ON THE SPACE Q_p AND ITS DYADIC COUNTERPART

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ABSTRACT. We define a dyadic counterpart Q_p^d of Q_p, and study some relations between Q_p and Q_p^d. For example, a function on $\mathbb{T}$ belongs to Q_p if and only if (almost) all its translates belong to Q_p^d. Conversely, functions in Q_p may be obtained by averaging translates of functions in Q_p^d.

1. INTRODUCTION

The space Q_p, $0 < p < 1$ was introduced in [1] as the Banach space of all analytic functions in the unit disc Δ satisfying

$$\sup_{w\in\Delta} \iint_{\Delta} |f'(z)|^2 g(z,w)^p \, dx \, dy < \infty, \tag{1}$$

where $z = x + iy$ and $g(z,w) = \log|(1 - \bar{w}z)/(w - z)|$. As is shown in [1], Q_p is a proper subspace of BMOA (which is obtained by taking $p = 1$ in (1)), and $Q_{p_1} \subsetneq Q_{p_2}$ if $0 < p_1 < p_2 < 1$.

Essén and Xiao [4] showed that an analytic function f, in H^1 say, belongs to Q_p if and only if its boundary values on the unit circle $\mathbb{T}$ satisfy

$$\sup_I |I|^{-p} \int_I \int_I \frac{|f(x) - f(y)|^2}{|x - y|^{2-p}} \, dx \, dy < \infty, \tag{2}$$

where I ranges over all intervals $\subseteq \mathbb{T}$.

We will in the sequel consider functions defined on $\mathbb{T}$ rather than Δ, and we redefine Q_p to be the space of all (measurable) functions on $\mathbb{T}$ that satisfy (2), analytic or not. (Analyticity is not important in (2); in fact, it was shown by Essén and Xiao [4, Corollary 3.2] that a real function satisfies (2) if and only if it is the real part of (the boundary values of) an analytic function in Q_p.) We define $\|f\|_{Q_p}$ to be the square root of the supremum in (2); this is a seminorm only, since $\|f\|_{Q_p} = 0$ if f is a.s. constant, and a proper norm is given by e.g. $\|f\|_{Q_p} + |\int_{\mathbb{T}} f|$.

Note that $Q_p \subseteq$ BMO. (BMO can be defined by taking $p = 2$ in (2). In fact, it follows easily using the John–Nirenberg inequality that any p with $1 < p \leq 2$ yields BMO.)

It is well-known that BMO has a dyadic counterpart BMOd. The purpose of this note is to define a corresponding dyadic version Q_p^d of Q_p, and to study some relations between Q_p and Q_p^d.

2. MAIN RESULTS

We will, for convenience, identify the circle $\mathbb{T}$ with the unit interval $[0, 1)$, keeping in mind that subintervals may wrap around 0. In particular, our circle has length 1. (The ambiguity in interpreting $|x - y|$ in (2) is harmless.)

A dyadic interval is an interval of the type $[(m-1)2^{-n}, m2^{-n})$; let $\mathcal{D} = \mathcal{D}(\mathbb{T})$ be the set of all dyadic subintervals of $\mathbb{T}$ (including $\mathbb{T}$ itself) and let $\mathcal{D}_n(\mathbb{T})$, $n \geq 0$, be the subset of the 2^n dyadic intervals of length 2^{-n}.

We define the dyadic distance $\delta(x,y)$ between two points in $\mathbb{T}$ by

$$\delta(x,y) = \inf\{|I| : x,y \in I \in \mathcal{D}(\mathbb{T})\}.$$

The space Q_p^d, $0 < p < 1$, may now be defined as the space of all (measurable) functions f on $\mathbb{T}$ such that

$$\sup_{I\in\mathcal{D}} |I|^{-p} \int_I \int_I \frac{|f(x)-f(y)|^2}{\delta(x,y)^{2-p}}\,dx\,dy < \infty. \tag{3}$$

Again we define a seminorm $\|f\|_{Q_p^d}$ as the square root of the supremum, and a norm as this seminorm + $|\int_{\mathbb{T}} f|$. Some equivalent definitions, and corresponding equivalent (semi)norms, are given in Section 4.

Since $\delta(x,y) \geq |x-y|$, and the supremum in (3) is over a subset of the set of intervals in (2), it follows immediately that $Q_p \subseteq Q_p^d$. The inclusion is strict; for example, it is easily seen that if $f(x) = \log x$, $0 < x < 1$, then $f \in Q_p^d$ (cf. the similar analytic example $\log(1-z)$ in [4]), but $f \notin \mathrm{BMO}$ because of the infinite jump at 0, and thus $f \notin Q_p$.

It is also immediate that $Q_p^d \subset \mathrm{BMOd}$, which is obtained by taking $p = 2$ in (3). (In fact, the John–Nirenberg theorem again shows that any p with $1 < p \leq 2$ in (3) defines BMOd.)

