Q SPACES OF SEVERAL REAL VARIABLES

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ABSTRACT. For $\alpha \in (-\infty, \infty)$, let $Q_\alpha(\mathbb{R}^n)$ be the space of all measurable functions with

$$\sup[\ell(I)]^{2\alpha-n}\int_I\int_I \frac{|f(x)-f(y)|^2}{|x-y|^{n+2\alpha}}\,dx\,dy < \infty,$$

where the supremum is taken over all cubes I with the edge length $\ell(I)$ and the edges parellel to the coordinate axes in $\mathbb{R}^n$. If $\alpha \in (-\infty, 0)$, then $Q_\alpha(\mathbb{R}^n) = BMO(\mathbb{R}^n)$, and if $\alpha \in [1, \infty)$, then $Q_\alpha(\mathbb{R}^n) = \{\text{constants}\}$. In the present paper, we discuss the case $\alpha \in [0, 1)$. These spaces are new subspaces of $BMO(\mathbb{R}^n)$ containing some special Besov spaces. We characterize functions in $Q_\alpha(\mathbb{R}^n)$ by means of the Poisson extension, p-Carleson measures, mean oscillation and wavelet coefficients, and give a dyadic counterpart. Finally, we pose some open problems.

CONTENTS

1. INTRODUCTION

For each $p \in (0, 1)$, the space Q_p was introduced in [3] as the Banach space of all analytic functions f in the unit disk $\triangle$ satisfying

$$\sup_{w\in\triangle}\iint_\triangle |f'(z)|^2[g(z,w)]^p dm(z) < \infty, \tag{1.1}$$

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where $g(z,w) = \log|(1-\overline{w}z)/(w-z)|$ is the Green function of $\triangle$ and m is the Lebesgue measure. As in [3], Q_p is a proper subspace of $BMOA$ (obtained by taking $p=1$ in (1.1)) and $Q_{p_1} \subsetneq Q_{p_2}$ if $0 < p_1 < p_2 < 1$.

Essén and Xiao [5] showed that an analytic function f in the Hardy space H^2 on the unit disk belongs to Q_p if and only if its boundary values on the unit circle $\partial\triangle$ satisfy

$$\sup_I |I|^{-p} \int_I \int_I \frac{|f(e^{i\theta}) - f(e^{i\varphi})|^2}{|e^{i\theta} - e^{i\varphi}|^{2-p}} \, d\theta \, d\varphi < \infty, \tag{1.2}$$

where the supremum is taken over all subarcs $I \subset \partial\triangle$. Janson [9] used (1.2) to define a dyadic analogue Q_p^d of Q_p and to prove that Q_p is the intersection of Q_p^d and $BMOA$. Observe that (1.2) makes sense even if f is not analytic and it becomes possible to consider an extension to *Harmonic Analysis* over Euclidean spaces. This is our starting point.

Throughout this paper, we always let $\mathbb{R}^n$ be n-dimensional Euclidean space, and let $\mathbb{R}^{n+1}_+$ be the upper half space based on $\mathbb{R}^n$. A cube means always a cube in $\mathbb{R}^n$ with edges parallel to the coordinate axes. We denote the edge length of a cube I by $\ell(I)$, and the Lebesgue measure of I by $|I|$; thus $\ell(I) = |I|^{1/n}$. Also, for $t > 0$, tI means the cube which has the same center as I and the edge length $t\ell(I)$. We let $|x|$ denote the usual Euclidean norm for $x \in \mathbb{R}^n$.

For $\alpha \in (-\infty, \infty)$, in analogy with (1.2), we define $Q_\alpha(\mathbb{R}^n)$ to be the space of all measurable functions on $\mathbb{R}^n$ that satisfy

$$\|f\|^2_{Q_\alpha(\mathbb{R}^n)} = \sup_I [\ell(I)]^{2\alpha-n} \int_I \int_I \frac{|f(x) - f(y)|^2}{|x-y|^{n+2\alpha}} \, dx \, dy < \infty, \tag{1.3}$$

where I ranges over all cubes in $\mathbb{R}^n$. Note that we have changed the parameter from p in (1.2) to α in (1.3); the relation between them is $p = 1 - 2\alpha$ (for $n = 1$ as in (1.2)). We will henceforth use $Q_\alpha(\mathbb{R}^n)$ as defined by (1.3) exclusively, hopefully avoiding any possible confusion.

Note that $\|f\|_{Q_\alpha(\mathbb{R}^n)} = 0$ if and only if f is constant a.e.; we thus regard $Q_\alpha(\mathbb{R}^n)$ as a Banach space of functions modulo constants. (It is immediate that $\|f\|_{Q_\alpha(\mathbb{R}^n)}$ is a norm; completeness also is easily verified, see Section 2.)

Remark 1.1. Since every cube I is contained in a cube J with dyadic edge length (i.e. $\ell(J) \in \{2^k : k \in \mathbb{Z}\}$) such that $\ell(J) < 2\ell(I)$, it is obvious that we obtain an equivalent definition, with an equivalent norm, if we consider only cubes of dyadic edge lengths in (1.3).

Similarly, one can consider balls instead of cubes (with $\ell(I)$ replaced by the radius).

We first observe that if $\alpha = -\frac{n}{2}$, then $Q_\alpha(\mathbb{R}^n) = BMO(\mathbb{R}^n)$, which can be defined for example as the space all functions in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ satisfying

$$\|f\|^2_{BMO(\mathbb{R}^n)} = \sup_I |I|^{-1} \int_I |f(x) - f(I)|^2 \, dx < \infty, \tag{1.4}$$

where the supremum is taken over all cubes I in $\mathbb{R}^n$ and

$$f(I) = |I|^{-1} \int_I f(x)\, dx$$

stands for the mean value of f over the cube I, cf. [10]. In fact, we will prove below that $Q_\alpha(\mathbb{R}^n) = BMO(\mathbb{R}^n)$ for all $\alpha \in (-\infty, 0)$.

It is well known that the important space $BMO(\mathbb{R}^n)$ can be described by Poisson integrals, Carleson measures, wavelet coefficients and dyadic cubes. The purpose of this paper is to give analogues for $Q_\alpha(\mathbb{R}^n)$, $\alpha \in (0,1)$, which are given in Sections 2–4 and 6–7. In Section 7 we also consider a dyadic version of $Q_\alpha(\mathbb{R}^n)$. In Section 5, we will provide a local analysis of $Q_\alpha(\mathbb{R}^n)$ which sheds further light on the relation between $Q_\alpha(\mathbb{R}^n)$ and $BMO(\mathbb{R}^n)$. Finally, in Section 8 we will pose some open problems.

Remark 1.2. One can similarly define Q_α spaces of functions defined on a cube in $\mathbb{R}^n$ or a torus $\mathbb{T}^n$. Many of the results below extend to these situations, but we leave the details to the reader.

Remark 1.3. It is also possible to study Q_p spaces in several complex variables, see [2] and [15].

Some notations. Throughout the paper, α is a fixed number in $(-\infty, \infty)$; usually we assume $\alpha \in (0,1)$. C and c will denote unspecified positive constants, possibly different at each occurence; the constants may depend on α and the dimension n, but not on the functions or cubes involved. (They may sometimes depend on other fixed parameters, for example, in Section 6, on the choice of wavelets.) We write $X \asymp Y$, meaning $cX \le Y \le CX$.

We sometimes consider dyadic cubes: Let $\mathcal{D}_0 = \mathcal{D}_0(\mathbb{R}^n)$ be the set of unit cubes whose vertices have integer coordinates, and let, for any integer $k \in \mathbb{Z}$, $\mathcal{D}_k = \mathcal{D}_k(\mathbb{R}^n) = \{2^{-k} I : I \in \mathcal{D}_0\}$; then the cubes in $\mathcal{D} = \bigcup_{-\infty}^{\infty} \mathcal{D}_k$ are called dyadic. Furthermore, if I is any cube, we let $\mathcal{D}_k(I)$, $k \ge 0$, denote the set of the 2^{kn} subcubes of edge length $2^{-k}\ell(I)$ obtained by k successive bipartitions of each edge of I. Moreover, put $\mathcal{D}(I) = \bigcup_0^\infty \mathcal{D}_k(I)$.

χ_E is the characteristic function of the set E.

We let, for $x \in \mathbb{R}^n$, $|x|_\infty$ be the l^∞-norm on $\mathbb{R}^n$: $|(x_1, \dots, x_n)|_\infty = \max_k |x_k|$.

2. Basic properties

This section is devoted to some simple properties of $Q_\alpha(\mathbb{R}^n)$ and to relations between $Q_\alpha(\mathbb{R}^n)$ and the Besov spaces.

We first observe that, by simple changes of variables in (1.3), $\|f\|_{Q_\alpha(\mathbb{R}^n)}$ is not affected by translations or dilations of $\mathbb{R}^n$, i.e. by replacing $f(x)$ by $f(x - x_0)$, $x_0 \in \mathbb{R}^n$ or $f(tx)$, $t > 0$; if we use the norm defined using balls, cf. Remark 1.1, the same holds for rotations. Thus,

Theorem 2.1. *$Q_\alpha(\mathbb{R}^n)$ is invariant under translations, rotations and dilations, and thus under all similarities of $\mathbb{R}^n$; moreover, there exists an equivalent norm on the space such that all similarities preserve the norm.* □

We note the following alternative characterization of $Q_\alpha(\mathbb{R}^n)$.

Lemma 2.2. *Let* $-\infty < \alpha < \infty$. *Then* $f \in Q_\alpha(\mathbb{R}^n)$ *if and only if*

$$\sup_I [\ell(I)]^{2\alpha-n} \int_{|y|<\ell(I)} \int_I |f(x+y) - f(x)|^2 \, dx \frac{dy}{|y|^{n+2\alpha}} < \infty. \tag{2.1}$$

Proof. If the double integrals in (1.3) and (2.1) are denoted by $A(I)$ and $B(I)$, respectively, then by the change of variable $y \to x + y$ and simple geometry one obtains $A(I) \le B(\sqrt{n}I)$ and $B(I) \le A(3I)$. □

The following properties indicate that we only need to pay attention to the case $\alpha \in [0,1)$ for $n > 1$, and to the case $\alpha \in [0, 1/2]$ for $n = 1$.

Theorem 2.3.
(i) $Q_\alpha(\mathbb{R}^n)$ *is decreasing in* α, *i.e.* $Q_\alpha(\mathbb{R}^n) \supseteq Q_\beta(\mathbb{R}^n)$ *if* $\alpha \le \beta$.
(ii) *If* $n \ge 2$ *and* $\alpha \ge 1$, *or if* $n = 1$ *and* $\alpha > 1/2$, *then* $Q_\alpha(\mathbb{R}^n)$ *contains only functions that are a.e. constant, and thus* $Q_\alpha(\mathbb{R}^n) = \{0\}$ *(as a Banach space).*
(iii) *If* $\alpha < 0$, *then* $Q_\alpha(\mathbb{R}^n) = BMO(\mathbb{R}^n)$.

Remark 2.4. The inclusion in (i) is strict if $\alpha < \beta$ except in the cases $1 \le \alpha < \beta$ $(n \ge 2)$, $1/2 < \alpha < \beta$ $(n = 1)$, and $\alpha < \beta < 0$, where equality holds by (ii) or (iii); see Example 2.10 and Remarks 2.8 and 2.11 below. In particular, if $n \ge 2$ and $0 \le \alpha < 1$ or $n = 1$ and $0 \le \alpha \le 1/2$, then $\{0\} \subsetneq Q_\alpha(\mathbb{R}^n) \subsetneq BMO(\mathbb{R}^n)$.

Proof. (i). Suppose $\alpha < \beta$. If $f \in Q_\beta(\mathbb{R}^n)$, then for any cube I in $\mathbb{R}^n$, we have

$$\begin{aligned}
&\int_I \int_I \frac{|f(x) - f(y)|^2}{|x-y|^{n+2\alpha}} \, dx \, dy \\
&= \int_I \int_I \frac{|f(x) - f(y)|^2}{|x-y|^{n+2\beta}} |x-y|^{2(\beta-\alpha)} \, dx \, dy \\
&\le C[\ell(I)]^{2(\beta-\alpha)} \int_I \int_I \frac{|f(x) - f(y)|^2}{|x-y|^{n+2\beta}} \, dx \, dy \\
&\le C[\ell(I)]^{n-2\alpha} \|f\|^2_{Q_\beta(\mathbb{R}^n)},
\end{aligned}$$

that is to say, $f \in Q_\alpha(\mathbb{R}^n)$. So, $Q_\beta(\mathbb{R}^n) \subseteq Q_\alpha(\mathbb{R}^n)$.

(ii). First assume $\alpha > n/2$. If $f \in Q_\alpha(\mathbb{R}^n)$, then by (1.3), for any cube I in $\mathbb{R}^n$,

$$\int_I \int_I \frac{|f(x) - f(y)|^2}{|x-y|^{n+2\alpha}} \, dx \, dy \le [\ell(I)]^{n-2\alpha} \|f\|^2_{Q_\alpha(\mathbb{R}^n)}.$$

Since $n - 2\alpha < 0$, letting I grow to $\mathbb{R}^n$ in the last inequality produces

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^2}{|x-y|^{n+2\alpha}} \, dx \, dy = 0.$$

So, f is constant a.e. on $\mathbb{R}^n$.

Secondly, assume $\alpha \ge 1$ and assume $f \in Q_\alpha(\mathbb{R}^n)$. Assume first that $f \in C^1(\mathbb{R}^n)$, and that f is non-constant. By considering either the real or imaginary part, we may further assume that f is real. Then there exists a point x_0 with $\nabla f(x_0) \ne 0$, and by the rotation invariance (Theorem 2.1) we may assume that $\nabla f(x_0)$ is directed along the positive x_1-axis. Then there exist $\delta > 0$ and a small cube I about x_0 on which $\partial f/\partial x_1 > 2\delta$ and $|\partial f/\partial x_k| < \delta$, $k \geq 2$. Let D be the cone $\{x : |x_2| + \cdots + |x_n| < x_1 < \ell(I)/2\}$; then if $x, y \in I$ and $x - y \in D$, by the mean value theorem, $f(x) - f(y) > \delta(x_1 - y_1)$. Consequently,

$$\begin{aligned}
&\int_I \int_I \frac{|f(x) - f(y)|^2}{|x - y|^{n+2\alpha}}\, dx\, dy \\
&\geq \int_{I/2} \int_{z \in D} \frac{\delta^2 z_1^2}{|z|^{n+2\alpha}}\, dz\, dx \\
&= c\delta^2 |I| \int_0^{\ell(I)/2} \frac{z_1^2}{z_1^{1+2\alpha}}\, dz_1 = \infty,
\end{aligned}$$

a contradiction. Thus, if $f \in Q_\alpha(\mathbb{R}^n) \cap C^1(\mathbb{R}^n)$, then f is constant.

