ω and

$$F_y(x) = F(x+iy). \tag{2}$$

It is well-known that the spaces $H^p(\Omega_\eta)$ are Banach spaces and that $H^\infty(\Omega_\eta)$ is a commutative Banach algebra. The translation $S_\delta : f \mapsto f(z - i\delta)$ is an isometry $H^p(\Omega_\eta) \to H^p(\Omega_{\eta+\delta})$, while the identity (or rather restriction) yields a continuous (but not isometric) inclusion $H^p(\Omega_{\eta_1}) \subset H^p(\Omega_{\eta_2})$ whenever $\eta_1 < \eta_2$.

Let $\Pi^p = \bigcup_{-\infty<\eta<\infty} H^p(\Omega_\eta)$, thus Π^p is the linear space of all functions which are holomorphic in some Ω_η and belong to H^p there, with two functions identified if they coincide on some Ω_η.

The space $e^{iz}H^p(\Omega_\eta) = \{e^{iz}F : F \in H^p(\Omega_\eta)\}$ is a closed subspace of $H^p(\Omega_\eta)$ (and an ideal if $p = \infty$). If $\eta_1 < \eta_2$, then the inclusion map $H^p(\Omega_{\eta_1}) \to H^p(\Omega_{\eta_2})$ induces a map $j : H^p(\Omega_{\eta_1})/e^{iz}H^p(\Omega_{\eta_1}) \to H^p(\Omega_{\eta_2})/e^{iz}H^p(\Omega_{\eta_2})$.

It turns out that j is a bijection for all $\eta_1 < \eta_2$; hence j is an isomorphism

$$H^p(\Omega_{\eta_1})/e^{iz}H^p(\Omega_{\eta_1}) \cong H^p(\Omega_{\eta_2})/e^{iz}H^p(\Omega_{\eta_2}) \tag{3}$$

and it follows that the inclusion $H^p(\Omega_\eta) \to \Pi^p$ induces an (algebraic) isomorphism

$$H^p(\Omega_\eta)/e^{iz}H^p(\Omega_\eta) \cong \Pi^p/e^{iz}\Pi^p, \qquad -\infty < \eta < \infty. \tag{4}$$

This result was proved by Frankfurt and Rovnyak (1975, 1978) in their study of finite convolution operators. It is easily seen that $H^p(\Omega_{\eta_1}) \cap e^{iz}H(\Omega_{\eta_2}) = e^{iz}H^p(\Omega_{\eta_1})$, which implies that j is an injection, so the main part of the assertion is that j is surjective, or equivalently that every $F \in H^p(\Omega_{\eta_2})$ equals some $G \in H^p(\Omega_{\eta_1})$ modulo $e^{iz}H^p(\Omega_{\eta_2})$. The purpose of this note is to give a simple constructive proof of this assertion. For notational convenience we assume $\eta_1 = 0$, which implies the general case by translations.

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Theorem. *Let $\eta > 0$. Assume that $F \in H^p(\Omega_\eta)$ for some p, $1 \le p \le \infty$, and define*

$$T_\eta F(z) = \frac{1}{2\pi i}\int_{\operatorname{Im}\zeta=-\eta}\left(\frac{1}{\zeta} - \frac{1}{\zeta+2i\eta}\right)e^{i\zeta}F(z-\zeta)\,d\zeta, \qquad z \in \Omega_0. \tag{5}$$

Then $T_\eta F \in H^p(\Omega_0)$ and $T_\eta F - F \in e^{iz}H^p(\Omega_\eta)$. Moreover, T_η is a bounded linear map $H^p(\Omega_\eta) \to H^p(\Omega_0)$, and it induces a map $H^p(\Omega_\eta)/e^{iz}H^p(\Omega_\eta) \to H^p(\Omega_0)/e^{iz}H^p(\Omega_0)$ which is the inverse of the map $H^p(\Omega_0)/e^{iz}H^p(\Omega_0) \to H^p(\Omega_\eta)/e^{iz}H^p(\Omega_\eta)$ induced by the inclusion $H^p(\Omega_0) \to H^p(\Omega_\eta)$.

Proof. Let

$$K(\zeta) = \frac{1}{2\pi i}\Big(\frac{1}{\zeta} - \frac{1}{\zeta+2i\eta}\Big)e^{i\zeta} = \frac{\eta}{\pi\zeta(\zeta+2i\eta)}e^{i\zeta}.$$

Then

$$\begin{aligned}||K_\eta||_{L^1} &= \int_{-\infty}^{\infty}|K(x-i\eta)|\,dx = \int_{-\infty}^{\infty}\frac{\eta}{\pi|x-i\eta||x+i\eta|}e^{\eta}\,dx \\ &= \frac{e^\eta}{\pi}\int_{-\infty}^{\infty}\frac{\eta}{x^2+\eta^2}\,dx = e^\eta < \infty.\end{aligned}$$

Since $H^p(\Omega_\eta) \subset H^\infty(\Omega_{\eta'})$ for every $\eta' > \eta$, it follows that the integral in (5) converges for every $z \in \Omega_0$ and $T_\eta F(z)$ is easily seen to be holomorphic. Moreover, $(T_\eta F)_y = K_{-\eta} * F_{y+\eta}$ and thus by Young's inequality,

$$||(T_n F)_y||_{L^p} \le ||K_{-\eta}||_{L^1}||F_{y+\eta}||_{L^p} \le e^\eta ||F||_{H^p(\Omega_\eta)}, \qquad y > 0.$$

Thus $T_\eta F \in H^p(\Omega_0)$.

