ON COMPLEX HYPERCONTRACTIVITY

SVANTE JANSON

ABSTRACT. We give a new proof of a hypercontractivity theorem for the Mehler transform with a complex parameter, earlier proved by Weissler [15] and Epperson [4]. The proof uses stochastic integrals and Itô calculus. The method also yields new proofs of some related results.

1. INTRODUCTION

Let μ be the standard Gaussian measure $(2\pi)^{-1/2}e^{-x^2/2}\,dx$ on $\mathbb{R}$, and let $h_0(x)$, $h_1(x)$, ... be the corresponding sequence of orthogonal polynomials (the Hermite polynomials). The Mehler transform M_z, where z is a complex number with $|z| \le 1$, can be defined by $M_z\left(\sum_0^\infty a_n h_n\right) = \sum_0^\infty a_n z^n h_n$; since $\{h_n\}_0^\infty$ is an orthogonal basis in $L^2(d\mu)$, M_z is a bounded linear operator $L^2(d\mu) \to L^2(d\mu)$ with norm 1.

Remark. As is well known, see e.g. [10] and [7], M_z can also, for $|z| < 1$ at least, be defined as an integral operator.

If z is real, $-1 \le z \le 1$, it is not difficult to show that M_z maps $L^p(d\mu)$ into itself with norm 1 for every $p \ge 1$; i.e. M_z is a contraction in $L^p(d\mu)$. (For $p < 2$ this, of course, entails extending M_z from $L^2(d\mu)$ by continuity.)

Nelson [11] proved the much stronger result, known as hypercontractivity, that if $1 \le p \le q < \infty$ and z is real with $z^2 \le (p-1)/(q-1)$, then M_z maps $L^p(d\mu)$ into $L^q(d\mu)$ with norm 1; conversely, if $z^2 > (p-1)/(q-1)$, then M_z is not even bounded $L^p(d\mu) \to L^q(d\mu)$. Many different proofs of this important result are known, see the survey by Gross [5] and the many references given there. We will in this paper study the extension of Nelson's result to complex z.

The case of an imaginary z was studied by Beckner [1], who showed that hypercontractivity holds for $z = i\sqrt{p-1}$ when $1 \le p \le 2$ and $q = p'$, the conjugate exponent. (As shown by Beckner, this is by a change of variable equivalent to the sharp form of the Hausdorff–Young inequality.)

General complex values of z were studied by Weissler [15] and Epperson [4], who characterized the set of z for which M_z is a contraction $L^p(d\mu) \to L^q(d\mu)$, and showed that it equals the set where M_z is bounded. (The special case $q = p'$ was also treated by Coifman, Cwikel, Rochberg, Sagher and Weiss [3].) Weissler's proof excludes the cases $3/2 < p \le q < 2$ and $2 < p \le q < 3$ while Epperson's covers the whole range $1 \le p \le q < \infty$. (Weissler [15] also gave some results for $p > q$; this case will not be considered here.) An alternative approach to this result was given by Lieb [9], who proved a very general theorem on integral operators with Gaussian kernels.

The main purpose of the present paper is to present a new proof of the result by Weissler and Epperson, which can be stated as follows.

Theorem 1. *Let $1 \le p \le q < \infty$. Then M_z is a contraction $L^p(d\mu) \to L^q(d\mu)$ if and only if*

$$(q-1)|\operatorname{Re} zw|^2 + |\operatorname{Im} zw|^2 \le (p-1)|\operatorname{Re} w|^2 + |\operatorname{Im} w|^2, \quad \text{for all } w \in \mathbb{C}. \tag{1}$$

Moreover, if (1) *fails (for some w), then M_z is not even bounded $L^p(d\mu) \to L^q(d\mu)$.*

Remark. By expressing the difference between the two sides in (1) as a quadratic form in $\operatorname{Re} w$ and $\operatorname{Im} w$, it is easily seen that (1) is equivalent to the two conditions

$$\begin{cases} |z|^2 \le p/q \\ (q-1)|z|^4 - (p+q-2)(\operatorname{Re} z)^2 - (pq-p-q+2)(\operatorname{Im} z)^2 + (p-1) \ge 0 \end{cases}$$

Consequently, the set of allowed z is bounded by a quartic curve in the complex plane.