Now, $Q_\alpha(\mathbb{R}^n)$ is translation invariant by Theorem 2.1, and it follows by Minkowski's inequality that if $f \in Q_\alpha(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$, then $f * g \in Q_\alpha(\mathbb{R}^n)$. In particular, if $g \in C_0^\infty(\mathbb{R}^n)$, then $f * g \in Q_\alpha(\mathbb{R}^n) \cap C^1(\mathbb{R}^n)$, and thus (assuming $\alpha \geq 1$) $f * g$ is constant. Finally, choosing a sequence $g_n \geq 0$ with $\int g_n = 1$ and $\operatorname{supp} g_n$ shrinking to 0, $f * g_n \to f$ a.e., and it follows that f is a.e. constant, which completes the proof of (ii). (Note that the first case uses large cubes and the second case small cubes to show that f has to be constant.)

(iii). Case 1: $-n/2 \leq \alpha < 0$. On the one hand, by (i), $Q_\alpha(\mathbb{R}^n) \subseteq Q_{-n/2}(\mathbb{R}^n) = BMO(\mathbb{R}^n)$. On the other hand, if $f \in BMO(\mathbb{R}^n)$ and I is a cube, then for every $y \in \mathbb{R}^n$ with $|y| < \ell(I)$

$$\begin{aligned}
&\int_I |f(x+y) - f(x)|^2\, dx \\
&\leq \int_I 2\big(|f(x+y) - f(2I)|^2 + |f(x) - f(2I)|^2\big)\, dx \\
&\leq 4 \int_{2I} |f(x) - f(2I)|^2\, dx \leq C|I| \|f\|_{BMO(\mathbb{R}^n)}^2
\end{aligned}$$

and thus, since $\alpha < 0$,

$$\begin{aligned}
&\int_{|y|<\ell(I)} \int_I |f(x+y) - f(x)|^2\, dx \frac{dy}{|y|^{n+2\alpha}} \\
&\leq C|I| \|f\|_{BMO(\mathbb{R}^n)}^2 \int_{|y|<\ell(I)} \frac{dy}{|y|^{n+2\alpha}} \\
&= C[\ell(I)]^{n-2\alpha} \|f\|_{BMO(\mathbb{R}^n)}^2.
\end{aligned}$$

By Lemma 2.2, $f \in Q_\alpha(\mathbb{R}^n)$ and hence $Q_\alpha(\mathbb{R}^n) = BMO(\mathbb{R}^n)$ for $\alpha \in [-n/2, 0)$.

Case 2: $\alpha \in (-\infty, -n/2]$. In this case, $BMO(\mathbb{R}^n) \subseteq Q_\alpha(\mathbb{R}^n)$ is known by (i). Now, let $f \in Q_\alpha(\mathbb{R}^n)$ and let I be a cube. If $x, y \in I$, then the set $\{z \in I : \min(|x - z|, |y - z|) > \frac{1}{8}\ell(I)\}$ has measure at least $\frac{1}{2}|I|$, and thus, using $-2\alpha - n \geq 0$,

$$\int_I \min(|x - z|^{-2\alpha-n}, |y - z|^{-2\alpha-n})\, dz > c\ell(I)^{-2\alpha-n}|I| = c\ell(I)^{-2\alpha}.$$

Consequently,

$$\begin{aligned}
&|I|^{-2}\int_I\int_I |f(x)-f(y)|^2\,dx\,dy\\
&\le C\ell(I)^{2\alpha-2n}\int_I\int_I\int_I \min\bigl(|x-z|^{-2\alpha-n},|y-z|^{-2\alpha-n}\bigr)|f(x)-f(y)|^2\,dx\,dy\,dz\\
&\le C\ell(I)^{2\alpha-2n}\int_I\int_I\int_I \min\bigl(|x-z|^{-2\alpha-n},|y-z|^{-2\alpha-n}\bigr)\\
&\qquad\qquad\times\bigl(|f(x)-f(z)|^2+|f(y)-f(z)|^2\bigr)\,dx\,dy\,dz\\
&\le C\ell(I)^{2\alpha-n}\left(\int_I\int_I\frac{|f(x)-f(z)|^2}{|x-z|^{2\alpha+n}}\,dx\,dz+\int_I\int_I\frac{|f(y)-f(z)|^2}{|y-z|^{2\alpha+n}}\,dy\,dz\right)\\
&= C\ell(I)^{2\alpha-n}\int_I\int_I\frac{|f(x)-f(y)|^2}{|x-y|^{2\alpha+n}}\,dx\,dy
\end{aligned}\tag{2.2}$$

Hence the left hand side of (2.2) is bounded as I ranges over all cubes, which means that $f\in BMO(\mathbb{R}^n)$, cf. (5.2). Thus $Q_\alpha(\mathbb{R}^n)\subseteq BMO(\mathbb{R}^n)$, and we have shown $Q_\alpha(\mathbb{R}^n)=BMO(\mathbb{R}^n)$. □

We may now easily verify that $Q_\alpha(\mathbb{R}^n)$ is a Banach space.

Theorem 2.5. *$Q_\alpha(\mathbb{R}^n)$ is complete, and thus a Banach space.*

Proof. Let $\{f_m\}$ be a Cauchy sequence in $Q_\alpha(\mathbb{R}^n)$. By Theorem 2.3 and its proof, $Q_\alpha(\mathbb{R}^n)\subseteq BMO(\mathbb{R}^n)$ with the inclusion map bounded. Hence, $\{f_m\}$ is a Cauchy sequence in $BMO(\mathbb{R}^n)$ too, and $f_m\to f$ in $BMO(\mathbb{R}^n)$ for some f. It follows easily, using Fatou's lemma, that for every $k\ge 1$, $\|f-f_k\|_{Q_\alpha(\mathbb{R}^n)}\le \limsup_{m\to\infty}\|f_m-f_k\|_{Q_\alpha(\mathbb{R}^n)}$, which implies that $f_k\to f$ in $Q_\alpha(\mathbb{R}^n)$ too. □

The following result relates Q_α spaces defined in different dimensions.

Theorem 2.6. *Let $-\infty<\alpha<\infty$. Let f be a function on $\mathbb{R}^n$, $n\ge 1$, and define F on $\mathbb{R}^{n+1}$ by $F(x,t)=f(x)$, $x\in\mathbb{R}^n$, $t\in\mathbb{R}$. Then $F\in Q_\alpha(\mathbb{R}^{n+1})\iff f\in Q_\alpha(\mathbb{R}^n)$.*

Proof. A cube in $\mathbb{R}^{n+1}$ can be written $I\times[a,a+\ell(I)]$ for a cube I in $\mathbb{R}^n$ and a real number a. Thus,

$$\begin{aligned}
&\|F\|^2_{Q_\alpha(\mathbb{R}^{n+1})}\\
&=\sup_{I,a}[\ell(I)]^{2\alpha-(n+1)}\int_I\int_I\int_a^{\ell(I)+a}\int_a^{\ell(I)+a}\frac{|f(x)-f(y)|^2}{|(x,t)-(y,u)|^{n+1+2\alpha}}\,dt\,du\,dx\,dy.
\end{aligned}$$

The multiple integral is independent of a, so we may take $a = 0$. Moreover, letting $s = t - u$, and assuming $\alpha > -n/2$, as we may by Theorem 2.3(iii),

$$\begin{aligned}
&\int_0^{\ell(I)} \int_0^{\ell(I)} \frac{dt\,du}{|(x,t)-(y,u)|^{n+1+2\alpha}} \\
&\asymp \int_0^{\ell(I)} \int_0^{\ell(I)} \frac{dt\,du}{|x-y|^{n+1+2\alpha} + |t-u|^{n+1+2\alpha}} \\
&\le \ell(I) \int_{-\infty}^{\infty} \frac{ds}{|x-y|^{n+1+2\alpha} + |s|^{n+1+2\alpha}} \\
&= C\ell(I)|x-y|^{-n-2\alpha},
\end{aligned}$$

while a similar opposite inequality follows by considering only t and u with $|t-u| < |x-y|$. Consequently,

$$\|F\|^2_{Q_\alpha(\mathbb{R}^{n+1})} \asymp \sup_I [\ell(I)]^{2\alpha-n} \int_I \int_I \frac{|f(x)-f(y)|^2}{|x-y|^{n+2\alpha}}\,dx\,dy = \|f\|^2_{Q_\alpha(\mathbb{R}^n)}.$$

$\square$

Connection with Besov spaces. Denote the homogeneous Besov spaces on $\mathbb{R}^n$ by $\Lambda^{p,q}_\alpha(\mathbb{R}^n)$. We refer to e.g. [16] for a general definition; if $0 < \alpha < 1$ and $1 \le p, q < \infty$, then $\Lambda^{p,q}_\alpha(\mathbb{R}^n)$ consists of all measurable functions f such that

$$\int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} |f(x+y) - f(x)|^p \,dx \right]^{q/p} \frac{dy}{|y|^{n+q\alpha}} < \infty, \tag{2.3}$$

and if $0 < \alpha < 1$ and $1 \le p < q = \infty$, then $\Lambda^{p,q}_\alpha(\mathbb{R}^n)$ consists of the functions f such that

$$\|f\|_{\Lambda^{p,\infty}_\alpha(\mathbb{R}^n)} = \sup_{y\in\mathbb{R}^n} |y|^{-\alpha} \left[\int_{\mathbb{R}^n} |f(x+y) - f(x)|^p \,dx \right]^{1/p} < \infty. \tag{2.4}$$

Theorem 2.7. *Let $n \ge 2$ and $0 < \alpha < 1$, or $n = 1$ and $0 < \alpha < 1/2$.*

(i) *If $q \le 2$, then $\Lambda^{n/\alpha,q}_\alpha(\mathbb{R}^n) \subseteq Q_\alpha(\mathbb{R}^n)$.*
(ii) *If $\beta > \alpha$ and $q \le \infty$, then $\Lambda^{n/\beta,q}_\beta(\mathbb{R}^n) \subseteq Q_\alpha(\mathbb{R}^n)$.*

Remark 2.8. In the case $n = 1$ and $\alpha = 1/2$, it is seen by (1.3) and (2.3) that

$$Q_{1/2}(\mathbb{R}) = \left\{ f : \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|f(x)-f(y)|^2}{|x-y|^2}\,dx\,dy < \infty \right\} = \Lambda^{2,2}_{1/2}(\mathbb{R}),$$

which coincides with the Sobolev space $L^2_{1/2}(\mathbb{R})$. Thus (i) holds in this case too, while (ii) holds only for $q \le 2$.

In particular, $Q_{1/2}(\mathbb{R})$ contains non-constant functions, and it is thus clear that the inclusion $Q_{1/2}(\mathbb{R}) \supset Q_\beta(\mathbb{R})$, $\beta > 1/2$, is strict (cf. Theorem 2.3 and Remark 2.4).

Proof. (i). Since $\Lambda_\alpha^{n/\alpha,q}(\mathbb{R}^n) \subset \Lambda_\alpha^{n/\alpha,2}(\mathbb{R}^n)$ [16, Chapter 3, Theorem 4], we may assume that $q=2$. Thus, assume that $f \in \Lambda_\alpha^{n/\alpha,2}(\mathbb{R}^n)$. By Hölder's inequality with exponents $n/2\alpha$ and $n/(n-2\alpha)$ we get, for any cube I in $\mathbb{R}^n$,

$$\begin{aligned}&\int_{|y|<\ell(I)}\int_I |f(x+y)-f(x)|^2\,dx\,\frac{dy}{|y|^{n+2\alpha}}\\&\le |I|^{\frac{n-2\alpha}{n}}\int_{\mathbb{R}^n}\left[\int_I |f(x+y)-f(x)|^{n/\alpha}\,dx\right]^{2\alpha/n}\frac{dy}{|y|^{n+2\alpha}},\end{aligned}$$

which gives $f \in Q_\alpha(\mathbb{R}^n)$ by (2.3) and Lemma 2.2.

(ii). Since $\Lambda_\beta^{n/\beta,q}(\mathbb{R}^n) \subset \Lambda_\gamma^{n/\gamma,q}(\mathbb{R}^n)$ for $\beta > \gamma$ [16, Chapter 3, Theorem 5], we may assume that $\alpha < \beta < 1$, and in the case $n=1$ further $\beta \le 1/2$. If $f \in \Lambda_\beta^{n/\beta,q}(\mathbb{R}^n) \subseteq \Lambda_\beta^{n/\beta,\infty}(\mathbb{R}^n)$, then for any cube I in $\mathbb{R}^n$ we apply Hölder's inequality with exponents $n/2\beta$ and $n/(n-2\beta)$ together with (2.4) to get

$$\begin{aligned}&\int_{|y|<\ell(I)}\int_I |f(x+y)-f(y)|^2\,dx\frac{dy}{|y|^{n+2\alpha}}\\&\le |I|^{\frac{n-2\beta}{n}}\int_{|y|<\ell(I)}\left[\int_I |f(x+y)-f(x)|^{n/\beta}\,dx\right]^{2\beta/n}\frac{dy}{|y|^{n+2\alpha}}\\&\le \ell(I)^{n-2\beta}\int_{|y|<\ell(I)}|y|^{2\beta-n-2\alpha}\,dy\,\|f\|^2_{\Lambda_\beta^{n/\beta,\infty}(\mathbb{R}^n)}\\&\le C\ell(I)^{n-2\alpha}\|f\|^2_{\Lambda_\beta^{n/\beta,\infty}(\mathbb{R}^n)}.\end{aligned}$$

So, $f \in Q_\alpha(\mathbb{R}^n)$ and the result follows. ☐

Remark 2.9. The inclusions in Theorem 2.7 are the only possible for these α. First, if $\Lambda_\beta^{p,q}(\mathbb{R}^n) \subseteq Q_\alpha(\mathbb{R}^n)$, then the inclusion mapping is bounded by the closed graph theorem, and since for any $t>0$, the norm of the dilation $f_t(x) = f(tx)$ satisfies $\|f_t\|_{Q_\alpha(\mathbb{R}^n)} = \|f\|_{Q_\alpha(\mathbb{R}^n)}$ by Theorem 2.1, while $\|f_t\|_{\Lambda_\beta^{p,q}(\mathbb{R}^n)} = t^{\beta-n/p}\|f\|_{\Lambda_\beta^{p,q}(\mathbb{R}^n)}$ (for a suitable choice of norm), we see that necessarily $\beta - n/p = 0$ and thus $p = n/\beta$.

Secondly, the following example shows that $\Lambda_\beta^{n/\beta,q}(\mathbb{R}^n) \not\subseteq Q_\alpha(\mathbb{R}^n)$ for $\beta < \alpha$ or $\beta = \alpha$ and $q > 2$.

Example 2.10. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be a fixed function such that $\hat\varphi$ has support in the unit ball and $\varphi \ne 0$ on the cube $[-3\pi, 3\pi]^n$. (Such functions are easily constructed, for example as a dilation $\varphi_1(\delta x)$ with $\hat\varphi_1 \in C_0^\infty$, $\hat\varphi_1 \ge 0$ and δ small.)