It is clear that one reason for the discrepancy between Q_p and Q_p^d is that Q_p is translation (i.e. rotation) invariant whereas Q_p^d is not. Indeed, the next theorems show that a function belongs to Q_p if and only if all its translates belong to Q_p^d.

Remark. Although the theorems are phrased in terms of translating the function, it is obviously equivalent, and sometimes more natural, to instead consider the function as fixed and translate the dyadic partition used to define Q_p^d.

Let τ_t denote the translation operator $\tau_t f(x) = f(x-t)$.

Theorem 1. *Let $0 < p < 1$. Then $f \in Q_p$ if and only if $\tau_t f \in Q_p^d$ for all $t \in \mathbb{T}$ and $\sup_t \|\tau_t f\|_{Q_p^d} < \infty$. Moreover, $\|f\|_{Q_p} \asymp \sup_t \|\tau_t f\|_{Q_p^d}$.*

(Here $\asymp$ means that the two sides are equivalent within constant factors that may depend on p.)

The condition that $\tau_t f \in Q_p^d$ for all t may be relaxed considerably.

Theorem 2. *Let $0 < p < 1$. Then the following are equivalent.*

(i) $f \in Q_p$
(ii) $\tau_t f \in Q_p^d$ *for all* $t \in \mathbb{T}$
(iii) $\tau_t f \in Q_p^d$ *for almost all* $t \in \mathbb{T}$
(iv) $\tau_t f \in Q_p^d$ *for* $t \in E \subseteq \mathbb{T}$, *with* $|E| > 0$.

It follows easily from the proof below that the median (or any other fixed quantile) of $t \mapsto \|\tau_t f\|_{Q_p^d}$ is an equivalent seminorm on Q_p.

We state two immediate corollaries of Theorem 2.

Corollary 3. *Let $0 < p < 1$ and let f be a function on $\mathbb{T}$. If $f \in Q_p$ then $\tau_t f \in Q_p^d$ for all $t \in \mathbb{T}$, while if $f \notin Q_p$ then $\tau_t f \notin Q_p^d$ for a.e. $t \in \mathbb{T}$.*

(The corresponding result for BMO is easy, see Lemma 7 below; the dual result for H^1 (where 'all' and 'a.e.' are interchanged) is given in [2].)

For $0 < q \leq \infty$, define $L^q(Q_p^d)$ as the space of all measurable functions F on $\mathbb{T} \times \mathbb{T}$, such that $F(t,\cdot) \in Q_p^d$ for a.e. t and $\|F(t,\cdot)\|_{Q_p^d} \in L^q(\mathbb{T})$.

Remark. This is not the usual Lebesgue space of Banach space valued functions, defined as the closure of simple functions in the obvious norm. The problem is that Q_p^d is not separable, and it is easily seen that if e.g. $F(t,x) = \mathbf{1}[0 < x < t]$ (where $\mathbf{1}$ denotes the indicator function), then F belongs to $L^q(Q_p^d)$ for any $q \le \infty$ with our definition, but $\|F(t,\cdot) - F(s,\cdot)\|_{Q_p^d} \ge 1/4$ for a.e. s and t, and thus there is no separable subspace of Q_p^d that contains $F(t,\cdot)$ for a.e. t, as required by the standard definition, see e.g. [3].

Corollary 4. *Let $0 < p < 1$ and $0 < q \le \infty$. Then $f \in Q_p$ if and only if $\tau_t f(x) \in L^q(Q_p^d)$.*

Again, an equivalent seminorm on Q_p is given by $\| \|\tau_t f\|_{Q_p^d}\|_{L^q}$.

Conversely, if $q > 1$, starting with any function in $L^q(Q_p^d)$, we may construct a function in Q_p as a suitable average.

Theorem 5. *Let $0 < p < 1$ and $1 < q \le \infty$. Suppose that $F(t,x) \in L^q(Q_p^d)$, and define $g(x) = \int_{\mathbb{T}} F(t, x+t)\,dt$. Then $g \in Q_p$.*

Theorem 5 is an extension to Q_p of a result by Garnett and Jones [5] for BMO. That result has important applications, since it enables several deep results for BMO to be shown as consequences of similar but much simpler results for BMOd [5]. We do not know if there are similar applications of Theorem 5.

If we introduce the linear operator $Tf(t,x) = f(x-t)$ (mapping functions on $\mathbb{T}$ to functions on $\mathbb{T}^2$), Corollary 4 shows that T defines an embedding of Q_p into $L^q(Q_p^d)$. Conversely, Theorem 5 shows if $q > 1$, then the adjoint averaging operator $T^* : F \mapsto \int_{\mathbb{T}} F(t, x+t)\,dt$ maps $L^q(Q_p^d)$ into Q_p. Moreover, T^*T is the identity and TT^* is a projection, which yields the following corollaries.

