LARGE DEVIATION INEQUALITIES FOR SUMS OF INDICATOR VARIABLES

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This paper was written in 1994, but was never published because I had overlooked some existing papers containing some of the inequalities. Because of some recent interest in one of the inequalities, which does not seem to be published anywhere else, it has now been lightly edited and made available here. 2 September, 2016

Abstract. A survey is given of some Chernoff type bounds for the tail probabilities $\mathrm{P}(X - EX \geq a)$ and $\mathrm{P}(X - EX \leq a)$ when X is a random variable that can be written as a sum of indicator variables that are either independent or negatively related. Most bounds are previously known and some comparisons are made.

1. Introduction and conclusions

The purpose of this paper is to give a survey of some simple upper bounds for the probabilities $\mathrm{P}(X - EX \leq -a)$ and $\mathrm{P}(X - EX \geq a)$, where X is a random variable that can be written as a sum $I_1 + \dots + I_n$ of 0–1 (indicator) random variables. We consider both independent and dependent variables I_i (with strong restrictions in the dependent case). Many of the inequalities extend to sums of more general bounded variables, but we consider for simplicty only the indicator case.

Most of the bounds are known, see in particular Bennett (1962) and Hoeffding (1963), but are included for comparison and (partial) completeness. A few versions seem to be new. Many of the inequalities appear in various places, for example Janson, Łuczak and Ruciński (2000), Chapter 2. See also the book Boucheron, Lugosi and Massart (2013) which presents several of these bounds and many extensions to other situation.

Independent identically distributed summands. The simplest case is when the indicator variables I_i are independent and identically distributed, $I_i \sim \mathrm{Be}(p)$, with $0 < p < 1$ (avoiding trivial cases); then X has the binomial distribution $\mathrm{Bi}(n, p)$. This case has been studied by many authors, giving bounds or asymptotic results (sometimes in greater generality); see for example Khintchine (1929), Cramér (1938), Feller (1943), Chernoff (1952), Bahadur and Rao (1960), Bennett (1962), Hoeffding (1963), Littlewood (1969), and the further references given in these papers.

We are here interested in explicit bounds for finite n rather than asymptotic results. One simple but powerful such bound was given by Chernoff (1952); since for every $t \geq 0$,

$$\mathrm{P}(X \geq \mathrm{E}\, X + a) \leq e^{-t(\mathrm{E}\, X + a)}\, \mathrm{E}\, e^{tX} \tag{1.1}$$

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and $Ee^{tX} = (1+p(e^t-1))^n$, we obtain by simple calculus, letting $\lambda = \operatorname{E} X$ and assuming $0 \le a \le n-\lambda$,

$$\begin{aligned}\operatorname{P}(X \ge \operatorname{E} X + a) &\le \inf_{t\ge 0} \exp\bigl(-at - npt + n\ln(1+p(e^t-1))\bigr)\\ &= \exp\Bigl(-(\lambda+a)\ln\frac{\lambda+a}{\lambda} - (n-\lambda-a)\ln\frac{n-\lambda-a}{n-\lambda}\Bigr).\end{aligned} \tag{1.2}$$

Similarly, for $0 \le a \le \lambda$,

$$\begin{aligned}\operatorname{P}(X \le \operatorname{E} X - a) &\le \inf_{t\ge 0} \exp\bigl(-at + npt + n\ln(1+p(e^{-t}-1))\bigr)\\ &= \exp\Bigl(-(\lambda-a)\ln\frac{\lambda-a}{\lambda} - (n-\lambda+a)\ln\frac{n-\lambda+a}{n-\lambda}\Bigr).\end{aligned} \tag{1.3}$$

Chernoff (1952) proved also that the estimates (1.2) and (1.3) are asymptotically sharp in the sense that if $n \to \infty$ with a/n and p fixed, then equality holds within factors $1 + o(1)$ in the exponent.

Simpler but (slightly) cruder bounds are easily obtained by finding suitable upper bounds for the right hand sides of (1.2) and (1.3), see for example Alon and Spencer (1992), Appendix A. We state here some, more or less well-known, such bounds. Proofs are given in Section 2.

Theorem 1. *Suppose that $X \sim \operatorname{Bi}(n,p)$ and let $q = 1-p$. Then, for every $a \ge 0$,*

$$\begin{aligned}\operatorname{P}(X \ge \operatorname{E} X + a) &\le \exp\Bigl(-\frac{a^2}{2(npq + a(q-p)/3)}\Bigr) \le \exp\Bigl(-\frac{a^2}{2(npq+a/3)}\Bigr)\\ &\le \exp\Bigl(-\frac{a^2}{2npq}\Bigl(1-\frac{a}{3npq}\Bigr)\Bigr),\end{aligned} \tag{1.4}$$

$$\operatorname{P}(X \ge \operatorname{E} X + a) \le \exp\Bigl(-np\Bigl(\Bigl(1+\frac{a}{np}\Bigr)\ln\Bigl(1+\frac{a}{np}\Bigr) - \frac{a}{np}\Bigr)\Bigr), \tag{1.5}$$

$$\operatorname{P}(X \ge \operatorname{E} X + a) \le \exp\Bigl(-\frac{a^2}{2np(1+a/3np)}\Bigr) \le \exp\Bigl(-\frac{a^2}{2np}\Bigl(1-\frac{a}{3np}\Bigr)\Bigr), \tag{1.6}$$

$$\begin{aligned}\operatorname{P}(X \le \operatorname{E} X - a) &\le \exp\Bigl(-\frac{a^2}{2(npq - a(q-p)/3)}\Bigr) \le \exp\Bigl(-\frac{a^2}{2(npq+a/3)}\Bigr)\\ &\le \exp\Bigl(-\frac{a^2}{2npq}\Bigl(1-\frac{a}{3npq}\Bigr)\Bigr),\end{aligned} \tag{1.7}$$

$$\operatorname{P}(X \le \operatorname{E} X - a) \le \exp\Bigl(-np\Bigl(\Bigl(1-\frac{a}{np}\Bigr)\ln\Bigl(1-\frac{a}{np}\Bigr) + \frac{a}{np}\Bigr)\Bigr), \tag{1.8}$$

$$\operatorname{P}(X \le \operatorname{E} X - a) \le \exp\Bigl(-\frac{a^2}{2np}\Bigr). \tag{1.9}$$

Moreover, if $0 \le p \le 1/2$, then

$$\operatorname{P}(X \le \operatorname{E} X - a) \le \exp\Bigl(-\frac{a^2}{2npq}\Bigr). \tag{1.10}$$

Remark 1. The estimates (1.4) and (1.7) (as well as (1.2) and (1.3)) are obvious "mirror images" of each other, and are equivalent by the substitution $X \to n - X$. On the other hand, the remaining estimates in Theorem 1 are asymmetric, and are useful mainly when p is small.