Let $(a_k)_1^\infty$ be a sequence real numbers with $\sum_{k=1}^\infty a_k^2 < \infty$, and define

$$g(x) = \sum_{k=1}^\infty a_k \exp(2^k x_1 i),$$

where x_1 is the first coordinate of x. Let $f = \varphi g$. It is then easily seen, by the definition [16, p. 51], that for every p, q and α

$$\|f\|_{\Lambda_\alpha^{p,q}(\mathbb{R}^n)} \asymp \|(2^{k\alpha} a_k)_1^\infty\|_{\ell^q} = \Big(\sum_{k=1}^\infty 2^{qk\alpha} |a_k|^q\Big)^{1/q}$$

(with the usual modification if $q = \infty$).

In particular, if $\sum_k 2^{2k\alpha} |a_k|^2 < \infty$, then $f \in \Lambda_\alpha^{n/\alpha,2}(\mathbb{R}^n)$, and thus $f \in Q_\alpha(\mathbb{R}^n)$ by Theorem 2.7, provided $n \geq 2$ and $0 < \alpha < 1$ or $n = 1$ and $0 < \alpha \leq 1/2$.

Conversely, if $f \in Q_\alpha(\mathbb{R}^n)$, then, choosing $I = [-\pi, \pi]^n$ in Lemma 2.2,

$$\int_{|y|<2\pi} \int_I |f(x+y) - f(x)|^2 \, dx \, \frac{dy}{|y|^{n+2\alpha}} < \infty.$$

Since

$$f(x+y) - f(x) = \varphi(x+y)\big(g(x+y) - g(x)\big) + g(x)\big(\varphi(x+y) - \varphi(x)\big),$$

and $|\varphi(x+y)| \geq c > 0$ for $x \in I$ and $|y| < 2\pi$,

$$\begin{aligned}
&|g(x+y) - g(x)|^2 \\
&\leq C\big|\varphi(x+y)\big(g(x+y) - g(x)\big)\big|^2 \\
&\leq C|f(x+y) - f(x)|^2 + C|g(x)|^2 |\varphi(x+y) - \varphi(x)|^2.
\end{aligned}$$

Moreover,

$$|\varphi(x+y) - \varphi(x)| \leq C|y|,$$

and thus

$$\begin{aligned}
&\int_{|y|<2\pi} \int_I |g(x)|^2 |\varphi(x+y) - \varphi(x)|^2 \, dx \, \frac{dy}{|y|^{n+2\alpha}} \\
&\leq C \int_{\mathbb{R}^n} |g(x)|^2 \int_{|y|<2\pi} |y|^{2-2\alpha-n} \, dy < \infty.
\end{aligned}$$

Consequently, using Parseval's formula and writing $y = (y_1, y')$, $y' \in \mathbb{R}^{n-1}$,

$$
\begin{aligned}
\infty &> \int_{|y|<1} \int_{x\in I} |g(x+y)-g(x)|^2\,dx\,\frac{dy}{|y|^{n+2\alpha}} \\
&= (2\pi)^n \int_{|y|<1} \sum_{k=1}^{\infty} \big|a_k(e^{2^k y_1 i}-1)\big|^2 \frac{dy}{|y|^{n+2\alpha}} \\
&\geq c \sum_{k=1}^{\infty} |a_k|^2 \int_{|y'|<y_1<2^{-k}} |2^k y_1|^2 \frac{dy}{y_1^{n+2\alpha}} \\
&\geq c \sum_{k=1}^{\infty} |a_k|^2 \int_0^{2^{-k}} 2^{2k} y_1^{1-2\alpha}\,dy_1 \\
&= c \sum_{k=1}^{\infty} 2^{2k\alpha} |a_k|^2.
\end{aligned}
$$

We have thus shown that, for α as in Theorem 2.7,

$$
f \in Q_\alpha(\mathbb{R}^n) \iff \sum_{k=1}^{\infty} 2^{2k\alpha}|a_k|^2 < \infty.
$$

Choosing e.g. $a_k = 2^{-k\alpha}k^{-1}$, we obtain a function $f \in Q_\alpha(\mathbb{R}^n) \setminus Q_\beta(\mathbb{R}^n)$ for every $\beta > \alpha$, justifying the claim in Remark 2.4. Similarly, if $\beta < \alpha$ or $\beta = \alpha$ and $q \geq 2$, suitable choices of a_k yields $f \in \Lambda_\beta^{n/\beta,q}(\mathbb{R}^n) \setminus Q_\alpha(\mathbb{R}^n)$ as asserted in Remark 2.9.

Remark 2.11. For $\alpha \leq 0$, it may be shown that if f is as in Example 2.10, then

$$
f \in Q_\alpha(\mathbb{R}^n) \iff \begin{cases} \sum_1^\infty |a_k|^2 < \infty, & \alpha < 0, \\ \sum_1^\infty k|a_k|^2 < \infty, & \alpha = 0. \end{cases}
$$

In particular, this shows that $Q_0(\mathbb{R}^n) \subsetneq Q_\alpha(\mathbb{R}^n) = BMO(\mathbb{R}^n)$, $\alpha < 0$.

3. Poisson extension

In this section, we discuss differences and similarities between $Q_\alpha(\mathbb{R}^n)$, $\alpha \in (0,1)$, and $BMO(\mathbb{R}^n)$ with respect to Poisson extensions to $\mathbb{R}^{n+1}_+$.

Let f be any measurable function on $\mathbb{R}^n$ that satisfies

$$
\int_{\mathbb{R}^n} \frac{|f(x)|}{1+|x|^{n+1}}\,dx < \infty. \tag{3.1}
$$

Its Poisson integral is defined by

$$
f(x,t) = \int_{\mathbb{R}^n} P_t(x-y)f(y)\,dy, \tag{3.2}
$$

where

$$
P_t(x) = \frac{c_n t}{(t^2+|x|^2)^{\frac{n+1}{2}}}, \qquad c_n = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}.
$$

Define the gradient of $f(x,t)$ by

$$|\nabla f(x,t)|^2 = \left|\frac{\partial f(x,t)}{\partial t}\right|^2 + \sum_{j=1}^{n}\left|\frac{\partial f(x,t)}{\partial x_j}\right|^2,$$

and the Carleson box based on a cube I by

$$S(I) = I \times (0,\ell(I)] = \{(x,t) \in \mathbb{R}^{n+1}_+ : x \in I, t \in (0,\ell(I)]\}.$$

We extend a lemma of Stegenga [18] from one dimension to higher dimensions.

Lemma 3.1. *Let I and J be cubes in $\mathbb{R}^n$ centered at x_0 with $\ell(J) \geq 3\ell(I)$ and let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. For $\alpha \in (0,1)$, there is a constant C independent of f, I and J such that*

$$\begin{aligned}
&\int_{S(I)} |\nabla f(x,t)|^2 t^{1-2\alpha}\,dx\,dt \\
&\leq C\int_J\int_J \frac{|f(x)-f(y)|^2}{|x-y|^{n+2\alpha}}\,dx\,dy \\
&+ C[\ell(I)]^{n+2(1-\alpha)}\left(\int_{\mathbb{R}^n\setminus\frac{2}{3}J}|f(x)-f_J|\frac{dx}{|x-x_0|^{n+1}}\right)^2. \qquad (3.3)
\end{aligned}$$

Proof. Without loss of generality, we assume that x_0 is the origin. Also, let φ be a function with $0 \leq \varphi \leq 1$ such that $\varphi = 1$ on $\frac{2}{3}J$, $\operatorname{supp}\varphi \subset \frac{3}{4}J$, and

$$|\varphi(x)-\varphi(y)| \leq C[\ell(J)]^{-1}|x-y|, \qquad x,y \in \mathbb{R}^n. \qquad (3.4)$$

Write $\tilde{\varphi} = 1-\varphi$. Then we have

$$f = f_J + (f-f_J)\varphi + (f-f_J)\tilde{\varphi} = f_1 + f_2 + f_3.$$

We also have

$$f(x,t) = f_1(x,t) + f_2(x,t) + f_3(x,t)$$

for the corresponding Poisson integrals. In the integral with the gradient square, f_1 contributes nothing since it is constant.

We have

$$\frac{\partial f(x,s)}{\partial s} = \int_{\mathbb{R}^n}\frac{\partial P_s(y)}{\partial s}[f(x+y)-f(x)]\,dy$$

and therefore by the elementary estimates

$$\left|\frac{\partial P_s(y)}{\partial s}\right| \leq cs^{-n-1}, \qquad \left|\frac{\partial P_s(y)}{\partial s}\right| \leq c|y|^{-n-1},$$

we see that

$$\begin{aligned}
&\left\|\frac{\partial f(\cdot,s)}{\partial s}\right\|_{L^2(\mathbb{R}^n)} \\
&\leq Cs^{-n-1}\int_{|y|\leq s}\|f(\cdot+y)-f\|_{L^2(\mathbb{R}^n)}\,dy \\
&+ C\int_{|y|>s}\|f(\cdot+y)-f\|_{L^2(\mathbb{R}^n)}\frac{dy}{|y|^{n+1}}.
\end{aligned}$$

Next set $y = r\xi \in \mathbb{R}^n$, with $r = |y|$, and $|\xi| = 1$. Then with

$$\Omega(r) = \int_{|\xi|=1} \|f(\cdot + r\xi) - f\|_{L^2(\mathbb{R}^n)} d\xi$$

we write

$$\begin{aligned} A &= \|\frac{\partial f(x,s)}{\partial s}\|_{L^2(\mathbb{R}^n)} \\ &\le Cs^{-n-1} \int_0^s \Omega(r) r^{n-1} dr + C \int_s^\infty \Omega(r) r^{-2} dr. \end{aligned}$$

Therefore, by Hardy's inequalities [19, p. 272],

$$\int_0^\infty s^{1-2\alpha} A^2 ds \le C \int_0^\infty [\Omega(r)]^2 r^{-1-2\alpha} dr.$$

Note that by Hölder's inequality,

$$[\Omega(r)]^2 \le C \int_{|\xi|=1} \|f(\cdot + r\xi) - f\|^2_{L^2(\mathbb{R}^n)} d\xi.$$

Substituting this in the above leads to the bound

$$\begin{aligned} &C \int_{|\xi|=1} \int_0^\infty \|f(\cdot + r\xi) - f\|^2_{L^2(\mathbb{R}^n)} r^{-1-2\alpha} dr d\xi \\ &= C \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} |f(x+y) - f(x)|^2 \, dx \right] \frac{dy}{|y|^{n+2\alpha}}. \end{aligned}$$

In the same way we can prove

$$\begin{aligned} &\int_0^\infty t^{1-2\alpha} \|\frac{\partial f(\cdot,t)}{\partial x_j}\|^2_{L^2(\mathbb{R}^n)} dt \\ &\le C \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} |f(x+y) - f(x)|^2 \, dx \right] \frac{dy}{|y|^{n+2\alpha}}. \end{aligned}$$

Now, we obtain

$$\begin{aligned} &\int_{S(I)} |\nabla f_2(x,t)|^2 t^{1-2\alpha} \, dx \, dt \\ &\le \int_0^\infty \left[\int_{\mathbb{R}^n} |\nabla f_2(x,t)|^2 t^{1-2\alpha} \, dx \right] dt \\ &\le C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f_2(x) - f_2(y)|^2}{|x-y|^{n+2\alpha}} \, dx \, dy \\ &= \iint_{x,y \in J} \cdots + \iint_{x \notin J, y \in \frac{3}{4}J} \cdots + \iint_{y \notin J, x \in \frac{3}{4}J} \cdots \\ &= B_1 + B_2 + B_3. \end{aligned}$$

As to B_1, we have using (3.4)

$$|f_2(x) - f_2(y)| \le |f(x) - f(y)| + C[\ell(J)]^{-1} |x-y| |f(y) - f_J|.$$

Thus, we need only estimate

$$
\begin{aligned}
&[\ell(J)]^{-2}\int_J\int_J \frac{|f(y)-f_J|^2}{|x-y|^{n+2\alpha-2}}\,dx\,dy\\
&=[\ell(J)]^{-2}\int_J |f(y)-f_J|^2\left[\int_J |x-y|^{2-n-2\alpha}\,dx\right]dy\\
&\le C[\ell(J)]^{-2\alpha}\int_J |f(y)-f_J|^2\,dy\\
&=C[\ell(J)]^{-2\alpha-n}\int_J\int_J |f(x)-f(y)|^2\,dx\,dy\\
&\le C\int_J\int_J \frac{|f(x)-f(y)|^2}{|x-y|^{n+2\alpha}}\,dx\,dy,
\end{aligned}
$$

cf. (5.2), which gives the estimate for B_1. The B_2 and B_3 terms are handled similarly as the last estimate, using $f_2(x)=0$ for $x\notin J$.

Moreover,

$$
\begin{aligned}
&|\nabla f_3(x,t)|\\
&\le \int_{\mathbb{R}^n} |\nabla P_t(x-y)|f_3(y)\,dy\\
&\le C\int_{\mathbb{R}^n\setminus\frac{2}{3}J} \frac{|f(y)-f_J|}{(t+|x-y|)^{n+1}}\,dy.
\end{aligned}
$$

If $(x,t)\in S(I)$ then for $y\in\mathbb{R}^n\setminus\frac{2}{3}J$,

$$
\frac{1}{(t+|x-y|)^{n+1}}\le\frac{C}{|y|^{n+1}},
$$

and hence

$$
\begin{aligned}
&\int_{S(I)} |\nabla f_3(x,t)|^2 t^{1-2\alpha}\,dx\,dt\\
&\le \left(\int_{S(I)} t^{1-2\alpha}\,dx\,dt\right)\left(\int_{\mathbb{R}^n\setminus\frac{2}{3}J}\frac{|f(x)-f_J|}{|x|^{n+1}}\,dx\right)^2\\
&\le C[\ell(I)]^{n+2(1-\alpha)}\left(\int_{\mathbb{R}^n\setminus\frac{2}{3}J}\frac{|f(x)-f_J|}{|x|^{n+1}}\,dx\right)^2.
\end{aligned}
$$

Combining the above inequalities, we obtain (3.3). □

With Lemma 3.1, we can characterize functions of $Q_\alpha(\mathbb{R}^n)$ in terms of the Poisson integral. Note that setting $\alpha=0$ in (3.5) yields a characterization of $BMO(\mathbb{R}^n)$ [6].

Theorem 3.2. *Let $\alpha\in(0,1)$ and let $f\in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ satisfy (3.1). Then $f\in Q_\alpha(\mathbb{R}^n)$ if and only if*

$$
\int_{S(I)} |\nabla f(x,t)|^2 t^{1-2\alpha}\,dx\,dt\le M[\ell(I)]^{n-2\alpha} \tag{3.5}
$$

for some $M < \infty$ and all cubes $I \subset \mathbb{R}^n$.