Corollary 6. *If $0 < p < 1$ and $1 < q \le \infty$, then Q_p is isomorphic to the complemented subspace $T(Q_p)$ of $L^q(Q_p^d)$.*

Corollary 7. *If $0 < p < 1$ and $1 < q \le \infty$, then T^* maps $L^q(Q_p^d)$ onto Q_p.*

The final theorem in this section gives a different relation between Q_p and Q_p^d. The example $\log x$ above of a function in $Q_p^d \setminus Q_p$ failed to be in Q_p because it did not belong to BMO; in fact, this is the only way to get such an example.

Theorem 8. *Let $0 < p < 1$. Then $Q_p = Q_p^d \cap \mathrm{BMO}$.*

3. Preliminaries

Throughout the paper, p is a fixed number with $0 < p < 1$. We let c and C denote unspecified positive constants that may depend on p (but not on anything else).

We define, for any interval I and integrable function f,

$$f(I) = \frac{1}{|I|} \int_I f,$$

the mean of f on I, and

$$\varphi_f(I) = \frac{1}{|I|} \int_I |f - f(I)|^2,$$

the square mean oscillation of f on I. Obviously, $\varphi_f(I) < \infty \Leftrightarrow f \in L^2(I)$; we extend the definition to all measurable functions f on I by letting $\varphi_f(I) = \infty$

when $f \notin L^1(I)$. Recall that $f \in \mathrm{BMO}$ if and only if $\sup_I \varphi_f(I) < \infty$. Note the well-known identities

$$\frac{1}{|I|}\int_I |f-a|^2 = \varphi_f(I) + |f(I)-a|^2 \tag{4}$$

for any complex number a, and

$$\frac{1}{|I|^2}\int_I\int_I |f(x)-f(y)|^2\,dx\,dy = 2\varphi_f(I). \tag{5}$$

Moreover, if I is divided into two subintervals I' and I'' of the same length $\frac12|I|$, then $|f(I)-f(I')| = |f(I)-f(I'')| = \frac12|f(I')-f(I'')|$ and

$$\begin{aligned}\varphi_f(I) &= \frac{1}{2|I'|}\int_{I'}|f-f(I)|^2 + \frac{1}{2|I''|}\int_{I''}|f-f(I)|^2\\ &= \tfrac12\bigl(\varphi_f(I') + |f(I)-f(I')|^2\bigr) + \tfrac12\bigl(\varphi_f(I'') + |f(I)-f(I'')|^2\bigr)\\ &= \tfrac12\varphi_f(I') + \tfrac12\varphi_f(I'') + \tfrac14|f(I')-f(I'')|^2. \end{aligned}\tag{6}$$

In particular,

$$|f(I)-f(I')| = \tfrac12|f(I')-f(I'')| \le \varphi_f(I)^{1/2}. \tag{7}$$

Furthermore, if $I \subset J$, then by (4),

$$\varphi_f(I) \le \frac{1}{|I|}\int_I |f-f(J)|^2 \le \frac{|J|}{|I|}\varphi_f(J). \tag{8}$$

We defined above dyadic intervals in $\mathbb{T}$. Similarly, if I is any interval, dyadic or not, we let $\mathcal{D}_n(I)$, $n \ge 0$, denote the set of the 2^n subintervals of length $2^{-n}|I|$ obtained by n successive bipartitions of I.

We define, for any interval I and a measurable function f on I,

$$\psi_f(I) = \sum_{k=0}^{\infty}\sum_{J\in\mathcal{D}_k(I)} 2^{-pk}\varphi_f(J). \tag{9}$$

By (9) and (5),

$$\begin{aligned}\psi_f(I) &= \sum_{k=0}^{\infty}\sum_{J\in\mathcal{D}_k(I)} 2^{-pk}\tfrac12\bigl(2^{-k}|I|\bigr)^{-2}\int_J\int_J |f(x)-f(y)|^2\,dx\,dy\\ &= \int_{\mathbb{T}}\int_{\mathbb{T}} \alpha_I(x,y)|f(x)-f(y)|^2\,dx\,dy, \end{aligned}\tag{10}$$

where

$$\alpha_I(x,y) = \tfrac12\sum_{k=0}^{\infty}\sum_{J\in\mathcal{D}_k(I)} 2^{(2-p)k}|I|^{-2}\mathbf{1}[x,y\in J]. \tag{11}$$

Since $x,y \in J \in \mathcal{D}_k(I)$ implies $|x-y| \le |J| = 2^{-k}|I|$, and thus $2^k \le |I|/|x-y|$,

$$\alpha_I(x,y) \le \sum_{2^k\le|I|/|x-y|} 2^{(2-p)k}|I|^{-2} \le C\Bigl(\frac{|I|}{|x-y|}\Bigr)^{2-p}|I|^{-2} = C|I|^{-p}|x-y|^{p-2};$$

furthermore $\alpha_I(x,y) = 0$ unless $x,y\in I$. Consequently,

$$\psi_f(I) \le C|I|^{-p}\int_I\int_I \frac{|f(x)-f(y)|^2}{|x-y|^{2-p}}\,dx\,dy. \tag{12}$$

The converse is less obvious, but will be proved in Lemma 3 below.