Remark 2. Note that (1.5), (1.6), (1.8) and (1.9) use n and p only in the combination $np = \lambda = \operatorname{E} X$.

Remark 3. It may also be observed that if we replace np by λ, then (1.5), (1.6), (1.8) and (1.9) are (e.g. by continuity) valid also when $X \sim \operatorname{Po}(\lambda)$; in fact, (1.5) and (1.8) then become the Chernoff bounds for the Poisson distribution. These Chernoff bounds too are asymptotically sharp in the sense, considering for example (1.5), that if $c > 0$ is fixed and $c_1 > (1+c)\ln(1+c) - c$, then $\operatorname{P}(X \geq \operatorname{E} X + c\lambda) > \exp(-c_1\lambda)$ for $X \sim \operatorname{Po}(\lambda)$ with λ large, and thus also for some $X \sim \operatorname{Bi}(n,p)$ with np large and p small. In particular this implies that the simple bound (1.9) is *not* valid for $\operatorname{P}(X \geq \operatorname{E} X + a)$. This implies further, by considering $n - X$, that (1.10) cannot hold without some restriction on p.

Independent summands with different distributions. The Chernoff bounds given above for the binomial distribution are easily extended to the case when the 0–1 variables I_i are independent with different distributions, $I_i \sim \operatorname{Be}(p_i)$. In fact, as is well-known (see for example Alon and Spencer (1992), Appendix A), if $\lambda = \operatorname{E} X = \sum_1^n p_i$, $p = \lambda/n$ (the average of $p_1, \dots, p_n$), and we let $X_0 \sim \operatorname{Bi}(n,p)$ be a binomially distributed random variable with the same n and expectation as X, then by Jensen's inequality for the convex function $x \mapsto -\ln\bigl(1 + x(e^t - 1)\bigr)$,

$$\operatorname{E} e^{tX} = \prod_1^n \bigl(1 + p_i(e^t - 1)\bigr) \leq \bigl(1 + p(e^t - 1)\bigr)^n = \operatorname{E} e^{tX_0}, \qquad -\infty < t < \infty. \tag{1.11}$$

Consequently,

$$\operatorname{P}(X \geq \operatorname{E} X + a) \leq e^{-ta - t\operatorname{E} X} \operatorname{E} e^{tX} \leq e^{-ta - t\operatorname{E} X_0} \operatorname{E} e^{tX_0}, \tag{1.12}$$

and thus every Chernoff type bound for the binomial variable X_0 derived from (1.1), applies also to X.

Theorem 2. *The bounds* (1.2)–(1.10) *hold also when* $X = \sum_1^n I_i$ *where* $I_i \sim \operatorname{Be}(p_i)$ *are independent indicator variables and* $p = \operatorname{E} X/n$, $q = 1 - p$. □

Remark 4. We do not claim that the actual tail probability $\operatorname{P}(X_0 \geq \operatorname{E} X + a)$ is larger than $\operatorname{P}(X \geq \operatorname{E} X + a)$, and indeed this is in general false as is shown by the example $n = 2$, $p_1 = 1/5$, $p_2 = 3/5$, where $\operatorname{P}(X \geq 1) = 17/25$ while $X_0 \sim \operatorname{Bi}(2, 2/5)$ and $\operatorname{P}(X_0 \geq 1) = 16/25$.

As mentioned above, the bounds (1.2) and (1.3) are asymptotically sharp for the binomial distribution, but that is no longer generally true when the 0–1 variables have different distributions. In fact, a Taylor expansion shows that the exponent in (1.2) or (1.3) is $-\frac{a^2}{2np(1-p)}\bigl(1 + o(1)\bigr)$ provided $a = o\bigl(np(1-p)\bigr)$, cf. (1.4) and (1.7). In the binomial case, this equals $-\frac{a^2}{2\sigma^2}\bigl(1 + o(1)\bigr)$, with $\sigma^2 = \operatorname{Var} X$, which is what one would expect from normal approximation heuristics; in general, however, $\sigma^2 = \operatorname{Var} X$ may be much smaller than $np(1-p)$, and it would be advantageous to have better bounds with exponents $-\frac{a^2}{2\sigma^2}\bigl(1 + o(1)\bigr)$ for moderately large a. This is achieved by *Bennett's inequality*, see Bennett (1962) and Hoeffding (1963), which we state as (1.13) in the next theorem; the simple consequence (1.14) is known as *Bernstein's inequality*, see Boucheron, Lugosi and Massart (2013). Note that these inequalitites give bounds depending on a and σ^2 only, with an exponent of the expected order for $a = o(\sigma^2)$. We give a proof in Section 3 using (1.1) as above, but doing a more careful estimation of $\operatorname{E} e^{tX}$ than (1.11).

Theorem 3. *Let X be a random variable and suppose that there exist independent* 0–1 *variables $I_i \sim \mathrm{Be}(p_i)$, $i = 1, \dots, n$, such that $X \overset{\mathrm{d}}{=} \sum_1^n I_i$. Let $\lambda = \mathrm{E}\, X$ and $\sigma^2 = \mathrm{Var}\, X$. Then*

(i) *If $a \geq 0$, then*

$$\mathrm{P}(X \geq \mathrm{E}\, X + a) \leq \exp\Big(-\sigma^2\Big(\Big(1 + \frac{a}{\sigma^2}\Big)\ln\Big(1 + \frac{a}{\sigma^2}\Big) - \frac{a}{\sigma^2}\Big)\Big) \tag{1.13}$$

(ii) *If $a \geq 0$, then*

$$\mathrm{P}(X \geq \mathrm{E}\, X + a) \leq \exp\Big(-\frac{a^2}{2\sigma^2}\Big/\Big(1 + \frac{a}{3\sigma^2}\Big)\Big) \leq \exp\Big(-\frac{a^2}{2\sigma^2}\Big(1 - \frac{a}{3\sigma^2}\Big)\Big). \tag{1.14}$$

(iii) *If $a \geq c\sigma^2$, with $c > 0$, then*

$$\mathrm{P}(X \geq \mathrm{E}\, X + a) \leq \exp\big(-((1 + c^{-1})\ln(1 + c) - 1)a\big). \tag{1.15}$$

The same estimates hold for $\mathrm{P}(X \leq \mathrm{E}\, X - a)$, and thus $\mathrm{P}(|X - \mathrm{E}\, X| \geq a)$ may be estimated by twice the right hand sides in (1.13)–(1.15).