Proof. First, suppose that $f \in Q_\alpha(\mathbb{R}^n)$. Then, by Theorem 2.3, $f \in BMO(\mathbb{R}^n)$ with $\|f\|_{BMO(\mathbb{R}^n)} \leq C\|f\|_{Q_\alpha(\mathbb{R}^n)}$. For convenience, we may assume that the cube I has the origin as its center. Now, let $J = 3I$. Then we have

$$\begin{aligned}
&\int_{\mathbb{R}^n \setminus \frac{2}{3}J} |f(x) - f_J| \frac{dx}{|x|^{n+1}} \\
&\leq \sum_{k=0}^{\infty} \int_{3^k J \setminus 3^{k-1} J} |f(x) - f_J| \frac{dx}{|x|^{n+1}} \\
&\leq C \sum_{k=0}^{\infty} [3^k \ell(J)]^{-(n+1)} \int_{3^k J} |f(x) - f_{3^k J}|\, dx + C \sum_{k=0}^{\infty} (3^k \ell(J))^{-1} |f_{3^k J} - f_J| \\
&\leq \frac{C}{\ell(I)} \left[\|f\|_{BMO(\mathbb{R}^n)} + \sum_{k=1}^{\infty} \frac{k}{3^k} \|f\|_{BMO(\mathbb{R}^n)} \right] \\
&\leq C[\ell(I)]^{-1} \|f\|_{BMO(\mathbb{R}^n)}.
\end{aligned}$$

So,

$$\begin{aligned}
&[\ell(I)]^{n+2(1-\alpha)} \left(\int_{\mathbb{R}^n \setminus \frac{2}{3}J} |f(x) - f_J|\, dx / |x - x_0|^{n+1} \right)^2 \\
&\leq C[\ell(I)]^{n-2\alpha} \|f\|^2_{Q_\alpha(\mathbb{R}^n)}.
\end{aligned}$$

This inequality, the definition of $Q_\alpha(\mathbb{R}^n)$ and Lemma 3.1 imply (3.5), which proves one implication.

Conversely, suppose that (3.5) holds for f. We claim that then

$$\int_{|y| < \ell(I)} \left[\int_I |f(x+y) - f(x)|^2\, dx \right] \frac{dy}{|y|^{n+2\alpha}} \leq CM[\ell(I)]^{n-2\alpha}. \tag{3.6}$$

First, by the triangle inequality,

$$\begin{aligned}
&|f(x+y) - f(x)| \\
&\leq |f(x+y) - f(x+y, |y|)| \\
&+ |f(x+y, |y|)) - f(x, |y|)| \\
&+ |f(x, |y|) - f(x)| \\
&= A_1 + A_2 + A_3.
\end{aligned}$$

For A_3, we employ Minkowski's inequality to get

$$\begin{aligned}&\left(\int_I |A_3|^2\,dx\right)^{\frac12}\\ &\le \int_0^{|y|}\left[\int_I \left|\frac{\partial f(x,t)}{\partial t}\right|^2\,dx\right]^{\frac12} dt\\ &\le \int_0^{|y|}\left[\int_I |\nabla f(x,t)|^2\,dx\right]^{\frac12} dt.\end{aligned}$$

By Hardy's inequality [19, p. 272], we further have

$$\begin{aligned}&\int_{|y|<\ell(I)} \frac{1}{|y|^{n+2\alpha}}\left(\int_I |A_3|^2\,dx\right)dy\\ &\le C\int_0^{\ell(I)}\left\{\int_0^s\left[\int_I |\nabla f(x,t)|^2\,dx\right]^{\frac12} dt\right\}^2 s^{-1-2\alpha}ds\\ &\le C\int_0^{\ell(I)}\left[\int_I |\nabla f(x,t)|^2\,dx\right] t^{1-2\alpha}dt\\ &\le CM[\ell(I)]^{n-2\alpha}.\end{aligned}$$

Since, for $|y|<\ell(I)$, $\int_I |A_1|^2\,dx = \int_{I+y}|A_3|^2\,dx \le \int_{3I}|A_3|^2\,dx$, we similarly obtain

$$\int_{|y|<\ell(I)} \frac{1}{|y|^{n+2\alpha}}\left(\int_I |A_1|^2\,dx\right)dy \le CM[\ell(I)]^{n-2\alpha}.$$

It remains to handle A_2. It is clear that

$$|A_2| \le \int_0^{|y|} |\nabla f(x+te_y,|y|)|dt, \qquad e_y = y/|y|.$$

If $|y|\le \ell(I)$, then an application of Minkowski's inequality gives

$$\begin{aligned}&\left(\int_I |A_2|^2\,dx\right)^{\frac12}\\ &\le \int_0^{|y|}\left[\int_I |\nabla f(x+te_y,|y|)|^2\,dx\right]^{\frac12} dt\\ &\le C\int_0^{|y|}\left[\int_{3I} |\nabla f(x,|y|)|^2\,dx\right]^{\frac12} dt\\ &= C|y|\left[\int_{3I} |\nabla f(x,|y|)|^2\,dx\right]^{\frac12}.\end{aligned}$$

Hence,

$$\begin{aligned}
&\int_{|y|\le \ell(I)} \frac{1}{|y|^{n+2\alpha}} \left(\int_I |A_2|^2\,dx\right)\,dy\\
&\le \int_{S(3I)} |\nabla f(x,t)|^2 t^{1-2\alpha}\,dx\,dt\\
&\le CM[\ell(I)]^{n-2\alpha}.
\end{aligned}$$

Putting these estimates on A_1, A_2 and A_3 together, we see that (3.6) holds, and thus, by Lemma 2.2, f belongs to $Q_\alpha(\mathbb{R}^n)$. □

4. Generalized Carleson measures

Theorem 3.2 shows that it is natural to introduce a generalized Carleson measure. Let $S(I) = I \times (0, \ell(I)]$ be the Carleson box based on the cube $I \subset \mathbb{R}^n$. Given $p > 0$ and a positive Borel measure μ on $\mathbb{R}^{n+1}_+$, we say that μ is a p-Carleson measure if

$$\mu(S(I)) \le M[\ell(I)]^{pn},$$

for some $M < \infty$ and all cubes $I \subset \mathbb{R}^n$. Of course, the case $p = 1$ gives the classical Carleson measures. It follows from Theorem 3.2 that whenever $\alpha \in (0,1)$, $f \in Q_\alpha(\mathbb{R}^n)$ if and only if $|\nabla f(x,t)|^2 t^{1-2\alpha}\,dxdt$ is a $(1-2\alpha/n)$-Carleson measure. In fact, the second half of the proof of Theorem 3.2 shows more generally that if there exists a differentiable extension $F(x,t)$ of f to $\mathbb{R}^{n+1}_+$ which satisfies the condition that $|\nabla F(x,t)|^2 t^{1-2\alpha}\,dxdt$ is a $(1-2\alpha/n)$-Carleson measure, then f is in $Q_\alpha(\mathbb{R}^n)$.

We next want to give a characterization of $Q_\alpha(\mathbb{R}^n)$ by means of an integral on $\mathbb{R}^{n+1}_+$. From now on, denote by $\delta(x)$ the distance of the point $x \in \mathbb{R}^{n+1}_+$ to the boundary $\partial\mathbb{R}^{n+1}_+$. Also, $\tilde{y}$ stands for the symmetric point of $y \in \mathbb{R}^{n+1}_+$ with respect to $\mathbb{R}^n$, that is to say, if $y = (y_1, \dots, y_{n+1})$, then $\tilde{y} = (y_1, \dots, -y_{n+1})$.

Lemma 4.1. *Let μ be a Borel measure on $\mathbb{R}^{n+1}_+$ and $p \in (0,\infty)$. Then μ is a p-Carleson measure if and only if*

$$\sup_{y\in\mathbb{R}^{n+1}_+} \int_{\mathbb{R}^{n+1}_+} \left(\frac{\delta(y)}{|x-\tilde{y}|^{n+1}}\right)^p d\mu(x) < \infty. \tag{4.1}$$

Proof. Sufficiency. Take y to be the center of the Carleson box $S(I)$. If $x \in S(I)$, then $|x-\tilde{y}| < C\ell(I)$ and hence

$$\begin{aligned}
&\int_{\mathbb{R}^{n+1}_+} \left(\frac{\delta(y)}{|x-\tilde{y}|^{n+1}}\right)^p d\mu\\
&\ge \int_{S(I)} \left(\frac{\delta(y)}{|x-\tilde{y}|^{n+1}}\right)^p d\mu\\
&\ge \frac{c\mu(S(I))}{[\ell(I)]^{pn}}.
\end{aligned}$$

Thus, if (4.1) holds, then μ is a p-Carleson measure.

Necessity. Fix $y = (y', y_{n+1}) \in \mathbb{R}^{n+1}_+$. Let $I \subset \mathbb{R}^n$ be the cube with center y' and edge length $\delta(y)$, and for each positive integer $m = 1, 2, \ldots$, define E_m to be the Carleson box $S(2^m I)$. When $x \in E_1$, we have $|x - \tilde{y}| \geq \delta(y)$. Also, if $x \in E_{m+1} \backslash E_m$, then $c2^m\delta(y) \leq |x - \tilde{y}| \leq C2^{m+1}\delta(y)$. Thus, if μ is a p-Carleson measure,

$$\begin{aligned}
&\int_{\mathbb{R}^{n+1}_+} \left(\frac{\delta(y)}{|x-\tilde{y}|^{n+1}}\right)^p d\mu(x) \\
&= \left(\int_{E_1} + \sum_{m=1}^{\infty} \int_{E_{m+1}\backslash E_m}\right) \left(\frac{\delta(y)}{|x-\tilde{y}|^{n+1}}\right)^p d\mu(x) \\
&\leq C\frac{\mu(E_1)}{\delta(y)^{np}} + C\sum_{m=1}^{\infty} \frac{\mu(E_{m+1})}{2^{mp(n+1)}[\delta(y)]^{np}} \\
&\leq C.
\end{aligned}$$

The last constant C is independent of y. We are done. □

For convenience, we will from now on use the same notation to denote f on $\mathbb{R}^n$ and its Poisson extension to the upper half space $\mathbb{R}^{n+1}_+$. So, we have

Theorem 4.2. *Let $\alpha \in (0,1)$ and let $f \in L^2_{\text{loc}}(\mathbb{R}^n)$ satisfy (3.1). Then $f \in Q_\alpha(\mathbb{R}^n)$ if and only if its Poisson integral $f(x) = f(z,t)$ on $\mathbb{R}^{n+1}_+$ satisfies*

$$\sup_{y \in \mathbb{R}^{n+1}_+} \int_{\mathbb{R}^{n+1}_+} \left(\frac{\delta(y)}{|x-\tilde{y}|^{n+1}}\right)^{1-2\alpha/n} |\nabla f(x)|^2 [\delta(x)]^{1-2\alpha}\, dx < \infty. \tag{4.2}$$

Proof. If $1 - 2\alpha/n > 0$, then the proof is immediate by Theorem 3.2 and Lemma 4.1.

It remains only to treat the (less interesting) case $n = 1$ and $\alpha \geq 1/2$. First, if $n = 1$ and $\alpha = 1/2$, then the integral in (4.2) is independent of y and equals $\int_{\mathbb{R}^{n+1}_+} |\nabla f(x)|^2\, dx$, which is finite if and only if $f \in Q_{1/2}(\mathbb{R})$, cf. Remark 2.8.

Finally, the case $n = 1$ and $\alpha > 1/2$ is trivial. Then $f \in Q_\alpha(\mathbb{R})$ only if f is a.e. constant by Theorem 2.3(ii). Similarly, if (4.2) holds, it follows by using Fatou's lemma as $y \to 0$ that $\nabla f(x) = 0$ in $\mathbb{R}^{n+1}_+$, and thus f is a.e. constant. □

Green potential. Observe that Q_p was first defined by the Green potential. Can we characterize $Q_\alpha(\mathbb{R}^n)$ in terms of the Green function for a half space? When $n = 1$, the Green function of the upper half plane is

$$G(x,y) = \log\frac{|x-\tilde{y}|}{|x-y|}. \tag{4.3}$$

When $n > 1$, the Green function of the upper half space $\mathbb{R}^{n+1}_+$ is given by

$$G(x,y) = \frac{1}{|x-y|^{n-1}} - \frac{1}{|x-\tilde{y}|^{n-1}}. \tag{4.4}$$

(cf. [1, p. 65]). Indeed, this function is related to the quantity introduced in Lemma 4.1.

Lemma 4.3. *Let $x, y \in \mathbb{R}^{n+1}_+$. Then*

(i) $G(x,y) \geq \frac{2\delta(x)\delta(y)}{|x-\tilde{y}|^{n+1}}, \quad n \geq 1.$

(ii) $G(x,y) \leq C\frac{\delta(x)\delta(y)}{|x-\tilde{y}|^2|x-y|^{n-1}}, \quad n \geq 2.$

(iii) $G(x,y) \leq \frac{-4\log a}{1-a^2}\frac{\delta(x)\delta(y)}{|x-\tilde{y}|^2}, \quad a \leq \frac{|x-y|}{|x-\tilde{y}|}, \quad n = 1.$

Proof. See [1, p. 68] and [7, p. 289]. □

Remark 4.4. It was proved in [12] that $f \in BMO(\mathbb{R}^n)$ if and only if

$$\sup_{y\in\mathbb{R}^{n+1}_+}\int_{\mathbb{R}^{n+1}_+} G(x,y)|\nabla f(x)|^2\,dx < \infty. \tag{4.5}$$

The following characterization of $Q_\alpha(\mathbb{R}^n)$ by means of the Green potential will employ Lemma 4.3 and Remark 4.4.

Theorem 4.5. *Let $\alpha \in (0,1)$ and let $f \in L^2_{loc}(\mathbb{R}^n)$ satisfy (3.1). Then $f \in Q_\alpha(\mathbb{R}^n)$ if and only if its Poisson integral $f(x) = f(z,t)$ satisfies*

$$\sup_{y\in\mathbb{R}^{n+1}_+}\int_{\mathbb{R}^{n+1}_+}[\delta(x)]^{2(\frac{1}{n}-1)\alpha}[G(x,y)]^{1-2\alpha/n}|\nabla f(x)|^2\,dx < \infty. \tag{4.6}$$

Proof. If (4.6) holds and $1 - 2\alpha/n \geq 0$, then Lemma 4.3(i) and Theorem 4.2 imply $f \in Q_\alpha(\mathbb{R}^n)$. The exceptional case $n = 1$ and $\alpha > 1/2$ follows as in the proof of Theorem 4.2.