We will also, following Epperson [4], consider a generalization to the following situation. (See e.g. [7] for further details.)

Let H and H' be two Gaussian Hilbert spaces (i.e. real Hilbert spaces consisting of centred Gaussian variables), let $H_{\mathbb{C}}$ and $H'_{\mathbb{C}}$ be their complexifications and let $A\colon H_{\mathbb{C}} \to H'_{\mathbb{C}}$ be a complex linear operator of norm ≤ 1. Suppose further that H and H' are defined on probability spaces $(\Omega, \mathcal{F}, \mathrm{P})$ and $(\Omega', \mathcal{F}', \mathrm{P}')$. Then $\Gamma(A)\colon L^2(\Omega, \mathcal{F}(H), \mathrm{P}) \to L^2(\Omega', \mathcal{F}(H'), \mathrm{P}')$ is the (unique) bounded linear operator satisfying

$$\Gamma(A)(\,{:}\xi_1 \cdots \xi_n{:}\,) = \,{:}A\xi_1 \cdots A\xi_n{:}\,, \qquad \xi_1, \dots, \xi_n \in H \text{ (or } H_{\mathbb{C}}),$$

where ${:}\cdots{:}$ denotes the Wick product. We then have the following generalization of Theorem 1 [4]. (Theorem 1 is the special case when $H = H'$ is a one-dimensional Gaussian Hilbert space and A is multiplication by z.)

Theorem 2. *Let $1 \le p \le q < \infty$. With notations as above, $\Gamma(A)$ is a contraction $L^p(\mathcal{F}(H)) \to L^q(\mathcal{F}(H'))$ if and only if*

$$(q-1)\|\operatorname{Re} A\xi\|^2 + \|\operatorname{Im} A\xi\|^2 \le (p-1)\|\operatorname{Re}\xi\|^2 + \|\operatorname{Im}\xi\|^2, \qquad \xi \in H_{\mathbb{C}}. \tag{2}$$

Moreover, if (2) *fails (for some $\xi \in H_{\mathbb{C}}$), then $\Gamma(A)$ is not even bounded $L^p(\mathcal{F}(H)) \to L^q(\mathcal{F}(H'))$.*

(The norms in (2) are the norms in $H_{\mathbb{C}}$ and $H'_{\mathbb{C}}$, i.e. the L^2-norms.)

We observe that by combining (2) and the same inequality with ξ replaced by $i\xi$, it follows easily that $\|A\|^2 \le p/q$. In particular, our assumption $\|A\| \le 1$ follows from (2).

Remark. A simple proof of Nelson's hypercontractivity theorem (z real) using Itô calculus was given by Neveu [12]. His proof is quite different from the one given here and does not seem to generalize to complex z.

Our method applies also to the case of two *complex* Gaussian Hilbert spaces H and H' (i.e. two complex Hilbert spaces consisting of symmetric complex Gaussian variables) and a complex linear operator of norm ≤ 1 $A\colon H \to H'$; in this case $\Gamma(A)\colon \Gamma(H) \to \Gamma(H')$, where $\Gamma(H)$ is a subspace of $L^2(\Omega, \mathcal{F}(H), \mathrm{P})$. The following (simpler) analogue of Theorem 2 holds.

Theorem 3. *Let $0 < p \le q < \infty$. If $\|A\| \le \sqrt{p/q}$, then $\|\Gamma(A)X\|_q \le \|X\|_p$ for every $X \in \Gamma(H)$. Conversely, if $\|A\| > \sqrt{p/q}$, then $\sup_{X\in\Gamma(H)} \|\Gamma(A)X\|_q/\|X\|_p = \infty$.*

The special case with $H = H'$ one-dimensional can be expressed as follows. (The corresponding result on $\mathbb{C}^n$ follows as well.) For earlier proofs, see Janson [6], Carlen [2] and Zhou [16].