(iv) *Moreover, if $a \geq 0$ and $\sigma^2 \geq \lambda/2$, then*

$$\mathrm{P}(X \leq \mathrm{E}\, X - a) \leq \exp\Big(-\frac{a^2}{2\sigma^2}\Big). \tag{1.16}$$

Remark 5. The estimates (1.13) and (1.14) are very similar to (1.5) and (1.6); the only difference is that $np = \lambda$ is replaced by $\sigma^2 < \lambda$. It is easily seen that this always improves the bound in (1.5) and the first bound in (1.6). On the other hand, the bounds for $\mathrm{P}(X \leq \mathrm{E}\, X - a)$ (except (1.16)) are somewhat different from the corresponding bounds in Theorem 1, because of the symmetry of the bounds in Theorem 3.

Remark 6. It is easily seen (by approximating a Poisson distribution) that the constant in (iii) is best possible. In particular, it follows that (1.16) cannot hold without restriction.

Much more precise estimates of the tail probabilities for sums of independent, but not necessarily identically distributed, random variables were obtained by Feller (1943) using different methods (conjugated distributions as in Cramér (1938) together with a Berry–Esseen estimate), and it is interesting to compare our result with Feller's. Feller's result (for our case, using $\lambda_n = 1/\sigma$ in Feller (1943)) is, for $0 < a < \sigma^2/12$,

$$\mathrm{P}(X \geq \mathrm{E}\, X + a) = e^{-\frac{x^2}{2}Q(x)}\Big(1 - \Phi(x) + \frac{\theta(x)}{\sigma}e^{-x^2/2}\Big), \tag{1.17}$$

where $x = a/\sigma$, $|\theta(x)| < 9$, Φ is the normal distribution function and

$$Q(x) = \sum_{\nu=1}^{\infty} q_\nu x^\nu, \tag{1.18}$$

where q_ν depends on the first $\nu + 2$ moments of X and

$$|q_\nu| < \frac{1}{7}\Big(\frac{12}{\sigma}\Big)^\nu. \tag{1.19}$$

If, say, $\sigma \le a \le \sigma^2/24$, this yields, using $1-\Phi(x) \le (2\pi)^{-1/2}x^{-1}e^{-x^2/2}$ for $x>0$,

$$\begin{aligned}\mathrm{P}(X \ge \mathrm{E}\,X + a) &< \exp\Bigl(-\frac{x^2}{2}\bigl(1+Q(x)\bigr)\Bigr) = \exp\Bigl(-\frac{a^2}{2\sigma^2}\Bigl(1+Q\bigl(\frac{a}{\sigma}\bigr)\Bigr)\Bigr)\\ &< \exp\Bigl(-\frac{a^2}{2\sigma^2}\Bigl(1-\frac{24}{7}\frac{a}{\sigma^2}\Bigr)\Bigr). \qquad (1.20)\end{aligned}$$

For $\mathrm{P}(X \le \mathrm{E}\,X - a)$ we have the same estimates if we replace $Q(x)$ by $Q(-x)$ (and $\theta(x)$ by some $\theta'(x)$); this follows by considering $n-X$.

The bound (1.20) is similar to the ones given in Theorem 3, in particular (1.14). It is somewhat inferior to (1.14) since the constant in the second order term in the exponent is worse, and the range of a is restricted, but for applications they are essentially equivalent.

Note also that Feller's result has other advantages. First, (1.17) is an equality (although the exact value of $\theta(x)$ is unspecified), and it leads also to a lower bound similar to (1.20) and to asymptotic results. In particular, simple asymptotic results follow when $a/\sigma^2 \to 0$ and thus $Q(x) \to 0$. Secondly, Feller (1943) describes how the coefficients q_ν may be explicitly expressed in terms of the semi-invariants $\varkappa_j$ of X (and thus in terms of the moments); for example (the sign seems to be wrong in Feller (1943), (2.18)–(2.19)),

$$q_1 = -\frac{\varkappa_3}{3\sigma^3}, \qquad (1.21)$$

$$q_2 = -\frac{\varkappa_4}{12\sigma^4} + \frac{1}{4\sigma^6}\varkappa_3^2. \qquad (1.22)$$

(Thus, $q_1 = -\frac{1}{3}\gamma_1$ and $q_2 = -\frac{1}{12}\gamma_2 + \frac{1}{4}\gamma_1^2$, where γ_1 and γ_2 are the skewness and excess of X, respectively.) For example, using (1.21) for q_1 and (1.19) for q_ν, $\nu \ge 2$, we obtain for $\sigma \le a \le \sigma^2/24$, instead of (1.20),

$$\mathrm{P}(X \ge \mathrm{E}\,X + a) \le \exp\Bigl(-\frac{a^2}{2\sigma^2}\Bigl(1-\frac{288}{7}\frac{a^2}{\sigma^4}-\frac{a\varkappa_3}{3\sigma^4}\Bigr)\Bigr), \qquad (1.23)$$

which yields an improvement in cases when $\varkappa_3 = \mathrm{E}(X-\mathrm{E}\,X)^3$ is known and either negative or not to large positive.

Dependent summands.

Let us now consider the case of dependent 0–1 variables I_i. Of course any bounded non-negative integer valued random variable X can be written as a sum of dependent 0–1 variables, so nothing can be said in general. We will here consider only 0–1 variables that are *negatively related* in the following sense, cf. Barbour, Holst and Janson (1992).

(Note that large deviation bounds for a class of sums of *positively* related indicators are given in Janson (1990) and Barbour, Holst and Janson (1992), Theorem 2.S. In this case only the lower tail probabilities $\mathrm{P}(X \le \mathrm{E}\,X - a)$ have nice upper bounds.)

Definition. The indicator random variables $(I_i)_{i=1}^n$ (defined on the same probability space) are negatively related if for each $j \le n$ there exist further random variables $(J_{ij})_{i=1}^n$, defined on the same probability space (or an extension of it), such that the distribution of the random vector $(J_{ij})_{i=1}^n$ equals the conditional distribution of $(I_i)_{i=1}^n$ given $I_j = 1$, and, moreover, for every i with $i \ne j$, $J_{ij}^i \le I_i$.

Example 1. (Hypergeometric distribution.) Let m, n and N be given positive integers with $\max(m,n) \le N$. Given N urns, labelled $1,\dots,N$, and m balls, put the balls at random into m different urns (drawing without replacement), and let X be the total number of balls in urns $1,\dots,n$. Clearly $X = \sum_1^n I_i$, where I_i equals 1 if urn i contains a ball. In this case it is easy to show that the indicators I_i are negatively related by explicitly construction J_i, as follows. After randomly distributing the balls as above, and recording I_i, we ensure that there is a ball in urn j by "cheating": if urn j is empty we select one of the balls at random and move it to urn j. Let $J_{ij} = 1$ if urn i now contains a ball. It is clear that (J_{ij}) has the right distribution, and that $J_{ij} \le I_i$ for $i \ne j$.