Conversely, suppose $f \in Q_\alpha(\mathbb{R}^n)$. We have

$$\begin{aligned}
&\int_{\mathbb{R}^{n+1}_+}[\delta(x)]^{2(\frac{1}{n}-1)\alpha}[G(x,y)]^{1-2\alpha/n}|\nabla f(x)|^2\,dx\\
&= \left\{\int_{|x-y|/|x-\tilde{y}|\leq\frac{1}{2}} + \int_{|x-y|/|x-\tilde{y}|>\frac{1}{2}}\right\}[\delta(x)]^{2(\frac{1}{n}-1)\alpha}[G(x,y)]^{1-2\alpha/n}|\nabla f(x)|^2\,dx\\
&= I_1(y) + I_2(y).
\end{aligned}$$

Since $|x-y| \leq \frac{1}{2}|x-\tilde{y}|$ implies $|x-y| \leq 2\delta(x)$ and, by (4.3) or (4.4), $G(x,y) \geq c/|x-y|^{n-1}$,

$$\begin{aligned}
&I_1(y)\\
&\leq C\int_{|x-y|/|x-\tilde{y}|\leq\frac{1}{2}}\frac{|x-y|^{\frac{2(n-1)\alpha}{n}}}{[\delta(x)]^{\frac{2\alpha(n-1)}{n}}}G(x,y)|\nabla f(x)|^2\,dx\\
&\leq C\int_{|x-y|\leq 2\delta(x)}\left[\frac{|x-y|}{\delta(x)}\right]^{\frac{2\alpha(n-1)}{n}}G(x,y)|\nabla f(x)|^2\,dx\\
&\leq C\int_{\mathbb{R}^{n+1}_+}G(x,y)|\nabla f(x)|^2\,dx.
\end{aligned}$$

We know that $Q_\alpha(\mathbb{R}^n) \subset BMO(\mathbb{R}^n)$, and it follows from the last estimate and Remark 4.4 that $\sup_{y\in\mathbb{R}^{n+1}_+} I_1(y) < \infty$.

As to $I_2(y)$, we apply Lemma 4.3(ii) or (iii) to derive

$$
\begin{aligned}
& I_2(y) \\
& \leq C \int_{|x-y|/|x-\tilde{y}|\geq\frac{1}{2}} |\nabla f(x)|^2 \left[\frac{\delta(x)\delta(y)}{|x-\tilde{y}|^2|x-y|^{n-1}}\right]^{1-2\alpha/n} [\delta(x)]^{2\alpha(\frac{1}{n}-1)} dx \\
& \leq C \int_{|x-y|/|x-\tilde{y}|\geq\frac{1}{2}} |\nabla f(x)|^2 [\delta(x)]^{2\alpha(\frac{1}{n}-1)} \left[\frac{\delta(x)\delta(y)}{|x-\tilde{y}|^{n+1}}\right]^{1-2\alpha/n} dx \\
& \leq C \int_{\mathbb{R}^{n+1}_+} |\nabla f(x)|^2 [\delta(x)]^{1-2\alpha} \left[\frac{\delta(y)}{|x-\tilde{y}|^{n+1}}\right]^{1-2\alpha/n} dx.
\end{aligned}
$$

With the help of Theorem 4.2, we deduce that $\sup_{y\in\mathbb{R}^{n+1}_+} I_2(y) < \infty$. $\square$

5. Mean oscillation

In this section we give an alternative characterization of $Q_\alpha(\mathbb{R}^n)$ in terms of the (square) mean oscillation over cubes. We follow the one-dimensional case given in [9].

We define, for any cube I and an integrable function f on I,

$$f(I) = \frac{1}{|I|}\int_I f(x)dx,$$

the mean of f on I, and

$$\Phi_f(I) = \frac{1}{|I|}\int_I |f(x) - f(I)|^2 dx,$$

the square mean oscillation of f on I. Obviously, $\Phi_f(I) < \infty \Leftrightarrow f \in L^2(I)$; we may extend the definition to all measurable functions f on I by letting $\Phi_f(I) = \infty$ when $f \notin L^2(I)$. Recall that $f \in BMO(\mathbb{R}^n)$ if and only if $\sup_I \Phi_f(I) < \infty$ [10]. Note the well-known identities

$$\frac{1}{|I|}\int_I |f(x) - a|^2 dx = \Phi_f(I) + |f(I) - a|^2 \tag{5.1}$$

for any complex number a, and

$$\frac{1}{|I|^2}\int_I\int_I |f(x) - f(y)|^2\, dx\, dy = 2\Phi_f(I). \tag{5.2}$$

Furthermore, if $I \subset J$, then by (5.1),

$$\Phi_f(I) \leq \frac{1}{|I|}\int_I |f(x) - f(J)|^2 dx \leq \frac{|J|}{|I|}\Phi_f(J), \tag{5.3}$$

and, similarly,

$$|f(I) - f(J)|^2 \leq \frac{|J|}{|I|}\Phi_f(J). \tag{5.4}$$

We define, for any cube I and a measurable function f on I, recalling the notation $\mathcal{D}_k(I)$ for the successive dyadic partitions of I,

$$\Psi_{f,\alpha}(I) = \sum_{k=0}^{\infty} \sum_{J \in \mathcal{D}_k(I)} 2^{(2\alpha-n)k} \Phi_f(J). \tag{5.5}$$

We will prove below that $Q_\alpha(\mathbb{R}^n)$ can be characterized by $\sup_I \Psi_{f,\alpha}(I) < \infty$.

We begin with some simple preliminary lemmas.

Lemma 5.1. *Let $-\infty < \alpha < \infty$. For any cube I and $f \in L^2(I)$, with J ranging over the 2^n subcubes in $\mathcal{D}_1(I)$,*

$$\Phi_f(I) = 2^{-n} \sum_{J \in \mathcal{D}_1(I)} \Phi_f(J) + 2^{-n} \sum_{J \in \mathcal{D}_1(I)} |f(J) - f(I)|^2 \tag{5.6}$$

and

$$\Psi_{f,\alpha}(I) \asymp \sum_{J \in \mathcal{D}_1(I)} \Psi_{f,\alpha}(J) + \sum_{J \in \mathcal{D}_1(I)} |f(J) - f(I)|^2. \tag{5.7}$$

Proof. By (5.1),

$$\begin{aligned}
&\Phi_f(I) \\
&= |I|^{-1} \sum_{J \in \mathcal{D}_1(I)} \int_J |f - f(I)|^2 \\
&= 2^{-n} \sum_{J \in \mathcal{D}_1(I)} \big(\Phi_f(J) + |f(J) - f(I)|^2\big),
\end{aligned}$$

which is (5.6).

Next, this and the definition (5.5) yield, since $\mathcal{D}_k(I) = \bigcup_{J \in \mathcal{D}_1(I)} \mathcal{D}_{k-1}(J)$ for $k \geq 1$,

$$\begin{aligned}
&\Psi_{f,\alpha}(I) \\
&= \Phi_f(I) + \sum_{k=1}^{\infty} \sum_{J \in \mathcal{D}_1(I)} \sum_{K \in \mathcal{D}_{k-1}(J)} 2^{(2\alpha-n)k} \Phi_f(K) \\
&= \Phi_f(I) + \sum_{J \in \mathcal{D}_1(I)} 2^{2\alpha-n} \Psi_{f,\alpha}(J) \\
&\asymp \sum_{J \in \mathcal{D}_1(I)} \big(\Psi_{f,\alpha}(J) + \Phi_f(J) + |f(J) - f(I)|^2\big),
\end{aligned}$$

which yields (5.7), since $\Psi_{f,\alpha}(J) + \Phi_f(J) \asymp \Psi_{f,\alpha}(J)$. □

Lemma 5.2. *If $\alpha < 0$, then $\Psi_{f,\alpha}(I) \asymp \Phi_f(I)$.*

Proof. By Lemma 5.1 and induction, $\sum_{J \in \mathcal{D}_k(I)} 2^{-nk} \Phi_f(J) \leq \Phi_f(I)$, and hence

$$\Phi_f(I) \leq \Psi_{f,\alpha}(I) \leq \sum_{k=0}^{\infty} 2^{2\alpha k} \Phi_f(I).$$

□

Lemma 5.3. *Let $-\infty < \alpha < \infty$. Then, for any cube I and $f \in L^2(I)$,*

$$\Psi_{f,\alpha}(I) \le C[\ell(I)]^{2\alpha-n} \int_I \int_I \frac{|f(x)-f(y)|^2}{|x-y|^{n+2\alpha}}\, dx\, dy.$$

Proof. By (5.5) and (5.2),

$$\begin{aligned}
&\Psi_{f,\alpha}(I)\\
&= \sum_{k=0}^{\infty} \sum_{J\in\mathcal{D}_k(I)} 2^{(2\alpha-n)k} \tfrac12 \big(2^{-nk}|I|\big)^{-2} \int_J \int_J |f(x)-f(y)|^2\, dx\, dy\\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \kappa_I(x,y) |f(x)-f(y)|^2\, dx\, dy, \qquad (5.8)
\end{aligned}$$

where

$$\kappa_I(x,y) = \tfrac12 \sum_{k=0}^{\infty} \sum_{J\in\mathcal{D}_k(I)} 2^{(2\alpha+n)k} |I|^{-2} \chi_J(x)\chi_J(y). \qquad (5.9)$$

First assume that $\alpha > -n/2$. Since $x, y \in J \in \mathcal{D}_k(I)$ implies $|x-y|_\infty \le \ell(J) = 2^{-k}\ell(I)$, and thus $2^k \le \ell(I)/|x-y|_\infty \le C\ell(I)/|x-y|$, we then have

$$\begin{aligned}
&\kappa_I(x,y)\\
&\le \sum_{2^k \le \ell(I)/|x-y|_\infty} 2^{(2\alpha+n)k} |I|^{-2}\\
&\le C\left(\frac{\ell(I)}{|x-y|}\right)^{2\alpha+n} |I|^{-2}\\
&= C[\ell(I)]^{2\alpha-n} |x-y|^{-2\alpha-n};
\end{aligned}$$

furthermore $\kappa_I(x,y) = 0$ unless $x, y \in I$. Consequently, by (5.8), the inequality holds.

The case $\alpha \le -n/2$ follows by Lemma 5.2 and (2.2). ∎

We will prove the converse inequality for $\alpha < 1/2$ in Lemma 5.8 below, but first we give a slightly weaker, but also more general, converse.

Lemma 5.4. *Let $-\infty < \alpha < \infty$. For any cube I and $f \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$,*

$$\begin{aligned}
&[\ell(I)]^{2\alpha-n} \int_I \int_I \frac{|f(x)-f(y)|^2}{|x-y|^{n+2\alpha}}\, dx\, dy\\
&\le \frac{C}{|I|} \int_{|t|_\infty < \ell(I)} \Psi_{f,\alpha}(I+t)\, dt + C\Psi_{f,\alpha}(I)\\
&\le C \sup_{|t|_\infty < \ell(I)} \Psi_{f,\alpha}(I+t).
\end{aligned}$$

Proof. By (5.8) and Fubini's theorem,

$$\begin{aligned}&\frac{1}{|I|}\int_{|t|_\infty<\ell(I)}\Psi_{f,\alpha}(I+t)\,dt\\&=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{1}{|I|}\int_{|t|_\infty<\ell(I)}\kappa_{I+t}(x,y)\,dt\,|f(x)-f(y)|^2\,dx\,dy.\end{aligned}$$

This and (5.8) show that it suffices to verify

$$\begin{aligned}&\frac{1}{|I|}\int_{|t|_\infty<\ell(I)}\kappa_{I+t}(x,y)\,dt+\kappa_I(x,y)\\&\geq c\ell(I)^{2\alpha-n}|x-y|^{-2\alpha-n},\qquad x,y\in I.\end{aligned}\tag{5.10}$$

First, suppose that $x,y\in I$ with $|x-y|_\infty\leq\frac{1}{2}\ell(I)$ and let $l\geq 0$ be such that

$$2^{-l-2}\ell(I)<|x-y|_\infty\leq 2^{-l-1}\ell(I).$$

Then, by (5.9), and noting that $x\notin I+t$ and thus $\kappa_{I+t}(x,y)=0$ when $|t|_\infty>\ell(I)$,

$$\begin{aligned}&\frac{1}{|I|}\int_{|t|_\infty<\ell(I)}\kappa_{I+t}(x,y)\,dt\\&\geq\frac{1}{|I|}\int_{\mathbb{R}^n}\frac{1}{2}\sum_{J\in\mathcal{D}_l(I+t)}2^{(2\alpha+n)l}|I|^{-2}\chi_J(x)\chi_J(y)\,dt\\&=\frac{2^{(2\alpha+n)l}}{2|I|^3}\sum_{J\in\mathcal{D}_l(I)}\int_{\mathbb{R}^n}\chi_{J+t}(x)\chi_{J+t}(y)\,dt\\&\geq c|x-y|^{-2\alpha-n}\ell(I)^{2\alpha-2n}\sum_{J\in\mathcal{D}_l(I)}\int_{\mathbb{R}^n}\chi_{J+t}(x)\chi_{J+t}(y)\,dt.\end{aligned}$$

Now, $\chi_{J+t}(x)\chi_{J+t}(y)=\chi_{J-x}(-t)\chi_{J-y}(-t)$. Thus the final integral equals the volume of $(J-x)\cap(J-y)$, which for each J is a rectangular box with edges at least $\ell(J)-|x-y|_\infty\geq\frac{1}{2}\ell(J)$, and thus volume at least $2^{-n}|J|$. Consequently, the sum over J is at least $2^{-n}|I|$, and (5.10) holds for $|x-y|\leq\frac{1}{2}\ell(I)$.

Finally, if $x,y\in I$ with $|x-y|>\frac{1}{2}\ell(I)$, then, taking $k=0$ in (5.9),

$$\kappa_I(x,y)\geq\tfrac{1}{2}|I|^{-2}\geq c\ell(I)^{2\alpha-n}|x-y|^{-2\alpha-n}$$

and (5.10) holds in this case too. □

As an immediate consequence of Lemmas 5.3 and 5.4, we obtain our alternative characterization of $Q_\alpha(\mathbb{R}^n)$.

Theorem 5.5. *Let $-\infty<\alpha<\infty$. Then $Q_\alpha(\mathbb{R}^n)$ equals the space of all measurable functions f on $\mathbb{R}^n$ such that $\sup_I\Psi_{f,\alpha}(I)$ is finite, where I ranges over all cubes in $\mathbb{R}^n$. Moreover, the square root of this supremum is a norm on $Q_\alpha(\mathbb{R}^n)$, equivalent to $\|f\|_{Q_\alpha(\mathbb{R}^n)}$ as defined above.* □

In order to prove a full converse to the inequality in Lemma 5.3, we begin with two further preliminary lemmas, which also may have independent interest.