Theorem 4. *Let $d\nu = (2\pi)^{-1} e^{-|z|^2/2}\,dx\,dy$ be the standard Gaussian measure on $\mathbb{C}$, and let B^p be the space of all entire functions belonging to $L^p(\mathbb{C}, d\nu)$ (with the subspace norm). Further, let $M_z f(w) = f(zw)$. If $0 < p \le q < \infty$, then M_z is a contraction $B^p \to B^q$ if and only if $|z| \le \sqrt{p/q}$. Moreover, if $|z| > \sqrt{p/q}$, then M_z is not even bounded $B^p \to B^q$.*

2. Proofs

We first prove Theorem 2; as explained above Theorem 1 is the special case with $H = H'$ one-dimensional and $A\xi = z\xi$. (The reader is urged to consider primarily this case, where the vectors and matrices below have a single component only and the matrix and scalar multiplications reduce to ordinary multiplication.)

The proofs of Theorems 3 and 4 follow the same path, and we indicate only the differences at the end of the section.

Proof of Theorem 2. Although it is possible to work with (Hermite) polynomials, we prefer to use the Wick exponentials defined by

$$:e^{\xi}: = \sum_0^\infty \tfrac{1}{n!} :\xi^n: = \exp(\xi - \mathrm{E}\,\xi^2/2)$$

whenever ξ is a random variable with a real or complex centred normal distribution. (Again, see [7] for further details.) We then have the simple formula $\Gamma(A) :e^{\xi}: = :e^{A\xi}:$, $\xi \in H_{\mathbb{C}}$.

First assume that $\Gamma(A)$ is bounded from L^p into L^q, with $\|\Gamma(A)\|_{p,q} = C$. Since a simple calculation yields, for every $\xi \in H_{\mathbb{C}}$,

$$\| :e^{\xi}: \|_p = e^{p\,\mathrm{E}(\operatorname{Re}\xi)^2/2 - \operatorname{Re}\mathrm{E}\,\xi^2/2} = e^{(p-1)\|\operatorname{Re}\xi\|^2/2 + \|\operatorname{Im}\xi\|^2/2},$$

it follows that

$$(q-1)\|\operatorname{Re} A\xi\|^2/2 + \|\operatorname{Im} A\xi\|^2/2 \le \ln C + (p-1)\|\operatorname{Re}\xi\|^2/2 + \|\operatorname{Im}\xi\|^2/2.$$

Replacing ξ by $t\xi$, multiplying by $2/t^2$ and letting $t \to \infty$ (with t real), we obtain (2).

Suppose now that (2) holds; we want to prove that $\|\Gamma(A)Z\|_q \le \|Z\|_p$ for every $Z \in L^2(\Omega, \mathcal{F}(H), \mathrm{P})$. Since finite sums $\sum_1^N a_j :e^{\eta_j}:$ with $a_j \in \mathbb{C}$ and $\eta_j \in H$ are dense in $L^p(\Omega, \mathcal{F}(H), \mathrm{P})$ and $L^2(\Omega, \mathcal{F}(H), \mathrm{P})$, it suffices to show the inequality for such sums, i.e.

$$\Big\|\sum_1^N a_j :e^{A\eta_j}:\Big\|_q \le \Big\|\sum_1^N a_j :e^{\eta_j}:\Big\|_p. \tag{3}$$

Thus, suppose that $a_j \in \mathbb{C}$ and $\eta_j \in H$, $1 \le j \le N$ are given. Let $H_1 \subseteq H$ be the linear span of $\{\eta_j\}_1^N$ and let $H_1' \subseteq H'$ be the linear span of $\{\operatorname{Re} A\eta_j, \operatorname{Im} A\eta_j\}_1^N$; let further $\{\xi_k\}_1^n$ and $\{\xi_l'\}_1^m$ be orthonormal bases in H_1 and H_1', respectively. In these bases, $A\colon H_1 \to H_1'$ is given by a complex $m \times n$ matrix which we also denote by A; (2) implies that

$$(q-1)|\operatorname{Re} Az|^2 + |\operatorname{Im} Az|^2 \le (p-1)|\operatorname{Re} z|^2 + |\operatorname{Im} z|^2, \tag{4}$$

for every vector $z \in \mathbb{C}^n$.