Example 2. Distribute m balls into n urns, but this time put the balls one by one at random, independently of the other choices of urn (drawing with replacement). Let X be the number of empty urns. Clearly $X = \sum_1^n I_i$, where $I_i = 1$ if urn i is empty. These indicators are negativlely related; this follow by a construction very similar to the one in Example 1, removing all balls (if any) in urn i and redistributing them (repeating if necessary).

Further examples of negatively related variables are given in Barbour, Holst and Janson (1992), where also some general results are established. In particular, it is proven (a special case of Corollary 2.D.1) that the variables I_i, $i = 1, \dots, n$, are negatively related if and only if I_j and $\phi(I_1, \dots, I_{j-1}, I_{j+1}, \dots, I_n)$ are negatively correlated for every j and every indicator function ϕ that is increasing in each variable. (Pairwise negative correlation of the I_i is not enough.) It follows immediately that the variables $(1 - I_i)_{i=1}^n$ are negatively related if $(I_i)_{i=1}^n$ are. It follows also that variables are negatively related if they are negatively associated in the sense of Joag-Dev and Proschan (1983).

Theorem 4. *Suppose that $X \overset{\mathrm{d}}{=} \sum_1^n I_i$, where $I_i \sim \mathrm{Be}(p_i)$ are negatively related indicator variables. Let $\widetilde{I}_i$, $i = 1, \dots, n$, be independent indicator variables with $\widetilde{I}_i \sim \mathrm{Be}(p_i)$, and put $\widetilde{X} = \sum_1^n \widetilde{I}_i$. Then, for every real t,*

$$\mathrm{E}\, e^{tX} \le \mathrm{E}\, e^{t\widetilde{X}}.$$

Consequently, any Chernoff type bound for $\widetilde{X}$ applies also to X. In particular, (1.2)–(1.10) *hold with $p = \mathrm{E}\, X/n$ and $q = 1 - p$; for example, with $\lambda = \mathrm{E}\, X$,*

$$\begin{aligned} \mathrm{P}(X \ge \mathrm{E}\, X + a) &\le \exp\Big(-\frac{a^2}{2\lambda(1 + a/3\lambda)}\Big) \le \exp\Big(-\frac{a^2}{2\lambda}\Big(1 - \frac{a}{3\lambda}\Big)\Big), \\ \mathrm{P}(X \le \mathrm{E}\, X - a) &\le \exp\Big(-\frac{a^2}{2\lambda}\Big). \end{aligned}$$

Of course, also the bounds in Theorem 3 (applied to $\widetilde{X}$) apply to X. The problem is that we have to use $\sigma^2 = \mathrm{Var}(\widetilde{X})$ instead of $\mathrm{Var}(X)$, which may be much smaller. In fact, the bounds in Theorem 3 are in general false with $\sigma^2 = \mathrm{Var}\, X$ in the dependent case; the following theorem implies that it is impossible to have a general bound that is, say, $\exp(-a^2/3\sigma^2)$ when $a = 4\sigma$.

Theorem 5. *Let $\alpha > 0$, $0 < c < 1/e$ and $A < \infty$. There exists a random variable X which is a finite sum of negatively related indicators such that $\sigma^2 = \mathrm{Var}\, X > A$ and, with $a = \alpha\sigma$,*

$$\mathrm{P}(X > \mathrm{E}\, X + a) > ce^{-a/\sigma}. \tag{1.24}$$

Nevertheless, there are cases where it is possible to do better. A striking example is based on the result by Vatutin and Mikhailov (1982) that certain random variables that occur in some occupancy problems, and have natural representations as sums of negatively related indicators (with the same expectation), also can be represented as sums of *independent* indicators with different expectations. (The proof is algebraic, and based on showing that the probability generating function has only real roots; there is no (known) probabilistic interpretation of these indicators, which in general have irrational expectations.) Their result includes the variables in Examples 1 and 2 (using in their notation $s_1 = N - n$, $s_2 = N - m$ for Example 1 and $s_1 = \cdots = s_m = 1$ for Example 2).

Consequently, the variables in Examples 1 and 2 actually satisfy the hypothesis of Theorem 3 (although we do not know the p_i explicitly). Hence we can apply Theorem 3; note that the bounds in Theorem 3 involve only σ^2 and possibly λ, and not the unknown p_i. (In fact, this application was one of the motivations for finding bounds of the form given in Theorem 3.)

Theorem 6. *Let X be either hypergeometric as in Example 1, or as in Example 2. Then the conclusions of Theorem 3 hold, with $\lambda = \operatorname{E} X$ and $\sigma^2 = \operatorname{Var} X$.* □

Example 2, cont. For the occupancy problem described above,

$$\lambda = \operatorname{E} X = n\Bigl(1 - \frac{1}{n}\Bigr)^m,$$
$$\sigma^2 = \operatorname{Var} X = n\Bigl(1 - \frac{1}{n}\Bigr)^m + n(n-1)\Bigl(1 - \frac{2}{n}\Bigr)^m - n^2\Bigl(1 - \frac{1}{n}\Bigr)^{2m};$$

estimates of the tail probabilities are obtained by using these values in any of the formulas (1.2)–(1.16), letting $p = \operatorname{E} X/n$ and $q = 1 - n$.

For asymptotical results in the case $m/n \to r > 0$, we easily find

$$\lambda = \operatorname{E} X \sim ne^{-r},$$
$$\sigma^2 = \operatorname{Var} X \sim n\bigl(e^{-r} - (1+r)e^{-2r}\bigr) = ne^{-2r}(e^r - 1 - r).$$

The asymptotics for the tail probabilites in this case have been studied in detail by Kamath, Motwani, Palem and Spirakis (1994).

Remark 7. A comparison of Theorems 3 and 5 shows that not every random variable that is a sum of negatively related indicators can be represented as a sum of independent indicators; the Vatutin–Mikhailov result depends on some further structure. The first example of such a variable was found by Andrew Barbour (personal communication): Let $\operatorname{P}(X = 3) = 4/13$, $\operatorname{P}(X = 4) = 5/13$, $\operatorname{P}(X = 5) = 4/13$. Then $X = \sum_1^5 I_i$, with I_i indicators and the distribution of $(I_1, \dots, I_5)$ uniform given X; and these I_i are easily verified to be negatively related. On the other hand, it is easily seen that X is not the sum of any number of independent indicators, since the probability generating function has non-real roots.