Lemma 5.6. *Let $\alpha < 1/2$. Let I, I' and I'' be three cubes of equal size, $|I| = |I'| = |I''|$, such that I' and I'' are adjacent and $I \subset I' \cup I''$. Then, for any $f \in L^1(I' \cup I'')$,*

$$\Phi_f(I) \le \Phi_f(I') + \Phi_f(I'') + |f(I') - f(I'')|^2, \tag{5.11}$$

$$\Psi_{f,\alpha}(I) \le C\big(\Psi_{f,\alpha}(I') + \Psi_{f,\alpha}(I'') + |f(I') - f(I'')|^2\big). \tag{5.12}$$

Proof. It follows from (5.1) that

$$\begin{aligned}
&\Phi_f(I)\\
&\le |I|^{-1}\int_I |f(x) - (f(I') + f(I''))/2|^2\,dx\\
&\le |I|^{-1}\int_{I'\cup I''} |f(x) - (f(I') + f(I''))/2|^2\,dx\\
&= \Phi_f(I') + \Phi_f(I'') + \tfrac{1}{2}|f(I') - f(I'')|^2,
\end{aligned}$$

proving (5.11).

For (5.12), we assume for simplicity that $I' = [0,1)^n$ and $I'' = I' + e_1$, where e_1 is the unit vector $(1, 0, \dots, 0)$; this is no loss of generality by homogeneity. Note that by assumption then $I = I' + te_1$ for some $t \in [0,1]$. For each $j \ge 0$, let $\mathcal{D}_j^* = \mathcal{D}_j(I') \cup \mathcal{D}_j(I'')$ be the set of the 2^{nj+1} dyadic cubes with side 2^{-j} contained in $I' \cup I''$. If $I_j \in \mathcal{D}_j(I)$, then $I_j \subset J \cup (J + 2^{-j}e_1)$ for some $J \in \mathcal{D}_j^*$, and thus by (5.11) applied to J,

$$\Phi_f(I_j) \le \Phi_f(J) + \Phi_f(J + 2^{-j}e_1) + |f(J) - f(J + 2^{-j}e_1)|^2.$$

The 2^{nj} different choices of $I_j \in \mathcal{D}_j(I)$ yield different $J \in \mathcal{D}_j^*$, and summing over all j and I_j we thus obtain,

$$\begin{aligned}
&\Psi_{f,\alpha}(I)\\
&= \sum_{j=0}^{\infty} \sum_{I_j \in \mathcal{D}_j(I)} 2^{(2\alpha-n)j}\Phi_f(I_j)\\
&\le 2\sum_{j=0}^{\infty} \sum_{J \in \mathcal{D}_j^*} 2^{(2\alpha-n)j}\Phi_f(J) + \sum_{j=0}^{\infty} \sum_{J \in \mathcal{D}_j^0} 2^{(2\alpha-n)j}|f(J) - f(J + 2^{-j}e_1)|^2,
\end{aligned} \tag{5.13}$$

where $\mathcal{D}_j^0 = \{J \in \mathcal{D}_j^* : J + 2^{-j}e_1 \in \mathcal{D}_j^*\}$.

The first double sum on the right hand side of (5.13) is just $\Psi_{f,\alpha}(I') + \Psi_{f,\alpha}(I'')$. In order to estimate the final sum, consider a cube $J \in \mathcal{D}_j^0$ for some $j \ge 0$. Let I^* be the smallest dyadic cube that contains $J \cup (J + 2^{-j}e_1)$, and let the edge length of I^* be 2^{-j+m}, where $m \ge 1$. Moreover, for $0 \le l \le m$, let J_l and K_l be the dyadic cubes of edge length 2^{-j+l} that contain J and $J + 2^{-j}e_1$, respectively; thus $J = J_0 \subset J_1 \subset \cdots \subset J_m = I^*$ and $J + 2^{-j}e_1 = K_0 \subset \cdots \subset K_m = I^*$. Using the Cauchy–Schwarz inequality and (5.6), we obtain

$$
\begin{aligned}
&|f(J) - f(J + 2^{-j}e_1)|^2 \\
&\le \Big(\sum_{l=1}^{m} |f(J_{l-1}) - f(J_l)| + \sum_{l=1}^{m} |f(K_l) - f(K_{l-1})|\Big)^2 \\
&\le \Big(2\sum_{l=1}^{\infty} l^{-2}\Big)\Big(\sum_{l=1}^{m} l^2 |f(J_l) - f(J_{l-1})|^2 + \sum_{l=1}^{m} l^2|f(K_l) - f(K_{l-1})|^2\Big) \\
&\le C\sum_{l=1}^{m} l^2\big(\Phi_f(J_l) + \Phi_f(K_l)\big). \qquad (5.14)
\end{aligned}
$$

If $J \cup (J + 2^{-j}e_1) \subseteq I'$ or I'', then $|I^*| \le 1$ and $m \le j$. If $J \subseteq I'$ and $J + 2^{-j}e_1 \subseteq I''$, however, then $I^* = [0, 2)^n$ and $m = j + 1$; in this case we modify (5.14) by observing that $J_j = I'$ and $K_j = I''$ which by the same argument yields

$$
\begin{aligned}
&|f(J) - f(J + 2^{-j}e_1)|^2 \\
&\le C\sum_{l=1}^{j} l^2\big(\Phi_f(J_l) + \Phi_f(K_l)\big) + C|f(I') - f(I'')|^2. \qquad (5.15)
\end{aligned}
$$

We now keep $j \ge 0$ fixed and sum (5.14) or (5.15) for $J \in \mathcal{D}_j^0$. We observe that the cubes J_l and K_l that appear belong to $\mathcal{D}^*_{j-l}$, with $1 \le l \le j$. Moreover, each dyadic cube $J' = \prod_1^n [a_i, b_i)$ in $\mathcal{D}^*_{j-l}$ appears as a J_l or a K_l only for the $J \in \mathcal{D}^*_j$ that are adjacent to either the face $x_1 = a_1$, the face $x_1 = b_1$ or the mid plane $x_1 = (a_1 + b_1)/2$, and there are thus at most $3 \cdot 2^{(n-1)l}$ such J. Consequently, since (5.15) is used for $2^{(n-1)j}$ cubes J,

$$
\begin{aligned}
&\sum_{\mathcal{D}_j^0} |f(J) - f(J + 2^{-j}e_1)|^2 \\
&\le C\sum_{l=1}^{j} \sum_{J\in\mathcal{D}^*_{j-l}} 2^{(n-1)l} l^2 \Phi_f(J) + C2^{(n-1)j}|f(I') - f(I'')|^2.
\end{aligned}
$$

Summing over j we finally obtain, substituting $j = k + l$ and observing that $\sum_{l=1}^{\infty} l^2 2^{(2\alpha-1)l} < \infty$,

$$
\begin{aligned}
&\sum_{j=0}^{\infty} \sum_{J\in\mathcal{D}_j^0} 2^{(2\alpha-n)j} |f(J) - f(J + 2^{-j}e_1)|^2 \\
&\le C\sum_{l=1}^{\infty}\sum_{k=0}^{\infty} \sum_{J\in\mathcal{D}_k(I')\cup\mathcal{D}_k(I'')} 2^{(2\alpha-n)k+(2\alpha-n)l} 2^{(n-1)l} l^2 \Phi_f(J) \\
&+ C\sum_{j=0}^{\infty} 2^{(2\alpha-n)j} 2^{(n-1)j} |f(I') - f(I'')|^2 \\
&= C\Psi_{f,\alpha}(I') + C\Psi_{f,\alpha}(I'') + C|f(I') - f(I'')|^2,
\end{aligned}
$$

which by (5.13) completes the proof of (5.12). □

Lemma 5.7. *Let $-\infty < \alpha < 1/2$. If I and J are any cubes with $I \subset J$ and $\ell(I) = \frac{1}{2}\ell(J)$, then $\Psi_{f,\alpha}(I) \le C\Psi_{f,\alpha}(J)$.*

Proof. Suppose, for notational convenience, that $J = [0,2)^n$ and that $I = \prod_1^n [a_i, a_i + 1)$, with $0 \le a_i \le 1$. Let $w(I) = \mathrm{Card}\{a_i : 0 < a_i < 1\}$; we prove the result by induction on $w(I)$.

If $w(I) = 0$, then I is one of the subcubes in $\mathcal{D}_1(J)$, and the result follows by Lemma 5.1. Otherwise, choose i such that $0 < a_i < 1$ and let I' and I'' be the cubes obtained from I by replacing $[a_i, a_i + 1)$ by $[0,1)$ and $[1,2)$, respectively, keeping the other coordinates unchanged. Then I' and I'' are adjacent, $I \subset I' \cup I'' \subset J$ and $w(I') = w(I'') = w(I) - 1$. By Lemma 5.6, $\Psi_{f,\alpha}(I) \le C\big(\Psi_{f,\alpha}(I') + \Psi_{f,\alpha}(I'') + |f(I') - f(I'')|^2\big)$. Since $\Psi_{f,\alpha}(I'), \Psi_{f,\alpha}(I'') \le C\Psi_{f,\alpha}(J)$ by the induction hypothesis and $|f(I') - f(I'')|^2 \le C\Phi_f(J) \le C\Psi_{f,\alpha}(J)$ by (5.4), the result follows. □

Lemma 5.8. *If $-\infty < \alpha < 1/2$, then, for any cube I and $f \in L^2(I)$,*

$$\Psi_{f,\alpha}(I) \asymp [\ell(I)]^{2\alpha-n} \int_I \int_I \frac{|f(x) - f(y)|^2}{|x-y|^{n+2\alpha}}\,dx\,dy.$$

Proof. One inequality is shown in Lemma 5.3.

To show the other, we may assume that f is extended to $\mathbb{R}^n$ with f constant $= f(I)$ outside I. If $t \in \mathbb{R}^n$ with $|t|_\infty < \ell(I)$, then $I + t$ is contained in a cube J with $\ell(J) = 2\ell(I)$ such that $I \in \mathcal{D}_1(J)$. Since the extended f is constant on the $2^n - 1$ other cubes in $\mathcal{D}_1(J)$, Lemma 5.1 yields $\Psi_{f,\alpha}(J) \le C\Psi_{f,\alpha}(I)$, and then Lemma 5.7 yields

$$\Psi_{f,\alpha}(I + t) \le C\Psi_{f,\alpha}(J) \le C\Psi_{f,\alpha}(I).$$

Finally, Lemma 5.4 yields

$$[\ell(I)]^{2\alpha-n} \int_I \int_I \frac{|f(x) - f(y)|^2}{|x-y|^{n+2\alpha}}\,dx\,dy \le C\Psi_{f,\alpha}(I).$$

□

Remark 5.9. We do not know whether any of Lemmas 5.6, 5.7 and 5.8 holds for $\alpha \ge 1/2$.

6. Wavelets

The purpose of this section is to observe that the well known characterization of $BMO(\mathbb{R}^n)$ in terms of an orthonormal wavelet basis, see Meyer [13, p. 154], extends to $Q_\alpha(\mathbb{R}^n)$. In the sequel, we let

$$\psi_{j,\hat{k},l}(x) = 2^{jn/2}\psi_l(2^j x - \hat{k}), \qquad j \in \mathbb{Z},\ \hat{k} \in \mathbb{Z}^n,\ l = 1, \dots, 2^n - 1.$$

be a 1-regular orthonormal wavelet basis as in [13, Chapter 3]. We adopt the shorter notation $\psi_{j,\hat{k},l} = \psi_\lambda$, with $\lambda \in \Lambda = \mathbb{Z} \times \mathbb{Z}^n \times \{1, \dots, 2^n - 1\}$. For simplicity we consider only the case of wavelets of compact support, and assume thus that the wavelets satisfy the conditions in [13, p. 108]; in particular

$\operatorname{supp}\psi_\lambda \subseteq mI(\lambda)$, where m is a constant (fixed throughout this section) and for every $\lambda = (j, \hat{k}, l) \in \Lambda$ we use $I(\lambda)$ to denote the dyadic cube

$$I(\lambda) = \{x : 2^j x - \hat{k} \in [0,1)^n\}. \tag{6.1}$$

We write $\ell(\lambda) = \ell(I(\lambda))$.

Recall that $\mathcal{D}$ is the set of all dyadic cubes in $\mathbb{R}^n$. For $I \in \mathcal{D}$ and a sequence $\boldsymbol{a} = \big(a(\lambda)\big)_{\lambda\in\Lambda}$, let

$$T_{\boldsymbol{a},\alpha}(I) = |I|^{-1} \sum_{I(\lambda)\subseteq I} \Big(\frac{\ell(I)}{\ell(\lambda)}\Big)^{2\alpha} |a(\lambda)|^2.$$

Lemma 6.1. *If $\alpha > 0$, then for every dyadic cube I and sequence $\boldsymbol{a}$,*

$$T_{\boldsymbol{a},\alpha}(I) \asymp \sum_{k=0}^{\infty} 2^{(2\alpha-n)k} \sum_{J\in\mathcal{D}_k(I)} T_{\boldsymbol{a},0}(J).$$

Proof. The right hand side equals

$$\begin{aligned}
&\sum_{J\in\mathcal{D}(I)} \sum_{I(\lambda)\subseteq J} \Big(\frac{\ell(J)}{\ell(I)}\Big)^{n-2\alpha} |J|^{-1} |a(\lambda)|^2 \\
&= \ell(I)^{2\alpha-n} \sum_{I(\lambda)\subseteq I} |a(\lambda)|^2 \sum_{\substack{J\in\mathcal{D}(I)\\ J\supseteq I(\lambda)}} \ell(J)^{-2\alpha} \\
&\asymp \ell(I)^{2\alpha-n} \sum_{I(\lambda)\subseteq I} |a(\lambda)|^2 \ell(\lambda)^{-2\alpha}.
\end{aligned}$$

□

Observe that the wavelet coefficients of functions in $BMO(\mathbb{R}^n)$ are characterized by

$$\sup_{I\in\mathcal{D}(\mathbb{R}^n)} T_{\boldsymbol{a},0}(I) < \infty$$

[13]. This extends to $Q_\alpha(\mathbb{R}^n)$ as follows.

Theorem 6.2. *Let $0 < \alpha < 1$. If $f \in Q_\alpha(\mathbb{R}^n)$, then the sequence of its wavelet coefficients*

$$a(\lambda) = (f, \psi_\lambda) = \int_{\mathbb{R}^n} f(x)\overline{\psi_\lambda(x)}\, dx,$$

satisfies

$$\sup_{I\in\mathcal{D}} T_{\boldsymbol{a},\alpha}(I) < \infty. \tag{6.2}$$

Conversely, every sequence $a(\lambda)$ satisfying (6.2) is the sequence of wavelet coefficients of a unique $f \in Q_\alpha(\mathbb{R}^n)$; moreover, $\|f\|_{Q_\alpha(\mathbb{R}^n)} \asymp \sup_{I\in\mathcal{D}} T_{\boldsymbol{a},\alpha}(I)^{1/2}$.