If Ξ and Ξ' denote the (column) vectors $(\xi_1,\dots,\xi_n)$ and $(\xi'_1,\dots,\xi'_m)$, there exist vectors $b_j \in \mathbb{R}^n \subset \mathbb{C}^n$ such that $\eta_j = b_j\cdot\Xi$ and thus $A\eta_j = Ab_j\cdot\Xi'$; hence (3) can be written

$$\|\sum_1^N a_j :e^{Ab_j\cdot\Xi'}:\|_q \le \|\sum_1^N a_j :e^{b_j\cdot\Xi}:\|_p. \tag{5}$$

Let $X(t) = \bigl(X_1(t),\dots,X_n(t)\bigr)$ and $Y(t) = \bigl(Y_1(t),\dots,Y_m(t)\bigr)$, $t\ge 0$, be independent multi-dimensional Brownian motions (i.e. all components X_k and Y_l are independent Brownian motions), and define the random function

$$F(s,t) = \sum_j a_j :e^{b_j\cdot X(s)+Ab_j\cdot Y(t)}: = \sum_j a_j :e^{b_j\cdot X(s)}:\,:e^{Ab_j\cdot Y(t)}:.$$

We assume that X and Y are defined on different probability spaces $(\Omega_X,\mathcal{F}_X,\mathrm{P}_X)$ and $(\Omega_Y,\mathcal{F}_Y,\mathrm{P}_Y)$; thus F is defined on $\Omega_X\times\Omega_Y$. Let E_X and E_Y denote the integrals over Ω_X and Ω_Y, respectively.

Note that $F(s,t)$ is (a.s.) continuous in s and t, and that the random variable $\sup_{0\le s,t\le 1}|F(s,t)|$ has all moments finite; this holds also if we fix X (i.e. fix $\omega_X\in\Omega_X$) and regard $F(s,t)$ as a random function defined on $(\Omega_Y,\mathcal{F}_Y,\mathrm{P}_Y)$, or fix Y and regard $F(s,t)$ as a random function defined on $(\Omega_X,\mathcal{F}_X,\mathrm{P}_X)$. The same properties are easily verified for the random functions introduced below, which justifies our uses of Fubini's theorem and dominated convergence.

Since $\Xi \overset{\mathrm{d}}{=} X(1)$ and $\Xi' \overset{\mathrm{d}}{=} Y(1)$ (these random vectors have independent standard normal components), the sought inequality (5) is equivalent to $\|F(0,1)\|_q \le \|F(1,0)\|_p$, which can be written

$$\mathrm{E}_X\bigl(\mathrm{E}_Y|F(0,1)|^q\bigr)^{p/q} \le \mathrm{E}_X\bigl(\mathrm{E}_Y|F(1,0)|^q\bigr)^{p/q}. \tag{6}$$

For technical reasons we fix $\varepsilon>0$ and define the random function

$$Q(s,t) = |F(s,t)|^2+\varepsilon^2$$

and the function

$$\varphi(s,t) = \mathrm{E}_X\bigl(\mathrm{E}_Y|Q(s,t)|^{q/2}\bigr)^{p/q}, \qquad s,t\ge 0.$$

Note that φ is continuous by dominated convergence.

We will show that $t\mapsto\varphi(1-t,t)$ is non-increasing on $[0,1]$; thus $\varphi(0,1)\le\varphi(1,0)$. Letting $\varepsilon\to 0$, we obtain (using dominated convergence again) (6), and the proof will be completed.