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2. Proof of Theorem 1

The first inequality in (1.4) is trivial for $a > nq$. For $0 \le a \le nq$, let $x = a/n \in [0, q]$. Then the bound (1.2) may be written

$$\operatorname{P}(X \ge \operatorname{E} X + a) \le \exp\Bigl(-np\Bigl(1 + \frac{x}{p}\Bigr)\ln\Bigl(1 + \frac{x}{p}\Bigr) - nq\Bigl(1 - \frac{x}{q}\Bigr)\ln\Bigl(1 - \frac{x}{q}\Bigr)\Bigr). \tag{2.1}$$

Let, for $0 \le x \le q$,

$$f(x) = p\Bigl(1+\frac{x}{p}\Bigr)\ln\Bigl(1+\frac{x}{p}\Bigr) + q\Bigl(1-\frac{x}{q}\Bigr)\ln\Bigl(1-\frac{x}{q}\Bigr) - \frac{x^2}{2\bigl(pq+x(q-p)/3\bigr)}.$$

Then $f(0) = f'(0) = 0$, and an elementary calculation yields

$$\begin{aligned} f''(x) &= \frac{1}{x+p} + \frac{1}{q-x} - \frac{p^2q^2}{\bigl(pq+x(q-p)/3\bigr)^3} \\ &= \frac{\frac13 pq(q-p)^2x^2 + \frac{1}{27}(q-p)^3x^3 + p^2q^2x^2}{(x+p)(q-x)\bigl(pq+x(q-p)/3\bigr)^3} \ge 0 \end{aligned}$$

for $0 \le x \le q$. Hence $f(x) \ge 0$ in this interval, and thus

$$\mathrm{P}(X \ge \mathrm{E}\,X + a) \le \exp\Bigl(-n\frac{x^2}{2\bigl(pq+x(q-p)/3\bigr)}\Bigr).$$

This proves the first inequality in (1.4). The second follows from $q-p \le 1$ and the third from

$$\frac{1}{1+a/3npq} \ge 1 - \frac{a}{3npq}.$$

Inequality (1.5) follows directly from (2.1) and

$$-nq\Bigl(1-\frac{x}{q}\Bigr)\ln\Bigl(1-\frac{x}{q}\Bigr) = n(q-x)\ln\Bigl(1+\frac{x}{q-x}\Bigr) \le nx. \tag{2.2}$$

The inequalities (1.6) follow from (1.4) and

$$-\frac{1}{npq+a/3} \le -\frac{1}{np(1+a/3np)} \le -\frac{1}{np}\Bigl(1-\frac{a}{3np}\Bigr);$$

alternatively, they follow easily from (1.5), cf. (3.10).

The inequalities (1.7) follow from (1.3) by an argument similar to the one given above for (1.4), or (simpler) by applying (1.4) to $n-X \sim \mathrm{Bi}(n,q)$; (1.8) follows from (1.3), using (2.2) with $x = -a/n \le 0$; (1.9) follows from (1.7) and, assuming (as we may) $a \le np$, $npq - a(q-p)/3 \le npq + ap/3 \le npq + np^2/3 \le np$. Finally, also (1.10) follows from (1.7) since we now assume $a(q-p)/3 \ge 0$. □

3. Proof of Theorem 3

We may assume that $X = \sum_1^n I_i$ where $I_i \sim \mathrm{Be}(p_i)$ are independent. Note that

$$\begin{aligned} \lambda &= \mathrm{E}\,X = \sum_1^n p_i \\ \sigma^2 &= \mathrm{Var}\,X = \sum_1^n p_i(1-p_i) = \lambda - \sum_1^n p_i^2 \end{aligned}$$

and thus

$$\sum_1^n p_i^2 = \lambda - \sigma^2.$$

We assume, to avoid trivialities, that at least one $p_i \ne 0, 1$. Thus $0 < \lambda < n$ and $0 < \sigma^2 < \lambda$.

We begin with a real analysis lemma. It is an analogue of Jensen's inequality but with a condition on the sign of the third derivative instead of the second.

Lemma 1. *Suppose that μ is a finite positive measure on $[0,1]$, and define*

$$\begin{aligned} m &= \mu([0,1]), \\ x_0 &= \int_0^1 x^2 d\mu \Big/ \int_0^1 x\, d\mu, \\ \alpha_0 &= \Big(\int_0^1 x\, d\mu\Big)^2 \Big/ \int_0^1 x^2 d\mu, \\ x_1 &= 1 - \int_0^1 (1-x)^2 d\mu \Big/ \int_0^1 (1-x) d\mu, \\ \alpha_1 &= \Big(\int_0^1 (1-x) d\mu\Big)^2 \Big/ \int_0^1 (1-x)^2 d\mu. \end{aligned}$$

(We here let $0/0 = 0$; this occurs in the degenerate cases where μ is a point mass at 0 or 1.) If f is a three times continuously differentiable real function on $[0,1]$ with $f''' \ge 0$, then

$$(m-\alpha_0)f(0) + \alpha_0 f(x_0) \le \int_0^1 f\, d\mu \le (m-\alpha_1)f(1) + \alpha_1 f(x_1). \tag{3.1}$$

If instead $f''' \le 0$ on $[0,1]$, then these inequalities are reversed.

Proof. We will show the left inequality of (3.1); the right inequality then follows by symmetry, considering the function $\widetilde{f}(x) = -f(1-x)$, which satisfies $\widetilde{f}'''(x) = f'''(1-x) \ge 0$, and the similarly reflected measure $\widetilde{\mu}(A) = \mu(\{1-x : x \in A\})$. Similarly, the statement for $f''' \le 0$ follows by considering $-f$.

Let ν be the measure $(m-\alpha_0)\delta_0 + \alpha_0\delta_{x_0}$; thus the sought inequality is $\int f\, d\nu \le \int f\, d\mu$, while the choice of α_0 and x_0 yields $\int 1\, d\nu = m = \int 1\, d\mu$, $\int x\, d\nu = \alpha_0 x_0 = \int x\, d\mu$, and $\int x^2 d\mu = \alpha_0 x_0^2 = \int x^2 d\mu$. (In fact, it is easily seen that ν is the unique measure concentrated on a two-point set $\{0, x\}$ for some $x \in [0,1]$, such that $\int x^k d\nu = \int x^k d\mu$ for $k = 0, 1, 2$.)

We now use Taylor's formula

$$f(x) = f(0) + f'(0)x + \tfrac{1}{2} f''(0)x^2 + \tfrac{1}{2}\int_0^x (x-t)^2 f'''(t)\, dt \tag{3.2}$$

and integrate against the signed measure $\mu - \nu$. Since as we just have shown,

$$\int_0^1 x^k d(\mu-\nu) = 0, \qquad k = 0, 1, 2, \tag{3.3}$$

we obtain from (3.2) and Fubini's theorem,

$$\begin{aligned} \int_0^1 f(x)\, d(\mu-\nu) &= \int_0^1 \int_0^x \tfrac{1}{2}(x-t)^2 f'''(t)\, dt\, d(\mu-\nu)(x) \\ &= \int_0^1 \int_t^1 \tfrac{1}{2}(x-t)^2 d(\mu-\nu)(x) f'''(t)\, dt \\ &= \int_0^1 \varphi(t) f'''(t) dt, \end{aligned} \tag{3.4}$$