If I is a dyadic interval, and $x, y \in I$, then $x, y \in J$ for some $J \in \mathcal{D}_k(I)$ if and only if $\delta(x,y) \le 2^{-k}|I|$, and thus, by (11),

$$\alpha_I(x,y) = \tfrac{1}{2} \sum_{2^k \le |I|/\delta(x,y)} 2^{(2-p)k}|I|^{-2} \asymp \Bigl(\frac{|I|}{\delta(x,y)}\Bigr)^{2-p}|I|^{-2} = |I|^{-p}\delta(x,y)^{p-2}.$$

Consequently, if I is a dyadic interval,

$$\psi_f(I) \asymp |I|^{-p} \int_I \int_I \frac{|f(x)-f(y)|^2}{\delta(x,y)^{2-p}}\,dx\,dy. \tag{13}$$

4. The dyadic space

The dyadic space Q_p^d was defined above by (3). We give here several equivalent definitions.

Theorem 9. *Let $0 < p < 1$. Then $f \in Q_p^d$ if and only if $\sup_{I\in\mathcal{D}} \psi_f(I) < \infty$. Moreover, $\sup_{I\in\mathcal{D}} \psi_f(I)^{1/2}$ is a seminorm on Q_p^d, equivalent to $\|f\|_{Q_p^d}$ as defined above.*

Proof. Immediate by the definition and (13). □

We let $\mathcal{F}_n$ denote the σ-field generated by the partition $\mathcal{D}_n(\mathbb{T})$; thus, if $f \in L^1(\mathbb{T})$, then $\mathrm{E}(f \mid \mathcal{F}_n)$ is the function that is constant $f(I)$ on each dyadic interval $I \in \mathcal{D}_n(\mathbb{T})$.

Theorem 10. *Let $0 < p < 1$. If $f \in L^1(\mathbb{T})$ and $f_n = \mathrm{E}(f \mid \mathcal{F}_n)$, then the following are equivalent.*

(i) $f \in Q_p^d$.

(ii) *For some $M < \infty$ and every $n \ge 0$,*

$$\sum_{k=0}^{\infty} 2^{(1-p)k}\,\mathrm{E}\bigl(|f - f_{n+k}|^2 \bigm| \mathcal{F}_n\bigr) \le M \qquad a.s.$$

(iii) *For some $M < \infty$ and every $n \ge 0$,*

$$\sum_{k=0}^{\infty} 2^{(1-p)k}\,\mathrm{E}\bigl(|f_{n+k+1} - f_{n+k}|^2 \bigm| \mathcal{F}_n\bigr) \le M \qquad a.s.$$

Proof. If $I \in \mathcal{D}_n(\mathbb{T})$ and $J \in \mathcal{D}_k(I) \subseteq \mathcal{D}_{n+k}$, then $\varphi_f(J) = |J|^{-1}\int_J |f - f_{n+k}|^2$, and $|J| = 2^{-k}|I|$. Hence, by the definition (9),

$$\psi_f(I) = \sum_{k=0}^{\infty} 2^{-pk}\frac{2^k}{|I|}\int_I |f - f_{n+k}|^2 = \sum_{k=0}^{\infty} 2^{(1-p)k}\,\mathrm{E}\bigl(|f - f_{n+k}|^2 \bigm| \mathcal{F}_n\bigr)(x)$$

for $x \in I$, which together with Theorem 9 shows (i) $\Leftrightarrow$ (ii).

Furthermore,

$$\mathrm{E}\bigl(|f - f_{n+k}|^2 \bigm| \mathcal{F}_n\bigr) = \sum_{j=k}^{\infty} \mathrm{E}\bigl(|f_{n+j+1} - f_{n+j}|^2 \bigm| \mathcal{F}_n\bigr),$$

and thus, interchanging the order of summation,

$$\begin{aligned}\sum_{k=0}^{\infty} 2^{(1-p)k}\,\mathrm{E}\bigl(|f - f_{n+k}|^2 \bigm| \mathcal{F}_n\bigr) &= \sum_{j=0}^{\infty}\sum_{k=0}^{j} 2^{(1-p)k}\,\mathrm{E}\bigl(|f_{n+j+1} - f_{n+j}|^2 \bigm| \mathcal{F}_n\bigr)\\ &\asymp \sum_{j=0}^{\infty} 2^{(1-p)j}\,\mathrm{E}\bigl(|f_{n+j+1} - f_{n+j}|^2 \bigm| \mathcal{F}_n\bigr),\end{aligned}$$

which yields the equivalence of (ii) and (iii). $\square$

Again, equivalent seminorms on Q_p^d may defined from (ii) and (iii), by taking the square roots of the smallest possible M.