We now begin in earnest. First, as is well-known, Itô's formula shows that $s\mapsto {:e^{b_j\cdot X(s)}:}$ is a martingale with $d{:e^{b_j\cdot X(s)}:} = {:e^{b_j\cdot X(s)}:}\, b_j\cdot dX(s)$ and thus, for fixed t and Y, $s\mapsto F(s,t)$ is a martingale with

$$dF(s,t) = \sum_j a_j :e^{b_j\cdot X(s)}:\,:e^{Ab_j\cdot Y(t)}:\, b_j\cdot dX(s) = G(s,t)\cdot dX(s),$$

defining the random vector $G(s,t) = \sum_j a_j :e^{b_j\cdot X(s)}:\,:e^{Ab_j\cdot Y(t)}:\, b_j$. (See e.g. [13] for the stochastic integration theory used here.)

Similarly, for fixed s and X, $t\mapsto F(s,t)$ is a martingale with

$$dF(s,t) = \sum_j a_j :e^{b_j\cdot X(s)}:\,:e^{Ab_j\cdot Y(t)}:\, Ab_j\cdot dY(t) = AG(s,t)\cdot dY(t).$$

(The filtrations are, here and below, the ones defined by $X(s)$ and $Y(t)$, respectively.)

Consider first the case of fixed s and X. By Itô's formula again, $t \mapsto Q(s,t)$ and $t \mapsto Q(s,t)^{q/2}$ are continuous semimartingales with

$$\begin{aligned} dQ(s,t) &= d|F(s,t)|^2 = \overline{F}(s,t)dF(s,t) + F(s,t)d\overline{F}(s,t) + d[\overline{F},F] \\ &= \overline{F}(s,t)AG(s,t)\cdot dY(t) + F(s,t)\overline{AG(s,t)}\cdot dY(t) + |AG(s,t)|^2\,dt \\ &= K(s,t)\cdot dY(t) + |AG(s,t)|^2\,dt, \end{aligned}$$

where $K = \overline{F}AG + F\overline{AG}$, and (using $Q \geq \varepsilon^2 \geq 0$)

$$\begin{aligned} dQ(s,t)^{q/2} &= \frac{q}{2}Q^{q/2-1}dQ + \frac{1}{2}\frac{q}{2}\Big(\frac{q}{2}-1\Big)Q^{q/2-2}|K|^2\,dt \\ &= \frac{q}{2}Q^{q/2-1}K\cdot dY + \frac{q}{2}Q^{q/2-1}|AG|^2\,dt + \frac{q(q-2)}{8}Q^{q/2-2}|K|^2\,dt. \end{aligned}$$

In other words,

$$t \mapsto Q(s,t)^{q/2} - \int_0^t \Big(\frac{q}{2}Q^{q/2-1}|AG|^2 + \frac{q(q-2)}{8}Q^{q/2-2}|K|^2\Big)\,du$$

is a continuous local martingale with quadratic variation $\int_0^t (\frac{q}{2})^2 Q^{q-2}|K|^2 du$, which has finite expectation; thus this local martingale is a square integrable martingale, and hence it has the same expectation for every t, which yields

$$\mathrm{E}_Y\, Q(s,t)^{q/2} = \mathrm{E}_Y\, Q(s,0)^{q/2} + \int_0^t \mathrm{E}_Y\Big(\frac{q}{2}Q^{q/2-1}|AG|^2 + \frac{q(q-2)}{8}Q^{q/2-2}|K|^2\Big)du.$$

Let $\Phi(s,t) = \mathrm{E}_Y\,|Q(s,t)|^{q/2}$; we have shown that $t \mapsto \Phi(s,t)$ is continuously differentiable with

$$\frac{\partial}{\partial t}\Phi(s,t) = \frac{q}{2}\,\mathrm{E}_Y\big(Q^{q/2-1}|AG|^2\big) + \frac{q(q-2)}{8}\,\mathrm{E}_Y\big(Q^{q/2-2}|K|^2\big).$$