where

$$\varphi(t) = \tfrac{1}{2}\int_t^1 (x-t)^2 d(\mu-\nu)(x).$$

We claim that $\varphi(t) \geq 0$ on [0,1]; this implies $\int_0^1 f(x)d(\mu-\nu) \geq 0$ by (3.4) which is the required result. Note that $\varphi(1)=0$ and $\varphi(0)=\frac{1}{2}\int_0^1 x^2 d(\mu-\nu)=0$. Moreover, again by (3.3), $\int_0^1 (x-t)^2 d(\mu-\nu)(x)=0$ and thus

$$\varphi(t) = -\frac{1}{2}\int_0^t (x-t)^2 d(\mu-\nu)(x).$$

Using Fubini again, and letting $F(x)=\mu([0,x])-\nu([0,x])$,

$$\varphi(t) = -\iiint_{0\leq x\leq y\leq z\leq t} dy\,dz\,d(\mu-\nu)(x) = -\int_0^t\int_0^z F(y)\,dy\,dz,$$

and thus φ is continuously differentiable with

$$\varphi'(z) = -\int_0^z F(y)\,dy. \tag{3.5}$$

In particular $\varphi'(0)=0$ and, using Fubini a last time,

$$\varphi'(1) = -\int_0^1 F(y)\,dy = -\int_0^1\int_0^y d(\mu-\nu)(x)\,dy = -\int_0^1 (1-x)\,d(\mu-\nu)(x) = 0.$$

On the interval $[0,x_0)$, $\nu([0,x])$ is constant $m-\alpha_0$, and thus $F(x)$ is increasing; hence there exists $x_2\in[0,x_0]$ such that $F(x)\leq 0$ on $(0,x_2)$ and $F(x)\geq 0$ on (x_2,x_0). It follows by (3.5) that φ' is increasing and thus φ is convex on $[0,x_2]$, while φ is concave on $[x_2,x_0]$. Since $\varphi(0)=\varphi'(0)=0$, this implies that $\varphi\geq 0$ on $[0,x_2]$. Similarly, on the interval $[x_0,1]$, we have $\nu([0,x])=m$ and thus

$$F(x) = \mu([0,x])-\nu([0,x]) = \mu([0,x])-m \leq 0,$$

which implies that φ is convex on $[x_0,1]$. Moreover, $\varphi(1)=\varphi'(1)=0$ and thus $\varphi\geq 0$ on $[x_0,1]$. Finally, on the interval $[x_2,x_0]$, φ is concave so it attains it minimum at one of the endpoints, but we have already shown $\varphi(x_2),\varphi(x_0)\geq 0$ and thus $\varphi\geq 0$ also on $[x_2,x_0]$, which completes the proof. ☐

We apply this lemma to estimate the moment generating function of X.

Lemma 2. *Let X be as above. If $0\leq t\leq 1$, then*

$$\mathrm{E}(1-t)^X \leq \Bigl(1-t\Bigl(1-\frac{\sigma^2}{\lambda}\Bigr)\Bigr)^{\lambda^2/(\lambda-\sigma^2)}$$

or

$$\ln \mathrm{E}(1-t)^X \leq \frac{\lambda^2}{\lambda-\sigma^2}\ln\Bigl(1-t\Bigl(1-\frac{\sigma^2}{\lambda}\Bigr)\Bigr). \tag{3.6}$$

Proof. Since the I_i are independent, and $\mathrm{E}(1-t)^{I_i}=1-p_i+p_i(1-t)=1-p_it$,

$$\mathrm{E}(1-t)^X = \mathrm{E}\prod_{i=1}^n (1-t)^{I_i} = \prod_{i=1}^n \mathrm{E}(1-t)^{I_i} = \prod_{i=1}^n (1-p_it),$$

and thus

$$\ln \mathrm{E}(1-t)^X = \sum_{i=1}^n \ln(1-p_i t) = \int_0^1 \ln(1-tx)\,d\mu(x),$$

where μ is the measure $\sum_1^n \delta_{p_i}$ consisting of n point masses at the (possibly coinciding) points p_i. Note that

$$\int_0^1 x\,d\mu = \sum_{i=1}^n p_i = \lambda$$

and

$$\int_0^1 x^2 d\mu = \sum_{i=1}^n p_i^2 = \lambda - \sigma^2.$$

We may assume that $t<1$ (the case $t=1$ follows then by continuity); then the function $f(x)=\ln(1-tx)$ is infinitely differentiable on $[0,1]$ with $f'''(x) = -2t^3/(1-tx)^3 \le 0$. Hence Lemma 1 yields

$$\ln \mathrm{E}(1-t)^X = \int_0^1 \ln(1-tx)\,d\mu(x) \le (m-\alpha_0)f(0) + \alpha_0 f(x_0) = \alpha_0 \ln(1-tx_0),$$

where $m=n$, $\alpha_0 = \lambda^2/(\lambda-\sigma^2)$ and $x_0 = (\lambda-\sigma^2)/\lambda$, which is the required estimate. $\square$

Remark 8. For $t \ge 0$, a similar argument yields

$$\begin{aligned}\ln \mathrm{E}(1+t)^X &\le (n-\alpha_1)\ln(1+t) + \alpha_1 \ln(1+tx_1)\\ &= \frac{n\lambda-\lambda^2-n\sigma^2}{n-\lambda-\sigma^2}\ln(1+t) + \frac{(n-\lambda)^2}{n-\lambda-\sigma^2}\ln\Bigl(1+t\frac{\sigma^2}{n-\lambda}\Bigr).\end{aligned}\tag{3.7}$$

This inequality could be used instead of (3.6) below, giving the same results. We prefer to use (3.6), which does not involve n explicitly.

Remark 9. Estimates of $\mathrm{E}\, e^{sX}$ are, of course, obtained by substituting $t = 1-e^s$ in (3.6) for $s \le 0$ and $t = e^s - 1$ in (3.7) for $s \ge 0$.

We can now obtain our basic estimate.