Proof. First, let $f \in Q_\alpha(\mathbb{R}^n)$ and $I \in \mathcal{D}$. For $J \in \mathcal{D}_k(I)$, we write

$$f = f_{mJ} + (f - f_{mJ})\chi_{mJ} + (f - f_{mJ})\chi_{\mathbb{R}^n\setminus mJ} = f_1 + f_2 + f_3.$$

Since $\operatorname{supp}\psi_\lambda \subseteq mI(\lambda)$, $(f_3,\psi_\lambda) = 0$ if $I(\lambda) \subseteq J$. On the other hand, the integral of each wavelet ψ_λ is zero. So $(f,\psi_\lambda) = (f_2,\psi_\lambda)$, and, using (5.2),

$$\begin{aligned}
&\sum_{I(\lambda)\subseteq J} |(f,\psi_\lambda)|^2 \\
&\le \sum_\lambda |(f_2,\psi_\lambda)|^2 \\
&= \|f_2\|^2_{L^2(\mathbb{R}^n)} \\
&= |mJ|\Phi_f(mJ) \\
&= \frac{1}{2|mJ|}\int_{mJ}\int_{mJ} |f(x)-f(y)|^2\,dx\,dy.
\end{aligned}$$

This gives that for $J \in \mathcal{D}_k(I)$,

$$\begin{aligned}
&T_{\boldsymbol{a},0}(J) \\
&= \frac{1}{|J|}\sum_{I(\lambda)\subseteq J} |a(\lambda)|^2 \\
&\le \frac{1}{|J||mJ|}\int_{mJ}\int_{mJ} |f(x)-f(y)|^2\,dx\,dy.
\end{aligned}$$

Using Lemma 6.1, we obtain in the same manner as for Lemma 5.3

$$\begin{aligned}
&T_{\boldsymbol{a},\alpha}(I) \\
&\le C\sum_{k=0}^{\infty} 2^{(2\alpha-n)k} \sum_{J\in\mathcal{D}_k(I)} T_{\boldsymbol{a},0}(J) \\
&\le C\int_{mI}\int_{mI} |f(x)-f(y)|^2 \sum_{k=0}^{\infty}\sum_{J\in\mathcal{D}_k(I)} 2^{(2\alpha-n)k}|J|^{-2}\chi_{mJ}(x)\chi_{mJ}(y)\,dx\,dy \\
&\le C\ell(I)^{2\alpha-n}\int_{mI}\int_{mI} \frac{|f(x)-f(y)|^2}{|x-y|^{n+2\alpha}}\,dx\,dy \\
&\le C\|f\|^2_{Q_\alpha(\mathbb{R}^n)}.
\end{aligned}$$

Thus (6.2) follows.

Conversely, suppose that (6.2) holds; multiplying f by a constant, we may assume that $T_{\boldsymbol{a},\alpha}(I) \le 1$ for every dyadic cube I. In particular, $T_{\boldsymbol{a},0}(I) \le T_{\boldsymbol{a},\alpha}(I) \le 1$ for every dyadic cube I, and thus, by [13, Section 5.6],

$$f = \sum_\lambda a(\lambda)\psi_\lambda \in BMO(\mathbb{R}^n),$$

with the sum converging e.g. in the weak* topology on $BMO(\mathbb{R}^n)$. We will show that $f \in Q_\alpha(\mathbb{R}^n)$, with $\|f\|_{Q_\alpha(\mathbb{R}^n)} \le C$.

Fix a (not necessarily dyadic) cube I of dyadic edge length and consider a subcube $J \in \mathcal{D}(I)$. Let $\Lambda_0(J) = \{\lambda \in \Lambda : mI(\lambda) \cap J \ne \emptyset\}$ and partition this set into

$$\begin{aligned}\Lambda_1 = \Lambda_1(J) &= \{\lambda \in \Lambda_0(J) : |I(\lambda)| \le |J|\},\\ \Lambda_2 = \Lambda_2(J) &= \{\lambda \in \Lambda_0(J) : |J| < |I(\lambda)| \le |I|\},\\ \Lambda_3 = \Lambda_3(J) &= \{\lambda \in \Lambda_0(J) : |I| < |I(\lambda)|\}.\end{aligned}$$

Since $\psi_\lambda = 0$ on J unless $\lambda \in \Lambda_0$ we have, on J, $f = f_1 + f_2 + f_3$, where

$$f_i = \sum_{\lambda \in \Lambda_i} a(\lambda)\psi_\lambda.$$

Hence, using the Cauchy–Schwarz inequality,

$$\Phi_f(J) \le 3\big(\Phi_{f_1}(J) + \Phi_{f_2}(J) + \Phi_{f_3}(J)\big). \tag{6.3}$$

We treat the three terms separately. First,

$$\Phi_{f_1}(J) \le \frac{1}{|J|}\|f_1\|_{L^2}^2 = \frac{1}{|J|}\sum_{\lambda\in\Lambda_1} |a(\lambda)|^2. \tag{6.4}$$

Secondly, $|\nabla\psi_\lambda| \le C\ell(\lambda)^{-1}|I(\lambda)|^{-1/2}$, and thus

$$|f_2(x) - f_2(y)| \le C\sum_{\lambda\in\Lambda_2} |a(\lambda)|\ell(\lambda)^{-1}|I(\lambda)|^{-1/2}|x-y|.$$

Consequently, letting $\varepsilon = 1-\alpha$ (for example) and using the Cauchy–Schwarz inequality,

$$\begin{aligned}&\Phi_{f_2}(J)\\ &\le C\ell(J)^2\Big(\sum_{\lambda\in\Lambda_2}|a(\lambda)|\ell(\lambda)^{-1}|I(\lambda)|^{-1/2}\Big)^2\\ &\le C\ell(J)^2\sum_{\lambda\in\Lambda_2}|a(\lambda)|^2\ell(\lambda)^{-2}|I(\lambda)|^{-1}\Big(\frac{\ell(\lambda)}{\ell(J)}\Big)^{\varepsilon}\sum_{\lambda\in\Lambda_2}\Big(\frac{\ell(J)}{\ell(\lambda)}\Big)^{\varepsilon}.\end{aligned}$$

If $\lambda \in \Lambda_2$, then $I(\lambda)$ is a dyadic cube contained in a cube with the same center as J and edge length $(m+1)\ell(\lambda)+\ell(J) \le (m+2)\ell(J)$. Hence, for each $k \ge 1$, there are at most $(m+2)^n$ such cubes $I(\lambda)$ with $\ell(\lambda) = 2^k\ell(J)$. Moreover, there is constant (viz. 2^n-1) number of different λ for each such cube, and thus $\operatorname{Card}\{\lambda\in\Lambda_2 : \ell(\lambda) = 2^k\ell(J)\} \le C$ for each $k \ge 1$. Consequently,

$$\sum_{\lambda\in\Lambda_2}\Big(\frac{\ell(J)}{\ell(\lambda)}\Big)^{\varepsilon} \le \sum_{k=1}^{\infty} C2^{-k\varepsilon} \le C,$$

and thus

$$\Phi_{f_2}(J) \le C\sum_{\lambda\in\Lambda_2}|a(\lambda)|^2\Big(\frac{\ell(J)}{\ell(\lambda)}\Big)^{2-\varepsilon}|I(\lambda)|^{-1}. \tag{6.5}$$

Thirdly, we similarly have

$$\begin{aligned}
&|f_3(x)-f_3(y)| \\
&\le C\sum_{\lambda\in\Lambda_3}|a(\lambda)|\ell(\lambda)^{-1}|I(\lambda)|^{-1/2}|x-y| \\
&\le C|x-y|\sum_{\lambda\in\Lambda_3}\ell(\lambda)^{-1}
\end{aligned}$$

using

$$|a(\lambda)||I(\lambda)|^{-1/2}\le T^{1/2}_{\boldsymbol{a},\alpha}\big(I(\lambda)\big)\le 1.$$

Again, there is a bounded number of terms for each $\ell(\lambda)$, and now $\ell(\lambda)=2^k\ell(I)$, $k\ge 1$; hence

$$|f_3(x)-f_3(y)|\le C|x-y|\ell(I)^{-1}$$

and thus

$$\Phi_{f_3}(J)\le C\ell(J)^2\ell(I)^{-2}. \tag{6.6}$$

Consequently, by (6.3), (6.4), (6.5) and (6.6),

$$\begin{aligned}
&\Phi_f(J) \\
&\le C\frac{1}{|J|}\sum_{\lambda\in\Lambda_1}|a(\lambda)|^2 \\
&+C\sum_{\lambda\in\Lambda_2}|a(\lambda)|^2\Big(\frac{\ell(J)}{\ell(\lambda)}\Big)^{2-\varepsilon}|I(\lambda)|^{-1} \\
&+C\ell(J)^2\ell(I)^{-2}.
\end{aligned}$$

Summing over $J\in\mathcal{D}(I)$ we obtain, cf. (5.5),

$$\begin{aligned}
&\Psi_{f,\alpha}(I) \\
&=\sum_{J\in\mathcal{D}(I)}\Big(\frac{\ell(J)}{\ell(I)}\Big)^{n-2\alpha}\Phi_f(J) \\
&\le C\sum_{J\in\mathcal{D}(I)}\sum_{\lambda\in\Lambda_1(J)}|a(\lambda)|^2\ell(J)^{-2\alpha}\ell(I)^{-n+2\alpha} \\
&+C\sum_{J\in\mathcal{D}(I)}\sum_{\lambda\in\Lambda_2(J)}|a(\lambda)|^2\ell(J)^{n-2\alpha+2-\varepsilon}\ell(\lambda)^{\varepsilon-2-n}\ell(I)^{2\alpha-n} \\
&+C\sum_{J\in\mathcal{D}(I)}\Big(\frac{\ell(J)}{\ell(I)}\Big)^{n-2\alpha+2}.
\end{aligned} \tag{6.7}$$

The final sum equals

$$\sum_{j=0}^{\infty}2^{nj}\big(2^{-j}\big)^{n-2\alpha+2}=\sum_{j=0}^{\infty}2^{-(2-2\alpha)j}=C.$$

In the two double sums, we interchange the order of summation. If λ occurs there, then $\ell(\lambda)\le\ell(I)$ and $mI(\lambda)\cap I\ne\emptyset$; thus, if we let $\mathcal{E}(I)$ be the set of dyadic dubes I' of the same size as I with $mI' \cap I \neq \emptyset$, it follows that $I(\lambda) \in \mathcal{D}(I')$ for some $I' \in \mathcal{E}(I)$.

Fix one such λ, with $\ell(\lambda) = 2^{-k}\ell(I)$. For each $j \le k$, there are at most C cubes $J \in \mathcal{D}_j(I)$ with $\lambda \in \Lambda_1(J)$, each contributing $2^{2\alpha j}|I|^{-1}|a(\lambda)|^2$ to the first double sum in (6.7). Similarly, for each $j > k$, there are at most $C2^{(j-k)n}$ cubes $J \in \mathcal{D}_j(I)$ with $\lambda \in \Lambda_2(J)$, each contributing $2^{-j(n-2\alpha+2-\varepsilon)-k(\varepsilon-2-n)}|I|^{-1}|a(\lambda)|^2$ to the second double sum. Together these yield at most

$$\begin{aligned}
&C|I|^{-1}|a(\lambda)|^2\Big(\sum_{j=0}^{k} 2^{2\alpha j} + \sum_{j=k+1}^{\infty} 2^{-(2-2\alpha-\varepsilon)(j-k)+2\alpha k}\Big)\\
&\le C2^{2\alpha k}|I|^{-1}|a(\lambda)|^2.
\end{aligned}$$

As a consequence, (6.7) yields

$$\begin{aligned}
&\Psi_{f,\alpha}(I)\\
&\le C\sum_{I'\in\mathcal{E}(I)}\sum_{k=0}^{\infty}\sum_{I(\lambda)\in\mathcal{D}_k(I')} 2^{2\alpha k}|I|^{-1}|a(\lambda)|^2 + C\\
&= C\sum_{I'\in\mathcal{E}(I)} T_{\boldsymbol{a},\alpha}(I') + C \le C.
\end{aligned}$$

We have proved that $\Psi_{f,\alpha}(I) \le C$ for every cube I of dyadic edge length. Since the same estimate applies to every translate $I+t$, Lemma 5.4 shows that for every cube I of dyadic edge length,

$$[\ell(I)]^{2\alpha-n}\int_I\int_I \frac{|f(x)-f(y)|^2}{|x-y|^{n+2\alpha}}\,dx\,dy \le C,$$

finally, Remark 1.1 yields $f \in Q_\alpha(\mathbb{R}^n)$ and $\|f\|_{Q_\alpha(\mathbb{R}^n)} \le C$.

Uniqueness of f follows from the uniqueness in $BMO(\mathbb{R}^n)$; if $f, g \in Q_\alpha(\mathbb{R}^n) \subset BMO(\mathbb{R}^n)$ have the same wavelet coefficients, then they define the same linear functional on $H^1(\mathbb{R}^n)$ and thus $f = g$ as elements of $BMO(\mathbb{R}^n)$ (i.e. modulo constants), see again [13, Section 5.6]. □

7. The dyadic counterpart

We define a dyadic analogue of $Q_\alpha(\mathbb{R}^n)$ and give some results relating the two spaces.

Once again, recall that $\mathcal{D}$ is the set of all dyadic cubes in $\mathbb{R}^n$, and define the dyadic distance $\delta(x,y)$ between two points in $\mathbb{R}^n$ by

$$\delta(x,y) = \inf\{\ell(I) : x, y \in I \in \mathcal{D}\}.$$

(The dyadic distance is infinite between points in different octants.) Note that

$$|x-y| \le \sqrt{d}\,\delta(x,y). \tag{7.1}$$

The space $Q^d_\alpha(\mathbb{R}^n)$ is defined as the space of all (measurable) functions f on $\mathbb{R}^n$ such that

$$\|f\|^2_{Q^d_\alpha(\mathbb{R}^n)} = \sup_{I\in\mathcal{D}}[\ell(I)]^{2\alpha-n}\int_I\int_I \frac{|f(x)-f(y)|^2}{\delta(x,y)^{n+2\alpha}}\,dx\,dy < \infty. \tag{7.2}$$

Note that $\|f\|_{Q^d_\alpha(\mathbb{R}^n)} = 0$ if and only if f is a.e. constant in each octant. $Q^d_\alpha(\mathbb{R}^n)$ is a Banach space if we regard it as a space of functions modulo such functions.

We have analogues of Lemma 5.8 and Theorem 5.5.

Lemma 7.1. *Let $-\infty < \alpha < \infty$. Then, for any cube $I \in \mathcal{D}$ and $f \in L^2(I)$,*

$$\Psi_{f,\alpha}(I) \asymp [\ell(I)]^{2\alpha-n}\int_I\int_I \frac{|f(x)-f(y)|^2}{\delta(x,y)^{n+2\alpha}}\,dx\,dy.$$

Proof. Suppose first $\alpha > -n/2$. If I is a dyadic cube and $x, y \in I$, then $x, y \in J$ for some $J \in \mathcal{D}_k(I)$ if and only if $\delta(x,y) \le 2^{-k}\ell(I)$, and thus, by (5.9),

$$\begin{aligned}
&\kappa_I(x,y)\\
&= \tfrac12 \sum_{2^k\le \ell(I)/\delta(x,y)} 2^{(2\alpha+n)k}|I|^{-2}\\
&\asymp \Big(\frac{\ell(I)}{\delta(x,y)}\Big)^{2\alpha+n}|I|^{-2}\\
&= \ell(I)^{2\alpha-n}\delta(x,y)^{-2\alpha-n}.
\end{aligned} \tag{7.3}$$

The result follows by (5.8).