5. Proofs

The proofs are based on the following fundamental lemma. Note that this lemma and the two following ones are valid for (finite) intervals in $\mathbb{R}$ as well as in $\mathbb{T}$; we will pass between the two cases without further comment.

Lemma 1. *Let I be an interval and let I' and I'' be two intervals such that I' and I'' are adjacent, $|I'|=|I''|=|I|$ and $I\subset I'\cup I''$. Then, for any $f\in L^1(I'\cup I'')$,*

$$\varphi_f(I)\le\varphi_f(I')+\varphi_f(I'')+|f(I')-f(I'')|^2, \tag{14}$$

$$\psi_f(I)\le C\big(\psi_f(I')+\psi_f(I'')+|f(I')-f(I'')|^2\big). \tag{15}$$

Proof. It follows from (8) and (6) that

$$\varphi_f(I)\le\frac{|I'\cup I''|}{|I|}\varphi_f(I'\cup I'')=\varphi_f(I')+\varphi_f(I'')+\tfrac12|f(I')-f(I'')|^2,$$

proving (14).

For (15), we assume for simplicity that $I'=[0,1)$ and $I''=[1,2)$; this is no loss of generality by homogeneity. For each $j\ge0$, let $\{I_{j,i}\}_{i=1}^{2^{j+1}}$ be the 2^{j+1} dyadic intervals of length 2^{-j} contained in $I'\cup I''$ (i.e., $\mathcal{D}_j(I')\cup\mathcal{D}_j(I'')$), arranged in the natural order. If $J\in\mathcal{D}_j(I)$, then $J\subset I_{j,i}\cup I_{j,i+1}$ for some i, and thus by (14) applied to J,

$$\varphi_f(J)\le\varphi_f(I_{j,i})+\varphi_f(I_{j,i+1})+|f(I_{j,i})-f(I_{j,i+1})|^2.$$

The 2^j different $J\in\mathcal{D}_j(I)$ yield different values of i, and summing over all j and J we thus obtain

$$\begin{aligned}\psi_f(I)&=\sum_{j=0}^\infty\sum_{J\in\mathcal{D}_j(I)}2^{-pj}\varphi_f(J)\\&\le2\sum_{j=0}^\infty\sum_{i=1}^{2^{j+1}}2^{-pj}\varphi_f(I_{j,i})+\sum_{j=0}^\infty\sum_{i=1}^{2^{j+1}-1}2^{-pj}|f(I_{j,i})-f(I_{j,i+1})|^2.\end{aligned}\tag{16}$$

The first double sum on the right hand side of (16) is just $\psi_f(I')+\psi_f(I'')$. In order to estimate the final sum, consider a pair (j,i) with $j\ge0$ and $1\le i<2^{j+1}-1$. Let I^* be the smallest dyadic interval that contains $I_{j,i}\cup I_{j,i+1}$, and let the length of I^* be 2^{-j+m}, where $m\ge1$. (Recall that $|I_{j,i}|=2^{-j}$.) Moreover, for $0\le l\le m$, let J_l and K_l be the dyadic intervals of length 2^{-j+l} that contain $I_{j,i}$ and $I_{j,i+1}$, respectively; thus $I_{j,i}=J_0\subset J_1\subset\dots\subset J_m=I^*$ and $I_{j,i+1}=K_0\subset\dots\subset K_m=I^*$. Using the Cauchy–Schwarz inequality and (7), we obtain

$$\begin{aligned}|f(I_{j,i})-f(I_{j,i+1})|^2&\le\Big(\sum_{l=1}^m|f(J_{l-1})-f(J_l)|+\sum_{l=1}^m|f(K_l)-f(K_{l-1})|\Big)^2\\&\le\Big(2\sum_{l=1}^\infty l^{-2}\Big)\Big(\sum_{l=1}^m l^2|f(J_l)-f(J_{l-1})|^2+\sum_{l=1}^m l^2|f(K_l)-f(K_{l-1})|^2\Big)\\&\le C\sum_{l=1}^m l^2\big(\varphi_f(J_l)+\varphi_f(K_l)\big).\end{aligned}\tag{17}$$

If $i \neq 2^j$, then $I_{j,i} \cup I_{j,i+1} \subseteq I'$ or I'', and thus $|I^*| \leq 1$ and $m \leq j$. If $i = 2^j$, however, then $I^* = [0,2)$ and $m = j+1$; in this case we modify (17) by observing that $J_j = I'$ and $K_j = I''$ and thus

$$|f(I_{j,i}) - f(I_{j,i+1})| \leq \sum_{l=1}^{j} |f(J_{l-1}) - f(J_l)| + \sum_{l=1}^{j} |f(K_l) - f(K_{l-1})| + |f(I') - f(I'')|$$

which by the same argument yields

$$|f(I_{j,i}) - f(I_{j,i+1})|^2 \leq C \sum_{l=1}^{j} l^2 \big(\varphi_f(J_l) + \varphi_f(K_l)\big) + C|f(I') - f(I'')|^2. \tag{18}$$