Lemma 3. *Let X be as above. If $0 \le a \le \lambda$, then*

$$\ln \mathrm{P}(X \le \lambda - a) \le -\frac{\lambda}{\lambda-\sigma^2}(a+\sigma^2-a\sigma^2/\lambda)\ln\Bigl(1+\frac{a}{\sigma^2}-\frac{a}{\lambda}\Bigr) - (\lambda-a)\ln\Bigl(1-\frac{a}{\lambda}\Bigr).\tag{3.8}$$

(*When $a=\lambda$, we define $(\lambda-a)\ln(1-a/\lambda)=0$.*)

Proof. For any t with $0 \le t \le 1$,

$$\mathrm{E}(1-t)^X \ge (1-t)^{\lambda-a}\,\mathrm{P}(X \le \lambda-a),$$

and thus, using Lemma 2,

$$\begin{aligned}\ln \mathrm{P}(X \le \lambda-a) &\le \ln \mathrm{E}(1-t)^X - (\lambda-a)\ln(1-t)\\ &\le \frac{\lambda^2}{\lambda-\sigma^2}\ln\Bigl(1-t\Bigl(1-\frac{\sigma^2}{\lambda}\Bigr)\Bigr) - (\lambda-a)\ln(1-t).\end{aligned}$$

Choosing

$$t = \frac{a}{a + \sigma^2 - a\sigma^2/\lambda}$$

(which minimizes the right hand side), this yields

$$\begin{aligned}
\ln \mathrm{P}(X \le \lambda - a) &\le \frac{\lambda^2}{\lambda - \sigma^2} \ln \frac{\sigma^2}{a + \sigma^2 - a\sigma^2/\lambda} - (\lambda - a) \ln \frac{\sigma^2(1 - a/\lambda)}{a + \sigma^2 - a\sigma^2/\lambda} \\
&= \Big(\frac{\lambda^2}{\lambda - \sigma^2} - \lambda + a\Big) \ln \frac{\sigma^2}{a + \sigma^2 - a\sigma^2/\lambda} - (\lambda - a) \ln\Big(1 - \frac{a}{\lambda}\Big) \\
&= -\frac{\lambda\sigma^2 + \lambda a - a\sigma^2}{\lambda - \sigma^2} \ln \frac{\sigma^2 + a - a\sigma^2/\lambda}{\sigma^2} - (\lambda - a) \ln\Big(1 - \frac{a}{\lambda}\Big),
\end{aligned}$$

which yields the sought result. □

While the estimate in Lemma 3 may be useful for numerical evaluation in applications, it is too complicated to be of much other direct use. Hence we will use it to derive the simpler (but slightly weaker) estimates in Theorem 3.

For notational convenience, let $\sigma^2 = x\lambda$ and $a = y\sigma^2 = xy\lambda$, where $0 < x < 1$ and $0 \le y \le 1/x$. Then (3.8) may be written

$$\ln \mathrm{P}(X \le \lambda - a) \le -\sigma^2 g(x, y), \tag{3.9}$$

where

$$g(x, y) = \frac{1}{1 - x}(1 + y - xy) \ln(1 + y - xy) + \frac{1}{x}(1 - xy) \ln(1 - xy).$$

Lemma 4. *$g(x, y)$ is an increasing function of x in the region*

$$U = \{(x, y) : 0 < x < 1,\ 0 \le y \le 1/x\}.$$

Proof. We want to show that $\partial g/\partial x \ge 0$ in the region $\{(x, y) : 0 < x < 1,\ 0 \le y < 1/x\}$; note that g is well-defined and infinitely differentiable in the larger region $V = \{(x, y) : 0 < x < 1,\ -1/(1 - x) < y < 1/x\}$, and continuous on $\bar{V} \cap \{(x, y) : 0 < x < 1\} \supset U$.

Instead of estimating $\partial g/\partial x$ directly, we first compute

$$\frac{\partial g}{\partial y} = \ln(1 + y - xy) - \ln(1 - xy)$$

and

$$\frac{\partial^2 g}{\partial x \partial y} = \frac{\partial^2 g}{\partial y \partial x} = \frac{-y}{1 + y - xy} - \frac{-y}{1 - xy} = \frac{y^2}{(1 + y - xy)(1 - xy)} \ge 0.$$

Hence $\partial g/\partial x$ is an increasing function of y. Moreover, taking $y = 0$, we find $g(x, 0) = 0$ and thus

$$\frac{\partial g}{\partial x}(x, 0) = 0, \qquad 0 < x < 1.$$

Hence $\frac{\partial g}{\partial x}(x, y) \ge 0$ for all $(x, y) \in V$ with $y \ge 0$. □

Proof of Theorem 3. If $x \searrow 0$ and $y \ge 0$ is fixed, then, as is easily seen,

$$g(x, y) \to (1 + y) \ln(1 + y) - y.$$

Lemma 4 thus yields, for $0 < x < 1$ and $0 \le y \le 1/x$,

$$g(x,y) \ge (1+y)\ln(1+y) - y,$$

and thus by (3.9), for $0 \le a \le \lambda$,

$$\ln \mathrm{P}(X \le \lambda - a) \le -\sigma^2\big((1+y)\ln(1+y) - y\big),$$

with $y = a/\sigma^2$, which is the analogue of (1.13) for $\mathrm{P}(X \le \mathrm{E}\, X - a)$; note that this estimate trivially holds for $a > \lambda$. In order to obtain (1.13), we consider the variable $X^* = n - X \overset{\mathrm{d}}{=} \sum_1^n (1 - I_i)$, and observe that $\mathrm{P}(X \ge \mathrm{E}\, X + a) = \mathrm{P}(X^* \le \mathrm{E}\, X^* - a)$ and $\mathrm{Var}(X^*) = \mathrm{Var}\, X$. The estimates (1.14) and (1.15) and their analogues for $\mathrm{P}(X \le \mathrm{E}\, X - a)$ now follow from the elementary estimates, defining $h(y) = (1+y)\ln(1+y) - y$,

$$h(y) \ge \frac{y^2}{2(1+y/3)} \ge \frac{y^2}{2}(1 - y/3), \qquad y \ge 0, \tag{3.10}$$

and

$$h(y) \ge yh(c)/c, \qquad y \ge c. \tag{3.11}$$

The estimate (3.10) may be verified by observing that $h(y) - y^2/2(1+y/3)$ vanishes together with its first derivative at 0, while the second derivative equals $(9y^2+y^3)/(1+y)(3+y)^3 \ge 0$. Similarly, (3.11) follows by the convexity of h. We omit the details.

Finally, if $\sigma^2 \ge \lambda/2$, then $x \ge 1/2$ and thus by Lemma 4 (assuming as we may that $a \le \lambda$),

$$g(x,y) \ge g(\tfrac{1}{2}, y) = (2+y)\ln(1+\tfrac{y}{2}) + (2-y)\ln(1-\tfrac{y}{2}) \ge \tfrac{1}{2}y^2,$$

which together with (3.9) yields (1.16). □

4. Proof of Theorem 4.

Let $Y = \sum_2^n I_i$ and $Z = \sum_2^n J_{i1}$, where $(J_{i1})_{i=1}^n$ are as in the definition of negatively related variables. Then $X = I_1 + Y$ and $(Y \mid I_1 = 1) \overset{d}{=} Z$. Moreover, since $J_{i1} \le I_i$ for $i \ge 2$, we have $Z \le Y$. Consequently, for any real t,

$$\begin{aligned}
\mathrm{E}\, e^{tX} - \mathrm{E}\, e^{tY} &= \mathrm{E}(e^{tI_1} - 1)e^{tY} = \mathrm{E}(e^t - 1)I_1 e^{tY} \\
&= (e^t - 1)p_1\, \mathrm{E}(e^{tY} \mid I_1 = 1) \\
&= (e^t - 1)p_1\, \mathrm{E}\, e^{tZ} \\
&\le (e^t - 1)p_1\, \mathrm{E}\, e^{tY} \\
&= \mathrm{E}(e^{tI_1} - 1)\, \mathrm{E}\, e^{tY}
\end{aligned}$$

and thus

$$\mathrm{E}\, e^{tX} \le \mathrm{E}\, e^{tI_1}\, \mathrm{E}\, e^{tY}.$$

Induction yields

$$\mathrm{E}\, e^{tX} \le \prod_1^n e^{tI_i} = \mathrm{E}\, e^{t\widetilde{X}}. \qquad \square$$