The kernel $K(\zeta)$ has simple poles at $\zeta = 0$ and $\zeta = -2i\eta$ and the residue at 0 is $1/2\pi i$. Hence, if $\delta > 0$ and $y = \operatorname{Im} z > \eta + \delta$, Cauchy's integral formula (with simple estimates for large $|\zeta|$) yields

$$T_\eta F(z) = \int_{\operatorname{Im}\zeta=-\eta}K(\zeta)F(z-\zeta)\,d\zeta = \int_{\operatorname{Im}\zeta=\delta}K(\zeta)F(z-\zeta)\,d\zeta + F(z).$$

Assume for simplicity $y > 2\eta$ and let $\delta = y - 2\eta$. Then

$$\begin{aligned}T_\eta F(z) - F(z) &= \int_{\operatorname{Im}\zeta=y-2\eta}K(\zeta)F(z-\zeta)\,d\zeta = -\int_{\operatorname{Im}\zeta=2\eta}K(z-\zeta)F(\zeta)\,d\zeta \\ &= e^{iz}G(z),\end{aligned}$$

with

$$G(z) = -\int_{\operatorname{Im}\zeta=2\eta}e^{-iz}K(z-\zeta)F(\zeta)\,d\zeta = -\int_{\operatorname{Im}\zeta=2\eta}\frac{\eta}{\pi(z-\zeta)(z-\zeta+2i\eta)}e^{-i\zeta}F(\zeta)\,d\zeta.$$

Young's inequality yields

$$\begin{aligned}||G_y||_{L^p} &\le \int_{-\infty}^{\infty}\frac{\eta}{\pi|t+i(y-2\eta)||t+iy|}\,dt\,||e^{2\eta}F_{2\eta}||_{L^p} \\ &\le \int\frac{\eta}{\pi(t^2+(y-2\eta)^2)}\,dt\,e^{2\eta}||F_{2\eta}||_{L^p} = \frac{\eta}{y-2\eta}e^{2\eta}||F_{2\eta}||_{L^p}, \qquad y > 2\eta.\end{aligned}$$

Hence $G \in H^p(\Omega_{3\eta})$ and

$$T_\eta F(z) - F(z) = e^{iz} G(z) \in H^p(\Omega_\eta) \cap e^{iz} H^p(\Omega_{3\eta}) = e^{iz} H^p(\Omega_\eta).$$

The remaining assertions follow easily. $\square$

The theorem can also be derived by Fourier analysis (this is in fact how the formula (5) was found). We sketch the argument which is based on well-known extensions of the Paley–Wiener theorem to $p \neq 2$. For $p > 2$, the Fourier transforms are interpreted as distributions on $\mathbf{R}$. Cf. for example Garnett (1981), Chapter II (in particular Exercise 2) for some background.

If $F \in \mathcal{H}^p(\Omega_\eta)$, then $\widehat{F}_y(\xi) = 0$ for $\xi < 0$, $y > \eta$ and $\widehat{F}_{y_2}(\xi) = e^{-(y_2-y_1)\xi} \widehat{F}_{y_1}(\xi)$ for $y_2 > y_1 > \eta$. Hence $\widehat{F}(\xi) = e^{y\xi} \widehat{F}_y(\xi)$, $y > \eta$, is well-defined. The mapping $F \mapsto \widehat{F}$ is an isomorphism of $H^p(\Omega_0)$ onto $\widehat{H^p} = \{f \in \widehat{L^p}(\mathbf{R}) : \operatorname{supp} f \subset [0, \infty)\}$ and, more generally, of $H^p(\Omega_\eta)$ onto $e^{\eta\xi}\widehat{H^p} = \{e^{\eta\xi} f(\xi) : f \in \widehat{H^p}\}$, $-\infty < \eta < \infty$.

If $F \in H^p(\Omega_\eta)$, then $\widehat{e^{iz} F}(\xi) = \widehat{F}(\xi - 1)$, and thus $F \mapsto \widehat{F}$ maps $e^{iz} H^p(\Omega_\eta)$ onto $\{e^{\eta\xi} f(\xi - 1) : f \in \widehat{H^p}\} = \{e^{\eta\xi} f(\xi) : f \in \widehat{L^p}, \operatorname{supp} f \subset [1, \infty)\}$. Hence $e^{iz} H^p(\Omega_\eta) = \{f \in H^p(\Omega_\eta) : \operatorname{supp} \widehat{f} \subset [1, \infty)\}$.