We now keep $j \geq 0$ fixed and sum (17) or (18) (when $i = 2^j$) for $1 \leq i \leq 2^{j+1} - 1$. We observe that the intervals J_l and K_l that appear belong to $\mathcal{D}_{j-l}(I') \cup \mathcal{D}_{j-l}(I'')$, with $1 \leq l \leq j$. Moreover, each dyadic interval J in $\mathcal{D}_{j-l}(I') \cup \mathcal{D}_{j-l}(I'')$ appears at most four times as a J_l or a K_l (viz. when, in $\mathcal{D}_l(J)$, $I_{j,i}$ is the rightmost interval, $I_{j,i+1}$ is the leftmost interval or $I_{j,i}$ and $I_{j,i+1}$ are the two middle intervals). Consequently, since (18) is used only once,

$$\sum_{i=1}^{2^{j+1}-1} |f(I_{j,i}) - f(I_{j,i+1})|^2 \leq C \sum_{l=1}^{j} \sum_{J \in \mathcal{D}_{j-l}(I') \cup \mathcal{D}_{j-l}(I'')} l^2 \varphi_f(J) + C|f(I') - f(I'')|^2.$$

Summing over j we finally obtain, substituting $j = k + l$,

$$\begin{aligned}
\sum_{j=0}^{\infty} \sum_{i=1}^{2^{j+1}-1} & 2^{-pj} |f(I_{j,i}) - f(I_{j,i+1})|^2 \\
&\leq C \sum_{l=1}^{\infty} \sum_{k=0}^{\infty} \sum_{J \in \mathcal{D}_k(I') \cup \mathcal{D}_k(I'')} 2^{-pk-pl} l^2 \varphi_f(J) + C \sum_{j=0}^{\infty} 2^{-pj} |f(I') - f(I'')|^2 \\
&= C\psi_f(I') + C\psi_f(I'') + C|f(I') - f(I'')|^2,
\end{aligned}$$

which by (16) completes the proof of (15). □

We proceed with further results on $\psi_f(I)$.

Lemma 2. *For any interval I and $f \in L^1(I)$,*

$$|I|^{-p} \int_I \int_I \frac{|f(x) - f(y)|^2}{|x-y|^{2-p}} \, dx \, dy \leq C \frac{1}{2|I|} \int_{-|I|}^{|I|} \psi_f(I+t) \, dt + C\psi_f(I).$$

Proof. By (10) and Fubini,

$$\frac{1}{2|I|} \int_{-|I|}^{|I|} \psi_f(I+t) \, dt = \int_{\mathbb{T}} \int_{\mathbb{T}} \frac{1}{2|I|} \int_{-|I|}^{|I|} \alpha_{I+t}(x,y) \, dt \, |f(x) - f(y)|^2 \, dx \, dy.$$

This and (10) show that it suffices to verify

$$\frac{1}{2|I|} \int_{-|I|}^{|I|} \alpha_{I+t}(x,y) \, dt + \alpha_I(x,y) \geq c|I|^{-p}|x-y|^{p-2}, \qquad x, y \in I. \tag{19}$$

First, suppose that $x, y \in I$ with $|x-y| \le \frac{1}{2}|I|$ and let $l \ge 0$ be such that $2^{-l-2}|I| < |x-y| \le 2^{-l-1}|I|$. Then, by (11),

$$\begin{aligned}
\frac{1}{2|I|}\int_{-|I|}^{|I|} \alpha_{I+t}(x,y)\,dt &\ge \frac{1}{2|I|}\int_{-|I|}^{|I|} \frac{1}{2} \sum_{J\in\mathcal{D}_l(I+t)} 2^{(2-p)l}|I|^{-2}\mathbf{1}[x,y\in J]\,dt \\
&= \frac{2^{(2-p)l}}{4|I|^3} \sum_{J\in\mathcal{D}_l(I)} \int_{-|I|}^{|I|} \mathbf{1}[x,y\in J+t]\,dt \\
&\ge c|x-y|^{p-2}|I|^{2-p-3} \sum_{J\in\mathcal{D}_l(I)} \int_{-|I|}^{|I|} \mathbf{1}[x,y\in J+t]\,dt.
\end{aligned}$$

It is easily seen that the final integral, for each J, equals $|J|-|x-y| \ge \frac{1}{2}|J|$, and thus the sum over J is at least $\frac{1}{2}|I|$. Hence (19) holds for $|x-y| \le \frac{1}{2}|I|$.