(The right hand side is continuous by dominated convergence.) By ordinary calculus, this implies

$$\begin{aligned} \frac{\partial}{\partial t}\Phi(s,t)^{p/q} &= \frac{p}{q}\Phi(s,t)^{p/q-1}\frac{\partial}{\partial t}\Phi(s,t) \\ &= \frac{p}{2}\Phi(s,t)^{p/q-1}\,\mathrm{E}_Y\big(Q^{q/2-1}|AG|^2\big) + \frac{p(q-2)}{8}\Phi(s,t)^{p/q-1}\,\mathrm{E}_Y\big(Q^{q/2-2}|K|^2\big). \end{aligned}$$

Applying E_X, wee see, using dominated convergence, that $t \mapsto \varphi(s,t) = \mathrm{E}_X\,\Phi(s,t)^{p/q}$ is continuously differentiable, with

$$\begin{aligned} \frac{\partial}{\partial t}\varphi(s,t) &= \frac{p}{2}\,\mathrm{E}_X\Big(\Phi(s,t)^{p/q-1}\,\mathrm{E}_Y\big(Q^{q/2-1}|AG|^2\big)\Big) \\ &\qquad + \frac{p(q-2)}{8}\,\mathrm{E}_X\Big(\Phi(s,t)^{p/q-1}\,\mathrm{E}_Y\big(Q^{q/2-2}|K|^2\big)\Big) \\ &= \frac{p}{2}\,\mathrm{E}_X\Big(\Phi(s,t)^{p/q-1}\,\mathrm{E}_Y\big(Q^{q/2-2}(Q|AG|^2 + \frac{q-2}{4}|K|^2)\big)\Big). \end{aligned} \tag{7}$$

Next, we keep t and Y fixed and obtain similarly that $s \mapsto Q(s,t)^{q/2}$ is a continuous semimartingale with

$$dQ(s,t)^{q/2} = \frac{q}{2}Q^{q/2-1}H\cdot dX + \frac{q}{2}Q^{q/2-1}|G|^2\,ds + \frac{q(q-2)}{8}Q^{q/2-2}|H|^2\,ds,$$

where $H=\overline{F}G+F\overline{G}$. We can here apply E_Y. (This is easily justified by rewriting the equation as a stochastic integral equation and using Fubini.) Thus $s\mapsto\Phi(s,t)$ is a continuous semimartingale with

$$d\Phi(s,t)=\frac{q}{2}\,\mathrm{E}_Y(Q^{q/2-1}H)\cdot dX+\frac{q}{2}\,\mathrm{E}_Y(Q^{q/2-1}|G|^2)\,ds \\ +\frac{q(q-2)}{8}\,\mathrm{E}_Y(Q^{q/2-2}|H|^2)\,ds.$$

A final application of Itô's formula yields

$$d\Phi(s,t)^{p/q}=\frac{p}{q}\Phi^{p/q-1}d\Phi+\frac{1}{2}\frac{p}{q}\Big(\frac{p}{q}-1\Big)\Phi^{p/q-2}\Big(\frac{q}{2}\Big)^2|\,\mathrm{E}_Y(Q^{q/2-1}H)|^2\,ds$$

and it follows by taking the expectation, arguing as for $t\mapsto Q(s,t)^{q/2}$ above, that $s\mapsto\varphi(s,t)=\mathrm{E}_X\,\Phi(s,t)^{p/q}$ is continuously differentiable, with