If $\alpha \le -n/2$, this argument yields the inequality

$$\Psi_{f,\alpha}(I) \ge C[\ell(I)]^{2\alpha-n}\int_I\int_I \frac{|f(x)-f(y)|^2}{\delta(x,y)^{n+2\alpha}}\,dx\,dy.$$

The opposite inequality follows by Lemma 5.3 and (7.1). □

Theorem 7.2. *Let $-\infty < \alpha < \infty$. Then $f \in Q^d_\alpha(\mathbb{R}^n)$ if and only if*

$$\sup_{I\in\mathcal{D}} \Psi_{f,\alpha}(I) < \infty.$$

Moreover, $\sup_{I\in\mathcal{D}}[\Psi_{f,\alpha}(I)]^{1/2}$ is a norm on $Q^d_\alpha(\mathbb{R}^n)$, equivalent to $\|f\|_{Q^d_\alpha(\mathbb{R}^n)}$ as defined above.

Proof. Immediate by Lemma 7.1. □

We have also an analogue of Theorem 2.3. Let $BMO^d(\mathbb{R}^n)$ be the dyadic BMO space defined by $\{f \in L^2_{\mathrm{loc}}(\mathbb{R}^n) : \sup_{I\in\mathcal{D}}\Phi_f(I) < \infty\}$.

Theorem 7.3.

(i) *$Q^d_\alpha(\mathbb{R}^n)$ is decreasing in α, i.e. $Q^d_\alpha(\mathbb{R}^n) \supseteq Q^d_\beta(\mathbb{R}^n)$ if $\alpha < \beta$.*

(ii) *If $\alpha > n/2$, then $Q^d_\alpha(\mathbb{R}^n)$ contains only functions that are a.e. constant in each octant, i.e. $Q^d_\alpha(\mathbb{R}^n) = \{0\}$ (as a Banach space).*

(iii) *If $\alpha < 0$, then $Q^d_\alpha(\mathbb{R}^n) = BMO^d(\mathbb{R}^n)$.*

Proof. (i) follows directly by $\delta(x,y) \le \ell(I)$ for $x, y \in I \in \mathcal{D}$.

For (ii), suppose that $f \in Q^d_\alpha(\mathbb{R}^n)$, with $\alpha > n/2$. If J is a dyadic cube, and J_k the dyadic cube containing J with $\ell(J_k) = 2^k \ell(J)$, then $\Psi_{f,\alpha}(J_k) \ge 2^{(2\alpha-n)k}\Phi_f(J)$, and by letting $k \to \infty$ we obtain $\Phi_f(J) = 0$, $J \in \mathcal{D}$.

Finally, (iii) follows by Lemmas 7.1 and 5.2. □

Remark 7.4. The inclusion in (i) is strict unless $n/2 < \alpha < \beta$ or $\alpha < \beta < 0$, as can be seen by the following example.

Example 7.5. For a cube I, let I_- and I_+ denote its left and right halves, separated by a hyperplane $x_1 = a$ through the center of I. Let $\{a_k\}_1^\infty$ be a sequence with $\sum_1^\infty |a_k|^2 < \infty$ and define $f = \sum_{k=1}^\infty \sum_{J \in \mathcal{D}_k(I_0)} a_k(\chi_{J_+} - \chi_{J_-})$, where I_0 is any fixed cube. It is then easily seen (cf. Theorem 7.11 below) that

$$f \in Q^d_\alpha(\mathbb{R}^n) \iff \begin{cases} \sum_1^\infty |a_k|^2 < \infty, & \alpha < 0, \\ \sum_1^\infty k|a_k|^2 < \infty, & \alpha = 0, \\ \sum_1^\infty 2^{2\alpha k}|a_k|^2 < \infty, & 0 < \alpha \le n/2, \\ \text{every } a_k = 0, & n/2 < \alpha. \end{cases}$$

In particular, $Q^d_{n/2}(\mathbb{R}^n) \ne \{0\}$, so there is no cut-off at $\alpha = 1$ as for the non-dyadic spaces (Theorem 2.3).

Remark 7.6. If one instead considers Q^d_α defined on a cube, cf. Remark 1.2, then the cut-off at $\alpha = n/2$ in Theorem 7.3(ii) disappears, and the space is non-trivial for arbitrarily large α.

Relations between $Q_\alpha(\mathbb{R}^n)$ and $Q^d_\alpha(\mathbb{R}^n)$. We first observe that if we consider all cubes I with dyadic edge lengths in Theorem 7.2, instead of just dyadic cubes, we obtain instead $Q_\alpha(\mathbb{R}^n)$.

Lemma 7.7. *Let $-\infty < \alpha < \infty$. Then $Q_\alpha(\mathbb{R}^n)$ equals the space of all functions f on $\mathbb{R}^n$ such that $\sup_I \Psi_{f,\alpha}(I) < \infty$, where I ranges over the set of all cubes in $\mathbb{R}^n$ with dyadic edge lengths. Moreover, $\|f\|_{Q_\alpha(\mathbb{R}^n)} \asymp \sup_I \Psi_{f,\alpha}(I)^{1/2}$.*

Proof. Immediate by Remark 1.1 and Lemmas 5.3 and 5.4. □

Let τ_t denote the translation operator $\tau_t f(x) = f(x-t)$.

Theorem 7.8. *Let $-\infty < \alpha < \infty$. Then $f \in Q_\alpha(\mathbb{R}^n)$ if and only if $\tau_t f \in Q^d_\alpha(\mathbb{R}^n)$ for all $t \in \mathbb{R}^n$ and $\sup_t \|\tau_t f\|_{Q^d_\alpha(\mathbb{R}^n)} < \infty$. Moreover, $\|f\|_{Q_\alpha(\mathbb{R}^n)} \asymp \sup_t \|\tau_t f\|_{Q^d_\alpha(\mathbb{R}^n)}$.*

Proof. Since every cube of dyadic edge length is the translate of a dyadic cube, Lemma 7.7 and Theorem 7.2 show that

$$\begin{aligned} &\|f\|^2_{Q_\alpha(\mathbb{R}^n)} \\ &\asymp \sup_{t \in \mathbb{R}^n} \sup_{I \in \mathcal{D}} \Psi_{f,\alpha}(I - t) \\ &= \sup_{t \in \mathbb{R}^n} \sup_{I \in \mathcal{D}} \Psi_{\tau_t f,\alpha}(I) \\ &\asymp \sup_{t \in \mathbb{R}^n} \|\tau_t f\|^2_{Q^d_\alpha(\mathbb{R}^n)}, \end{aligned}$$

and the result follows. □

Theorem 7.9. *Let* $-\infty < \alpha < 1/2$. *Then* $Q_\alpha(\mathbb{R}^n) = Q_\alpha^d(\mathbb{R}^n) \cap BMO(\mathbb{R}^n)$.

Proof. The inclusion $Q_\alpha(\mathbb{R}^n) \subseteq Q_\alpha^d(\mathbb{R}^n) \cap BMO(\mathbb{R}^n)$ follows directly from the definitions, (7.1) and Theorem 2.3(iii).

Conversely, suppose that $f \in Q_\alpha^d(\mathbb{R}^n) \cap BMO(\mathbb{R}^n)$. Let I be a cube of dyadic edge length. Then there exist a family of 2^n dyadic cubes I_i of the same size as I, such that the union $\bigcup_i I_i$ is a cube J with $\ell(J) = 2\ell(I)$ and $I \subset J$. (J is not necessarily a dyadic cube.) By Lemma 5.7 and (5.5),

$$\begin{aligned}\Psi_{f,\alpha}(I) &\le C\Psi_{f,\alpha}(J)\\ &= C\sum_{i=1}^{2^n}\Psi_{f,\alpha}(I_i) + C\Phi_f(J)\\ &\le C\|f\|^2_{Q_\alpha^d(\mathbb{R}^n)} + C\|f\|^2_{BMO(\mathbb{R}^n)}.\end{aligned}$$

Hence $\Psi_{f,\alpha}(I)$ is bounded uniformly for all cubes I of dyadic edge length, and the result follows by Lemma 7.7. □

Remark 7.10. We do not know whether the condition $\alpha < 1/2$ in Theorem 7.9 is necessary. The theorem fails at least for $1 \le \alpha \le n/2$, since then, cf. Example 7.5, $\chi_{I_+} - \chi_{I_-} \in Q_\alpha^d(\mathbb{R}^n) \cap BMO(\mathbb{R}^n)$ for any cube I, while $Q_\alpha(\mathbb{R}^n) = \{0\}$ by Theorem 2.3.

The Haar system. Let h_λ, $\lambda \in \Lambda$, be the n-dimensional Haar system. Functions in $BMO^d(\mathbb{R}^n)$ can be described by the Haar system, and the characterization coincides with one given above of $BMO(\mathbb{R}^n)$ using a wavelet basis [13, p. 157]. Similarly, for $Q_\alpha^d(\mathbb{R}^n)$ we have

Theorem 7.11. *Let* $\alpha > 0$. *If* $f \in Q_\alpha^d(\mathbb{R}^n)$, *then the sequence of its Haar coefficients* $a(\lambda) = (f, h_\lambda)$, $\lambda \in \Lambda$, *satisfies*

$$\sup_{I\in\mathcal{D}} T_{\boldsymbol{a},\alpha}(I) < \infty. \tag{7.4}$$

Conversely, every sequence $a(\lambda)$ *satisfying* (7.4) *is the sequence of Haar coefficients of a unique* $f \in Q_\alpha^d(\mathbb{R}^n)$.

Proof. If $a(\lambda) = (f, h_\lambda)$ and I is a dyadic cube, then

$$(f - f(I))\chi_I = \sum_{I(\lambda)\subseteq I} a(\lambda)h_\lambda$$

and thus

$$\Phi_f(I) = |I|^{-1}\sum_{I(\lambda)\subseteq I}|a(\lambda)|^2 = T_{\boldsymbol{a},0}(I).$$

It follows by the definition (5.5) and Lemma 6.1 that

$$\Psi_{f,\alpha}(I) \asymp T_{\boldsymbol{a},\alpha}(I),$$

and the result follows by Theorem 7.2. □

Let U be the isometry of $L^2(\mathbb{R}^n)$ given by $U(\psi_\lambda) = h_\lambda$. Then Theorems 6.2 and 7.11 show that the operator U can be extended to an isomorphism between $Q_\alpha(\mathbb{R}^n)$ and $Q_\alpha^d(\mathbb{R}^n)$.

Corollary 7.12. *Let $0 < \alpha < 1$. Then the Banach spaces $Q_\alpha(\mathbb{R}^n)$ and $Q_\alpha^d(\mathbb{R}^n)$ are isomorphic; more precisely, the map $U\big(\sum a(\lambda)\psi_\lambda\big) = \sum a(\lambda)h_\lambda$ is an isomorphism (with the sums interpreted formally or as converging in suitable weak topologies).* □

8. Some problems

In this section, we would like to mention some open problems.

John–Nirenberg type inequality. The John–Nirenberg distribution inequality [10] says that if $f \in BMO(\mathbb{R}^n)$ then for any cube I and any $t > 0$,

$$m_I(t) = |\{x \in I : |f(x) - f(I)| > t\}| \le C|I| \exp(\frac{-ct}{\|f\|_{BMO(\mathbb{R}^n)}}). \tag{8.1}$$

We hope to explore the following

Problem 8.1. *Let $\alpha \in (0,1)$. Give a John–Nirenberg type inequality for $Q_\alpha(\mathbb{R}^n)$.*

With (8.1) and the local behaviour of $Q_\alpha(\mathbb{R}^n)$, we may conjecture that if $f \in Q_\alpha(\mathbb{R}^n)$ then

$$\sum_{k=0}^{\infty} 2^{(2\alpha-n)k} \sum_{J\in\mathcal{D}_k(I)} \frac{m_J(t)}{|J|} \le Ct^{-1} \exp(\frac{-ct}{\|f\|_{Q_\alpha(\mathbb{R}^n)}}). \tag{8.2}$$

Duality. Corresponding to the important result $BMO(\mathbb{R}^n)=[H^1(\mathbb{R}^n)]^*$, a problem of $Q_\alpha(\mathbb{R}^n)$-type is

Problem 8.2. *Let $\alpha \in (0,1)$. Find a predual of $Q_\alpha(\mathbb{R}^n)$.*

Fefferman–Stein type decomposition. Fefferman and Stein [6] proved that $f \in BMO(\mathbb{R}^n)$ if and only if there are $\varphi_j \in L^\infty(\mathbb{R}^n)$ such that

$$f = \varphi_0 + \sum_{j=1}^{n} R_j(\varphi_j),$$

where $R_j\varphi = \varphi * (x_j/|x|^{n+1})$ are the Riesz transforms. So, we naturally pose

Problem 8.3. *Let $\alpha \in (0,1)$. Give a Fefferman–Stein type decomposition of $Q_\alpha(\mathbb{R}^n)$.*

Note that the space $Q_p(\partial\triangle)$ has a decomposition of Fefferman–Stein type [14, Theorem 1.2].

Quasiconformal homeomorphism. Reimann [17] proved that if φ is an ACL-homeomorphism from $\mathbb{R}^n$, $n \ge 2$, to itself, then the composition operator C_φ, defined by $C_\varphi f = f \circ \varphi$, is bounded on $BMO(\mathbb{R}^n)$ if and only if φ is quasiconformal. On the line, it was noted by P. Jones that the quasi-conformal ACL-homeomorphisms φ are those which satisfy the A_∞-condition for $\log\varphi'$ (cf. p.15 in [4]). With Theorem 2.1, we therefore have the following

Problem 8.4. *Let $\alpha \in (0,1)$. Prove or disprove C_φ is bounded on $Q_\alpha(\mathbb{R}^n)$.*

Quasiconformal extension. We can introduce the concept of $Q_\alpha(\Omega)$—a Q space on a general domain $\Omega \subseteq \mathbb{R}^n$, by considering only cubes $I \subseteq \Omega$ in (1.3). Jones's theorem [11] tells us that for a Jordan domain $\Omega \subset R^2$, $BMO(\Omega) = BMO(R^2)|_\Omega$ (the restriction of $BMO(\mathbb{R}^n)$ to Ω) if and only if $\partial\Omega$ (the boundary of Ω) is a quasi-circle. So, our question is

Problem 8.5. *Let $\alpha \in (0,1)$. Find a geometric property of $\partial\Omega$ such that $Q_\alpha(\Omega) = Q_\alpha(R^2)|_\Omega$.*

Further problems can be posed via [4].

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