5. Proof of Theorem 5.

Given n, p, k, with $1 \le k \le n$ and $0 < p < 1$, let $U \sim \mathrm{Bi}(n,p)$ and let X_{npk} be the random variable U conditioned on $U \ge k$. Since $U = \sum_1^n I_i$, with $I_i \sim \mathrm{Be}(p)$ independent, $X_{npk} = \sum_1^n (I_i \mid \sum_1^n I_i \ge k)$, and it follows from Barbour, Holst and Janson (1992) Proposition 2.2.10 and Theorem 2.I that X_{npk} is a sum of negatively related indicators. We claim that, for any α, c, A as in Theorem 5, some variable X_{npk} satisfies (1.24).

Suppose not. Then for each n, p, k either $\mathrm{Var}(X_{npk}) \le A$ or

$$\mathrm{P}(X_{npk} - \mathrm{E}\, X_{npk} > \alpha\sigma) \le ce^{-\alpha}. \tag{5.1}$$

Fix $p \in (0,1)$, let $q = 1-p$, choose ε with $0 < \varepsilon < q$, take $k = \lfloor n(p+\varepsilon) \rfloor + 1$, and let $n \to \infty$. Then, for fixed $i \ge 0$, with $r = p(q-\varepsilon)/(p+\varepsilon)q < 1$,

$$\frac{\mathrm{P}(X_{npk} = k+i+1)}{\mathrm{P}(X_{npk} = k+i)} = \frac{p}{q} \cdot \frac{n-k-i}{k+i+1} \to \frac{p}{q} \cdot \frac{q-\varepsilon}{p+\varepsilon} = r \qquad \text{as } n \to \infty, \tag{5.2}$$

and

$$\mathrm{P}(X_{npk} = k+i) \le r^i\, \mathrm{P}(X_{npk} = k) \le r^i. \tag{5.3}$$

It follows that $X_{npk} - k$ converges in distribution to a random variable Y_r with geometric distribution $\mathrm{Ge}(1-r)$: $\mathrm{P}(Y_r = i) = (1-r)r^i$, $i \ge 0$. Moreover, by (5.3), every moment $\mathrm{E}(X_{npk} - k)^m$ stays bounded, which implies that the moments converge to the corresponding moments of Y_r.

The variance of Y_r equals $r/(1-r)^2$. If $\mathrm{Var}(Y_r) > A$, then also $\mathrm{Var}(X_{npk}) > A$ for large n, so by our assumption (5.1) holds and taking the limit as $n \to \infty$ we obtain

$$\mathrm{P}(Y_r - \mathrm{E}\, Y_r > \alpha\sqrt{\mathrm{Var}\, Y_r}) \le ce^{-\alpha}. \tag{5.4}$$

Now, let $\varepsilon \to 0$ (keeping p fixed). Then $r \to 1$ and $\mathrm{Var}(Y_r) = r/(1-r)^2 \to \infty$, so (5.4) holds when r is close to 1. Moreover, it is easily seen that as $r \to 1, (1-r)Y_r$ converges in distribution to an exponential variable $Z \sim \mathrm{Exp}(1)$, again with convergence of all moments. Consequently we may take the limit again and obtain from (5.4)

$$\mathrm{P}(Z - \mathrm{E}\, Z > \alpha\sqrt{\mathrm{Var}\, Z}) \le ce^{-\alpha}. \tag{5.5}$$

But $\mathrm{E}\, Z = \mathrm{Var}\, Z = 1$, so the left hand side of (5.5) equals $\mathrm{P}(Z > 1+\alpha) = e^{-1-\alpha} > ce^{-\alpha}$, and we have obtained a contradiction. $\square$

References

N. Alon and J. H. Spencer (1992), *The Probabilistic Method*, Wiley, New York.

R. Bahadur and R. Ranga Rao (1960), On deviations of the sample mean, *Ann. Math. Stat.* **31**, 1015–1027.

A. D. Barbour, L. Holst and S. Janson (1992), *Poisson Approximation*, Oxford Univ. Press, Oxford.

G. Bennett (1962), Probability inequalities for the sum of independent random variables, *J Amer. Stat. Assoc.* **57**, 33–45.

S. Boucheron, G. Lugosi and P. Massart (2013), *Concentration inequalities*, Oxford Univ. Press, Oxford.

H. Chernoff (1952), A measure of asymptotic efficiency for tests of a hypothesis based on the sum of observations, *Ann. Math. Stat.* **23**, 493–507.

H. Cramér (1938), Sur un nouveau théorème-limite de la théorie des probabilités, *Actualités Scientifiques et Industrielles* **736**, 5–23.

W. Feller (1943), Generalization of a probability limit theorem of Cramér, *Trans. Amer. Math. Soc.* **54**, 361–372.

W. Hoeffding (1963), Probability inequalities for sums of bounded random variables, *J Amer. Stat. Assoc.* **58**, 13–30.

S. Janson (1990), Poisson approximation for large deviations, *Random Struct. Alg.* **1**, 221–230.

S. Janson, T. Łuczak and A. Ruciński (2000), *Random Graphs*, Wiley, New York.

K. Joag-Dev and F. Proschan (1983), Negative association of random variables, with applications, *Ann. Stat.* **11**, 286–295.

A. Kamath, R. Motwani, K. Palem and P. Spirakis (1994), Tail bounds for occupancy and the satisfiability threshold conjecture, *Proceedings of the 35th Annual IEEE Symposium on Foundations of Computer Science*.

A. Khintchine (1929), Über einen neuen Grenzwertsatz der Wahrscheinlichkeitsrechnung, *Math. Ann.* **101**, 745–752.

J.E. Littlewood (1969), On the probability in the tail of a binomial distribution, *Adv. Appl. Probab.* **1**, 43–72.

V.A. Vatutin and V.G. Mikhailov (1982), Limit theorems for the number of empty cells in an equiprobable scheme for group allocation of particles, *Teor. Veroyatnost. i Primenen.* **27**, 684–692 (Russian); English transl., *Theor. Probab. Appl.* **27**, 734–743.

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