If $\eta > 0$ and $F \in H^p(\Omega_\eta)$, then $\widehat{F}(\xi) = e^{\eta\xi} \widehat{g}(\xi)$ for some $\widehat{g} \in \widehat{H^p}$. In general, $\widehat{F}$ is, of course, not an element of $\widehat{H^p}$, but if we modify $\widehat{F}$ on $[1, \infty)$, for example to $\widehat{G}(\xi) = e^{\eta \min(\xi, 2-\xi)} \widehat{g}(\xi) = e^{\eta - \eta|\xi - 1|} \widehat{g}(\xi)$, then $\operatorname{supp}(\widehat{F} - \widehat{G}) \subset [1, \infty)$ and $\widehat{G} \in \widehat{H^p}$ because $G = k * g$ where $k(x) = e^\eta e^{ix} P_\eta(x) \in L^1$ (P_η is the Poisson kernel) since $\widehat{k}(\xi) = e^\eta \widehat{P_\eta}(\xi - 1) = e^{\eta - \eta|\xi - 1|}$. Thus $G \in H^p(\Omega_0)$ and $F - G \in e^{iz} H^p(\Omega_\eta)$ as required.

In fact, it is easily seen that $k(x) = K(x - i\eta)$ and $G = T_\eta F$ as given in (5). We also obtain $\widehat{G}(\xi) = e^{\eta - \eta|\xi-1|} \widehat{g}(\xi) = e^{\eta - \eta|\xi - 1| - \eta\xi} \widehat{F}(\xi)$, or

$$\widehat{T_\eta F}(\xi) = m_\eta(\xi) \widehat{F}(\xi), \tag{6}$$

with

$$m_\eta(\xi) = \begin{cases} 1 & \xi \leq 1 \\ e^{-2\eta(\xi - 1)} & \xi \geq 1. \end{cases} \tag{7}$$

Remark 1. Since $m_{\eta_1}(\xi) m_{\eta_2}(\xi) = m_{\eta_1 + \eta_2}(\xi)$ we obtain the semigroup property $T_{\eta_1} T_{\eta_2} = T_{\eta_1 + \eta_2}$.

Remark 2. The proof above shows that $||T_\eta||_{H^p(\Omega_\eta), H^p(\Omega_0)} \leq e^\eta$. In fact, considering F such that $\widehat{F} \in C_0^\infty$ (say) with $\operatorname{supp} \widehat{F} \subset [1 - \varepsilon, 1 - \varepsilon + \delta]$, where $0 < \delta < \varepsilon < 1$, it is easily seen that

$$||T_\eta||_{H^p(\Omega_\eta), H^p(\Omega_0)} = e^\eta, \qquad 1 \leq p \leq \infty. \tag{8}$$

This can easily be extended to show that any (not necessarily linear) operator $T : H^p(\Omega_\eta) \to H^p(\Omega_0)$ such that $TF - F \in e^{iz} H^p(\Omega_\eta)$ for every F, has norm at least e^η. Hence T_η has minimal norm among all such operators.

Remark 3. If $F \in \Pi^p$, then $T_\eta F$ is defined for every η that is large enough. These functions are in general different elements of $H^p = H^p(\Omega_0)$, although they all represent the same element of $H^p / e^{iz} H^p$. In order to study what happens when $\eta \to \infty$, let us for simplicity assume $F \in H^p$. We identify a function $F \in H^p = H^p(\Omega_0)$ with its boundary values in $L^p = L^p(\mathbf{R})$.

If $1 < p < \infty$, then it is an immediate consequence of Riesz' theorem on boundedness of harmonic conjugation that the operators I_a defined for $-\infty < a < \infty$ by $(I_a f)^\wedge = \chi_{(-\infty,a)}\widehat{f}$ are bounded operators on L^p, with norm independent of a. Moreover, $a \mapsto I_a f$ is a continuous map into L^p for every $f \in L^p$. It follows from (6) and (7) that

$$T_\eta F = \int_1^\infty 2\eta e^{-2\eta(a-1)} I_a F \, da.$$

Hence $||T_\eta||_{H^p} \le ||I_0||_{L^p} = C_p$, uniformly in η, and $T_\eta F \to I_1 F$ in H^p as $\eta \to \infty$ for every $F \in H^p$ (and thus for every $F \in \Pi^p$).

On the other hand, if $p = 1$ or $p = \infty$, then I_a is not bounded on L^p and I_1 is not bounded on H^p. It follows easily that there exists $F \in H^p$ such that $||T_\eta F||_{H^p}$ is unbounded as $\eta \to \infty$, and $||T_\eta||_{H^p} \to \infty$ as $\eta \to \infty$.

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