ON THE TRAVELING FLY PROBLEM

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Gargano, LoSacco and Gargano [**1**] have studied the following problem: "If a fly starts at a random point inside a sphere of radius one in R^n and flies in a straight line in a random direction until it reaches the boundary, what is the average distance the fly travels?"

They gave exact answers for $n = 1, 2, 3$ and approximate answers for $4 \leq n \leq 9$, leaving the general case as an open problem. This problem is solved here.

Theorem. *Let $n \geq 1$ and l_n be the average distance traveled by the n-dimensional fly. Then*

$$l_n = \frac{2}{\sqrt{\pi}} \frac{\Gamma(n/2+1)}{\Gamma(n/2+3/2)} = \begin{cases} 2\frac{n!!}{(n+1)!!}, & \text{if } n \text{ is odd;} \\ \frac{4}{\pi}\frac{n!!}{(n+1)!!}, & \text{if } n \text{ is even.} \end{cases}$$

For example, $l_1 = 1$, $l_2 = 8/3\pi$, $l_3 = 3/4$, $l_4 = 32/15\pi$, $l_5 = 5/8$, $l_6 = 64/35\pi$, $l_7 = 35/64$; moreover, it follows that $l_n \sim \sqrt{8/\pi n}$ as $n \to \infty$.

Proof. By symmetry, we may assume that the fly flies in a fixed direction. Assume $n \geq 2$, let B_n be the unit ball in R^n, write the starting point $x \in B_n$ as (x', x_n) with $x' \in B_{n-1}$ and assume that the fly flies in the x_n-direction; then the flown distance is $(1 - |x'|^2)^{1/2} - x_n$.

Since the average of x_n over B_n vanishes by symmetry, the average distance l_n equals the average of $(1-|x'|^2)^{1/2}$ over $B_n = \{(x', x_n) : x' \in B_{n-1}, -(1-|x'|^2)^{1/2} \leq x_n \leq (1-|x'|^2)^{1/2}\}$, and thus

$$l_n = \frac{2\int_{B_{n-1}} (1-|x'|^2)\,dx'}{2\int_{B_{n-1}} (1-|x'|^2)^{1/2}\,dx'}.$$

Polar coordinates followed by the change of variables $r^2 = t$ yield

$$l_n = \frac{\int_0^1 (1-r^2) r^{n-2}\,dr}{\int_0^1 (1-r^2)^{1/2} r^{n-2}\,dr} = \frac{\int_0^1 t^{(n-3)/2}(1-t)\,dt}{\int_0^1 t^{(n-3)/2}(1-t)^{1/2}\,dt}.$$

This exhibits l_n as the ratio $B\big((n-1)/2, 2\big)/B\big((n-1)/2, 3/2\big)$ of two beta integrals, and the result follows.

Reference

[1] M.L. Gargano, F. LoSacco and M.R. Gargano; The traveling fly problem, *Graph Theory Notes of New York* **XXX**, New York Academy of Sciences, 47 (1996).