$$\begin{aligned}\frac{\partial}{\partial s}\varphi(s,t)&=\frac{p}{2}\,\mathrm{E}_X\Big(\Phi^{p/q-1}\,\mathrm{E}_Y\big(Q^{q/2-1}|G|^2\big)\Big)\\&\qquad+\frac{p(q-2)}{8}\,\mathrm{E}_X\Big(\Phi^{p/q-1}\,\mathrm{E}_Y\big(Q^{q/2-2}|H|^2\big)\Big)\\&\qquad+\frac{p(p-q)}{8}\,\mathrm{E}_X\big(\Phi^{p/q-2}|\,\mathrm{E}_Y(Q^{q/2-1}H)|^2\big)\\&=\frac{p}{2}\,\mathrm{E}_X\Big(\Phi^{p/q-1}\,\mathrm{E}_Y\Big(Q^{q/2-2}\big(Q|G|^2+\frac{p-2}{4}|H|^2\big)\Big)\Big)\\&\qquad+\frac{p(q-p)}{8}\,\mathrm{E}_X\Big(\Phi^{p/q-2}\Big(\Phi\,\mathrm{E}_Y(Q^{q/2-2}|H|^2)-|\,\mathrm{E}_Y(Q^{q/2-1}H)|^2\Big)\Big).\end{aligned}\tag{8}$$

By Hölder's inequality and the definition $\Phi=\mathrm{E}_Y\,Q^{q/2}$, the final term is ≥ 0. (This is where we use $p\leq q$.) Thus, (8) yields

$$\frac{\partial}{\partial s}\varphi(s,t)\geq\frac{p}{2}\,\mathrm{E}_X\Big(\Phi^{p/q-1}\,\mathrm{E}_Y\Big(Q^{q/2-2}\big(Q|G|^2+\frac{p-2}{4}|H|^2\big)\Big)\Big).\tag{9}$$

Combining (9) and (7), we obtain

$$\begin{aligned}&\frac{\partial}{\partial s}\varphi(s,t)-\frac{\partial}{\partial t}\varphi(s,t)\\&\quad\geq\frac{p}{2}\,\mathrm{E}_X\Big(\Phi^{p/q-1}\,\mathrm{E}_Y\Big(Q^{q/2-2}\big(Q|G|^2+\frac{p-2}{4}|H|^2-Q|AG|^2-\frac{q-2}{4}|K|^2\big)\Big)\Big).\end{aligned}\tag{10}$$

Now, $H=\overline{F}G+F\overline{G}=2\operatorname{Re}(\overline{F}G)$ and similarly $K=2\operatorname{Re}(\overline{F}AG)=2\operatorname{Re}(A\overline{F}G)$. Thus, recalling $Q=|F|^2+\varepsilon^2$ and using (4) (with $z=\overline{F}G$) and $\|A\|\leq 1$,

$$\begin{aligned}&Q|G|^2+\frac{p-2}{4}|H|^2-Q|AG|^2-\frac{q-2}{4}|K|^2\\&\qquad=(|F|^2+\varepsilon^2)(|G|^2-|AG|^2)+(p-2)|\operatorname{Re}(\overline{F}G)|^2-(q-2)|\operatorname{Re}(A\overline{F}G)|^2\\&\qquad=\varepsilon^2(|G|^2-|AG|^2)\\&\qquad\qquad+|\overline{F}G|^2+(p-2)|\operatorname{Re}(\overline{F}G)|^2-|A\overline{F}G|^2-(q-2)|\operatorname{Re}(A\overline{F}G)|^2\\&\qquad\geq 0.\end{aligned}$$

Consequently, (10) yields $\frac{\partial}{\partial s}\varphi(s,t)-\frac{\partial}{\partial t}\varphi(s,t)\geq 0$ for all $s,t>0$. Since, as shown above, φ is continuously differentiable, this implies $\frac{d}{dt}\varphi(1-t,t)\leq 0$. Hence

$s \mapsto \varphi(1-t,t)$ is nonincreasing on $[0,1]$, and $\varphi(0,1) \le \varphi(1,0)$, which completes the proof. ☐