Finally, if $x, y \in I$ with $|x-y| > \frac{1}{2}|I|$, then, taking $k=0$ in (11),

$$\alpha_I(x,y) \ge \tfrac{1}{2}|I|^{-2} \ge c|I|^{-p}|x-y|^{p-2}$$

and (19) holds in this case too. □

Lemma 3. *For any interval I and $f \in L^1(I)$,*

$$c\psi_f(I) \le |I|^{-p}\int_I\int_I \frac{|f(x)-f(y)|^2}{|x-y|^{2-p}}\,dx\,dy \le C\psi_f(I).$$

Proof. The left inequality is just (12).

For the right inequality, we may assume that f is defined on $\mathbb{R}$ with f constant $= f(I)$ outside I. Let I_- and I_+ be the two intervals of the same length as I that are adjacent to I on the left and right, respectively. Note that then $\psi_f(I_-) = \psi_f(I_+) = 0$ and that $f(I_-) = f(I_+) = f(I)$.

For every t with $|t| < |I|$, either $I+t \subset I_- \cup I$ or $I+t \subset I \cup I_+$, and in both cases Lemma 1 yields $\psi_f(I+t) \le C\psi_f(I)$. The result follows by Lemma 2. □

An immediate consequence of Lemma 3 is another characterization of Q_p.

Lemma 4. *Q_p equals the space of all functions f on $\mathbb{T}$ such that $\sup_{I\subseteq\mathbb{T}}\psi_f(I)$ is finite.* □

We will use the fact that it here suffices to consider intervals I with dyadic lengths. (Note that we do not require I to be dyadic, only its length. Restriction to dyadic intervals would give Q_p^d by Theorem 9.)

Lemma 5. *Q_p equals the space of all functions f on $\mathbb{T}$ such that $\sup_I \psi_f(I) < \infty$, where I ranges over the set of all intervals in $\mathbb{T}$ with dyadic length 2^{-n}, $n = 0, 1, \dots$*

Proof. Since every interval I is contained in an interval J with dyadic length $|J| < 2|I|$, it is obvious that it suffices to consider intervals of dyadic lengths in (2). The proof is completed by Lemma 3. □

Proof of Theorem 1. Since every interval of dyadic length is the translate of a dyadic interval, Lemma 5 shows that $f \in Q_p$ if and only if $\sup_{t\in\mathbb{T}}\sup_{I\in\mathcal{D}}\psi_f(I-t) < \infty$. Moreover, using Theorem 9,

$$\|f\|_{Q_p}^2 \asymp \sup_{t\in\mathbb{T}}\sup_{I\in\mathcal{D}}\psi_f(I-t) = \sup_{t\in\mathbb{T}}\sup_{I\in\mathcal{D}}\psi_{\tau_t f}(I) \asymp \sup_{t\in\mathbb{T}}\|\tau_t f\|_{Q_p^d}^2,$$

and the result follows. □

Proof of Theorem 8. As remarked above, the inclusion $Q_p \subseteq Q_p^d \cap \mathrm{BMO}$ follows directly from the definitions.

Conversely, suppose that $f \in Q_p^d \cap \mathrm{BMO}$. Let I be an interval of dyadic length. Then there exist two adjacent dyadic intervals I' and I'' of the same length $|I|$ such that $I \subset I' \cup I''$. Lemma 1 yields

$$\psi_f(I) \le C\psi_f(I') + C\psi_f(I'') + C|f(I') - f(I'')|^2.$$

The first two terms on the right hand side are bounded by $C\|f\|^2_{Q_p^d}$ by Theorem 9, and the last term is bounded by $C\varphi_f(I' \cup I'') \le C\|f\|^2_{\mathrm{BMO}}$ by (7). Hence $\psi_f(I)$ is bounded uniformly for all intervals I of dyadic length, and the result follows by Lemma 5. ☐

Lemma 6. *Let $I \subset \mathbb{T}$ be an interval of length 2^{-n}, $n \ge 1$, and let $m(t)$, for $t \in \mathbb{T}$, be the smallest integer such that the translated interval $I + t$ is contained in a dyadic interval of length $2^{-n+m(t)}$. Then*

$$|\{t : m(t) > M\}| \le 2^{-M}, \qquad M = 0, 1, \dots \tag{20}$$

In particular, for every $r < \infty$,

$$\int_{\mathbb{T}} m(t)^r \, dt \le \sum_{m=1}^{\infty} m^r 2^{1-m} < \infty.$$

Proof. Clearly, $m(t) \le n$, so (20) is trivial for $M \ge n$. For $0 \le M < n$, it is easily seen that $\{t : m(t) > M\}$ consists of 2^{n-M} intervals of length 2^{-n}, and thus there is equality in (20).