Remark. Our proof is inspired by the one in Epperson [4], and can be regarded as a continuous version of the latter. In fact, the argument in [4] shows that, with $\psi(t) = \varphi(1-t,t)$,

$$\psi(k/N) \le \bigl(1+o(N^{-1})\bigr)\psi((k-1)/N) + o(N^{-1}), \qquad 1 \le k \le N, \tag{11}$$

which easily yields $\psi(1) \le \psi(0)$, by induction on k and letting $N \to \infty$. Our proof replaces the finite differences by (Itô) differentials, thus eliminating all higher order terms. (The proof in [4] is not quite complete, since the sets of 'regular configurations' defined there do not have the asserted symmetry properties. A simple remedy, which yields (11), is to use a different set of 'regular configurations' for each step in the induction.)

Proof of Theorem 3. The necessity of $\|A\| \le \sqrt{p/q}$ follows as above, using $\|{:}e^{\xi}{:}\|_p = e^{p\|\xi\|^2/4}$ when ξ is symmetric complex Gaussian.

For sufficiency, we argue as above, now letting $X(t)$ and $Y(t)$ be multi-dimensional complex Brownian motions (i.e., the real and complex parts are independent Brownian motions). This introduces minor differences each time we apply Itô's formula; we now obtain, for fixed s and X,

$$dQ(s,t) = d|F(s,t)|^2 = \overline{F}AG \cdot dY(t) + F\overline{AG} \cdot d\overline{Y}(t) + 2|AG|^2\,dt.$$

and eventually (leaving the details to the reader), instead of (7),

$$\begin{aligned}\frac{\partial}{\partial t}\varphi(s,t) &= p\,\mathrm{E}_X\Bigl(\Phi(s,t)^{p/q-1}\,\mathrm{E}_Y\bigl(Q^{q/2-2}(Q|AG|^2 + \frac{q-2}{2}|FAG|^2)\bigr)\Bigr)\\ &= p\,\mathrm{E}_X\Bigl(\Phi(s,t)^{p/q-1}\,\mathrm{E}_Y\bigl(Q^{q/2-2}(\varepsilon^2 + \tfrac{q}{2}|F|^2)|AG|^2\bigr)\Bigr).\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{\partial}{\partial s}\varphi(s,t) &= p\,\mathrm{E}_X\Bigl(\Phi^{p/q-1}\,\mathrm{E}_Y\bigl(Q^{q/2-1}|G|^2\bigr)\Bigr)\\ &\qquad + \frac{p(q-2)}{2}\,\mathrm{E}_X\Bigl(\Phi^{p/q-1}\,\mathrm{E}_Y\bigl(Q^{q/2-2}|FG|^2\bigr)\Bigr)\\ &\qquad + \frac{p(p-q)}{2}\,\mathrm{E}_X\bigl(\Phi^{p/q-2}|\,\mathrm{E}_Y(Q^{q/2-1}\overline{F}G)|^2\bigr)\\ &\ge p\,\mathrm{E}_X\Bigl(\Phi^{p/q-1}\,\mathrm{E}_Y\bigl(Q^{q/2-2}\bigl(\varepsilon^2 + \tfrac{p}{2}|F|^2\bigr)|G|^2\bigr)\Bigr).\end{aligned}$$

These equations imply $\frac{\partial}{\partial s}\varphi(s,t) - \frac{\partial}{\partial t}\varphi(s,t) \ge 0$, and the result follows. ☐

Proof of Theorem 4. Let $H = H'$ be the one-dimensional complex Gaussian Hilbert space consisting of the linear functions on $(\mathbb{C}, \mathcal{B}, \nu)$, and let $Aw = zw$. Then $\Gamma(H) = B^2$ and $\Gamma(A) = M_z$, and the result follows from Theorem 3 and the fact that linear combinations of exponential functions are dense in B^p (see e.g. [8, Theorem 8.2] and [14, Theorem 3.1]). ☐

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Department of Mathematics, Uppsala University, PO Box 480, S-751 06 Uppsala, Sweden

E-mail address: svante.janson@math.uu.se