In particular, $|\{t : m(t) = M\}| \le 2^{1-M}$, and the final estimate follows. ☐

Lemma 7. *Suppose that $\tau_t f \in \mathrm{BMOd}$ for $t \in E$, where $E \subseteq \mathbb{T}$ is a set with positive measure. Then $f \in \mathrm{BMO}$.*

Proof. Let $M \ge 1$ be such that $2^{-M} < |E|$. Then, since $t \mapsto \|\tau_t f\|_{\mathrm{BMOd}}$ is measurable, there exists a number $A < \infty$ and a subset $E' \subset E$ with $|E'| > 2^{-M}$ such that $\|\tau_t f\|_{\mathrm{BMOd}} \le A$ for $t \in E'$.

Suppose that I is an interval of dyadic length 2^{-n} with $n \ge M$, and let $m(t)$ be as in Lemma 6. By (20) and our assumptions, $|\{t : m(t) > M\}| \le 2^{-M} < |E'|$, and thus there exists a $t \in E'$ such that $m(t) \le M$. Then $\|\tau_t f\|_{\mathrm{BMOd}} \le A$ and $I + t$ is contained in a dyadic interval J with $|J|/|I| = 2^{m(t)} \le 2^M$, and thus, using (8),

$$\varphi_f(I) = \varphi_{\tau_t f}(I + t) \le \frac{|J|}{|I|}\varphi_{\tau_t f}(J) \le 2^M \|\tau_t f\|^2_{\mathrm{BMOd}} \le 2^M A^2.$$

Consequently, $\varphi_f(I)$ is uniformly bounded for all I of dyadic length $\le 2^{-M}$; this easily implies, using (6) and (8), that $\varphi_f(I)$ is uniformly bounded for all intervals $I \subseteq \mathbb{T}$, i.e. $f \in \mathrm{BMO}$. ☐

Proof of Theorem 2. By Theorem 1, it remains only to show that (iv) $\Rightarrow$ (i). Hence, assume that (iv) holds. Since $Q_p^d \subset \mathrm{BMOd}$, Lemma 7 shows that $f \in \mathrm{BMO}$.

Now, choose some $t \in E$. Since BMO is translation invariant, also $\tau_t f \in \mathrm{BMO}$; furthermore, $\tau_t f \in Q_p^d$ by assumption. Hence $\tau_t f \in Q_p$ by Theorem 8, and thus $f \in Q_p$ since Q_p is translation invariant. ☐

Proof of Theorem 5. We write $f_t(x) = F(t, x)$ and $h_t(x) = F(t, x + t)$.

Suppose that I is an interval of dyadic length 2^{-n}, $n \geq 1$. We fix $t \in T$ and let (ignoring the case when $I + t$ is dyadic) I' and I'' be the two dyadic intervals of length 2^{-n} that intersect I. Then $I \subset I' \cup I''$ and Lemma 1 yields

$$\begin{aligned}\psi_{h_t}(I)^{1/2} = \psi_{f_t}(I+t)^{1/2} &\leq C\big(\psi_{f_t}(I')^{1/2} + \psi_{f_t}(I'')^{1/2} + |f_t(I') - f_t(I'')|\big)\\ &\leq C\big(\|f_t\|_{Q_p^d} + |f_t(I') - f_t(I'')|\big). \end{aligned} \tag{21}$$

Let $m(t)$ be as in Lemma 6, and let, for $l = 0, \dots, m(t)$, J_l and K_l be the dyadic intervals of length 2^{-n+l} that contain I' and I'', respectively. Then $J_{m(t)} = K_{m(t)}$ and, using (7),

$$\begin{aligned}|f_t(I') - f_t(I'')| &\leq \sum_{l=1}^{m(t)} \big(|f(J_{l-1}) - f(J_l)| + |f(K_l) - f(K_{l-1})|\big)\\ &\leq \sum_{l=1}^{m(t)} \big(\varphi_{f_t}(J_l)^{1/2} + \varphi_{f_t}(K_l)^{1/2}\big)\\ &\leq m(t)\|f_t\|_{Q_p^d}.\end{aligned}$$

Consequently, (21) yields

$$\psi_{h_t}(I)^{1/2} \leq Cm(t)\|f_t\|_{Q_p^d}.$$

Since $g = \int_{\mathbb{T}} h_t\,dt$ and $\psi_f(I)^{1/2}$ may be regarded as an L^2 norm, we may use Minkowski's inequality and obtain

$$\psi_g(I)^{1/2} \leq \int_{\mathbb{T}} \psi_{h_t}(I)^{1/2}\,dt \leq C\int_{\mathbb{T}} m(t)\|f_t\|_{Q_p^d}\,dt.$$

Consequently, by Hölder's inequality, choosing $r < \infty$ such that $1/r + 1/q = 1$,

$$\psi_g(I)^{1/2} \leq C\|m(t)\|_{L^r(\mathbb{T})}\|F\|_{L^q(Q_p^d)}.$$

By Lemma 6, this shows that $\psi_g(I)$ is uniformly bounded when I is an interval of dyadic length $\leq 1/2$. The case $I = \mathbb{T}$ follows easily (we omit the details), and thus $g \in Q_p$ by Lemma 5. ∎

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